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Computing Sunspot Solutions to Rational Expectations Models with Timing Restrictions

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Abstract
Rational expectations (RE) frameworks featuring informational constraints are becoming increasingly popular in macroeconomic research. A recent strand of literature has explored the analytics of RE models with informational subperiods, in which the occurrence of exogenous shocks is period-specific and decision makers thus condition their own choices and expectations upon a sequence of nested information sets (timing restrictions). Assuming the unrestricted (full information) RE model satisfies saddle-path stability, this paper provides (i) necessary and sufficient conditions for existence of an uncountably infinite set of linearly perturbed solutions to its restricted (informationally constrained) counterpart, and (ii) an algorithm for computing the full set of (sunspot) solutions when equilibrium indeterminacy occurs.

Keywords: Rational expectations; Timing restrictions; Perturbation theory

JEL Classification: C62; C63

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1 Introduction

Several contributions to the literature on rational expectations (RE henceforth) models have analyzed the consequences of departing from standard assumptions about the conventional, full information structure. Under imperfect or partial information, economic agents may lack up-to-date knowledge about current endogenous and/or exogenous variables, and typically employ filtering techniques to extract information from current observables when rationally forming expectations (e.g. Pearlman et al., 1986; Carravetta and Sorge, 2010; Baxter et al., 2011; Shibayama, 2011). Notably, recent research in the field has emphasized on the ability of informational frictions in otherwise standard RE frameworks to account for a number of empirical facts, which are left unexplained by the standard theory of full information RE (e.g. Piazzesi and Schneider, 2008; Bacchetta et al., 2009).

As an alternative to King and Watson (2002)'s solution algorithm for linearized timing models under RE, Kormilitsina (2013) develops a perturbation-based approach able to handle a general class of discrete-time RE models with informational subperiods. The framework of analysis is one in which, upon facing exogenous shocks that fulfill a well-defined set of timing restrictions, model-consistent expectations — hence, equilibrium decision rules — are conditioned upon an increasing sequence of nested information sets. More specifically, timing restrictions therefore specify an information structure under which different variables can be chosen based on different observables, i.e. they allow to relate different variables to different information sets.[1] Kormilitsina (2013)'s algorithm relies on constructing a set of linear maps from the solution to the standard, unrestricted RE model to its informationally constrained counterpart. As claimed in Kormilitsina (2013), a useful implication of this linear relationship across solution sets is that the former inherits the property of equilibrium (in)determinacy — that is, of local (non)uniqueness of

[1]This peculiar feature of RE models with timing restrictions makes them different from sticky information environments (e.g. Mankiw and Reis, 2002; Meyer-Gohde, 2010), which rather relate equations to available information, as well as from the partial information setting of Pearlman et al. (1986), in which all the expectation operators are conditioned on a limited information set.

the linearly perturbed solution around the non-stochastic steady state — from the latter.

Hespeler and Sorge (2018) challenge this conclusion by providing a simple counter-example, namely a standard saddle-path stable Fisher-type inflation model, which fails to admit any dynamically stable equilibrium path when the monetary policy authority is informationally constrained. While thought-provoking, Hespeler and Sorge (2018)'s piece appears to suggest that informational constraints ought to be designed consistent with the timing of endogenous choices and expectation formation, for the underlying economy not to suffer from a non-existence pathology — see Kormilitsina (2018)'s reaction to Hespeler and Sorge (2018). Moreover, it offers little insight into whether, and under what conditions, informational constraints may serve as an independent source of equilibrium indeterminacy, even when the model's parametrization would support a unique equilibrium under full information. In fact, the occurrence of timing restrictions induces an enlarged state space as well as an increased degree of backward dependence in policy functions; thus, differently from Shibayama (2011), the dynamic matrices of the state-space representation of the first-order approximate solution are altered with respect to full information. Shedding light on this issue appears to be of particular relevance, for welfare and policy analysis in informationally constrained RE models is typically affected by the underlying equilibrium regime (e.g. Lubik et al., 2018).

