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Abstract

The *Shafer and Sonnenshein convexity* of preferences is a key property in game theory. Previous research has shown that, in case of decisions under uncertainty, the compliance with this property (jointly) depends on the concavity/convexity of the imprecise probabi- listic model with respect to the decision variable and on the attitudes towards imprecision of the decision maker. The present paper deepens the analysis by looking at set-valued imprecise probabilistic models that encompass sets of probability distributions and sets of almost desirable gambles. Moreover, it is shown that the required Shafer and Sonnenshein convexity property is obtained also in case the *imprecise probability correspondences* satisfy quasi-concavity/convexity with respect to the decision variable so that the set of admissible probabilistic models is significantly broadened. It is well known that sets of probability distributions and sets of almost desirable gambles are general models of representation of uncertainty that are connected to each other; moreover, they are both related to another model known as *lower expectation*. Therefore, the second part of this work explores the links between the (quasi-)concavity/convexity properties accross the three different models so as to understand to what extent the Shafer and Sonnenshein convexity results hold.

Keywords: Convex preferences, Imprecise probabilities, quasi-concavity/convexity, set-valued maps

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1 Introduction

The property of convexity of preferences has always been a key assumption in decision and game theory; not only it has a clear behavioral interpretation, but, above all, it plays a crucial role in the existence of equilibria in general economic equilibrium models and in noncooperative games. The seminal paper by Shafer and Sonnenshein ([25]) shows that existence of economic equilibria and Nash equilibria can be obtained in case each agent's preference satisfies a minimal convexity assumption (with respect to his own action/strategy), sometimes called *Shafer and Sonnenshein convexity*, which, roughly speaking, requires that each alternative x cannot be the convex combination of other two alternatives that are strictly preferred to x. The present paper looks at decisions and strategic form games under imprecise probabilities and focuses on the conditions on the representation of uncertainty and on the attitudes towards uncertainty that are required so as to guarantee that the Shafer and Sonnenshein convexity assumption holds; in particular, the results in this paper identify (imprecise) probabilistic models and preferences for imprecision that have a clear and simple interpretation. The present paper considers and extends the model presented in [8] in which each agent is endowed with an *ambiguous belief correspondence* that maps the set of mixed strategy profiles to the set of all subsets of probability distributions over the outcomes of the game: given a strategy profile chosen by the set of players, the belief correspondence of player i provides the set of probability distributions that player i perceives to be feasible and consistent with the strategy profile. This approach follows a strand of literature in decision theory in which alternatives are compared by looking at the the set of lotteries over consequences that they induce, (see [1], [21], [23], [26], [27], [6] and references therein), and turns to be particularly useful to study ambiguity in games as it encompasses classical models in which agents have *multiple priors* on a state space (see [16], [4], [3] for incomplete information under ambiguity, just to quote a few), as well as the so called models of *strategic ambiguity* (see, for instance, [11], [19], [17], [12], [20], [18], [22] and [5] and references therein). In [8], games are studied without requiring completeness and transitivity of preferences[1], so as to understand what is truly needed for the existence of equilibria, regardless of the assumptions that must be imposed in the representation theorems for preferences. It is shown that the required Shafer and Sonnenshein convexity property is obtained in case agents have a minimally pessimistic attitude towards imprecision (therein called *imprecision aversion*) when it is combined with the property of *convexity* of their ambiguous belief correspondences (with respect their own strategies); similarly, it is also obtained when agents have a (specular) minimally optimistic attitude (*imprecision loving*) when it is combined with the *concavity* property of their belief correspondences. The imprecision aversion (resp. loving) assumption says that an agent would not prefer a set of probabilities to another one if the former was a subset of the latter (resp. if the latter was a subset of the former)[2]. It turns out that these properties are minimal as every possible attitude (of a decision maker), towards the inclusion relation between sets of probability distributions, implies either

[1]Indeed, the game model under ambiguous belief correspondences has been firstly introduced and studied in case of complete and transitive preferences in [9] and [10].

[2]These behavioral traits can be reconducted to the idea of aversion (resp. inclination) towards imprecision presented in [13] and [14]. In those papers, such behevioral traits are used to axiomatize maxmin-like preferences, in decision models with complete and transitive preferences on uncertain acts under partial information about possible probability distributions.

imprecision aversion or imprecision loving[3]. Convexity (respectively concavity) of the ambiguous belief correspondences is a generalization to set-valued maps of the classical definition of linearity for classical (single-valued) valued maps as it requires that the ambiguous belief associated to the convex combination of two alternatives *is contained in* (respectively *contains*) the convex combination of the ambiguous belief associated to the two alternatives.

The present paper extends [8] in two features. Firstly, it takes into account a more general framework in which the information available to each player is represented by an *imprecise probability correspondence* that maps mixed strategy profiles to (set-valued) imprecise probabilities over the outcomes of the game. This approach encompasses the ambiguous belief correspondence model as well as the *almost desirable gambles correspondence* model. An almost desirable gambles correspondence maps each alternative to an induced set of almost desirable gambles, where a gamble Y is said to be *almost desirable* for the decision maker if he accepts all gambles $Y + \varepsilon$ with $\varepsilon > 0$ (see for instance [28], [29]). In this more general context, the implication that characterize imprecision aversion (resp. imprecision loving) will be called *superset aversion* (resp. *subset aversion*). In fact, while it clear that a set of probabilities is more imprecise than its subsets, the information becomes less imprecise as the set of almost desirable gambles enlarges to an half space. The second, (and more relevant), new feature that is investigated in this paper concerns the assumptions that are imposed on the *imprecise probability correspondences*. It is shown in this paper that they are now allowed to satisfy one of the assumption of *quasi-concavity/convexity* (see [24] for a survey) in place of concavity/convexity. This result has different advantages: on the one hand, it enlarges the family of probabilistic models that can be taken into account; on the other hand, it allows to skip the assumption of *semi-strict convexity* on the attitudes towards imprecison that, instead, is required in case of concave/convex imprecise probability correspondences.

It is well known from the theory of imprecise probabilities that sets of probability distributions and sets of almost desirable gambles are related concepts as a set of probability distributions implictly defines, in a natural way, a set of almost desirable gambles and viceversa. Then, it is relevant to understand to what extent an ambiguous belief (resp. almost desirable gambles) correspondence derived from an almost desirable gambles (resp. ambiguous belief) correspondence inherits the (quasi-)concavity/convexity properties of the latter one. This issue is addressed in this paper; the results show that the (quasi-)concavity (resp. (quasi-)convexity) of the former is related to the (quasi-)convexity (resp. (quasi-)concavity) of the latter. However, counterexamples show that there are exceptions that depend also on the fact that (quasi-)convexity and (quasi-)concavity for set-valued maps are not symmetric concepts.

Finally, there is another representation of uncertainty that plays a relevant role in the theory of imprecise probabilities, known as *lower expectation*; this concept specifies the supremum buying price for each possible gamble. It is well known that a set of probability distributions or a set of almost desirable gambles implicitly defines a lower expectation and viceversa. As the set of probability distributions or the set of almost desirable gambles depends, in this paper, on the alternative chosen by the decision maker; it is clear that, fixed a gamble Y, the lower expectation of Y is a real-valued function of the alternative. Now, the question is whether the (quasi-)concavity/convexity of the lower expectations of each gamble, as real-valued maps, is related to the (quasi-)concavity/convexity of ambiguous belief or almost desirable gambles correspondences.

[3]A complete analysis of all the possible attitudes towards the inclusion relation and their relations is given in [8].

This issue is here addressed and the results show that (quasi-)concavity (resp. (quasi-)convexity) of lower expectations is related to (quasi-)convexity (resp. (quasi-)concavity) of the ambiguous belief correspondence and to (quasi-)concavity (resp. (quasi-)convexity) of the almost desirable gambles correspondence. However, even in this case, counterexamples show that there are different exceptions.

Summarizing, the paper is organized as follows: Section 2 presents the game model and the issue of existence of equilibria that provides the main motivation for the problem addressed in this paper. Then, a precise formulation of the problem of convexity of preferences is provided. Section 3 presents (quasi-)concave/convex set-valued maps and their properties. The attitudes towards uncertainty are investigated in Section 4. Section 5 builds upon the definitions and the properties of the previous sections and gives sufficient conditions for the required property of convexity of preferences. Section 6 studies the relation between (quasi-)concavity (resp. (quasi-) convexity) of ambiguous belief correspondences and almost desirable gambles correspondences. In Sections 7 and 8, lower expectations are investigated; in particular, the link between (quasi-)concavity (resp. (quasi-)convexity) of lower expectations and ambiguous belief correspondences is studied in Section 7, while Section 8 addresses to the same analysis between lower expectations and almost desirable gambles correspondences.

2 Motivation and Problem Formulation

2.1 Games and equilibria

Strategies and outcomes

Consider a game where $I = \{1, \dots, n\}$ is the set of players. The strategy set of each player is a nonempty, compact and convex set $X_i \subset \mathbb{R}^{k_i}$. Each strategy of player i is denoted with $x_i \in X_i$ while $X = \prod_{j=1}^n X_j$ denotes the set of strategy profiles[4].

The finite set of outcomes of the game is denoted with[5] $\Omega = \{\omega_1, \dots, \omega_m\}$. The set of all the probability distributions over Ω, is

$$\mathcal{P} = \left\{ \varrho = \big(\varrho(\omega_1), \dots, \varrho(\omega_m)\big) \in \mathbb{R}^m \;\middle|\; i)\; \sum_{\omega \in \Omega} \varrho(\omega) = 1,\; ii)\; \varrho(\omega) \geqslant 0\; \forall \omega \in \Omega \right\},$$

where $\varrho(\omega_h)$ is the probability of state ω_h. Therefore, belief over the outcomes of the game are represented by subsets of $\mathcal{P}$. belief are unambiguous if they are singletons, they are ambiguous otherwise.

The the set of gambles over Ω is $\mathcal{L} = \{Y \,|\, Y : \Omega \to \mathbb{R}\}$; with an abuse of notation, each $Y \in \mathcal{L}$ is identified by the vector $Y = \big(Y(\omega_1), \dots, Y(\omega_m)\big) \in \mathbb{R}^m$ where $Y(\omega_h)$ is the payoff of the gamble Y when state $\omega_h \in \Omega$ occurs. Finally, $E_\varrho[Y]$ denotes the expectation of Y under ϱ, i.e. $E_\varrho[Y] = \sum_{\omega \in \Omega} \varrho(\omega) Y(\omega)$.

[4]The case of mixed strategies over a finite set of pure strategies is obviously included in this framework. In fact, in this case, if Ψ_i is the finite strategy set of player i, with $|\Psi_i| = k_i$, then X_i is the set of mixed strategies of player i, where each $x_i \in X_i$ is a vector $x_i = (x_i(\psi_i))_{\psi_i \in \Psi_i}$ s.t. $i)$ $\sum_{\psi_i \in \Psi_i} x_i(\psi_i) = 1$ and $ii)$ $x_i(\psi_i) \geqslant 0$, $\forall \psi_i \in \Psi_i$.

[5]In previous papers is a subset of $\mathbb{R}^n$ so that the i-th component ω_i of ω represents the payoff of player i when outcome $\omega \in \Omega$ is realized. In this paper, Ω does not have necessarily any topological or algebraic structure.

Strategic games revised

From a different perspective, a classical strategic form game can be regarded as follows:

- Each player i is endowed with a function $p_i : X \to \mathcal{P}$ that provides, to player i, the information about the possible outcomes of the game. Therefore, $p_i(x)$ is the probability distribution over Ω that is *consistent* with x in view of player i. In most (but not all) models, it is assumed that $p_1 = \cdots = p_n$.

- Each player i is endowed with a preference $\succsim_{i,p}$ over $\mathcal{P}$. In most (but not all) models, it is assumed that $\succsim_{i,p}$ is represented by a von Neumann-Morgenstern expected utility function.

- $\succsim_{i,p}$ in $\mathcal{P}$ induces a preference $\succsim_i$ in X in the obvious way:

$$x \succsim_i x' \iff p_i(x) \succsim_{i,p} p_i(x')$$

  Therefore, the game is $\Gamma = \{I; (X_i)_{i \in I}; (\succsim_i)_{i \in I}\}$.

Introducing Imprecision

The key feature of the models investigated in this paper is that the function $p_i : X \to \mathcal{P}$ is replaced by imprecise probability correspondences. In particular, the focus is on two well known imprecise probability representations: *i)* set of probability distributions, *ii)* set of almost desirable gambles:

i) The function $p_i : X \to \mathcal{P}$ is replaced by $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$, called *ambiguous belief correspondence* of player i: For every $x \in X$, $\mathcal{B}_i(x) \neq \emptyset$ is the set of probability distributions over Ω that are feasible and consistent, in view of player i, with the strategy profile x. In the remainder of this paper it will not be assumed in general that $\mathcal{B}_i(x)$ is a closed and convex subset of $\mathcal{P}$, even if it is a classical condition used in many applications[6].

In games without ambiguity (or imprecision), $\mathcal{B}_i$ is single-valued and gives back the function p_i. Moreover, previous literature shows that multiple models with different sources of ambiguity have a formulation in terms of belief correspondences, for instance:

a) (Symmetric) Incomplete information games under multiple priors

b) Games under strategic ambiguity, that is games in which players have ambiguous expectations about opponents' behavior

ii) The function $p_i : X \to \mathcal{P}$ is replaced by $\mathcal{D}_i : X \rightsquigarrow \mathcal{L}$, called *almost desirable gambles correspondence* of player i, with $\mathcal{D}_i(x) \neq \emptyset$ for every $x \in X$, where $Y \in \mathcal{D}_i(x)$ means that agent i accepts all gambles $Y + \varepsilon$ with $\varepsilon > 0$. In the remainder of this paper it will not be required in general that $\mathcal{D}_i(x)$ is a closed convex cone in $\mathbb{R}^m$ containing all the the positive gambles, even if this is a relevant condition from a theoretical point of view[7].

[6]This condition characterizes coherence of (imprecise) probabilities that does not play, in general, a key role in this paper.

[7]This condition characterizes coherence as well.

Summarizing, the information (about the possible outcomes) available to the decision maker i is summarized by an exogenous set-valued map $\mathcal{C}_i : X \rightsquigarrow \mathcal{T} \subseteq \mathbb{R}^m$, called *imprecise probability correspondence*, which gives to the decision maker i and for every strategy profile $x \in X$, the information[8] $\mathcal{C}_i(x) \subseteq \mathcal{T}$. In particular we obtain:

Ambiguous belief correspondences when $\mathcal{T} = \mathcal{P}$ and $\mathcal{C}_i = \mathcal{B}_i$

Almost desirable gambles correspondences when $\mathcal{T} = \mathcal{L}$ and $\mathcal{C}_i = \mathcal{D}_i$

Preferences and games

Preferences of agent i over strategy profiles are constructed as follows:

Agent i is endowed with a *reflexive* preference $\succsim_{i,\mathcal{T}}$ over the set $\mathbb{K}(\mathcal{T})$ of all subsets of $\mathcal{T}$:

- Given $A, B \in \mathbb{K}(\mathcal{T})$ then $A \succsim_{i,\mathcal{T}} B$ means that A is *at least as good as* B for player i
- No completeness or transitivity assumptions are imposed on $\succsim_{i,\mathcal{T}}$
- $\succ_{i,\mathcal{P}}$ and $\sim_{i,\mathcal{P}}$ denote respectively the strict preference and the indifference relation induced by $\succsim_{i,\mathcal{P}}$.
- The strict upper level set correspondence $\mathcal{U}_{i,\mathcal{T}} : \mathbb{K}(\mathcal{T}) \rightsquigarrow \mathbb{K}(\mathcal{T})$ for $\succsim_{i,\mathcal{P}}$ is defined by:

$$\mathcal{U}_{i,\mathcal{T}}(A) = \{B \in \mathbb{K}(\mathcal{T}) \mid B \succ_{i,\mathcal{T}} A\} \quad \forall A \in \mathbb{K}(\mathcal{T})$$

The preference relation $\succsim_i$ of player i over X, is naturally defined as follows

$$x \succsim_i x' \iff \mathcal{C}_i(x) \succsim_{i,\mathcal{T}} \mathcal{C}_i(x')$$

- $x \succsim_i y$ means that the strategy profile x is *at least as good as* the strategy profile y, for player i.
- $\succ_i$ and $\sim_i$ are induced by $\succsim_i$ in the classical way.
- $\mathcal{U}_i : X \rightsquigarrow X_i$ defined by

$$\mathcal{U}_i(x_i, x_{-i}) = \{x'_i \in X_i \mid (x'_i, x_{-i}) \succ_i (x_i, x_{-i})\} \quad \forall (x_i, x_{-i}) \in X$$

  is the strict upper level set correspondence for $\succsim_i$.

Therefore, the game is $\Gamma = \{I; (X_i)_{i\in I}; (\succsim_i)_{i\in I}\}$. This is a generalized game[9] as defined by Shafer and Sonnenshein in [25]. Generalized games have a natural equilibrium notion that, in the framework of the present paper, we call equilibrium under imprecise probability correspondences.

Definition 2.1: A strategy profile $\overline{x} \in X$ is an *equilibrium under imprecise probability correspondences* $\mathcal{C}_i$ of the game Γ if $\mathcal{U}_i(\overline{x}) = \emptyset$ for every $i \in I$.

[8] In this view, the strategy set X has a double use: first it represents the set of objects of choice of the decision maker but, at the same time, it stands for the set of variables that parameterize the belief of the decision maker.

[9] This game can be regarded also as a set-valued game under generalized preferences (see also [15] for set-valued optimization and games in a more classical framework).

Remark 2.2: The previous concept is the natural generalization of the concept of Nash equilibrium. In an equilibrium under imprecise probability correspondences x:

i) The information about the consequences of each strategy x_i' is provided to player i by the imprecise probability $\mathcal{C}_i(x_i', x_{-i})$.

ii) The strategy x_i is *maximal* to player i with respect to his preference $\succsim_i$ and the information available to player i.

Existence of equilibria follows directly from [25]:

Theorem 2.3: *Assume that for every $i \in I$,*

(1) $\mathcal{U}_i$ has an open graph[10]*,*

(2) $x_i \notin co(\mathcal{U}_i(x_i, x_{-i}))$ for every $(x_i, x_{-i}) \in X$.

Then, the game Γ has a least an equilibrium.

Problem 2.4: The purpose of this paper is to find explicit conditions on $\mathcal{C}_i$ and $\succsim_{i,\mathcal{T}}$ which guarantee that the assumption (2) of the previous Theorem hold.

2.2 Problem Formulation

To simplify notation we consider the case of a single agent:

- $\mathcal{C} : X \rightsquigarrow \mathcal{T}$ is the imprecise probability correspondence of the decision maker.

- $\succsim_{\mathcal{T}}$ is the preference over $\mathbb{K}(\mathcal{T})$ of the decision maker and $\succsim$ is the preference over X induced by $\succsim_{\mathcal{T}}$.

  The strict upper level correspondences are respectively:

  a) $\mathcal{U}_{\mathcal{T}} : \mathbb{K}(\mathcal{T}) \rightsquigarrow \mathbb{K}(\mathcal{T})$ defined by

  $$\mathcal{U}_{\mathcal{T}}(A) = \{B \in \mathbb{K}(\mathcal{T}) \mid B \succ_{\mathcal{T}} A\} \quad \forall A \in \mathbb{K}(\mathcal{T})$$

  b) $\mathcal{U} : X \rightsquigarrow X$ defined by

  $$\mathcal{U}(x) = \{x' \in X \mid x' \succ x\}$$

- *Problem Statement:* Find explicit conditions on $\mathcal{C}$ and $\succsim_{\mathcal{T}}$ which guarantee that:

  $$x \notin co(\mathcal{U}(x)) \quad \forall x \in X$$

- *Approach:* Different results are proposed. In each of them, the condition imposed on $\mathcal{C}$ is one of the (quasi-)concavity/convexity notions for set-valued maps and the conditions imposed on $\succsim_{\mathcal{T}}$ include superset or subset aversion, depending on the assumption imposed on $\mathcal{C}$.

[10] That is, the graph of $\mathcal{U}_i$ is an open subset of $X \times X_i$.

3 Convexity/concavity and quasi-convexity/concavity of functions and correspondences

This section recalls the basic definitions of (quasi)-convex/concave set-valued maps that play a key role for the purpose of this paper.

3.1 (Quasi-)convex/concave functions

Firstly, it is useful to recall well known definitions for single-valued maps. Given a convex subset X of a finite dimensional space and let $f : X \to \mathbb{R}$ then

Definition 3.1: f is said to be convex if, for every $x', x' \in X$ and $t \in]0,1[$, it follows that

$$f(tx' + (1-t)x'') \leqslant tf(x') + (1-t)f(x'').$$

f is said to be concave if $-f$ is convex.

Moreover

Definition 3.2: f is said to be quasi-convex if, for every $x', x' \in X$ and $t \in]0,1[$, it follows that one of the following equivalent conditions[11] is satisfied:

$$\begin{array}{rc} (i) & f(tx' + (1-t)x'') \leqslant \max\{f(x'), f(x'')\} \\ (ii) & f(tx' + (1-t)x'') \leqslant f(x') \quad \text{or} \quad f(tx' + (1-t)x'') \leqslant f(x'') \\ (iii) & \underline{S}_\alpha = \{x \mid f(x) \leqslant \alpha\} \quad \text{is a convex set} \ \ \forall \alpha \in \mathbb{R}. \end{array}$$

f is said to be quasi-concave if $-f$ is quasi-convex.

3.2 Convex and quasi-convex set valued maps

In this subsection, the definitions of convex/concave and quasi-convex/concave set-valued maps are given. These latter concepts appear sporadically and extemporaneously in the literature even if they play a relevant role in optimization theory. In this paper, it is shown that they play a key role in the problem of existence of equilibria in strategic games.

The recent paper [24] provides a systematization of the definitions of quasi-convex/concave set-valued maps and of the relations that exist among them. The main concepts and results, that are useful for our purposes, are presented below; most of the results could be derived from the more general ones that are presented in [24]. For the sake of completeness, self-contained proofs of all results and counterexamples are presented in the Appendix.

Quasi-convexity

Definition 3.3: The set valued map $\mathcal{C} : X \rightsquigarrow \mathcal{T}$ is said to be:

M1) convex if

$$\mathcal{C}(tx' + (1-t)x'') \subseteq t\mathcal{C}(x') + (1-t)\mathcal{C}(x'') \quad \forall\, x', x'' \in X,\ \forall\, t \in]0,1[. \tag{1}$$

[11] The conditions are obviously equivalent, but they are written explicitly in order to better understand the differences with the set-valued case.

M2) quasi-convex if

$$\mathcal{C}(tx' + (1-t)x'') \subseteq \mathcal{C}(x') \cup \mathcal{C}(x'') \quad \forall\, x', x'' \in X,\ \forall\, t \in]0,1[. \tag{2}$$

M3) strongly quasi-convex if

$$\mathcal{C}(tx' + (1-t)x'') \subseteq \mathcal{C}(x') \quad \text{or} \quad \mathcal{C}(tx' + (1-t)x'') \subseteq \mathcal{C}(x'') \quad \forall\, x', x'' \in X,\ \forall\, t \in]0,1[. \tag{3}$$

M4) weakly quasi-convex if

$$\mathcal{C}(tx' + (1-t)x'') \subseteq co\,(\mathcal{C}(x') \cup \mathcal{C}(x'')) \quad \forall\, x', x'' \in X,\ \forall\, t \in]0,1[. \tag{4}$$

Finally, $\mathcal{C} : X \rightsquigarrow \mathcal{T}$ is said to have *convex images* if $\mathcal{C}(x)$ is a convex subset of $\mathcal{T}$ for every $x \in X$.

It is useful to give the following characterizations. The proofs are given in the Appendix.

PROPOSITION 3.4: *The set-valued map $\mathcal{C} : X \rightsquigarrow \mathcal{T}$ is quasi-convex if and only if for every $A \subseteq \mathcal{T}$, with $A \neq \emptyset$, the set*

$$\underline{L}_A = \{x \,|\, \mathcal{C}(x) \subseteq A\} \tag{5}$$

is a convex subset of X.

PROPOSITION 3.5: *The set-valued map $\mathcal{C} : X \rightsquigarrow \mathcal{T}$ is weakly quasi-convex if and only if for every convex subset $A \subseteq \mathcal{T}$, with $A \neq \emptyset$, the set*

$$\underline{L}_A = \{x \,|\, \mathcal{C}(x) \subseteq A\} \tag{6}$$

is a convex subset of X.

REMARK 3.6: Note that quasi-convexity falls into definition *u3* in [24], strong quasi-convexity in definition *u4*, while, in light of the previous proposition 3.5, weak quasi-concavity falls into definition *u1* .

The next result is derived directly from proposition 3.6 in [24]. A self contained proof is given in the Appendix.

PROPOSITION 3.7: *Given the set valued map $\mathcal{C} : X \rightsquigarrow \mathcal{T}$, then*

i) If $\mathcal{C}$ is strongly quasi-convex then $\mathcal{C}$ is quasi-convex.

ii) If $\mathcal{C}$ is quasi-convex then $\mathcal{C}$ is weakly quasi-convex.

iii) If $\mathcal{C}$ is convex then $\mathcal{C}$ is weakly quasi-convex.

The converse statements of the implications *i), ii), iii)* do not hold as shown by counterexamples given in the Appendix.

Concave and quasi-concave set valued maps

Definition 3.8: The set valued map $\mathcal{C} : X \rightsquigarrow \mathcal{T}$ is said to be:

N1) concave if

$$t\mathcal{C}(x') + (1-t)\mathcal{C}(x'') \subseteq \mathcal{C}(tx' + (1-t)x'') \quad \forall\, x', x'' \in X,\ \forall\, t \in]0,1[. \tag{7}$$

N2) quasi-concave if

$$\mathcal{C}(x') \cap \mathcal{C}(x'') \subseteq \mathcal{C}(tx' + (1-t)x'') \quad \forall\, x', x'' \in X,\ \forall\, t \in]0,1[. \tag{8}$$

N3) strongly quasi-concave if

$$\mathcal{C}(x') \subseteq \mathcal{C}(tx' + (1-t)x'') \quad \text{or} \quad \mathcal{C}(x'') \subseteq \mathcal{C}(tx' + (1-t)x'') \quad \forall\, x', x'' \in X,\ \forall\, t \in]0,1[. \tag{9}$$

The next results are derived directly from Proposition 3.4 in [24]. Self contained proofs are given in the Appendix.

Proposition 3.9: *Let $\mathcal{C} : X \rightsquigarrow \mathcal{T}$ be a set-valued map with convex images, then $\mathcal{C}$ is quasi-concave if and only if for every convex $A \subseteq \mathcal{T}$, with $A \neq \emptyset$, the set*

$$\overline{L}_A = \{x \mid A \subseteq \mathcal{C}(x)\} \tag{10}$$

is a convex subset of X.

Proposition 3.10: *Given the set valued map $\mathcal{C} : X \rightsquigarrow \mathcal{T}$, then*

i) If $\mathcal{C}$ is strongly quasi-concave then $\mathcal{C}$ is quasi-concave.

ii) If $\mathcal{C}$ is concave then $\mathcal{C}$ is quasi-concave.

In the Appendix, counterexamples are given showing that the converse statements of the implications *i), ii)* do not hold.

Remark 3.11: Note that quasi-concavity falls into definition *l3* in [24], strong quasi-concavity in definition *l4*. Proposition 3.9 shows that a definition of weakly quasi-concavity would be equivalent to quasi-concavity. Indeed, even [24] shows that quasi-concavity and quasi convexity are not symmetric concepts as there exists nine distinct definitions of quasi-convexity and only seven distinct definitions of quasi concavity.

3.2.1 A particular model

In [8], two well known game models are presented and studied in the framework of ambiguous belief correspondences: a game model under strategic ambiguity and a game model of incomplete information in case of multiple priors. Here it is presented a decision model that encompasses both of them; the analysis of the (quasi-)convexity/concavity properties of the associated ambiguous belief correspondences is given in order to highlight the applicability of the results presented in this paper.

Consider the case in which $\Omega = \Omega_1 \times \Omega_2$. Let $\mathcal{P}$ denote the set of probability distributions over Ω, $\mathcal{P}_1$ the set of probability distributions over Ω_1 and $\mathcal{P}_2$ the set of probability distributions over Ω_2. Note that any pair $(\varrho_1, \varrho_2) \in \mathcal{P}_1 \times \mathcal{P}_2$ induces a unique probability distribution $\varrho \in \mathcal{P}$ in the obvious way:

$$\varrho(\omega_1, \omega_2) = \varrho_1(\omega_1)\varrho_2(\omega_2) \quad \forall(\omega_1, \omega_2) \in \Omega.$$

Let $h : X \to \mathcal{P}_1$ be a linear function and $K \subseteq \mathcal{P}_2$. Denote with $h(x) = h_x$, then consider the correspondence $\mathcal{B} : X \rightsquigarrow \mathcal{P}$ defined by

$$\mathcal{B}(x) = \{\varrho \in \mathcal{P} \mid \varrho(\omega_1, \omega_2) = h_x(\omega_1)\varrho_2(\omega_2) \ \forall(\omega_1, \omega_2) \in \Omega, \text{ where } \varrho_2 \in K\} \tag{11}$$

Then

Lemma 3.12: *Let $\mathcal{B} : X \rightsquigarrow \mathcal{P}$ be the correspondence defined by (11), then:*

i) $\mathcal{B}$ is convex;

ii) $\mathcal{B}$ is quasi-concave.

Proof. i) Let $\varrho \in \mathcal{B}(tx' + (1-t)x'')$, then there exist $\varrho_2 \in K$ such that

$$\varrho(\omega_1, \omega_2) = h_{tx'+(1-t)x''}(\omega_1)\varrho_2(\omega_2) \quad \forall(\omega_1, \omega_2) \in \Omega.$$

Since h is linear, it follows that

$$= h_{tx'+(1-t)x''}(\omega_1)\varrho_2(\omega_2) = th_{x'}(\omega_1)\varrho_2(\omega_2) + (1-t)h_{x''}(\omega_1)\varrho_2(\omega_2) \quad \forall(\omega_1, \omega_2) \in \Omega$$

Given the two probability distributions ϱ' and ϱ'' defined for every $(\omega_1, \omega_2) \in \Omega$ by

$$\varrho'(\omega_1, \omega_2) = h_{x'}(\omega_1)\varrho_2(\omega_2) \quad \text{and} \quad \varrho''(\omega_1, \omega_2) = h_{x''}(\omega_1)\varrho_2(\omega_2),$$

then it follows that $\varrho' \in \mathcal{B}(x')$ and $\varrho'' \in \mathcal{B}(x'')$. So

$$\varrho = t\varrho' + (1-t)\varrho'' \in t\mathcal{B}(x') + (1-t)\mathcal{B}(x'')$$

and we get the assertion.

ii) Let $\varrho \in \mathcal{B}(x') \cap \mathcal{B}(x'')$, then there exists $\varrho_2' \in K$ and $\varrho_2'' \in K$ such that

$$\varrho(\omega_1, \omega_2) = h_{x'}(\omega_1)\varrho_2'(\omega_2) = h_{x''}(\omega_1)\varrho_2''(\omega_2) \quad \forall(\omega_1, \omega_2) \in \Omega.$$

So

$$h_{x'}(\omega_1) = \sum_{\omega_2 \in \Omega_2} h_{x'}(\omega_1)\varrho_2'(\omega_2) = \sum_{\omega_2 \in \Omega_2} h_{x''}(\omega_1)\varrho_2''(\omega_2) = h_{x''}(\omega_1) \quad \forall \omega_1 \in \Omega_1.$$

This immediately implies that $\varrho_2' = \varrho_2'' = \varrho_2$. So

$$\varrho(\omega_1, \omega_2) = [th_{x'}(\omega_1) + (1-t)h_{x''}(\omega_1)]\,\varrho_2(\omega_2) = h_{tx'+(1-t)x''}(\omega_1)\varrho_2(\omega_2) \quad \forall(\omega_1, \omega_2) \in \Omega$$

So $\varrho \in \mathcal{B}(tx' + (1-t)x'')$ and the assertion follows. □

Lemma 3.13: *Let $\mathcal{B} : X \rightsquigarrow \mathcal{P}$ be the correspondence defined by (11):*

i) If the function h is not constant, then $\mathcal{B}$ is not quasi-convex;

ii) If the function h is not constant and the cardinality $|K| > 1$, then $\mathcal{B}$ is not concave.