The present paper fills this gap by providing (i) necessary and sufficient conditions for existence and multiplicity of linearly perturbed solutions to a general, multivariate RE model with two-subperiod informational constraints, and (ii) an algorithm for computing all (sunspot) solutions in the presence of equilibrium indeterminacy, which only requires knowledge of the (unique) solution to the unrestricted version of the model. Inspection of such conditions reveals that the solution invariance property advocated in Kormilitsina (2013) holds generically in well-designed informationally constrained RE models, for its failure requires that a real square matrix be rank deficient. This implication is consistent with Shibayama (2011)'s analytical findings, according to which informational constraints modifies the quantitative (dynamic) properties of a given model, without impinging on its qualitative nature.

The paper proceeds as follows. Section 2 briefly reviews the basic framework of analysis. In section 3, the properties of the model's linearly perturbed solution in the two-subperiod setting is studied, and our main propositions delivered. Section 4 concludes.

2 Framework of analysis

Following the canonical approach of Schmitt-Grohé and Uribe (2004), let a standard RE model be described by a dynamic system of n_F expectational equations in n_F variables

$$E\left[f\left(Y', X', Y, X; \sigma\right)\right] = 0 \tag{1}$$

where all zero-mean (square-integrable) random variables (Y', X', Y, X) are defined on a properly filtered probability space, the prime superscript denotes one-step ahead variables, and $E[\cdot]$ is the standard (conditional) expectation operator associated with the underlying probability measure. The n_Y-dimensional vector Y collects the model's endogenous jump variables, whereas the n_X-dimensional vector X contains endogenous predetermined variables as well as exogenous states. Finally, the scalar $\sigma \geq 0$ is assumed to scale the size of aggregate uncertainty surrounding the economy.

2.1 Full information

In the standard unrestricted case, policy functions of all endogenous variables depend on all the state variables X. Time-invariant, analytic solutions to (1) are in the form

$$Y = \bar{G}(X, \sigma), \quad X' = \bar{H}(X, \sigma) + \sigma\epsilon' \tag{2}$$

where the elements of the n_X-dimensional vector ϵ are i.i.d. zero-mean, unit variance innovations (e.g. structural shocks).

As shown in Schmitt-Grohé and Uribe (2004), up to first order certainty equivalence holds generically, and therefore σ does not enter the linearly perturbed model's dynamics for endogenous variables, i.e. one has

$$Y = \bar{G}_X X, \quad X' = \bar{H}_X X + \sigma \epsilon' \tag{3}$$

where $\bar{G}_X$ and $\bar{H}_X$ are conformable matrices of first-order derivatives of the maps $\bar{G}(X, \sigma)$ and $\bar{H}(X, \sigma)$ with respect to X, evaluated at the non-stochastic steady state $(\bar{Y}, \bar{X})$ solving (1) when $\sigma = 0$.[2]

2.2 Informational constraints

As mentioned above, Kormilitsina (2013) examines models in which timing restrictions on information affect agent's decisions and/or expectations as based on observable shocks, which may well fail to hit simultaneously. A simple modeling strategy to capture these informational constraints is to allow for several informational subperiods, each starting with realizations of some shocks, with choices/expectations for endogenous variables made in different subperiods conditional on different (nested) information sets.

In the simple case with two informational subperiods only, the control and state vectors are partitioned as follows

$$Y = [y;\, z]\,; \quad X = [x;\, \theta] \tag{4}$$

where the n_x-dimensional vector x consists of endogenous predetermined as well as exogenous variables which materialize in the beginning of the first subperiod, θ contains n_θ exogenous variables with realizations in the second subperiod, y is the n_y-dimensional vector of fully endogenous jump variables, i.e. endogenous variables which are conditioned on all the state variables X. Finally, the n_z-dimensional vector z collects partially endogenous variables, which are decided upon in the first subperiod, when realizations of only a subset of state variables are known.

Kormilitsina (2013)'s solution approach requires that the RE system (1)

[2] Perturbation-based approaches (e.g. Schmitt-Grohé and Uribe, 2004) conventionally assume that there exists an arbitrarily small neighborhood of $(\bar{Y}, \bar{X})$ for which the maps $(\bar{G}, \bar{H})$ are unique and over which $(f, \bar{G}, \bar{H})$ are smooth.

be partitioned as follows

$$f = \left[f^0;\ f^1;\ f^\theta\right] \tag{5}$$

so that the sub-system f^0 includes n_z equations determining endogenous variables z, the sub-system f^1 includes n_y equations that determine endogenous variables y and n_x equations determining the dynamics of the states x, and the sub-system f^θ describes the evolution of exogenous shocks θ, represented as a first-order stationary autoregressive process

$$\theta' = P\theta + \sigma\epsilon'_\theta, \quad \epsilon_\theta \sim i.i.d.N(0, V_{\epsilon_\theta}) \tag{6}$$

where P is a stable square matrix of autoregressive coefficients, and ϵ'_θ collects the n_θ shocks associated with the states θ.

Letting $\mathcal{E}$ denote the (conditional) expectation operator accounting for timing restrictions, the RE system with informational subperiods can be rewritten as

$$\mathcal{E}\left[f\left(Y', X', Y, X\right)\right] = 0 \tag{7}$$

and its recursive solution represented in general form as

$$y = g(x, \theta, \theta_{-1}, \sigma), \quad z = j(x, \theta_{-1}, \sigma), \quad x' = h(x, \theta, \theta_{-1}, \sigma) + \sigma\epsilon'_x \tag{8}$$

Endogenous (jump) variables in z only react to the conditional forecast of states in θ (a function of previous period variables θ_{-1}), as the latter do not belong in the first subperiod information set. Notice the solution to the trivial filtering problem associated with the autoregressive process (6) is already embedded in the h function. By the same token, endogenous (jump) variables in y are a function of z — a state variable in the second informational subperiod — and thus of lagged states θ_{-1}. Notice that the timing restrictions only involve exogenous variables θ which are uncorrelated with other exogenous variables in x; also, all the θ variables are not observed in the first subperiod, hence the filtering problem does not require using the variance-covariance matrix of the ϵ_θ shocks in order to compute an optimal (in the mean-square sense) estimate of unobserved states.

Following Kormilitsina (2013), the first-order approximation to (8) is

$$y = g_x x + g_\theta \theta + g_{\theta_{-1}} \theta_{-1} + g_\sigma \sigma,$$

$$z = j_x x + j_{\theta_{-1}} \theta_{-1} + j_\sigma \sigma, \tag{9}$$

$$x' = h_x x + h_\theta \theta + h_{\theta_{-1}} \theta_{-1} + h_\sigma \sigma + \sigma \epsilon'_x$$

where matrices $\left(g_x, g_\theta, g_{\theta_{-1}}, g_\sigma, j_x, j_{\theta_{-1}}, j_\sigma, h_x, h_\theta, h_{\theta_{-1}}, h_\sigma\right)$ are to be determined. In the following, we investigate conditions under which multiple (infinitely many) choices for these matrices exist, which do not depend on the model's structural parameters held by the f equations, and yet are indexed by non-structural (sunspot) parameters (*parametric indeterminacy*, e.g. Broze and Szafarz, 1991).

By showing that $(g_\sigma, j_\sigma, h_\sigma)$ are zero vectors, Kormilitsina (2013) recasts the linearly perturbed solution in compact form as follows

$$Y = G_X \begin{pmatrix} x \\ \theta \\ \theta_{-1} \end{pmatrix}, \quad X' = H_X \begin{pmatrix} x \\ \theta \\ \theta_{-1} \end{pmatrix} + \sigma \epsilon' \tag{10}$$

where

$$G_X = \begin{pmatrix} g_x & g_\theta & g_{\theta_{-1}} \\ j_x & 0_{n_z \times n_\theta} & j_{\theta_{-1}} \end{pmatrix}, \quad H_X = \begin{pmatrix} h_x & h_\theta & h_{\theta_{-1}} \\ 0_{n_\theta \times n_\theta} & P & 0_{n_\theta \times n_\theta} \end{pmatrix} \tag{11}$$