Proof. i) Suppose that $\mathcal{B}$ is quasi-convex. Let $x', x'' \in X$ such that $h_{x'} \neq h_{x''}$ and $t \in]0,1[$. Since h is linear, it follows that

$$h_{tx'+(1-t)x''} \neq h_{x'} \quad \text{and} \quad h_{tx'+(1-t)x''} \neq h_{x''}.$$

Let $\varrho \in \mathcal{B}(tx' + (1-t)x'')$, then there exists $\varrho_2 \in K$ such that

$$\varrho(\omega_1, \omega_2) = h_{tx'+(1-t)x''}(\omega_1)\varrho_2(\omega_2) \quad \forall(\omega_1, \omega_2) \in \Omega.$$

On the other hand, the quasi-convexity of $\mathcal{B}$ implies $\mathcal{B}(tx'+(1-t)x'') \subseteq \mathcal{B}(x') \cup \mathcal{B}(x'')$ so $\varrho(\omega_1, \omega_2) \in \mathcal{B}(x') \cup \mathcal{B}(x'')$. Without loss of generality, assume that $\varrho(\omega_1, \omega_2) \in \mathcal{B}(x')$; then there exists $\varrho'_2 \in K$ such that

$$\varrho(\omega_1, \omega_2) = h_{x'}(\omega_1)\varrho'_2(\omega_2) \quad \forall(\omega_1, \omega_2) \in \Omega.$$

It follows that

$$\varrho(\omega_1, \omega_2) = h_{tx'+(1-t)x''}(\omega_1)\varrho_2(\omega_2) = h_{x'}(\omega_1)\varrho'_2(\omega_2) \quad \forall(\omega_1, \omega_2) \in \Omega.$$

So,

$$h_{tx'+(1-t)x''}(\omega_1) = \sum_{\omega_2 \in \Omega_2} h_{tx'+(1-t)x''}(\omega_1)\varrho_2(\omega_2) = \sum_{\omega_2 \in \Omega_2} h_{x'}(\omega_1)\varrho'_2(\omega_2) = h_{x'}(\omega_1) \quad \forall \omega_1 \in \Omega_1,$$

but this is impossible because $h_{tx'+(1-t)x''} \neq h_{x'}$. Therefore we get a contradiction and $\mathcal{B}$ is not quasi-convex.

ii) Suppose that $\mathcal{B}$ is concave. Let $x', x'' \in X$ such that $h_{x'} \neq h_{x''}$ and $t \in]0,1[$. Since h is linear, it follows that

$$h_{tx'+(1-t)x''} \neq h_{x'} \quad \text{and} \quad h_{tx'+(1-t)x''} \neq h_{x''}.$$

Since $|K| > 1$, let $\varrho'_2, \varrho''_2 \in K$ be such that $\varrho'_2 \neq \varrho''_2$. Let $\varrho \in t\mathcal{B}(x') + (1-t)\mathcal{B}(x'')$, be such that

$$\varrho(\omega_1, \omega_2) = th_{x'}(\omega_1)\varrho'_2(\omega_2) + (1-t)h_{x''}(\omega_1)\varrho''_2(\omega_2) \quad \forall(\omega_1, \omega_2) \in \Omega.$$

Since $\mathcal{B}$ is concave then it follows that $\varrho \in \mathcal{B}(tx' + (1-t)x'')$; so, there exists $\varrho_2 \in K$ such that

$$\varrho(\omega_1, \omega_2) = h_{tx'+(1-t)''}(\omega_1)\varrho_2(\omega_2) = th_{x'}(\omega_1)\varrho_2(\omega_2) + (1-t)h_{x''}(\omega_1)\varrho_2(\omega_2) \quad \forall(\omega_1, \omega_2) \in \Omega.$$

So

$$th_{x'}(\omega_1)\varrho'_2(\omega_2) + (1-t)h_{x''}(\omega_1)\varrho''_2(\omega_2) = th_{x'}(\omega_1)\varrho_2(\omega_2) + (1-t)h_{x''}(\omega_1)\varrho_2(\omega_2) \quad \forall(\omega_1, \omega_2) \in \Omega \tag{12}$$

Now, suppose that $\overline{\omega}_2$ is such that $\varrho'_2(\overline{\omega}_2) \neq \varrho_2(\overline{\omega}_2)$, from the previous equation (12) it follows that $h_{x''}(\omega_1) > 0$ for every ω_1 and

$$\frac{t}{1-t}\frac{h_{x'}(\omega_1)}{h_{x''}(\omega_1)} = \frac{\varrho_2(\overline{\omega}_2) - \varrho''_2(\overline{\omega}_2)}{\varrho'_2(\overline{\omega}_2) - \varrho_2(\overline{\omega}_2)} \quad \forall \omega_1 \in \Omega_1$$

Since, $h_{x'}$ and $h_{x''}$ are probability distributions with $h_{x'} \neq h_{x''}$, then there exist ω_1^* and ω_1^{**} such that

$$h_{x'}(\omega_1^*) > h_{x''}(\omega_1^*) \quad \text{and} \quad h_{x'}(\omega_1^{**}) < h_{x''}(\omega_1^{**}).$$

Hence

$$\frac{\varrho_2(\overline{\omega}_2) - \varrho_2''(\overline{\omega}_2)}{\varrho_2'(\overline{\omega}_2) - \varrho_2(\overline{\omega}_2)} = \frac{h_{x'}(\omega_1^*)}{h_{x''}(\omega_1^*)} > \frac{h_{x'}(\omega_1^{**})}{h_{x''}(\omega_1^{**})} = \frac{\varrho_2(\overline{\omega}_2) - \varrho_2''(\overline{\omega}_2)}{\varrho_2'(\overline{\omega}_2) - \varrho_2(\overline{\omega}_2)},$$

but this is a contradiction. So $\varrho_2(\omega_2) = \varrho_2'(\omega_2)$ for every $\omega_2 \in \Omega_2$. Similarly, we get a contradiction if $\overline{\omega}_2$ is such that $\varrho_2''(\overline{\omega}_2) \neq \varrho_2(\overline{\omega}_2)$; so $\varrho_2(\omega_2) = \varrho_2''(\omega_2)$ for every $\omega_2 \in \Omega_2$.

Since $\varrho_2' \neq \varrho_2''$ we get a contradiction and $\mathcal{B}$ is not concave. □

4 Attitudes towards uncertainty

In [8], it has been shown that the attitudes of the agents towards the inclusion relation between sets of probability distributions over outcomes plays a key role in the convexity of preferences. In particular a minimally pessimistic (resp. optimistic) attitude towards ambiguity is identified: an agent would not prefer a set of probability distributions to another one if the former is a subset of the latter (resp. if the latter is a subset of the former). These attitudes are called, respectively, imprecision aversion and imprecision loving. In this paper, the same approach is taken into account in the more general context of set-valued imprecise probabilities. In this more general framework, the implication that characterize imprecision aversion (resp. imprecision loving) will be called *superset aversion* (resp. *subset aversion*). The reason for this change of notation is that, while it clear that superset of probabilities are more imprecise, when uncertainty is represented by sets of desirable gambles, the information becomes less imprecise as the set of desirable gambles enlarges to the half space.

Definition 4.1: The preference relation $\succsim_{\mathcal{T}}$ is said to be

a) superset averse if

$$A \subseteq B \implies B \not\succ_{\mathcal{P}} A;$$

b) subset averse if

$$A \subseteq B \implies A \not\succ_{\mathcal{P}} B.$$

Remark 4.2: It is clear that there are many properties, in terms of the set inclusion relation, that characterize preferences within subsets of a given set. A detailed analysis is presented in [8], in the case of sets of probability distributions. However, it can be easily observed that the definitions and relations among definitions in [8] can be immediately generalized to arbitrary sets. Moreover, every definition given in [8] implies either imprecision aversion or imprecision loving (superset aversion or subset aversion). Hence superset aversion and subset aversion can be considered minimal properties in this sense.

Two further attitudes towards uncertainty, that will be relevant, are given below.

Definition 4.3: The preference relation $\succsim_{\mathcal{T}}$ is said to be

u) sup-consistent if

$$A \succ_{\mathcal{T}} C \, \mathit{and} \, B \succ_{\mathcal{T}} C \implies A \cup B \succ_{\mathcal{T}} C;$$

l) inf-consistent if

$$A \succ_{\mathcal{T}} C,\, B \succ_{\mathcal{T}} C \, and \, A \cap B = \emptyset \implies A \cap B \succ_{\mathcal{T}} C.$$

Moreover, the following convexity assumption for preferences will be relevant:

Definition 4.4: The preference relation $\succsim_{\mathcal{T}}$ is said to be

c1) semi-strictly convex if

$$\mathcal{U}_{\mathcal{T}}(A) = \{B \subseteq \mathcal{T} \,|\, B \succ_{\mathcal{T}} A\} \, is \, a \, convex \, subset \, of \, \mathbb{K}(\mathcal{T}),\ \forall A \subseteq \mathcal{T}.$$

c2) convex if

$$\mathcal{S}_{\mathcal{T}}(A) = \{B \subseteq \mathcal{T} \,|\, B \succsim_{\mathcal{T}} A\} \, is \, a \, convex \, subset \, of \, \mathbb{K}(\mathcal{T}),\ \forall A \subseteq \mathcal{T}.$$

Special cases

Here we present classical models of preferences in case of imprecise probabilities, (see, for instance, [28],[23] or [27] just to quote a few), such as maximin preferences or maximax preferences (that are rational preferences) and interval dominance (which is instead not complete). While in [8] it is shown that all these preferences are imprecision averse/loving (superset/subset averse) and convex, here the sup/inf-consistency is analyzed. For the sake of completeness, it is shown below that these preferences are semi-strictly convex[12].

We consider the case in which $\mathcal{T} = \mathcal{P}$ and preferences are binary relations in the set $\mathbb{K}(\mathcal{P})$ of all compact subset of $\mathcal{P}$.

Let $f : \mathcal{P} \to \mathbb{R}$ be a continuous function which gives, to the decision maker, the utility $f(\varrho)$ of every lottery $\varrho \in \mathcal{P}$[13]. Then

Definition 4.5: For every $A \in \mathbb{K}(\mathcal{P})$, let $F(A) = \min_{\varrho \in A} f(\varrho)$ and $G(A) = \max_{\varrho \in A} f(\varrho)$. Then,

Min) The preference $\succsim_{\mathcal{P}}^{m}$, defined for every $(A, B) \in \mathbb{K}(\mathcal{P}) \times \mathbb{K}(\mathcal{P})$, by

$$A \succsim_{\mathcal{P}}^{m} B \iff F(A) \geqslant F(B),$$

is a maximin preference

Max) The preference $\succsim_{\mathcal{P}}^{M}$, defined for every $(A, B) \in \mathbb{K}(\mathcal{P}) \times \mathbb{K}(\mathcal{P})$, by

$$A \succsim_{\mathcal{P}}^{M} B \iff G(A) \geqslant G(B),$$

is a maximax preference.

ID) The preference $\succsim_{\mathcal{P}}^{ID}$, defined for every $(A, B) \in \mathbb{K}(\mathcal{P}) \times \mathbb{K}(\mathcal{P})$, by

$$A \succsim_{\mathcal{P}}^{ID} B \iff F(A) \geqslant G(B),$$

is an interval dominance preference.

[12] The proofs show that semi-strict convexity is a simple consequence of convexity in this particular case.

[13] The function f can be the classical expected utility, but it can also be something different as in the variational preference model.

Note maximax and maximin preferences are complete and transitive while interval dominance is not a complete preference, while it is transitive. Now the properties of these preferences are investigated.

PROPOSITION 4.6: *Given $f : \mathcal{P} \to \mathbb{R}$, let $\succsim_{\mathcal{P}}^{m}$ be the corresponding maximin preference. Then,*

i) $\succsim_{\mathcal{P}}^{m}$ *is imprecision averse.*

ii) If f is a concave function then $\succsim_{\mathcal{P}}^{m}$ is a semi-strictly convex preference.

iii) $\succsim_{\mathcal{P}}^{m}$ *is sup-consistent and inf-consistent*

Proof. i) ii) See Proposition 4.9 in [8].

ii) Let f be a concave function. For every $t \in [0,1]$, $F(tA + (1-t)B) = f(t\varrho_A + (1-t)\varrho_B)$ for some $\varrho_A \in A$ and $\varrho_B \in B$. By definition, it follows that $f(\varrho_A) \geqslant F(A)$ and $f(\varrho_B) \geqslant F(B)$. Since f a concave function, it follows that $f(t\varrho_A + (1-t)\varrho_B) \geqslant tf(\varrho_A) + (1-t)f(\varrho_B)$. Summarizing:

$$F(tA + (1-t)B) = f(t\varrho_A + (1-t)\varrho_B) \geqslant tf(\varrho_A) + (1-t)f(\varrho_B) \geqslant tF(A) + (1-t)F(B).$$

Now, if $A \succ_{\mathcal{P}}^{m} C$ and $B \succ_{\mathcal{P}}^{m} C$ then $F(A) > F(C)$ and $F(B) > F(C)$. Therefore, $F(tA+(1-t)B) > F(C)$ and $tA + (1-t)B \succ_{\mathcal{P}}^{m} C$. So $\succsim_{\mathcal{P}}^{m}$ is semi-strictly convex.

iii) Let A, B, C be subsets of $\mathbb{K}(\mathcal{P})$ such that

$$F(A) = f(\varrho_A) > F(C),\ F(B) = f(\varrho_B) > F(C) \quad \text{where } \varrho_A \in A,\ \varrho_B \in B$$

It follows that

$$F(A \cup B) = \min\{f(\varrho_A), f(\varrho_B)\} > F(C).$$

so that $\succsim_{\mathcal{P}}^{m}$ is sup-consistent. Moreover, if A and B are also such that $A \cap B \neq \emptyset$ then it follows since

$$F(A \cap B) = \min_{\varrho \in A \cap B} f(\varrho) \geqslant \max\left\{\min_{\varrho \in A} f(\varrho), \min_{\varrho \in B} f(\varrho)\right\} = \max\left\{F(A), F(B)\right\}.$$

Therefore, $F(A \cap B) > F(C)$ and $\succsim_{\mathcal{P}}^{m}$ is inf-consistent. □

PROPOSITION 4.7: *Given $f : \mathcal{P} \to \mathbb{R}$, let $\succsim_{\mathcal{P}}^{M}$ be the corresponding maximax preference. Then,*

i) $\succsim_{\mathcal{P}}^{M}$ *is imprecision loving.*

ii) If f is a quasi-concave function then $\succsim_{\mathcal{P}}^{M}$ is a semi-strictly convex preference.

iii) $\succsim_{\mathcal{P}}^{M}$ *is sup-consistent*

Proof. i) See Proposition 4.11 in [8].

ii) Let $\varrho_A \in A$ and $\varrho_B \in B$ be such that $G(A) = g(\varrho_A)$ and $G(B) = g(\varrho_B)$. Fix $t \in [0,1]$, then $t\varrho_A + (1-t)\varrho_B \in tA + (1-t)B$ which implies that $G(tA + (1-t)B) \geqslant f(t\varrho_A + (1-t)\varrho_B)$. Since f is quasi-concave, it follows that

$$G(tA + (1-t)B) \geqslant f(t\varrho_A + (1-t)\varrho_B) \geqslant \min\{f(\varrho_A), f(\varrho_B)\} = \min\{G(A), G(B)\}.$$

Now, if $A \succ_{\mathcal{P}}^{M} C$ and $B \succ_{\mathcal{P}}^{M} C$ then $G(A) > G(C)$ and $G(B) > G(C)$. Therefore, $G(tA+(1-t)B) > G(C)$ and $tA + (1-t)B \succ_{\mathcal{P}}^{M} C$. So $\succsim_{\mathcal{P}}^{M}$ is semi-strictly convex.

iii) Let A, B, C be subsets of $\mathbb{K}(\mathcal{P})$ such that

$$G(A) = f(\varrho_A) > G(C),\ G(B) = f(\varrho_B) > G(C) \quad \text{where } \varrho_A \in A,\ \varrho_B \in B$$

It follows that

$$G(A \cup B) = \max\{f(\varrho_A), f(\varrho_B)\} > G(C).$$

so that $\succsim_{\mathcal{P}}^{M}$ is sup-consistent. □

PROPOSITION 4.8: *Given $f : \mathcal{P} \to \mathbb{R}$, let $\succsim_{\mathcal{P}}^{ID}$ be the corresponding interval dominance preference. Then,*

i) *$\succsim_{\mathcal{P}}^{ID}$ is imprecision averse and imprecision loving*

ii) *If f is a concave function, then $\succsim_{\mathcal{P}}^{ID}$ is a semi-strictly convex preference.*

i) *$\succsim_{\mathcal{P}}^{ID}$ is sup-consistent and inf-consistent.*

Proof. i) See Proposition 4.14 in [8].

ii) Suppose that $A \succ_{\mathcal{P}}^{D} C$ and $B \succ_{\mathcal{P}}^{D} C$, meaning that $F(A), F(B) > G(C)$. For every $t \in [0, 1]$, $F(tA + (1-t)B) = f(t\varrho_A + (1-t)\varrho_B)$ for some $\varrho_A \in A$ and $\varrho_B \in B$. By definition, it follows that $f(\varrho_A) \geqslant F(A)$ and $f(\varrho_B) \geqslant F(B)$. Since f a concave function, it follows that $f(t\varrho_A + (1-t)\varrho_B) \geqslant tf(\varrho_A) + (1-t)f(\varrho_B)$. Summarizing:

$$F(tA + (1-t)B) = f(t\varrho_A + (1-t)\varrho_B) \geqslant tf(\varrho_A) + (1-t)f(\varrho_B) \geqslant tF(A) + (1-t)F(B) > G(S).$$

Therefore, $tA + (1-t)B \succ_{\mathcal{P}}^{ID} C$. So $\succsim_{\mathcal{P}}^{ID}$ is semi-strictly convex.

iii) Let A, B, C be subsets of $\mathbb{K}(\mathcal{P})$ such that

$$F(A) = f(\varrho_A) > G(C),\ F(B) = f(\varrho_B) > G(C) \quad \text{where } \varrho_A \in A,\ \varrho_B \in B$$

It follows that

$$F(A \cup B) = \min\{f(\varrho_A), f(\varrho_B)\} > G(C).$$

so that $\succsim_{\mathcal{P}}^{ID}$ is sup-consistent. Moreover, if A and B are such that $A \cap B \neq \emptyset$ then

$$F(A \cap B) = \min_{\varrho \in A \cap B} f(\varrho) \geqslant \max\left\{\min_{\varrho \in A} f(\varrho), \min_{\varrho \in B} f(\varrho)\right\} = \max\{F(A), F(B)\} > G(C)$$

and $F(A \cap B) > G(C)$ and $\succsim_{\mathcal{P}}^{ID}$ is inf-consistent. □

5 Convexity of preferences

In this section, the concepts introduced in the previous ones are used in order to guarantee the Shafer-Sonnenschein relaxed convexity property of the preference relation $\succsim$ over X. In all the results presented, the assumption of superset aversion (respectively subset aversion) of the preference $\succsim_{\mathcal{T}}$ is combined with a (quasi-)convexity (respectively (quasi-)concavity) assumption on the imprecise probability correspondence $\mathcal{C}$. In particular, three results involve the superset aversion assumption and three others involve subset aversion. In the first result (Theorem 5.1), superset aversion is combined with the assumption of convexity of $\succsim$; these two assumptions alone are not enough as the Shafer-Sonnenschein relaxed convexity of $\mathcal{C}$ is obtained by imposing the additional assumption of semi-strict convexity on $\succsim_{\mathcal{T}}$. This latter assumption is, instead, completely removed in the second and third result in which the convexity of $\mathcal{C}$ is replaced by a version of quasi-convexity. In particular, in case of strong quasi-convexity of $\mathcal{C}$ (Theorem 5.2), no assumption other than superset aversion is imposed on $\succsim_{\mathcal{T}}$. In case strong quasi-convexity is (significatively) relaxed by quasi-convexity of $\mathcal{C}$, then the superset aversion assumption is combined with the sup-consistency assumption on $\succsim_{\mathcal{T}}$ (Theorem 5.3). Theorems 5.4, 5.5 and 5.6 involve, instead, the subset aversion assumption in place of superset aversion. In this case, the Shafer-Sonnenschein relaxed convexity property of $\succsim$ is obtained by replacing, in Theorems 5.1, 5.2 and 5.3, the assumptions of convexity or (strong) quasi-convexity of $\mathcal{C}$ with the assumptions of concavity or (strong) quasi-concavity of $\mathcal{C}$, respectively, and the assumption of sup-consistency with the assumption of inf-consistency. However, these latter results are not perfectly symmetric as in Theorem 5.6 an additional assumption must be imposed; this is not completely surprising because in section 2 it has already been shown that the notions of quasi-concavity and quasi-convexity for set-valued maps are not perfectly symmetric. As a final remark, note that Theorems 5.1, 5.4 are a straightforward extension of Propositions 4.3 (2) and 4.4 (2) in [8] to the present model; nevertheless, a self contained (and slightly different) proof of these results is given below, for the sake of completeness.

5.1 (Quasi-)convex correspondences and convexity of preferences

THEOREM 5.1: *Assume that the following conditions hold:*

$$\begin{cases} (M1) & \mathcal{C}\, is\, convex \\ (a) & \succsim_{\mathcal{T}}\, is\, superset\, averse \\ (c1) & \succsim_{\mathcal{T}}\, is\, semi-strictly\, convex \end{cases}.$$

Then, $x \notin co(\mathcal{U}(x))$.

Proof. Suppose that $x \in co(\mathcal{U}(x))$, then there exist x' and x'' in $\mathcal{U}(x)$ and $t \in]0,1[$ such that $x = tx' + (1-t)x''$. It follows that $\mathcal{C}(x') \succ_{\mathcal{T}} \mathcal{C}(x)$, $\mathcal{C}(x'') \succ_{\mathcal{T}} \mathcal{C}(x)$ and $\mathcal{C}(x) = \mathcal{C}(tx' + (1-t)x'')$. Denote with $S = t\mathcal{C}(x') + (1-t)\mathcal{C}(x'')$, then assumption (M1) implies that $\mathcal{C}(x) \subseteq S$. So $S \not\succ_{\mathcal{T}} \mathcal{C}(x)$ because of assumption (a). On the other hand, assumption (c) implies that $\mathcal{U}_{\mathcal{T}}(\mathcal{C}(x))$ is a convex set so that $S \in \mathcal{U}_{\mathcal{T}}(\mathcal{C}(x))$ and $S \succ_{\mathcal{T}} \mathcal{C}(x)$. Therefore, we get a contradiction and $x \notin co(\mathcal{U}(x))$. □

THEOREM 5.2: *Assume that the following conditions hold:*

$$\begin{cases} (M3) & \mathcal{C}\, is\, strongly\, quasi-convex \\ (a) & \succsim_{\mathcal{T}}\, is\, superset\, averse \end{cases}$$

Then $x \notin co(\mathcal{U}(x))$.

Proof. Suppose that $x \in co(\mathcal{U}(x))$, then there exist x' and x'' in $\mathcal{U}(x)$ and $t \in]0,1[$ such that $x = tx' + (1-t)x''$. It follows that $\mathcal{C}(x') \succ_{\mathcal{T}} \mathcal{C}(x)$, $\mathcal{C}(x'') \succ_{\mathcal{T}} \mathcal{C}(x)$ and $\mathcal{C}(x) = \mathcal{C}(tx' + (1-t)x'')$. From $(M3)$ it follows that $\mathcal{C}(tx' + (1-t)x'') \subseteq \mathcal{C}(x')$ or $\mathcal{C}(tx' + (1-t)x'') \subseteq \mathcal{C}(x'')$. If $\mathcal{C}(x) = \mathcal{C}(tx' + (1-t)x'') \subseteq \mathcal{C}(x')$, then (a) implies that $\mathcal{C}(x') \not\succ_{\mathcal{T}} \mathcal{C}(x)$ which is a contradiction. Similarly, if $\mathcal{C}(x) = \mathcal{C}(tx' + (1-t)x'') \subseteq \mathcal{C}(x'')$, then (a) implies that $\mathcal{C}(x'') \not\succ_{\mathcal{T}} \mathcal{C}(x)$ which is a contradiction as well. Hence, $x \notin co(\mathcal{U}(x))$. □

THEOREM 5.3: *Assume that the following conditions hold:*

$$\begin{cases} (M2) & \mathcal{C}\, is\, quasi-convex \\ (a) & \succsim_{\mathcal{T}}\ is\, superset\, averse \\ (u) & \succsim_{\mathcal{T}}\ is\, sup-consistent \end{cases}$$

Then $x \notin co(\mathcal{U}(x))$.

Proof. Suppose that $x \in co(\mathcal{U}(x))$, then there exist x' and x'' in $\mathcal{U}(x)$ and $t \in]0,1[$ such that $x = tx' + (1-t)x''$. It follows that $\mathcal{C}(x') \succ_{\mathcal{T}} \mathcal{C}(x)$, $\mathcal{C}(x'') \succ_{\mathcal{T}} \mathcal{C}(x)$ and $\mathcal{C}(x) = \mathcal{C}(tx' + (1-t)x'')$. Hence, assumption (u) implies that

$$\mathcal{C}(x') \cup \mathcal{C}(x'') \succ_{\mathcal{T}} \mathcal{C}(x).$$

On the other hand, assumption (M2) implies that

$$\mathcal{C}(x) = \mathcal{C}(tx' + (1-t)x'') \subseteq \mathcal{C}(x') \cup \mathcal{C}(x'').$$

Therefore, assumption (a) implies that $\mathcal{C}(x') \cup \mathcal{C}(x'') \not\succ_{\mathcal{T}} \mathcal{C}(x)$ which is a contradiction. So $x \notin co(\mathcal{U}(x))$. □

5.2 (Quasi-)concave correspondences and convexity of preferences

THEOREM 5.4: *Assume that the following conditions hold:*

$$\begin{cases} (N1) & \mathcal{C}\, is\, concave \\ (b) & \succsim_{\mathcal{T}}\ is\, subset\, averse \\ (c1) & \succsim_{\mathcal{T}}\ is\, semi-strictly\, convex \end{cases}.$$

Then, $x \notin co(\mathcal{U}(x))$.

Proof. Suppose that $x \in co(\mathcal{U}(x))$, then there exist x' and x'' in $\mathcal{U}(x)$ and $t \in]0,1[$ such that $x = tx' + (1-t)x''$. It follows that $\mathcal{C}(x') \succ_{\mathcal{T}} \mathcal{C}(x)$, $\mathcal{C}(x'') \succ_{\mathcal{T}} \mathcal{C}(x)$ and $\mathcal{C}(x) = \mathcal{C}(tx' + (1-t)x'')$. Denote with $S = t\mathcal{C}(x') + (1-t)\mathcal{C}(x'')$, then assumption (N1) implies that $S \subseteq \mathcal{C}(x)$. So $S \not\succ_{\mathcal{T}} \mathcal{C}(x)$ because of assumption (b). On the other hand, assumption (c1) implies that $\mathcal{U}_{\mathcal{T}}(\mathcal{C}(x))$ is a convex set so that $S \in \mathcal{U}_{\mathcal{T}}(\mathcal{C}(x))$ and $S \succ_{\mathcal{T}} \mathcal{C}(x)$. Therefore, we get a contradiction and $x \notin co(\mathcal{U}(x))$. □

THEOREM 5.5: *Assume that the following conditions hold:*

$$\begin{cases} (N3) & \mathcal{C}\, is\, strongly\, quasi-concave. \\ (b) & \succsim_{\mathcal{T}}\ is\, subset\, averse \end{cases}$$

Then $x \notin co(\mathcal{U}(x))$.

Proof. Suppose that $x \in co(\mathcal{U}(x))$, then there exist x' and x'' in $\mathcal{U}(x)$ and $t \in]0,1[$ such that $x = tx' + (1-t)x''$. It follows that $\mathcal{C}(x') \succ_{\mathcal{T}} \mathcal{C}(x)$, $\mathcal{C}(x'') \succ_{\mathcal{T}} \mathcal{C}(x)$ and $\mathcal{C}(x) = \mathcal{C}(tx' + (1-t)x'')$. Assumption $(N3)$ implies that $\mathcal{C}(x') \subseteq \mathcal{C}(tx' + (1-t)x'')$ or $\mathcal{C}(x'') \subseteq \mathcal{C}(tx' + (1-t)x'')$. If $\mathcal{C}(x') \subseteq C(tx' + (1-t)x'') = \mathcal{C}(x)$, then assumption (b) implies that $\mathcal{C}(x') \not\succ_{\mathcal{T}} \mathcal{C}(x)$ which is a contradiction. Similarly, if $\mathcal{C}(x'') \subseteq \mathcal{C}(tx' + (1-t)x'') = \mathcal{C}(x)$, then (b) implies that $\mathcal{C}(x'') \not\succ_{\mathcal{T}} \mathcal{C}(x)$ which is a contradiction as well. Hence, $x \notin co(\mathcal{U}(x))$.

□

Theorem 5.6: *Assume that the following conditions hold:*

$$\begin{cases} (N2) & \mathcal{C}\, is\, quasi-concave \\ (b) & \succsim_{\mathcal{T}}\ \ is\, subset\, averse \\ (l) & \succsim_{\mathcal{T}}\ \ is\, inf-consistent \\ (\eta) & \mathcal{C}(x') \succ_{\mathcal{T}} A\, and\, \mathcal{C}(x'') \succ_{\mathcal{T}} A\, for\, some\, A \subseteq \mathcal{T} \implies \mathcal{C}(x') \cap \mathcal{C}(x'') \neq \emptyset \end{cases}$$

Then $x \notin co(\mathcal{U}(x))$.

Proof. Suppose that $x \in co(\mathcal{U}(x))$, then there exist x' and x'' in $\mathcal{U}(x)$ and $t \in]0,1[$ such that $x = tx' + (1-t)x''$. It follows that $\mathcal{C}(x') \succ_{\mathcal{T}} \mathcal{C}(x)$, $\mathcal{C}(x'') \succ_{\mathcal{T}} \mathcal{C}(x)$ and $\mathcal{C}(x) = \mathcal{C}(tx' + (1-t)x'')$. Then, assumptions (l) and (η) imply that

$$\mathcal{C}(x') \cap \mathcal{C}(x'') \succ_{\mathcal{T}} \mathcal{C}(x)$$

On the other hand, assumption $(N2)$ implies that

$$\mathcal{C}(x') \cap \mathcal{C}(x'') \subseteq \mathcal{C}(tx' + (1-t)x'') = \mathcal{C}(x)$$

Therefore, assumption (b) implies that $\mathcal{C}(x') \cap \mathcal{C}(x'') \not\succ_{\mathcal{T}} \mathcal{C}(x)$ which is a contradiction. So $x \notin co(\mathcal{U}(x))$. □

Remark 5.7: The assumption (η) in the previous theorem is obviously satisfied when

$$\bigcap_{x \in X} \mathcal{C}(x) \neq \emptyset.$$

However, the previous condition is rather demanding. There are many other examples in which assumption (η) is satisfied. For instance, given $W \in \mathbb{K}(\mathcal{T})$, with $W \neq \emptyset$, consider a maxmin preference $\succsim_{\mathcal{T}}^m$ corresponding the function $f_W : \mathbb{K}(\mathcal{T}) \to \mathbb{R}$ defined by

$$f_W(K) = \begin{cases} 1 & if \quad W \subseteq K \\ 0 & otherwise \end{cases}$$

Now, if $\mathcal{C}(x') \succ_{\mathcal{T}} A$ and $\mathcal{C}(x'') \succ_{\mathcal{T}} A$ for some $A \subseteq \mathcal{T}$, then it follows that $f_W(\mathcal{C}(x')) = f_W(\mathcal{C}(x'')) = 1 > 0 = f_W(A)$. This implies that $\emptyset \neq W \subseteq \mathcal{C}(x') \cap \mathcal{C}(x'')$.

6 Ambiguous belief and Almost Desirable Gambles

It is well known that given a set of probability distributions, one can construct, in an natural way, a corresponding set of almost desirable gambles and, similarly, given a set of almost desirable gambles, one can construct a corresponding set of probability distributions (see for instance [28], [29], [2] or [7] and references therein). Therefore, we get an almost desirable gambles correspondence from an ambiguous belief correspondence and viceversa. More precisely, given the set-valued map $\mathcal{D} : X \rightsquigarrow \mathcal{L}$ of almost desirable gambles, the induced belief correspondence $\mathcal{B}_{\mathcal{D}} : X \rightsquigarrow \mathcal{P}$ is defined by

$$\mathcal{B}_{\mathcal{D}}(x) = \{\varrho \in \mathcal{P} \mid E_{\varrho}[Y] \geqslant 0,\ \forall Y \in \mathcal{D}(x)\},$$

where $E_\varrho[Y] = \sum_{\omega\in\Omega} \varrho(\omega)Y(\omega)$ is the expectation of Y under ϱ.

Conversely, given the ambiguous belief correspondence $\mathcal{B} : X \rightsquigarrow \mathcal{P}$, then the induced almost desirable gambles correspondence $\mathcal{D}_\mathcal{B} : X \rightsquigarrow \mathcal{L}$ is defined by

$$\mathcal{D}_\mathcal{B}(x) = \{Y \in \mathcal{L} \,|\, E_\varrho[Y] \geqslant 0, \, \forall P \in \mathcal{B}(x)\}.$$

Aim of this section is to investigate the relation between the (quasi-)convexity/concavity properties in the two representations.

6.1 Almost desirable gambles correspondences and the induced ambiguous belief

PROPOSITION 6.1: *The following implications hold:*

i) If $\mathcal{D}$ is convex in X, then $\mathcal{B}_\mathcal{D}$ is quasi-concave in X.

ii) If $\mathcal{D}$ is quasi-convex in X, then $\mathcal{B}_\mathcal{D}$ is quasi-concave in X.

iii) If $\mathcal{D}$ is strongly quasi-convex in X, then $\mathcal{B}_\mathcal{D}$ is strongly quasi-concave in X.

iv) If $\mathcal{D}$ concave in X and $0 \in \mathcal{D}(x)$ for every $x \in X$, then $\mathcal{B}_\mathcal{D}$ convex and strongly quasi-convex in X.

v) If $\mathcal{D}$ is strongly quasi-concave in X, then $\mathcal{B}_\mathcal{D}$ is strongly quasi-convex in X.

Proof. i) Given x', $x'' \in X$ and $t \in]0,1[$, we show that

$$\mathcal{D}(tx' + (1-t)x'') \subseteq t\mathcal{D}(x') + (1-t)\mathcal{D}(x'') \implies \mathcal{B}_\mathcal{D}(x') \cap \mathcal{B}_\mathcal{D}(x'') \subseteq \mathcal{B}_\mathcal{D}(tx' + (1-t)x'')$$

Assume that $\mathcal{D}(tx' + (1-t)x'') \subseteq t\mathcal{D}(x') + (1-t)\mathcal{D}(x'')$. Let $\varrho \in \mathcal{B}_\mathcal{D}(x') \cap \mathcal{B}_\mathcal{D}(x'')$ then for every $Y' \in \mathcal{D}(x')$ and for every $Y'' \in \mathcal{D}(x'')$ it follows that $E_\varrho[Y'] \geqslant 0$ and $E_\varrho[Y''] \geqslant 0$ which implies that

$$E_\varrho[tY' + (1-t)Y''] \geqslant 0. \tag{13}$$

Now, let $Y \in \mathcal{D}(tx' + (1-t)x'')$ then $Y \in t\mathcal{D}(x') + (1-t)\mathcal{D}(x'')$ so $Y = tY' + (1-t)Y''$ for some $Y' \in \mathcal{D}(x')$ and $Y'' \in \mathcal{D}(x'')$. Hence (13) implies that $E_\varrho[Y] \geqslant 0$. Therefore $\varrho \in \mathcal{B}_\mathcal{D}(tx' + (1-t)x'')$ and the assertion follows.

ii) Given x', $x'' \in X$ and $t \in]0,1[$, we show that

$$\mathcal{D}(tx' + (1-t)x'') \subseteq \mathcal{D}(x') \cup \mathcal{D}(x'') \implies \mathcal{B}_\mathcal{D}(x') \cap \mathcal{B}_\mathcal{D}(x'') \subseteq \mathcal{B}_\mathcal{D}(tx' + (1-t)x'').$$

Let $\varrho \in \mathcal{B}_\mathcal{D}(x') \cap \mathcal{B}_\mathcal{D}(x'')$ then it follows that $E_\varrho[Y] \geqslant 0$ for every $Y \in \mathcal{D}(x') \cup \mathcal{D}(x'')$; so $E_\varrho[Y] \geqslant 0$ for every $Y \in \mathcal{D}(tx' + (1-t)x'')$. and $\varrho \in \mathcal{B}_\mathcal{D}(tx' + (1-t)x'')$. So the assertion follows.

iii) Given x', $x'' \in X$ and $t \in]0,1[$, it immediately follows that

$$\mathcal{D}(tx' + (1-t)x'') \subseteq \mathcal{D}(x') \quad \text{(respectively} \quad \mathcal{D}(tx' + (1-t)x'') \subseteq \mathcal{D}(x''))$$

implies

$$\mathcal{B}_{\mathcal{D}}(x') \subseteq \mathcal{B}_{\mathcal{D}}(tx' + (1-t)x'') \quad \text{(respectively} \quad \mathcal{B}_{\mathcal{D}}(x'') \subseteq \mathcal{B}_{\mathcal{D}}(tx' + (1-t)x'')) .$$

and the assertion follows.