2.3 Example economies

Nearly all non-linear RE models can be arranged according to partition (5), in order to explore the impact of timing restrictions on shock observability. Shibayama (2011) presents several versions of an otherwise standard real business cycle (RBC) model, each encompassing specific assumptions about the timing of realization of innovations (technology shocks) and of agents' optimal decisions. Kormilitsina (2013) illustrates the properties of her approximation algorithm by means of a New Keynesian business cycle framework and a factor hoarding model, both subjected to informational constraints. Hespeler and Sorge (2018) study a simple Fisherian model of inflation determination with asymmetric (though nested) information sets between private agents and the monetary policy authority.

3 Computing sunspot solutions under timing restrictions

Kormilitsina (2013) shows that the first-order approximate solution (9) to the restricted model can be readily computed via linear transformations of the solution(s) (3) to its unrestricted counterpart. Several methods exist that can be used to derive the latter (e.g. Klein, 2000; Christiano, 2002; King and Watson, 2002; Sims, 2002). As a main implication, the two versions of the RE framework are claimed to share the same equilibrium (non)uniqueness properties. Hespeler and Sorge (2018) rather show that this property need not hold generically, and that its failure may result in a non-existence pathology.

Building on these two contributions, the following proposition characterizes existence of multiple (indeterminate) linearly perturbed solutions to the restricted RE model (7), on the assumption that the unrestricted model (1) admits a locally unique equilibrium path around the non-stochastic steady state.

Proposition 1. *Let* $\Delta(f^1)_{[x',y]} = \left[f^1_{Y'}G_x + f^1_{x'},\ f^1_y\right]$ *denote the Jacobian of the sub-system* f^1 *with respect to the vector* $[x', y]$*. Assume the full information RE equilibrium (3) is determinate. Then the restricted model (7) admits an uncountably infinite set of solutions if and only if*

$$rank\left[\Delta(f^1)_{[x',y]}\right] = rank\left[\Delta(f^1)_{[x',y]}\middle| f^1_z\bar{j}_\theta\right] < n_y + n_x \tag{12}$$

Proof. - See the Appendix. □

Proposition 1 establishes that equilibrium indeterminacy in RE models with timing restrictions requires that the square Jacobian of the f^1 equations pinning down the fully endogenous variables be rank deficient. This can occur when such equations fail to include the current values — other than expected ones — of these non-predetermined variables. In Hespeler and Sorge (2018)'s counter-example, informational constraints in fact prevent the Fisher equation from pinning down the current rate of inflation, when the monetary policy authority lacks the ability of observing the current real interest rate.[3] More generally, the following holds

Proposition 2. *Let $[f_y^1]_{*,i}$ denote the i^{th} column of the derivative matrix f_y^1. Assume the full information RE equilibrium (3) is determinate. Then the restricted model (7) admits a determinate solution only if there exists no $i \in [1, n_y]$ such that $[f_y^1]_{*,i} = 0$.*

Proof. - See the Appendix. □

By construction, in the simple economy studied by Hespeler and Sorge (2018), failure of solution invariance across information structures — full versus constrained — holds generically in the set of the model's structural parameters. The characterization (12) from Proposition 1 however reveals that in more involved settings, singularity of the Jacobian $\Delta(f^1)_{[x',y]}$ is more likely to result in equilibrium non-existence rather than indeterminacy, as conjectured in Kormilitsina (2018).

Based on Proposition 1, the following algorithm — which nests Kormilitsina (2013)'s as a special case — can be exploited to compute the full set of solutions to RE models with two-subperiod informational constraints:

Step 1. Sort the equilibrium conditions into vectors f^0, f^1 and f^θ, and arrange them into vector f.

Step 2. Arrange variables in Y and X according to partition (4);

Step 3. Obtain matrices $\bar{G}_X$ and $\bar{H}_X$ for the unrestricted RE model, and partition them accordingly;

[3] See the Appendix for an alternative computational perspective on this solution non-existence result.

Step 4. Assign $G_x = \bar{G}_x$, $h_x = \bar{h}_x$, $h_\sigma = 0$ and $G_\sigma = 0$;

Step 5a. If $\Delta(f^1)_{[x',y]}$ is non-singular, calculate $(h_\theta, g_\theta, h_{\theta_{-1}}, g_{\theta_{-1}}, j_{\theta_{-1}})$ from

$$\begin{pmatrix} h_{\theta_{-1}} \\ g_{\theta_{-1}} \end{pmatrix} = -\Delta(f^1)^{-1}_{[x',y]} f^1_z \bar{j}_\theta P \tag{13}$$

$$\begin{pmatrix} h_\theta \\ g_\theta \end{pmatrix} = \begin{pmatrix} \bar{h}_\theta \\ \bar{g}_\theta \end{pmatrix} + \Delta(f^1)^{-1}_{[x',y]} f^1_z \bar{j}_\theta \tag{14}$$

$$j_{\theta_{-1}} = \bar{j}_\theta P \tag{15}$$

Step 5b. If $\Delta(f^1)_{[x',y]}$ is singular, compute its Moore-Penrose pseudo-inverse matrix $\Delta(f^1)^{+}_{[x',y]}$ and check $\left(\Delta(f^1)_{[x',y]}\ \Delta(f^1)^{+}_{[x',y]}\right) f^1_z \bar{j}_\theta = f^1_z \bar{j}_\theta$. If this condition is violated, then the restricted model admits no dynamically stable solution. If fulfilled, calculate (h_θ, g_θ) from

$$\begin{pmatrix} h_\theta \\ g_\theta \end{pmatrix} = \begin{pmatrix} \bar{h}_\theta \\ \bar{g}_\theta \end{pmatrix} + \Delta(f^1)^{+}_{[x',y]} f^1_z \bar{j}_\theta + \left(I - \Delta(f^1)^{+}_{[x',y]} \Delta(f^1)_{[x',y]}\right) \omega \tag{16}$$

where ω is an $(n_y + n_x) \times n_\theta$ matrix of arbitrary (sunspot) parameters; then compute $(h_{\theta_{-1}}, g_{\theta_{-1}}, j_{\theta_{-1}})$ from

$$h_{\theta_{-1}} = (\bar{h}_\theta - h_\theta)P; \quad G_{\theta_{-1}} = \left(\bar{G}_\theta - G_\theta\right) P \tag{17}$$

4 Concluding remarks

This paper clarifies the conditions under which RE models with timing restrictions admit an indeterminate linearly perturbed solution, even when its unrestricted (full information) counterpart exhibits saddle-path stability. Equilibrium indeterminacy is shown to require that a well-defined square matrix be rank deficient, which in turn may induce non-existence issues; thus, the solution invariance property hinted at by Kormilitsina (2013) is generically supported. While fully consistent with previous literature dealing with linear RE models under time-specific imperfect information (e.g. Shibayama, 2011), this result has sharp implications for the modeling of informational constraints in RE frameworks, for it suggests that timing restrictions are bound to produce fairly different effects on the model's qualitative properties from those induced by limited information settings which involve noisy state observability and thus require solving a dynamic filtering problem, which may inject spurious indeterminacy in the underlying economy (e.g. Lubik et al., 2018).