iv) Given x', $x'' \in X$ and $t \in]0,1[$, we show that

$$t\mathcal{D}(x') + (1-t)\mathcal{D}(x'') \subseteq \mathcal{D}(tx' + (1-t)x'') \implies \mathcal{B}_{\mathcal{D}}(tx' + (1-t)x'') \subseteq \mathcal{B}_{\mathcal{D}}(x') \cap \mathcal{B}_{\mathcal{D}}(x'')$$

Assume that $t\mathcal{D}(x')+(1-t)\mathcal{D}(x'') \subseteq \mathcal{D}(tx'+(1-t)x'')$. Let $\varrho \in \mathcal{B}_{\mathcal{D}}(tx'+(1-t)x'')$. Then $E_\varrho[Y] \geqslant 0$ for every $Y \in \mathcal{D}(tx' + (1-t)x'')$; since $\mathcal{D}$ is concave then it follows that $E_\varrho[tY' + (1-t)tY''] \geqslant 0$ for every $Y' \in \mathcal{D}(x')$ and every $Y'' \in \mathcal{D}(x'')$. Hence

$$tE_\varrho[Y'] + (1-t)E_\varrho[Y''] \geqslant 0, \quad \forall Y' \in \mathcal{D}(x') \text{ and } \forall Y'' \in \mathcal{D}(x'').$$

Since $0 \in D(x')$ then it follows that $(1-t)E_\varrho[Y''] \geqslant 0$ for every $Y'' \in \mathcal{D}(x'')$. Therefore, $\varrho \in \mathcal{B}_{\mathcal{D}}(x'')$. Similarly, $\varrho \in \mathcal{B}_{\mathcal{D}}(x')$. Hence, $\mathcal{B}_{\mathcal{D}}(tx' + (1-t)x'') \subseteq \mathcal{B}_{\mathcal{D}}(x') \cap \mathcal{B}_{\mathcal{D}}(x'')$. It immediately follows that $\mathcal{B}_{\mathcal{D}}$ is strongly quasi-convex. Moreover, $\mathcal{B}_{\mathcal{D}}(x') \cap \mathcal{B}_{\mathcal{D}}(x'') \subseteq t\mathcal{B}_{\mathcal{D}}(x') + (1-t)\mathcal{B}_{\mathcal{D}}(x'')$ so that $\mathcal{B}_{\mathcal{D}}$ is convex.

v) Given x', $x'' \in X$ and $t \in]0,1[$, it immediately follows that

$$\mathcal{D}(x') \subseteq \mathcal{D}(tx' + (1-t)x'') \quad \text{(respectively} \quad \mathcal{D}(x'') \subseteq \mathcal{D}(tx' + (1-t)x''))$$

implies

$$\mathcal{B}_{\mathcal{D}}(tx' + (1-t)x'') \subseteq \mathcal{B}_{\mathcal{D}}(x') \quad \text{(respectively} \quad \mathcal{B}_{\mathcal{D}}(tx' + (1-t)x'') \subseteq \mathcal{B}_{\mathcal{D}}(x'')) .$$

and the assertion follows. □

Counterexamples

Below some counterexamples are given. They show that not every implication of (quasi-)convexity has a specular counterpart in terms of (quasi-)concavity and viceversa. Moreover, they show that strong quasi-concavity (quasi-convexity) of $\mathcal{D}$ does not imply convexity (concavity) of $\mathcal{B}_{\mathcal{D}}$.

Example 6.2: This example shows that if $\mathcal{D}$ is quasi-concave then $\mathcal{B}_{\mathcal{D}}$ is not necessarily quasi-convex.

Consider $\Omega = \{\omega_1, \omega_2\}$. Given $\varrho \in \mathcal{P}$ then $\varrho = (\varrho_1, \varrho_2)$; since $\varrho_2 = 1 - \varrho_1$, we identify ϱ with its first component ϱ_1 that, with an abuse of notation, is denoted with ϱ. Now let $X = [0,1]$ and $\mathcal{D} : [0,1] \rightsquigarrow \mathbb{R}^2$ be defined by

$$\mathcal{D}(x) = \begin{cases} \{(y_1, y_2) \,|\, y_2 \geqslant 0\} & \text{if} \quad x \in [0, 1/2[\\ \{(y_1, y_2) \,|\, y_1 + y_2 \geqslant 0\} & \text{if} \quad x = 1/2 \\ \{(y_1, y_2) \,|\, y_1 \geqslant 0\} & \text{if} \quad x \in]1/2, 1] \end{cases} .$$

This correspondence is clearly quasi-concave as for every x', $x'' \in X$, with $x' \neq x''$ and $t \in]0,1[$ it follows that $\mathcal{D}(x') \cap \mathcal{D}(x'') \subseteq \mathcal{D}(tx' + (1-t)x'')$. Now, the corresponding $\mathcal{B}_{\mathcal{D}} : X \rightsquigarrow \mathcal{P}$ is given by

$$\mathcal{B}_{\mathcal{D}}(x) = \begin{cases} \{0\} & \text{if} \quad x \in [0, 1/2[\\ \{1/2\} & \text{if} \quad x = 1/2 \\ \{1\} & \text{if} \quad x \in]1/2, 1] \end{cases} .$$

Now, for $x' = 0$, $x'' = 1$ and $t = 1/2$ we have:

$$\mathcal{B}_{\mathcal{D}}(0) = \{0\}\,,\ \mathcal{B}_{\mathcal{D}}(1) = \{1\}\,,\ \mathcal{B}_{\mathcal{D}}(1/2) = \{1/2\}$$

It immediately follows that

$$\{1/2\} = \mathcal{B}_{\mathcal{D}}(1/2) \not\subseteq \mathcal{B}_{\mathcal{D}}(0) \cup \mathcal{B}_{\mathcal{D}}(1) = \{0,1\}\,.$$

EXAMPLE 6.3: This example shows that if $\mathcal{D}$ is strongly quasi-concave then $\mathcal{B}_{\mathcal{D}}$ is not necessarily convex.

Consider $\Omega = \{\omega_1, \omega_2\}$. Given $\varrho \in \mathcal{P}$ then $\varrho = (\varrho_1, \varrho_2)$; since $\varrho_2 = 1 - \varrho_1$, we identify ϱ with its first component ϱ_1 that, with an abuse of notation, is denoted with ϱ. Let $X = [0,1]$ and $\mathcal{D} : [0,1] \rightsquigarrow \mathbb{R}^2$ be defined by

$$\mathcal{D}(x) = \begin{cases} \{(y_1,y_2)\,|\,(2x)y_1 + (1-2x)y_2 \geqslant 0 \text{ and } y_2 \geqslant 0\} & \text{if} \quad x \in [0,1/2[\\ \{(y_1,y_2)\,|\,y_1 \geqslant 0 \text{ and } y_2 \geqslant 0\} & \text{if} \quad x \in [1/2,1] \end{cases}.$$

Clearly, the correspondence $\mathcal{D}$ is strongly quasi-concave as for $x' < x''$ and $t \in]0,1[$, it follows that $\mathcal{D}(x'') \subseteq \mathcal{D}(tx' + (1-t)x'')$. The associated $\mathcal{B}_{\mathcal{D}} : X \rightsquigarrow \mathcal{P}$ is given by

$$\mathcal{B}_{\mathcal{D}}(x) = \begin{cases} [0,2x] & \text{if} \quad x \in [0,1/2[\\ [0,1] & \text{if} \quad x \in [1/2,1[\end{cases}.$$

Now, for $x' = 0$, $x'' = 1$ and $t = 1/2$ we have:

$$\mathcal{B}_{\mathcal{D}}(0) = \{0\}\,,\ \mathcal{B}_{\mathcal{D}}(1) = \mathcal{B}_{\mathcal{D}}(1/2) = [0,1]$$

It immediately follows that

$$[0,1] = \mathcal{B}_{\mathcal{D}}(1/2) \not\subseteq \frac{1}{2}\mathcal{B}_{\mathcal{D}}(0) + \frac{1}{2}\mathcal{B}_{\mathcal{D}}(1) = [0,1/2]$$

EXAMPLE 6.4: This example shows that if $\mathcal{D}$ is convex or strongly quasi-convex then $\mathcal{B}_{\mathcal{D}}$ is not necessarily concave.

Consider $\Omega = \{\omega_1, \omega_2\}$. Given $\varrho \in \mathcal{P}$ then $\varrho = (\varrho_1, \varrho_2)$; since $\varrho_2 = 1 - \varrho_1$, we identify ϱ with its first component ϱ_1 that, with an abuse of notation, is denoted with ϱ. Now let $X = [0,1]$ and $\mathcal{D} : [0,1] \rightsquigarrow \mathbb{R}^2$ be defined by

$$\mathcal{D}(x) = \begin{cases} \{(y_1,y_2)\,|\,y_2 \geqslant 0\} & \text{if} \quad x \in [0,1[\\ \{(y_1,y_2)\,|\,y_1 \geqslant 0 \text{ and } y_2 \geqslant 0\} & \text{if} \quad x = 1 \end{cases}.$$

This correspondence is clearly convex. In fact, for every x', $x'' \in X$ and $t \in]0,1[$ it follows that

$$t\mathcal{D}(x') + (1-t)\mathcal{D}(x'') = \{(y_1,y_2)\,|\,y_2 \geqslant 0\} = \mathcal{D}(tx' + (1-t)x'')$$

Moreover it is strongly quasi-convex as for $x' < x''$ and $t \in]0,1[$ it immediately follows that $\mathcal{D}(tx' + (1-t)x'') \subseteq \mathcal{D}(x')$.

Now, the corresponding $\mathcal{B}_{\mathcal{D}} : X \rightsquigarrow \mathcal{P}$ is given by

$$\mathcal{B}_{\mathcal{D}}(x) = \begin{cases} \{0\} & \text{if} \quad x \in [0,1[\\ [0,1] & \text{if} \quad x = 1 \end{cases} .$$

Now, for $x' = 0$, $x'' = 1$ and $t = 1/2$ we have:

$$\mathcal{B}_{\mathcal{D}}(0) = \{0\}\,, \mathcal{B}_{\mathcal{D}}(1) = [0,1],\, \mathcal{B}_{\mathcal{D}}(1/2) = \{0\}$$

It immediately follows that

$$\frac{1}{2}\mathcal{B}_{\mathcal{D}}(0) + \frac{1}{2}\mathcal{B}_{\mathcal{D}}(1) = [0,1/2] \nsubseteq \{0\} = \mathcal{B}_{\mathcal{D}}(1/2)$$

6.2 Ambiguous belief correspondences and the induced almost desirable gambles

Proposition 6.5: *The following implications hold:*

i) If $\mathcal{B}$ is convex in X, then $\mathcal{D}_{\mathcal{B}}$ is quasi-concave in X.

ii) If $\mathcal{B}$ is quasi-convex in X, then $\mathcal{D}_{\mathcal{B}}$ is quasi-concave in X.

iii) If $\mathcal{B}$ is strongly quasi-convex in X, then $\mathcal{D}_{\mathcal{B}}$ is strongly quasi-concave in X.

iv) If $\mathcal{B}$ is concave in X, then $\mathcal{D}_{\mathcal{B}}$ is quasi-convex and convex in X.

v) If $\mathcal{B}$ is strongly quasi-concave in X, then $\mathcal{D}_{\mathcal{B}}$ is convex and strongly quasi-convex in X.

Proof. i) Given x', $x'' \in X$ and $t \in]0,1[$, we show that

$$\mathcal{B}(tx' + (1-t)x'') \subseteq t\mathcal{B}(x') + (1-t)\mathcal{B}(x'')$$

implies that

$$\mathcal{D}_{\mathcal{B}}(x') \cap \mathcal{D}_{\mathcal{B}}(x'') \subseteq \mathcal{D}_{\mathcal{B}}(tx' + (1-t)x''). \tag{14}$$

Let $Y \in \mathcal{D}_{\mathcal{B}}(x') \cap \mathcal{D}_{\mathcal{B}}(x'')$, then

$$E_{\varrho'}[Y] \geqslant 0 \quad \forall \varrho' \in \mathcal{B}(x') \quad \text{and} \quad E_{\varrho''}[Y] \geqslant 0 \quad \forall \varrho'' \in \mathcal{B}(x'')$$

Therefore

$$E_{t\varrho' + (1-t)\varrho''}[Y] = tE_{\varrho'}[Y] + (1-t)E_{\varrho''}[Y] \geqslant 0 \; \forall \varrho' \in \mathcal{B}(x'), \forall \varrho'' \in \mathcal{B}(x'')$$

Since $\mathcal{B}$ is convex then every $\varrho \in \mathcal{B}(tx' + (1-t)x'')$ can be written as $\varrho = t\varrho' + (1-t)\varrho''$ for some $\varrho' \in \mathcal{B}(x')$ and $\varrho'' \in \mathcal{B}(x'')$, Then,

$$E_{\varrho}[Y] \geqslant 0 \quad \forall \varrho \in \mathcal{B}(tx' + (1-t)x'')$$

that implies that $Y \in \mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'')$ and (14) holds.

ii) Given x', $x'' \in X$ and $t \in]0,1[$, we show that

$$\mathcal{B}(tx' + (1-t)x'') \subseteq \mathcal{B}(x') \cup \mathcal{B}(x'')$$

implies

$$\mathcal{D}_{\mathcal{B}}(x') \cap \mathcal{D}_{\mathcal{B}}(x'') \subseteq \mathcal{D}_{\mathcal{B}}(tx' + (1-t)x''). \tag{15}$$

Let $Y \in \mathcal{D}_{\mathcal{B}}(x') \cap \mathcal{D}_{\mathcal{B}}(x'')$, then

$$E_{\varrho'}[Y] \geqslant 0 \; \forall \varrho' \in \mathcal{B}(x') \quad \text{and} \quad E_{\varrho''}[Y] \geqslant 0 \quad \forall \varrho'' \in \mathcal{B}(x'')$$

Therefore,

$$E_{\varrho}[Y] \geqslant 0 \quad \forall \varrho \in \mathcal{B}(x') \cup \mathcal{B}(x'').$$

$\mathcal{B}$ is quasi-convex, so

$$E_{\varrho}[Y] \geqslant 0 \quad \forall \varrho \in \mathcal{B}(tx' + (1-t)x'')$$

that finally implies

$$Y \in \mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'')$$

Hence, $\mathcal{D}_{\mathcal{B}}(x') \cap \mathcal{D}_{\mathcal{B}}(x'') \subseteq \mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'')$ and the assertion follows.

iii) Given x', $x'' \in X$ and $t \in]0,1[$, it immediately follows that

$$\mathcal{B}(tx' + (1-t)x'') \subseteq \mathcal{B}(x') \quad (\text{respectively} \quad \mathcal{B}(tx' + (1-t)x'') \subseteq \mathcal{B}(x''))$$

implies

$$\mathcal{D}_{\mathcal{B}}(x') \subseteq \mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'') \quad (\text{respectively} \quad \mathcal{D}_{\mathcal{B}}(x'') \subseteq \mathcal{D}_{\mathcal{B}}(tx' + (1-t)x''))\,.$$

and the assertion follows.

iv) Given x', $x'' \in X$ and $t \in]0,1[$, we show that

$$t\mathcal{B}(x') + (1-t)\mathcal{B}(x'') \subseteq \mathcal{B}(tx' + (1-t)x'')$$

implies

$$\mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'') \subseteq \mathcal{D}_{\mathcal{B}}(x') \cup \mathcal{D}_{\mathcal{B}}(x'')$$

and

$$\mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'') \subseteq t\mathcal{D}_{\mathcal{B}}(x') + (1-t)\mathcal{D}_{\mathcal{B}}(x'')$$

Let Y in $\mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'')$. It follows that

$$E_{\varrho}[Y] \geqslant 0 \quad \forall \varrho \in \mathcal{B}(tx' + (1-t)x'')$$

Since $\mathcal{B}$ is concave, it follows that

$$E_{t\varrho' + (1-t)\varrho''}[Y] = tE_{\varrho'}[Y] + (1-t)E_{\varrho''} \geqslant 0 \quad \forall \varrho' \in \mathcal{B}(x'), \quad \forall \varrho'' \in \mathcal{B}(x'').$$

Therefore, if there exists $\varrho'' \in \mathcal{B}(x'')$ such that $E_{\varrho''}[Y] < 0$ then it must follow that $E_{\varrho'}[Y] > 0$ for every $\varrho' \in \mathcal{B}(x')$, that is $Y \in \mathcal{D}_{\mathcal{B}}(x')$ that in particular gives $Y \in \mathcal{D}_{\mathcal{B}}(x') \cup \mathcal{D}_{\mathcal{B}}(x'')$. Moreover, if $Y \in \mathcal{D}_{\mathcal{B}}(x')$ then it is clear that $E_{\varrho'}\left[\frac{1}{t}Y\right] \geqslant 0$ for every $\varrho' \in \mathcal{B}(x')$, so that $\frac{1}{t}Y \in \mathcal{D}_{\mathcal{B}}(x')$. Since $0 \in \mathcal{D}_{\mathcal{B}}(x') \cap \mathcal{D}_{\mathcal{B}}(x'')$, then

$$Y = t\left(\frac{1}{t}Y\right) + (1-t)0 \in t\mathcal{D}_{\mathcal{B}}(x') + (1-t)\mathcal{D}_{\mathcal{B}}(x'').$$