Appendix

Proof of Proposition 1

Following Kormilitsina (2013), let the matrices defining the solution under full information be represented as follows

$$\bar{G}_X = \begin{pmatrix} \bar{G}_x & \begin{matrix} \bar{g}_\theta \\ \\ \bar{j}_\theta \end{matrix} \end{pmatrix}, \quad \bar{H}_X = \begin{pmatrix} \bar{h}_x & \bar{h}_\theta \\ 0 & P \end{pmatrix} \tag{18}$$

The entires of such matrices are pinned down by the following set of partial derivatives, once evaluated at the non-stochastic steady state:

$$E\bar{f}_x = f_{Y'}\bar{G}_x\bar{h}_x + f_{x'}\bar{h}_x + f_Y\bar{G}_x + f_x = 0 \tag{19}$$

$$E\bar{f}_\sigma = f_{Y'}\left[\bar{G}_X\bar{H}_\sigma + \bar{G}_\sigma\right] + f_{X'}\bar{H}_\sigma + f_Y\bar{G}_\sigma = 0, \tag{20}$$

$$E\bar{f}_\theta = f_{Y'}\left[\bar{G}_x\bar{h}_\theta + \bar{G}_\theta P\right] + f_{x'}\bar{h}_\theta + f_Y\bar{G}_\theta + f_{\theta'}P + f_\theta = 0 \tag{21}$$

The solution matrices (H_X, G_X) are by contrast pinned down by the following set of partial derivatives, once evaluated at the non-stochastic steady state:

$$\mathcal{E}\tilde{f}_x = 0, \tag{22}$$

$$\mathcal{E}\tilde{f}_\sigma = 0, \tag{23}$$

$$\mathcal{E}\left(\tilde{f}^0_\theta\right)P + \mathcal{E}\tilde{f}^0_{\theta_{-1}} = 0, \tag{24}$$

$$\mathcal{E}\tilde{f}^1_{\theta_{-1}} = 0, \tag{25}$$

$$\mathcal{E}\tilde{f}^1_\theta = 0 \tag{26}$$

For any P matrix (whether singular or not), any solution to system (22)-(26) must also be a solution to the system[4]

$$\mathcal{E}\tilde{f}_x = 0, \tag{27}$$

$$\mathcal{E}\tilde{f}_\sigma = 0, \tag{28}$$

$$\mathcal{E}\left(\tilde{f}_\theta\right)P + \mathcal{E}\tilde{f}_{\theta_{-1}} = 0, \tag{29}$$

$$\mathcal{E}\tilde{f}^1_{\theta_{-1}} = 0, \tag{30}$$

whereas any solution to (21) must also satisfy the (possibly non-invertible) linear transformation

$$E\bar{f}_\theta P = 0 \tag{31}$$

We next show that, provided $\Delta(f^1)_{[x',y]}$ is non-singular, the restricted model

[4]Equation (32) in Kormilitsina (2013, p. 547) misses post-multiplication of the term $G_\theta P + G_{\theta_{-1}}$ by P. Notice that P is assumed to be non-singular and stable, and thus cannot be idempotent. None of the results reported in Kormilitsina (2013) are however affected by this typo.

is saddle-path stable if and only if its unrestricted counterpart is.

Sufficiency. Assume the unrestricted model is saddle-path stable, i.e. a generically unique stable solution for $(\bar{h}_\theta, \bar{G}_\theta)$ exists. Consider first equations (27)-(28) versus equations (19)-(20). Given the block partitions $H_\sigma = [h_\sigma; 0]$ and $\bar{H}_\sigma = \left[\bar{h}_\sigma; 0\right]$, the terms $G_X H_\sigma$ and $\bar{G}_X \bar{H}_\sigma$ reduce to $G_x h_\sigma$ and $\bar{G}_x \bar{h}_\sigma$ respectively. Pair-wise comparison of same coefficients equations (22) versus (19) and (23) versus (20) reveals that for any solution $(\bar{h}_x, \bar{G}_x, \bar{h}_\sigma, \bar{G}_\sigma)$ to the unrestricted model, there exists a solution $(h_x, G_x, h_\sigma, G_\sigma)$ to the restricted model such that $G_x = \bar{G}_x$, $h_x = \bar{h}_x$ $h_\sigma = \bar{h}_\sigma$ and $G_\sigma = \bar{G}_\sigma$.