Similarly, if there exists $\varrho' \in \mathcal{B}(x')$ such that $E_{\varrho'}[Y] < 0$ then $Y \in \mathcal{D}_{\mathcal{B}}(x'')$ which implies that $Y \in \mathcal{D}_{\mathcal{B}}(x') \cup \mathcal{D}_{\mathcal{B}}(x'')$ and $Y \in t\mathcal{D}_{\mathcal{B}}(x') + (1-t)\mathcal{D}_{\mathcal{B}}(x'')$ as well. So conditions $\mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'') \subseteq \mathcal{D}_{\mathcal{B}}(x') \cup \mathcal{D}_{\mathcal{B}}(x'')$ and $\mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'') \subseteq t\mathcal{D}_{\mathcal{B}}(x') + (1-t)\mathcal{D}_{\mathcal{B}}(x'')$ hold and the assertion follows.

v) Given x', $x'' \in X$ and $t \in]0,1[$, we show that

$$\mathcal{B}(x') \subseteq \mathcal{B}(tx' + (1-t)x'') \quad \text{or} \quad \mathcal{B}(x'') \subseteq \mathcal{B}(tx' + (1-t)x'')$$

implies

$$\mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'') \subseteq \mathcal{D}_{\mathcal{B}}(x') \quad \text{or} \quad \mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'') \subseteq \mathcal{D}_{\mathcal{B}}(x'')$$

and

$$\mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'') \subseteq t\mathcal{D}_{\mathcal{B}}(x') + (1-t)\mathcal{D}_{\mathcal{B}}(x'')$$

Let Y in $\mathcal{D}_{\mathcal{B}}(tx'+(1-t)x'')$ and suppose that $\mathcal{B}(x') \subseteq \mathcal{B}(tx'+(1-t)x'')$. It follows that $E_\varrho[Y] \geqslant 0$ for all $\varrho \in \mathcal{B}(x')$. Therefore $Y \in \mathcal{D}_{\mathcal{B}}(x')$. Following the same steps in the proof of *iv)*, it follows that if $Y \in \mathcal{D}_{\mathcal{B}}(x')$ then $E_{\varrho'}\left[\frac{1}{t}Y\right] \geqslant 0$ for every $\varrho' \in \mathcal{B}(x')$, so that $\frac{1}{t}Y \in \mathcal{D}_{\mathcal{B}}(x')$. Since $0 \in \mathcal{D}_{\mathcal{B}}(x') \cap \mathcal{D}_{\mathcal{B}}(x'')$, then

$$Y = t\left(\frac{1}{t}Y\right) + (1-t)0 \in t\mathcal{D}_{\mathcal{B}}(x') + (1-t)\mathcal{D}_{\mathcal{B}}(x'').$$

Similar arguments hold when $\mathcal{B}(x'') \subseteq \mathcal{B}(tx' + (1-t)x'')$ as it follows that $Y \in \mathcal{D}_{\mathcal{B}}(x'')$ and

$$Y = t\,(0) + (1-t)\left(\frac{1}{1-t}Y\right) \in t\mathcal{D}_{\mathcal{B}}(x') + (1-t)\mathcal{D}_{\mathcal{B}}(x'').$$

So

$$\mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'') \subseteq \mathcal{D}_{\mathcal{B}}(x') \quad \text{or} \quad \mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'') \subseteq \mathcal{D}_{\mathcal{B}}(x'')$$

and

$$\mathcal{D}_{\mathcal{B}}(tx' + (1-t)x'') \subseteq t\mathcal{D}_{\mathcal{B}}(x') + (1-t)\mathcal{D}_{\mathcal{B}}(x''),$$

and the assertion follows. □

Counterexamples

Similarly as done in subsection 6.1, counterexamples are now given showing that (quasi-)concavity and (quasi-)convexity are not symmetric concepts even in the previous Proposition.

Example 6.6: This example shows that if $\mathcal{B}$ is convex then $\mathcal{D}_{\mathcal{B}}$ is not necessarily concave.

Consider $\Omega = \{\omega_1, \omega_2\}$. Given $\varrho \in \mathcal{P}$ then $\varrho = (\varrho_1, \varrho_2)$; since $\varrho_2 = 1 - \varrho_1$, we identify ϱ with its first component ϱ_1 that, with an abuse of notation, is denoted with ϱ. Now let $X = [0,1]$ and $\mathcal{B} : X \rightsquigarrow \mathcal{P}$ defined by

$$\mathcal{B}(x) = \{x\} \quad \forall x \in [0,1]$$

This correspondence is clearly convex.The corresponding correspondence $\mathcal{D}_{\mathcal{B}} : [0,1] \rightsquigarrow \mathbb{R}^2$ is defined by

$$\mathcal{D}_{\mathcal{B}}(x) = \{(y_1, y_2) \mid xy_1 + (1-x)\,y_2 \geqslant 0\} \quad \forall x \in [0,1].$$

Now, for $x' = 0$, $x'' = 1$ and $t = 1/2$ we have:

$$\mathcal{D}_{\mathcal{B}}(0) = \{(y_1, y_2) \mid y_2 \geqslant 0\}\,, \mathcal{D}_{\mathcal{B}}(1) = \{(y_1, y_2) \mid y_1 \geqslant 0\}\,,$$

$$\mathcal{D}_{\mathcal{B}}(1/2) = \left\{ (y_1, y_2) \;\middle|\; \frac{1}{2}y_1 + \frac{1}{2}y_2 \geqslant 0 \right\}.$$

It immediately follows that

$$\frac{1}{2}\mathcal{D}_{\mathcal{B}}(0) + \frac{1}{2}\mathcal{D}_{\mathcal{B}}(1) = \mathbb{R}^2 \not\subseteq \mathcal{D}_{\mathcal{B}}(1/2).$$

Example 6.7: This example shows that if $\mathcal{B}$ is strongly quasi-convex then $\mathcal{D}_{\mathcal{B}}$ is not necessarily concave.

Consider $\Omega = \{\omega_1, \omega_2\}$. Given $\varrho \in \mathcal{P}$ with $\varrho = (\varrho_1, \varrho_2)$; since $\varrho_2 = 1 - \varrho_1$, we identify ϱ with its first component ϱ_1 that, with an abuse of notation, is denoted with ϱ. Now let $X = [0,1]$ and $\mathcal{B} : X \rightsquigarrow \mathcal{P}$ defined by

$$\mathcal{B}(x) = \begin{cases} [0, 2x] & \text{if} \quad x \in [0, 1/2[\\ [0, 1] & \text{if} \quad x \in [1/2, 1] \end{cases}$$

This correspondence is clearly strongly quasi-convex as for every $t \in]0,1[$ and $x' < x''$ it follows that $\mathcal{B}(tx' + (1-t)x'') \subseteq \mathcal{B}(x'')$. The induced correspondence $\mathcal{D}_{\mathcal{B}} : [0,1] \rightsquigarrow \mathbb{R}^2$ is defined by

$$\mathcal{D}_{\mathcal{B}}(x) = \begin{cases} \{(y_1, y_2) \,|\, (2x)y_1 + (1-2x)\, y_2 \geqslant 0,\ y_2 \geqslant 0\} & \text{if} \quad x \in [0, 1/2[\\ \{(y_1, y_2) \,|\, y_1 \geqslant 0,\ y_2 \geqslant 0\} & \text{if} \quad x \in [1/2, 1] \end{cases}.$$

Now, for $x' = 0$, $x'' = 1$ and $t = 1/2$ we have:

$$\mathcal{D}_{\mathcal{B}}(0) = \{(y_1, y_2) \mid y_2 \geqslant 0\}\,,\ \mathcal{D}_{\mathcal{B}}(1) = \{(y_1, y_2) \mid y_1 \geqslant 0,\ y_2 \geqslant 0\}\,,$$
$$\mathcal{D}_{\mathcal{B}}(1/2) = \{(y_1, y_2) \mid y_1 \geqslant 0,\ y_2 \geqslant 0\}\,.$$

It immediately follows that

$$(1/2)\mathcal{D}_{\mathcal{B}}(0) + (1/2)\mathcal{D}_{\mathcal{B}}(1) = \{(y_1, y_2) \mid y_2 \geqslant 0\} \not\subseteq \mathcal{D}_{\mathcal{B}}(1/2)$$

so that $\mathcal{D}_{\mathcal{B}}$ is not concave.

Example 6.8: This example shows that if $\mathcal{B}$ is quasi-concave then $\mathcal{D}_{\mathcal{B}}$ is not necessarily quasi-convex.

Consider $\Omega = \{\omega_1, \omega_2\}$. Given $\varrho \in \mathcal{P}$ with $\varrho = (\varrho_1, \varrho_2)$; since $\varrho_2 = 1 - \varrho_1$, we identify ϱ with its first component ϱ_1 that, with an abuse of notation, is denoted with ϱ. Now let $X = [0,1]$ and $\mathcal{B} : X \rightsquigarrow \mathcal{P}$ defined by

$$\mathcal{B}(x) = \begin{cases} \frac{1}{4}x & \text{if} \quad x \in [0, 1/2[\\ \frac{15}{8} - \frac{7}{4}x & \text{if} \quad x \in [1/2, 1] \end{cases}$$

This (single-valued) correspondence is clearly quasi-concave as the images have only empty intersection. The corresponding correspondence $\mathcal{D}_{\mathcal{B}} : [0,1] \rightsquigarrow \mathbb{R}^2$ is defined by

$$\mathcal{D}_{\mathcal{B}}(x) = \begin{cases} \{(y_1, y_2) \,|\, \left(\frac{1}{4}x\right) y_1 + \left(1 - \frac{1}{4}x\right) y_2 \geqslant 0\} & \text{if} \quad x \in [0, 1/2[\\ \{(y_1, y_2) \,|\, \left(\frac{15}{8} - \frac{7}{4}x\right) y_1 + \left(\frac{7}{4}x - \frac{7}{8}\right) y_2 \geqslant 0\} & \text{if} \quad x \in [1/2, 1] \end{cases}.$$

Now, for $x' = 0$, $x'' = 1$ and $t = 1/2$ we have:

$$\mathcal{D}_{\mathcal{B}}(0) = \{(y_1, y_2) \mid y_2 \geqslant 0\}\,,\ \mathcal{D}_{\mathcal{B}}(1) = \left\{ (y_1, y_2) \;\middle|\; \frac{1}{8}y_1 + \frac{7}{8}y_2 \geqslant 0 \right\},\ \mathcal{D}_{\mathcal{B}}(1/2) = \{(y_1, y_2) \mid y_1 \geqslant 0\}\,.$$

It immediately follows that

$$\mathcal{D}_{\mathcal{B}}(1/2) \not\subseteq \mathcal{D}_{\mathcal{B}}(0) \cup \mathcal{D}_{\mathcal{B}}(1)$$

so that $\mathcal{D}_{\mathcal{B}}$ is not quasi-convex.

7 Lower Expectations and Ambiguous Beliefs

One of the most important concepts in the theory of imprecise probabilities is given by lower expectations (see for instance [28], [29], [2] or [7] and references therein). The theory shows that the representation of uncertainty in terms of lower expectations induces a representation in terms of sets of probability distributions and almost desirable gambles and viceversa. In this section, we study the relation among the (quasi-)convexity/concavity properties of lower expectations (of gambles $Y \in \mathcal{L}$) as real valued functions in the decision variable $x \in X$ and the (quasi-)convexity/concavity properties of the induced ambiguous belief correspondence with respect to the same decision variable x and viceversa. In the next section, a specular analysis will be presented involving sets of almost desirable gambles in place of ambiguous beliefs.

Let $\mathcal{K} \subseteq \mathcal{L}$, then the function $\Pi : \mathcal{K} \times X \to \mathbb{R}$ is called *lower expectation function* if, for every $x \in X$, $\Pi(\cdot, x)$ is a lower expectation, that is, $\Pi(Y, x)$ represents the supremum buying price of the gamble Y, for every $Y \in \mathcal{K}$.

Given the well known relations between sets of probability distributions and lower expectations, for a given the ambiguous belief correspondence $\mathcal{B} : X \rightsquigarrow \mathcal{P}$, the induced lower expectation function is defined as follows:

$$\Pi_{\mathcal{B}}(Y, x) = \inf_{\varrho \in \mathcal{B}(x)} E_\varrho[Y] \quad \forall Y \in \mathcal{K}, \tag{16}$$

Conversely, given a lower expectation function Π, the ambiguous belief correspondence $\mathcal{B}_\Pi : X \rightsquigarrow \mathcal{P}$ derived form Π is defined by

$$\mathcal{B}_\Pi(x) = \{\varrho \in \mathcal{P} \mid E_\varrho[Y] \geqslant \Pi(Y, x) \quad \forall Y \in \mathcal{K}\} \quad \forall x \in X, \tag{17}$$

7.1 Ambiguous belief correspondences and the induced lower expectations

The following results analyze the relation between the properties of (quasi) concavity/convexity of $\mathcal{B}$ with the properties of (quasi) concavity/convexity of $\Pi_{\mathcal{B}}$.

THEOREM 7.1: *The following implications hold*

i) If $\mathcal{B}$ is quasi-convex in X then, $\Pi_{\mathcal{B}}(Y, \cdot)$ is quasi-concave in X for all $Y \in \mathcal{K}$.

ii) If $\mathcal{B}$ is concave in X, then $\Pi_{\mathcal{B}}(Y, \cdot)$ is convex in X for all $Y \in \mathcal{K}$.

iii) If $\mathcal{B}$ is convex in X, then $\Pi_{\mathcal{B}}(Y, \cdot)$ is concave in X for all $Y \in \mathcal{K}$.

Proof. i) Let x', $x'' \in X$ and $t \in]0, 1[$, then, from the assumptions, it follows that

$$\mathcal{B}(tx' + (1 - t)x'') \subseteq \mathcal{B}(x') \cup \mathcal{B}(x'').$$

Let $(\varepsilon_\nu)_{\nu \in \mathbb{N}}$ be a sequence converging to 0. Fixed $Y \in \mathcal{K}$, let $(\varrho_\nu)_{\nu \in \mathbb{N}} \subset \mathcal{B}(tx' + (1 - t)x'')$ be a sequence such that

$$\Pi_{\mathcal{B}}(Y, tx' + (1 - t)x'') \leqslant E_{\varrho_\nu}[Y] \leqslant \Pi_{\mathcal{B}}(Y, tx' + (1 - t)x'') + \varepsilon_\nu \quad \forall \nu \in \mathbb{N}.$$

From the assumptions it follows that, for every $\nu \in \mathbb{N}$, $\varrho_\nu \in \mathcal{B}(x') \cup \mathcal{B}(x'')$. So

$$\Pi_{\mathcal{B}}(Y, x') \leqslant E_{\varrho_\nu}[Y] \quad \text{or} \quad \Pi_{\mathcal{B}}(Y, x'') \leqslant E_{\varrho_\nu}[Y]$$

then

$$\min\{\Pi_{\mathcal{B}}(Y,x'),\Pi_{\mathcal{B}}(Y,x'')\} \leqslant E_{\varrho_\nu}[Y]$$

So

$$\min\{\Pi_{\mathcal{B}}(Y,x'),\Pi_{\mathcal{B}}(Y,x'')\} \leqslant \Pi_{\mathcal{B}}(Y,tx'+(1-t)x'')+\varepsilon_\nu \quad \forall \nu \in \mathbb{N}.$$

As $\nu \to \infty$, we get

$$\min\{\Pi_{\mathcal{B}}(Y,x'),\Pi_{\mathcal{B}}(Y,x'')\} \leqslant \Pi_{\mathcal{B}}(Y,tx'+(1-t)x'')$$

and $\Pi_{\mathcal{B}}(Y,\cdot)$ is quasi-concave. Since Y is arbitrary, the assertion follows.

ii) Let x', $x'' \in X$ and $t \in]0,1[$, then, from the assumptions, it follows that

$$t\mathcal{B}(x')+(1-t)\mathcal{B}(x'') \subseteq \mathcal{B}(tx'+(1-t)x'')$$

Fix $Y \in \mathcal{K}$ and let $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ a sequence of positive numbers converging to 0. Since $\Pi_{\mathcal{B}}(Y,x)$ is defined by (16) for every x, then there exist $\varrho'_\nu \in \mathcal{B}(x')$ and $\varrho''_\nu \in \mathcal{B}(x'')$ such that

$$\Pi_{\mathcal{B}}(Y,x')+\varepsilon_\nu > E_{\varrho'_\nu}[Y], \quad \Pi_{\mathcal{B}}(Y,x'')+\varepsilon_\nu > E_{\varrho''_\nu}[Y].$$

Since

$$\varrho_\nu = t\varrho'_\nu+(1-t)\varrho''_\nu \in t\mathcal{B}(x')+(1-t)\mathcal{B}(x'') \subseteq \mathcal{B}(tx'+(1-t)x'')$$

It follows that $E_{\varrho_\nu}[Y] \geqslant \Pi_{\mathcal{B}}(Y,tx'+(1-t)x'')$. So summarizing

$$t\Pi_{\mathcal{B}}(Y,x')+(1-t)\Pi_{\mathcal{B}}(Y,x'')+\varepsilon_\nu > tE_{\varrho'_\nu}[Y]+(1-t)E_{\varrho''_\nu}[Y] = E_{\varrho}[Y] \geqslant \Pi_{\mathcal{B}}(Y,tx'+(1-t)x'').$$

In particular,

$$t\Pi_{\mathcal{B}}(Y,x')+(1-t)\Pi_{\mathcal{B}}(Y,x'')+\varepsilon_\nu > \Pi_{\mathcal{B}}(Y,tx'+(1-t)x'') \tag{18}$$

Taking the limit as $\nu \to \infty$ in (18) we get

$$t\Pi_{\mathcal{B}}(Y,x')+(1-t)\Pi_{\mathcal{B}}(Y,x'') \geqslant \Pi_{\mathcal{B}}(Y,tx'+(1-t)x'').$$

and the assertion follows.

iv) Let x', $x'' \in X$ and $t \in]0,1[$, then, from the assumptions, it follows that

$$\mathcal{B}(tx'+(1-t)x'') \subseteq t\mathcal{B}(x')+(1-t)\mathcal{B}x'')$$

Fix $Y \in \mathcal{K}$. Let $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ a sequence of positive numbers converging to 0. Since $\Pi_{\mathcal{B}}(Y,tx'+(1-t)x'')$ is defined by (16) then, for every $\nu \in \mathbb{N}$ there exists $\varrho_\nu \in \mathcal{B}(tx'+(1-t)x'')$ such that $E_{\varrho_\nu}[Y] < \Pi_{\mathcal{B}}(Y,tx'+(1-t)x'')+\varepsilon_\nu$. From the assumptions it follows that there exist $\varrho'_\nu \in \mathcal{B}(x')$ and $\varrho''_\nu \in \mathcal{B}(x'')$ such that $t\varrho'_\nu+(1-t)\varrho''_\nu = \varrho_\nu$. It follows that

$$\Pi_{\mathcal{B}}(Y,tx'+(1-t)x'')+\varepsilon_\nu = E_{\varrho_\nu}[Y] = tE_{\varrho'_\nu}[Y]+(1-t)E_{\varrho''_\nu}[Y] \geqslant t\Pi_{\mathcal{B}}(Y,x')+(1-t)\Pi_{\mathcal{B}}(Y,x'').$$

In particular

$$\Pi_{\mathcal{B}}(Y,tx'+(1-t)x'')+\varepsilon_\nu \geqslant t\Pi_{\mathcal{B}}(Y,x')+(1-t)\Pi_{\mathcal{B}}(Y,x''). \tag{19}$$

Taking the limit as $\nu \to \infty$ in (19) we get

$$\Pi_{\mathcal{B}}(Y,tx'+(1-t)x'') = E_{\varrho}[Y] \geqslant t\Pi_{\mathcal{B}}(Y,x')+(1-t)\Pi_{\mathcal{B}}(Y,x''),$$

and the assertion follows. □

Counterexample

In the next example we show that if $\mathcal{B}$ is quasi-concave then $\Pi_{\mathcal{B}}$ is not necessarily quasi-convex.