Notice that, given $G_x = \bar{G}_x$, a pair $(\widehat{h}_\theta, \widehat{G}_\theta)$ solves (29) if and only if there exist matrices $(h_\theta, G_\theta, h_{\theta_{-1}}, G_{\theta_{-1}})$ such that

$$\begin{aligned} h_\theta P + h_{\theta_{-1}} &= \widehat{h}_\theta P \\ G_\theta P + G_{\theta_{-1}} &= \widehat{G}_\theta P \end{aligned} \tag{32}$$

solve (31). As this set of linear restrictions by construction hold for $(\bar{h}_\theta, \bar{G}_\theta)$, for the latter to be part of the first-order approximate equilibrium under timing restrictions it must also fulfill (25) and (26). Since the full set of linear equations linking $(\bar{h}_\theta, \bar{G}_\theta)$ to (h_θ, G_θ) is in the form

$$\left[f^1_{Y'} G_x + f^1_{x'};\ f^1_{y'};\ f^1_y\right] \begin{pmatrix} h_\theta - \bar{h}_\theta \\ (g_\theta - \bar{g}_\theta)P + g_{\theta_{-1}} \\ g_\theta - \bar{g}_\theta \end{pmatrix} + f^1_{z'}(j_{\theta_{-1}} - \bar{j}_\theta P) - f^1_z \bar{j}_\theta = 0 \tag{33}$$

the linear restrictions (32) imply that (33) is equivalent to

$$\begin{aligned} \Delta(f^1)_{[x',y]} \begin{pmatrix} h_\theta \\ g_\theta \end{pmatrix} &= \Delta(f^1)_{[x',y]} \begin{pmatrix} \bar{h}_\theta \\ \bar{g}_\theta \end{pmatrix} + f^1_z \bar{j}_\theta; \\ G_\theta P + G_{\theta_{-1}} &= \bar{G}_\theta P \end{aligned} \tag{34}$$

Notice that non-singularity of the Jacobian $\Delta(f^1)_{[x',y]}$ requires that no column of f_y^1 features zero entries only. Assume this condition is fulfilled. Then, since equations (25) require

$$\Delta(f^1)_{[x',y]} \begin{pmatrix} h_{\theta_{-1}} \\ g_{\theta_{-1}} \end{pmatrix} = -f_z^1 j_{\theta_{-1}} \tag{35}$$

the latter are implied by (34) provided the linear restrictions (32) hold true. By uniqueness of $(\bar{h}_\theta, \bar{G}_\theta)$, system (34) and (35) uniquely determine the entries of $(h_\theta, g_\theta, G_{\theta_{-1}}, h_{\theta_{-1}})$. By contrast, when $\Delta(f^1)_{[x',y]}$ is singular — which occurs e.g. when there exists $i \in [1, n_y]$ such that $[f_y^1]_{*,i} = 0$ (Hespeler and Sorge, 2018) — then (34) delivers, if consistent, an uncountable infinity of solutions; consistency obtains if and only if the rank of $\Delta(f^1)_{[x',y]}$ coincides with the rank of the augmented matrix $\left[\Delta(f^1)_{[x',y]} \middle| f_z^1 \bar{j}_\theta\right]$, or equivalently if and only if

$$\Delta(f^1)_{[x',y]} \Delta(f^1)^+_{[x',y]} f_z^1 \bar{j}_\theta = f_z^1 \bar{j}_\theta$$

from which step 5b) of the solution algorithm follows.

Necessity. Assume now there exist unique matrices $(h_\theta, G_\theta, h_{\theta_{-1}}, G_{\theta_{-1}})$ solving (29)-(30). When P is non-singular, then the unique solution to (31) is $(\widehat{h}_\theta, \widehat{G}_\theta) = (\bar{h}_\theta, \bar{G}_\theta)$, and the assertion follows by inversion of restrictions (32). When, by contrast, P is singular, any solution to (31) is in the form

$$\begin{pmatrix} \widehat{h}_\theta \\ \widehat{G}_\theta \end{pmatrix} = \begin{pmatrix} \bar{h}_\theta \\ \bar{G}_\theta \end{pmatrix} + \Upsilon\Psi \tag{36}$$

where $\Upsilon := \left[f_{Y'}\bar{G}_x + f_{x'}; f_{Y'}; f_Y\right]^+$ and Ψ is a non-empty $(n_y+n_x) \times n_\theta$ matrix whose rows are right null vectors of P, i.e. $\Psi P = 0$.[5] Since (32) involve post-multiplication by P, for any choice of Ψ the solution $(\bar{h}_\theta, \bar{G}_\theta)$ to (21) is uniquely determined by $(h_\theta, G_\theta, h_{\theta_{-1}}, G_{\theta_{-1}})$ via the linear restrictions (32),

[5] The null-space of P can be determined via either the SVD or the QR decomposition, both of which are routinely implemented in matrix-oriented software packages such as GAUSS and MATLAB.

provided there exists no null column of f^1_y.

To inspect the stability properties of the restricted model, let $\Delta(f^1)_{[x',y]}$ be non-singular. Equilibrium state dynamics under timing restrictions are in the form

$$\begin{pmatrix} X' \\ \theta' \\ \theta \end{pmatrix} = \begin{pmatrix} \bar{h}_x & \bar{h}_\theta + (\Pi)_{n_x} & (\Pi)_{n_x} \\ 0 & P & 0 \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} X \\ \theta \\ \theta_{-1} \end{pmatrix} + \sigma \begin{pmatrix} \epsilon_x \\ \epsilon_\theta \\ 0 \end{pmatrix} \tag{37}$$

where $(\Pi)_{n_x}$ are the first n_x rows of the matrix $\Pi := \Delta(f^1)^{-1}_{[x',y]} f^1_z \bar{j}_\theta$. Since all the non-zero eigenvalues of the companion matrix in (37) are those of $\bar{h}_x$ and P, the assertion follows. An analogous argument easily shows that all the sunspot solutions in (16) are dynamically stable.

Solution non-existence in restricted Fisher inflation model

Hespeler and Sorge (2018) consider the following full information RE model of inflation determination

$$\begin{aligned} r_t &= i_t - \pi_{t+1} \\ i_t &= \alpha\pi_t; \qquad \alpha > 1 \\ r_t &= \rho r_{t-1} + \epsilon_t; \qquad |\rho| < 1 \\ \epsilon_t &\sim i.i.d.N(0,1) \end{aligned} \tag{38}$$

where i_t is the nominal interest rate, π_t is inflation and the real rate of interest r_t is exogenous and assumed to follow a covariance stationary AR(1) process. Notice that the Taylor principle is assumed to hold, and thus (38) admits a determinate equilibrium (Lubik and Schorfheide, 2003).

Hespeler and Sorge (2018) explore an informationally constrained version of (38) in which the nominal interest rate is chosen when only observing the real interest rate with a lag, i.e. on the basis of $\{r_{t-i}\}_{i\geq 1}$. By contrast, inflation expectations held by private agents rely on perfect observability of the current period state r_t. Let $\zeta_t := E_{t-1}[\pi_t]$, then the RE system under timing restrictions can be written as

$$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} E_t \begin{pmatrix} \zeta_{t+1} \\ \pi_{t+1} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} r_t \tag{39}$$

or in compact form

$$AE_t[s_{t+1}] = Bs_t + Cr_t \tag{40}$$

where $s_t = [\zeta_t,\, \pi_t]'$. The generalized Schur (QZ) decomposition of A and B into $QAZ = S$ and $QBZ = T$ (S and T upper-triangular, Q and Z unitary) then yields

$$Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{41}$$

and thus $Z_{11} = 0$, i.e.the top left block associated with the backward-looking variable ζ_t is zero valued. This violates the rank condition as required by Klein (2000)'s assumption 4.5.

Proof of Proposition 2

Assume there exists $i \in [1, n_y]$ such that $[f_y^1]_{*,i} = 0$. Then the Jacobian $\Delta(f^1)_{[x',y]}$ displays a null column and is thus singular, inducing (by force of Proposition 1) either non-existence or equilibrium indeterminacy.

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