EXAMPLE 7.2: Consider $\Omega = \{\omega_1, \omega_2\}$. Given $\varrho \in \mathcal{P}$ with $\varrho = (\varrho_1, \varrho_2)$; since $\varrho_2 = 1 - \varrho_1$, we identify ϱ with its first component ϱ_1 that, with an abuse of notation, is denoted with ϱ. Now let $X = [0,1]$ and $\mathcal{B} : X \rightsquigarrow \mathcal{P}$ defined by

$$\mathcal{B}(x) = \begin{cases} 0 & \text{if} \quad x \in [0, 1/2[\\ 1 & \text{if} \quad x = 1/2 \\ 1/2 & \text{if} \quad x \in]1/2, 1] \end{cases}$$

It is easy to check that $\mathcal{B}$ is quasi-concave but the intersection $\mathcal{B}(x') \cap \mathcal{B}(x'')$ might be empty for some x' and x''. Now we prove that the corresponding lower previson function $\Pi_{\mathcal{B}}$ is not quasi-convex for some $Y \in \mathcal{L}$.

Let $Y \in \mathcal{L}$ be the gamble defined by $(Y(\omega_1), Y(\omega_2)) = (1, 0)$, and let $x' = 0$, $x'' = 1$ and $t = 1/2$ then it follows that

$$1/2 = \max\{0, 1/2\} = \max\{\Pi_{\mathcal{B}}(Y, x'), \Pi_{\mathcal{B}}(Y, x''\} < \Pi_{\mathcal{B}}(Y, tx' + (1-t)x'') = 1$$

So $\Pi_{\mathcal{B}}(Y, \cdot)$ is not quasi-convex in X.

7.2 Lower expectation functions and the induced ambiguous beliefs

In this subsection, it is presented the relation between the properties of (quasi) concavity/convexity of Π with the properties of (quasi) concavity/convexity of $\mathcal{B}_{\Pi}$.

THEOREM 7.3: *The following implications hold*

i) If $\Pi(Y, \cdot)$ is quasi-convex in X for all $Y \in \mathcal{K}$, then $\mathcal{B}_{\Pi}$ is quasi-concave in X.

ii) If $\Pi(Y, \cdot)$ is convex in X for all $Y \in \mathcal{K}$, then $\mathcal{B}_{\Pi}$ is concave in X.

Proof. i) Let x', $x'' \in X$ and $t \in]0, 1[$, then, from the assumptions, it follows that

$$\Pi(Y, tx' + (1-t)x'') \leqslant \max\{\Pi(Y, x'), \Pi(Y, x'')\}, \, \forall Y \in \mathcal{K}.$$

Given $Y \in \mathcal{K}$, let $\varrho \in \mathcal{B}_{\Pi}(x') \cap \mathcal{B}_{\Pi}(x'')$, then $E_{\varrho}[Y] \geqslant \Pi(Y, x')$ and $E_{\varrho}[Y] \geqslant \Pi(Y, x'')$. So, it follows that $E_{\varrho}[Y] \geqslant \Pi(Y, tx' + (1-t)x'')$ for every $Y \in \mathcal{K}$ and $\varrho \in \mathcal{B}_{\Pi}(tx' + (1-t)x'')$.

ii) Let x', $x'' \in X$ and $t \in]0, 1[$, then, from the assumptions, it follows that

$$t\Pi(Y, x') + (1-t)\Pi(Y, x'') \geqslant \Pi(Y, tx' + (1-t)x''), \, \forall Y \in \mathcal{K}.$$

Let $\varrho' \in \mathcal{B}_{\Pi}(x')$ and $\varrho'' \in \mathcal{B}_{\Pi}(x'')$, $t \in]0, 1[$ and

$$\varrho = t\varrho' + (1-t)\varrho'' \in t\mathcal{B}_{\Pi}(x') + (1-t)\mathcal{B}_{\Pi}(x'').$$

Obviously, $\varrho \in \mathcal{P}$. For every random variable $Y \in \mathcal{K}$, it follows that

$$E_{\varrho}[Y] = tE_{\varrho'}[Y] + (1-t)E_{\varrho''}[Y] \geqslant \tag{20}$$

$$t\Pi(Y, x') + (1-t)\Pi(Y, x'') \geqslant \Pi(Y, tx' + (1-t)x'') \tag{21}$$

So, $E_\varrho[Y] \geqslant \Pi(Y, tx' + (1-t)x'')$ for every $Y \in \mathcal{K}$, which finally implies that

$$\varrho \in \mathcal{B}_\Pi(tx' + (1-t)x'').$$

Hence the assertion follows.

□

Counterexamples

The two examples below show respectively that, even if the lower expectation function is concave (hence quasi-concave) for every gamble $Y \in \mathcal{K}$:, then the induced ambiguos belief correspondence is not (1) convex; (2) quasi-convex.

EXAMPLE 7.4: Consider the case in which $X = [0,1]$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{K} = \{Y^1, Y^2, Y^3\}$, where

$$Y^1(\omega_1) = Y^1(\omega_2) = 1,\ Y^1(\omega_3) = 0 \quad \Pi(Y^1, x) = 1 - x^2 \tag{22}$$
$$Y^2(\omega_1) = Y^2(\omega_3) = 1,\ Y^2(\omega_2) = 0 \quad \Pi(Y^2, x) = 1 - x^2 \tag{23}$$
$$Y^3(\omega_1) = 1,\ Y^3(\omega_2) = Y^3(\omega_3) = 0 \quad \Pi(Y^3, x) = 1/2 \tag{24}$$

Clearly, $\Pi(Y^k, \cdot)$ is a concave function for $k = 1, 2, 3$. However, the associated belief correspondence $\mathcal{B}_\Pi(\cdot)$ is not a convex correspondence. In fact, $\mathcal{B}_\Pi(\cdot)$ is defined by

$$\mathcal{B}_\Pi(x) = \{(\varrho_1, \varrho_2, \varrho_3) \,|\, \varrho_1 \geqslant 1/2,\ \varrho_2, \varrho_3 \geqslant 0,\ \varrho_1 + \varrho_2 + \varrho_3 = 1,\ \varrho_1 + \varrho_2 \geqslant 1 - x^2,\ \varrho_1 + \varrho_3 \geqslant 1 - x^2\}$$

Consider $x' = 0$, $x'' = 1$ and $t = 1/2$; so that $x = x'/2 + x''/2 = 1/2$. It follows that $\mathcal{B}_\Pi(x') = \{(1,0,0)\}$. Now, it can be easily checked that $(1/2, 1/4, 1/4) \in \mathcal{B}_\Pi(x)$. Suppose that $(1/2, 1/4, 1/4) \in (1/2)\mathcal{B}_\Pi(x') + (1/2)\mathcal{B}_\Pi(x'')$ then there exists $(\varrho_1'', \varrho_2'', \varrho_3'') \in \mathcal{B}_\Pi(x'')$ such that $(1/2, 1/4, 1/4) = 1/2(1,0,0) + 1/2(\varrho_1'', \varrho_2'', \varrho_3'')$. It follows that

$$\frac{1 + \varrho_1''}{2} = \frac{1}{2} \implies \varrho_1'' = 0$$

which is a contradiction since $\varrho_1'' \geqslant 1/2$. So $(1/2, 1/4, 1/4) \notin (1/2)\mathcal{B}_\Pi(x') + (1/2)\mathcal{B}_\Pi(x'')$ and $\mathcal{B}_\Pi(\cdot)$ is not a convex set-valued map.

EXAMPLE 7.5: Consider a slight modification of the example above in which $X = [0,1]$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{K} = \{Y^1, Y^2, Y^3\}$, where

$$Y^1(\omega_1) = Y^1(\omega_2) = 1,\ Y^1(\omega_3) = 0 \quad \Pi(Y^1, x) = 1 - (x-1)^2 \tag{25}$$
$$Y^2(\omega_1) = Y^2(\omega_3) = 1,\ Y^2(\omega_2) = 0 \quad \Pi(Y^2, x) = 1 - x^2 \tag{26}$$
$$Y^3(\omega_1) = 1,\ Y^3(\omega_2) = Y^3(\omega_3) = 0 \quad \Pi(Y^3, x) = 1/2 \tag{27}$$

Clearly, $\Pi(Y^k, \cdot)$ is a concave (hence quasi-concave) function for $k = 1, 2, 3$. However, the associated belief correspondence $\mathcal{B}_\Pi(\cdot)$ is not a quasi-convex correspondence. In fact, $\mathcal{B}_\Pi(\cdot)$ is defined by

$$\mathcal{B}_\Pi(x) = \{(\varrho_1, \varrho_2, \varrho_3) \,|\, \varrho_1 \geqslant 1/2,\ \varrho_2, \varrho_3 \geqslant 0,\ \varrho_1 + \varrho_2 + \varrho_3 = 1,\ \varrho_1 + \varrho_2 \geqslant 1 - (x-1)^2,\ \varrho_1 + \varrho_3 \geqslant 1 - x^2\}$$

Consider $x' = 0$, $x'' = 1$ and $t = 1/2$; so that $x = x'/2 + x''/2 = 1/2$. It can be easily checked that $(1/2, 1/4, 1/4) \in \mathcal{B}_\Pi(1/2)$. Now, $(1/2, 1/4, 1/4) \notin \mathcal{B}_\Pi(0)$ because $\varrho_1 + \varrho_3 = 1/2 + 1/4 \ngeqslant 1$; similarly $(1/2, 1/4, 1/4) \notin \mathcal{B}_\Pi(1)$ because $\varrho_1 + \varrho_2 = 1/2 + 1/4 \ngeqslant 1$. It follows that

$$\mathcal{B}_\Pi(1/2) \nsubseteq \mathcal{B}_\Pi(0) \cup \mathcal{B}_\Pi(1)$$

and $\mathcal{B}_\Pi$ is not quasi-convex.

8 Lower Expectations and Almost Desirable Gambles

Similarly to the previous section, now we study the relation among the (quasi-)convexity/ concavity properties of lower expectations as real valued functions in the decision variable $x \in X$ and the (quasi-)convexity/concavity properties of the almost desirable gambles correspondence with respect to the same decision variable x.

Denote with $\mathbf{1} = (1, 1, \dots, 1) \in \mathcal{L}$, given the correspondence $\mathcal{D} : X \rightsquigarrow \mathcal{L}$, the induced lower expectation function is defined as follows:

$$\Pi_{\mathcal{D}}(Y, x) = \sup\{\mu \in \mathbb{R} \,|\, Y - \mu\mathbf{1} \in \mathcal{D}(x)\} \quad \forall Y \in \mathcal{L}, \tag{28}$$

Conversely, given the lower expectation function Π, the almost desirable gambles correspondence $\mathcal{D}_\Pi : X \rightsquigarrow \mathcal{L}$ derived form Π is defined by

$$\mathcal{D}_\Pi(x) = \{Y \in \mathcal{L} \mid \Pi(Y, x) \geqslant 0\} \quad \forall x \in X, \tag{29}$$

8.1 Lower expectation functions and the induced almost desirable gambles

The following results analyze the relation between the properties of (quasi) concavity/convexity of Π with the properties of (quasi) concavity/convexity of $\mathcal{D}_\Pi$.

Theorem 8.1: *The following implications hold*

i) If $\Pi(Y, \cdot)$ is quasi-convex in X for all $Y \in \mathcal{K}$, then $\mathcal{D}_\Pi$ is quasi-convex in X.

ii) If $\Pi(Y, \cdot)$ is quasi-concave in X for all $Y \in \mathcal{K}$, then $\mathcal{D}_\Pi$ is quasi-concave in X.

iii) If (1) $\Pi(Y, \cdot)$ is convex in X for all $Y \in \mathcal{K}$, (2) $\Pi(cY, x) = c\Pi(Y, x)$ for every $x \in X$, $Y \in \mathcal{K}$ and $c \geqslant 0$, then $\mathcal{D}_\Pi$ is quasi-concave in X.

Proof. i) Let x', $x'' \in X$, $t \in]0, 1[$ and $Y \in \mathcal{D}_\Pi(tx' + (1 - t)x'')$; by definition it follows that $\Pi(Y, tx' + (1 - t)x'') \geqslant 0$. Since $\Pi(Y, \cdot)$ is quasi-convex then

$$0 \leqslant \Pi(Y, tx' + (1 - t)x'') \leqslant \max\{\Pi(Y, x'), \Pi(Y, x'')\}$$

It follows that

$$\Pi(Y, x') \geqslant 0 \quad \text{or} \quad \Pi(Y, x'') \geqslant 0$$

Then $Y \in \mathcal{D}_\Pi(x')$ or $Y \in \mathcal{D}_\Pi(x'')$, implying that $Y \in \mathcal{D}_\Pi(x') \cup \mathcal{D}_\Pi(tx'')$. So,

$$\mathcal{D}_\Pi(tx' + (1 - t)x'') \subseteq \mathcal{D}_\Pi(x') \cup \mathcal{D}_\Pi(x'')$$

and $\mathcal{D}_\Pi$ is quasi-convex.

ii) Let x', $x'' \in X$, $t \in]0, 1[$ and $Y \in \mathcal{D}_\Pi(x') \cap \mathcal{D}_\Pi(x'')$. Then it follows that $\Pi(Y, x') \geqslant 0$ and $\Pi(Y, x'') \geqslant 0$. Being Π a quasi-concave function then it follows that

$$\Pi(Y, tx' + (1 - t)x'') \geqslant \min\{\Pi(Y, x'), \Pi(Y, x'')\} \geqslant 0.$$

So $Y \in \mathcal{D}_\Pi(tx' + (1-t)x'')$ and $\mathcal{D}_\Pi$ is quasi-concave in X.

iii) Let x', $x'' \in X$ and $t \in]0,1[$ and $Y \in \mathcal{D}_\Pi(tx' + (1-t)x'')$ then $\Pi(Y, tx' + (1-t)x'') \geqslant 0$. $\Pi(Y, \cdot)$ is convex in X, so

$$0 \leqslant \Pi(Y, tx' + (1-t)x'') \leqslant t\Pi(Y, x') + (1-t)\Pi(Y, x'')$$

Then $\Pi(Y, x') \geqslant 0$ or $\Pi(Y, x'') \geqslant 0$ (or both). Suppose that $\Pi(Y, x') \geqslant 0$, then from the assumption (2) it follows that

$$\Pi\left(\frac{1}{t}Y, x'\right) \geqslant 0 \implies \frac{1}{t}Y \in \mathcal{D}_\Pi(x').$$

Moreover assumption (2) implies also that $\Pi(0, x'') = 0$ so that $0 \in \mathcal{D}_\Pi(x'')$. So

$$Y = t\left(\frac{1}{t}Y\right) + (1-t)0 \in t\mathcal{D}_\Pi(x') + (1-t)\mathcal{D}_\Pi(x'').$$

Since, the same result is obtained when $\Pi(Y, x'') \geqslant 0$, it follows that

$$\mathcal{D}_\Pi(tx' + (1-t)x'') \subseteq t\mathcal{D}_\Pi(x') + (1-t)\mathcal{D}_\Pi(x'')$$

and $\mathcal{D}_\Pi$ is convex. □

Counterexample

In the next example it is shown that the concavity of the lower expectation function does not necessarily imply that the induced almost desirable gambles correspondence is concave.

Example 8.2: Let $\Omega = \{\omega_1, \omega_2\}$ and $X = [0,1]$. For each gamble $Y \in \mathcal{L}$, denote $(Y(\omega_1), Y(\omega_2)) = (y_1, y_2)$ and then define, for every $x \in X$,

$$\Pi(Y, x) = \begin{cases} x(y_1 - y_2) + y_2 & \text{if } y_2 \geqslant 0 \\ y_2 & \text{if } y_2 < 0 \end{cases} \tag{30}$$

It immediately follows that $\Pi(Y, \cdot)$ is concave in X for every $Y \in \mathcal{L}$. Now, by definition, it follows that

$$\mathcal{D}_\Pi(x) = \{(y_1, y_2) \,|\, xy_1 + (1-x)y_2 \geqslant 0 \text{ and } y_2 \geqslant 0\}$$

So

$$\mathcal{D}_\Pi(0) = \{(y_1, y_2) \,|\, y_2 \geqslant 0\} \quad \text{and} \quad \mathcal{D}_\Pi(1) = \{(y_1, y_2) \,|\, y_1 \geqslant 0 \text{ and } y_2 \geqslant 0\}$$

and

$$\frac{1}{2}\mathcal{D}_\Pi(0) + \frac{1}{2}\mathcal{D}_\Pi(1) = \{(y_1, y_2) \,|\, y_2 \geqslant 0\}$$

while

$$\mathcal{D}_\Pi(1/2) = \{(y_1, y_2) \,|\, y_1 + y_2 \geqslant 0 \text{ and } y_2 \geqslant 0\}$$

So

$$\frac{1}{2}\mathcal{D}_\Pi(0) + \frac{1}{2}\mathcal{D}_\Pi(1) \not\subseteq \mathcal{D}_\Pi(1/2)$$

and $\mathcal{D}_\Pi(\cdot)$ is not concave in X. As a final remark, note that in this example the function Π satisfies the assumption (2) in *iii)* of the previous proposition.

8.2 Almost desirable gambles correspondences and the induced lower expectations

In this subsection, it is presented the relation between the properties of (quasi) concavity/convexity of $\mathcal{D}$ with the properties of (quasi) concavity/convexity of $\Pi_{\mathcal{D}}$.

THEOREM 8.3: *Assume that $\mathbb{R}^m_+ \subseteq \mathcal{D}(x)$ for every $x \in X$. Then, the following implications hold:*

i) If $\mathcal{D}$ is concave in X, then $\Pi_{\mathcal{D}}(Y, \cdot)$ is concave in X for every $Y \in \mathcal{L}$.

ii) If $\mathcal{D}$ is quasi-convex in X, then $\Pi_{\mathcal{D}}(Y, \cdot)$ is quasi convex in X for every $Y \in \mathcal{L}$.

Proof. Firstly, since $\mathbb{R}^m_+ \subseteq \mathcal{D}(x)$ for every $x \in X$, then it immediately follows that, for every $(Y, x) \in \mathcal{L} \times X$, the subset $\{\mu \in \mathbb{R} \,|\, Y - \mu\mathbf{1} \in \mathcal{D}(x)\}$ is not empty. Therefore $\Pi_{\mathcal{D}}(Y, x) \in \mathbb{R}$ for every $(Y, x) \in \mathcal{L} \times X$.

i) Let x', $x'' \in X$ and $t \in]0, 1[$, and $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ be a sequence converging to 0. Let μ'_ν and μ''_ν be real numbers such that

$$\mu'_\nu > \Pi_{\mathcal{D}}(Y, x') - \varepsilon_\nu \quad \text{and} \quad Y - \mu'_\nu\mathbf{1} \in \mathcal{D}(x')$$

and

$$\mu''_\nu > \Pi_{\mathcal{D}}(Y, x'') - \varepsilon_\nu \quad \text{and} \quad Y - \mu''_\nu\mathbf{1} \in \mathcal{D}(x'').$$

Since $\mathcal{D}$ is concave, it follows that

$$Y - (t\mu'_\nu + (1-t)\mu''_\nu)\mathbf{1} = t(Y - \mu'_\nu\mathbf{1}) + (1-t)(Y - \mu''_\nu\mathbf{1}) \in \mathcal{D}(tx' + (1-t)x'')$$

then

$$t(\Pi_{\mathcal{D}}(Y, x'') - \varepsilon_\nu) + (1-t)(\Pi_{\mathcal{D}}(Y, x'') - \varepsilon_\nu) < t\mu'_\nu + (1-t)\mu''_\nu \leqslant \Pi_{\mathcal{D}}(Y, tx' + (1-t)x'').$$

In particular,

$$t\Pi_{\mathcal{D}}(Y, x'') + (1-t)\Pi_{\mathcal{D}}(Y, x'') - \varepsilon_\nu < \Pi_{\mathcal{D}}(Y, tx' + (1-t)x'').$$

As $\varepsilon_\nu \to 0$,

$$t\Pi_{\mathcal{D}}(Y, x'') + (1-t)\Pi_{\mathcal{D}}(Y, x'') \leqslant \Pi_{\mathcal{D}}(Y, tx' + (1-t)x'').$$

and we get the assertion.

ii) Let x', $x'' \in X$ and $t \in]0, 1[$, and let $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ be a sequence converging to 0. For every ν, let μ_ν be such that

$$\Pi_{\mathcal{D}}(Y, tx' + (1-t)x'') - \varepsilon_\nu < \mu_\nu \leqslant \Pi_{\mathcal{D}}(Y, tx' + (1-t)x'') \quad \text{and} \quad Y - \mu_\nu\mathbf{1} \in \mathcal{D}(tx' + (1-t)x'').$$

$\mathcal{D}$ is quasi-convex, so

$$Y - \mu_\nu\mathbf{1} \in \mathcal{D}(x') \cup \mathcal{D}(x'').$$

This implies that

$$\mu_\nu \leqslant \Pi_{\mathcal{D}}(Y, x') \quad \text{or} \quad \mu_\nu \leqslant \Pi_{\mathcal{D}}(Y, x'') \implies \mu_\nu \leqslant \max\{\Pi_{\mathcal{D}}(Y, x'), \Pi_{\mathcal{D}}(Y, x'')\}.$$

Hence,

$$\Pi_{\mathcal{D}}(Y, tx' + (1-t)x'') - \varepsilon_\nu < \max\{\Pi_{\mathcal{D}}(Y, x'), \Pi_{\mathcal{D}}(Y, x'')\}$$

and, as $\varepsilon_\nu \to 0$,

$$\Pi_{\mathcal{D}}(Y, tx' + (1-t)x'') \leqslant \max\{\Pi_{\mathcal{D}}(Y, x'), \Pi_{\mathcal{D}}(Y, x'')\}$$

and we get the assertion. □

Counterexamples

The two examples below show that the analogous of *i)* and *ii)* in the previous Proposition do not hold in case $\mathcal{D}$ is convex or quasi-concave. More precisely, the first example shows that there exists a correspondence $\mathcal{D}$ that is convex, while the induced $\Pi_{\mathcal{D}}(Y,\cdot)$ is not, for some Y. Similarly, in the second example, the correspondence $\mathcal{D}$ is quasi-concave, while the induced $\Pi_{\mathcal{D}}(Y,\cdot)$ is not, for some Y. Note that, even in this case, the *coherency* of the representation of uncertainty does not play any role as, in both the examples, the images $\mathcal{D}$ are closed and convex cones containing $\mathbb{R}^m_+$.

Example 8.4: Let $\Omega = (\omega_1, \omega_2)$ and $\mathcal{D} : [0,1] \rightsquigarrow \mathcal{L}$ be defined by

$$\mathcal{D}(x) = \{(y_1, y_2) \in \mathbb{R}^2 \,|\, y_1 + xy_2 \geqslant 0,\ y_2 \geqslant 0\}$$

Since $x' < x''$ implies that $\mathcal{D}(x') \subset \mathcal{D}(x'')$ then it immediately follows that $\mathcal{D}$ is convex in X. The corresponding lower expectation function is computed, for every $x \in X$, as the solution μ of the following equations

$$\begin{cases} (y_1-\mu) + x(y_2-\mu) = 0, & \text{if } y_2 \geqslant y_1 \\ y_2 - \mu = 0 & \text{if } y_2 < y_1 \end{cases}$$

Therefore,

$$\Pi_{\mathcal{D}}(Y,x) = \begin{cases} \frac{y_1+xy_2}{1+x}, & \text{if } y_2 \geqslant y_1 \\ y_2 & \text{if } y_2 < y_1 \end{cases}$$

Let $x' = 0$, $x'' = 1$ and $t = 1/2$; and $Y = (-1, 2)$, we get $\Pi_{\mathcal{D}}(Y, 1/2) = 0$ $\Pi_{\mathcal{D}}(Y, 1) = 1/2$ and $\Pi_{\mathcal{D}}(Y, 0) = -1$ that implies that

$$\Pi_{\mathcal{D}}(Y, 1/2) \nleqslant (1/2)\Pi_{\mathcal{D}}(Y, 1) + (1/2)\Pi_{\mathcal{D}}(Y, 0)$$

so $\Pi_{\mathcal{D}}$ is not convex.

Example 8.5: Let $\Omega = (\omega_1, \omega_2)$ and $\mathcal{D} : [0,1] \rightsquigarrow \mathcal{L}$ be defined by

$$\mathcal{D}(x) = \begin{cases} \{(y_1,y_2) \in \mathbb{R}^2 \,|\, 2xy_1 + (1-2x)y_2 \geqslant 0,\ y_2 \geqslant 0\} & \text{if } x \in [0,1/2] \\ \{(y_1,y_2) \in \mathbb{R}^2 \,|\, (2-2x)y_1 + (2x-1)y_2 \geqslant 0,\ y_2 \geqslant 0\} & \text{if } x \in]1/2,1] \end{cases}$$

By construction, for every $x' \neq x''$ it follows that $\mathcal{D}(x') \subset \mathcal{D}(x'')$ or $\mathcal{D}(x'') \subset \mathcal{D}(x')$. So $\mathcal{D}$ is quasi-concave in X. Similarly to the previous example, the corresponding lower expectation function is computed as the solution μ of the following equations

$$\begin{cases} 2x(y_1-\mu) + (1-2x)(y_2-\mu) = 0, & \text{if } y_2 \geqslant y_1 \\ y_2 - \mu = 0 & \text{if } y_2 < y_1 \end{cases} \quad \text{if; } x \in [0,1/2]$$

or

$$\begin{cases} (2-2x)(y_1-\mu) + (2x-1)(y_2-\mu) = 0, & \text{if } y_2 \geqslant y_1 \\ y_2 - \mu = 0 & \text{if } y_2 < y_1 \end{cases} \quad \text{if } x \in]1/2,1]$$

Therefore

$$\Pi_{\mathcal{D}}(Y,x) = \begin{cases} \begin{cases} 2x(y_1-y_2) + y_2, & \text{if } x \in [0,1/2] \\ (1-2x)(y_1-y_2) + y_1, & \text{if } x \in]1/2,1] \end{cases} & \text{if } y_2, \geqslant y_1 \\ y_2 & \text{if } y_2 < y_1 \end{cases}$$

and it immediately follows that $\Pi_{\mathcal{D}}(Y,\cdot)$ is not quasi concave for every $Y \in \mathcal{L}$. In fact, consider for instance $Y=(0,1)$, then

$$\Pi_{\mathcal{D}}(Y,x) = \begin{cases} -2x+1, & \text{if } x \in [0,1/2] \\ 2x-1, & \text{if } x \in]1/2,1] \end{cases}$$

that is not quasi-concave.

9 Appendix: Proofs of Section 3

Proposition (3.4): *The set-valued map $\mathcal{C} : X \rightsquigarrow \mathcal{T}$ is quasi-convex if and only if for every $A \subseteq \mathcal{T}$, with $A \neq \emptyset$, the set*

$$\underline{L}_A = \{x \,|\, \mathcal{C}(x) \subseteq A\} \tag{31}$$

is a convex subset of X.

Proof of Proposition 3.4. Assume first that for every subset $A \subseteq \mathcal{T}$, $\underline{L}_A$ is a convex subset of X. Given x' and x'', it immediately follows that $x', x'' \in \underline{L}_{\mathcal{C}(x')\cup\mathcal{C}(x'')}$ which is a convex set. Then, for every $t \in]0,1[$, $tx' + (1-t)x'' \in \underline{L}_{\mathcal{C}(x')\cup\mathcal{C}(x'')}$ implying that

$$\mathcal{C}(tx' + (1-t)x'') \subseteq \mathcal{C}(x') \cup \mathcal{C}(x''),$$

so $\mathcal{C}$ is quasi-convex.

Conversely, assume that $\mathcal{C}$ is quasi-convex. Given a subset $A \subseteq \mathcal{T}$, let $x', x'' \in \underline{L}_A$. By definition, $\mathcal{C}(x') \subseteq A$ and $\mathcal{C}(x'') \subseteq A$, then it follows that

$$\mathcal{C}(tx' + (1-t)x'') \subseteq \mathcal{C}(x') \cup \mathcal{C}(x'') \subseteq A.$$

So $tx' + (1-t)x'' \in \underline{L}_A$ an $\underline{L}_A$ is a convex subset of X. □

Proposition (3.5): *The set-valued map $\mathcal{C} : X \rightsquigarrow \mathcal{T}$ is weakly quasi-convex if and only if for every convex subset $A \subseteq \mathcal{T}$, with $A \neq \emptyset$, the set*

$$\underline{L}_A = \{x \,|\, \mathcal{C}(x) \subseteq A\} \tag{32}$$

is a convex subset of X.

Proof of Proposition 3.5. Assume first that for every convex subset $A \subseteq \mathcal{T}$, $\underline{L}_A$ is a convex subset of X. Given x' and x'', it immediately follows that $x', x'' \in \underline{L}_{co(\mathcal{C}(x')\cup\mathcal{C}(x''))}$ which is a convex set. Then, for every $t \in]0,1[$, $tx' + (1-t)x'' \in \underline{L}_{co(\mathcal{C}(x')\cup\mathcal{C}(x''))}$ implying that

$$\mathcal{C}(tx' + (1-t)x'') \subseteq co(\mathcal{C}(x') \cup \mathcal{C}(x'')),$$

so $\mathcal{C}$ is weakly quasi-convex.

Conversely, assume that $\mathcal{C}$ is weakly quasi-convex. Given a convex subset $A \subseteq \mathcal{T}$, let $x', x'' \in \underline{L}_A$. By definition, $\mathcal{C}(x') \subseteq A$ and $\mathcal{C}(x'') \subseteq A$, so $\mathcal{C}(x') \cup \mathcal{C}(x'') \subseteq A$. The subset A is convex so $co(\mathcal{C}(x') \cup \mathcal{C}(x'')) \subseteq A$ and then, weakly quasi-convexity of $\mathcal{C}$ implies that

$$\mathcal{C}(tx' + (1-t)x'') \subseteq co(\mathcal{C}(x') \cup \mathcal{C}(x'')) \subseteq A.$$

So $tx' + (1-t)x'' \in \underline{L}_A$ an $\underline{L}_A$ is a convex subset of X. □

PROPOSITION (3.7): *Given the set valued map $\mathcal{C} : X \rightsquigarrow \mathcal{T}$, then*

i) If $\mathcal{C}$ is strongly quasi-convex then $\mathcal{C}$ is quasi-convex

ii) If $\mathcal{C}$ is quasi-convex then $\mathcal{C}$ is weakly quasi-convex

iii) If $\mathcal{C}$ is convex then $\mathcal{C}$ is weakly quasi-convex

Proof of Proposition 3.7. i) It immediately follows from the definition.
ii) Since $\mathcal{C}(x') \cup \mathcal{C}(x'') \subseteq co(\mathcal{C}(x') \cup \mathcal{C}(x''))$ then the assertion immediately follows from the definitions.

iii) Assume that $\mathcal{C} : X \rightsquigarrow \mathcal{T}$ is convex and let A be a convex subset of $\mathcal{T}$. Let $x', x'' \in \underline{L}_A$. By definition, $\mathcal{C}(x') \subseteq A$ and $\mathcal{C}(x'') \subseteq A$. Since A is a convex set then $t\mathcal{C}(x') + (1-t)\mathcal{C}(x'') \subseteq A$. Being $\mathcal{C}$ a convex set valued map, it follows that

$$\mathcal{C}(tx' + (1-t)x'') \subseteq t\mathcal{C}(x') + (1-t)\mathcal{C}(x'') \subseteq A$$

which finally implies that $tx' + (1-t)x'' \in \underline{L}_A$ and $\underline{L}_A$ is a convex set. So $\mathcal{C}$ is weakly quasi-convex. □

EXAMPLE 9.1: Here it is shown that a strongly quasi-convex set-valued map is not necessarily convex. Consider the set-valued map $\mathcal{C} : [0,1] \rightsquigarrow [0,1]$ defined as follows

$$\mathcal{C}(x) = [0, 1 - x^2] \quad \forall x \in [0,1].$$

This set-valued map is obviously strongly quasi-convex (hence quasi-convex and weakly quasi-convex) as $x' < x'' \implies \mathcal{C}(x') \supset \mathcal{C}(x'')$. Moreover $\mathcal{C}$ has convex images. Nevertheless it is not convex. In fact, for $x = 1/2(0) + 1/2(1) = 1/2$, it follows that

$$\mathcal{C}(1/2) = [0, 3/4] \not\subseteq 1/2\mathcal{C}(0) + 1/2\mathcal{C}(1) = 1/2[0,1] + 1/2\{0\} = [0, 1/2].$$

EXAMPLE 9.2: Here it is shown that a convex set-valued map is not necessarily quasi-convex. Consider the set valued map $\mathcal{C} : [0, 1/5] \rightsquigarrow [0,1]$ defined as follows

$$\mathcal{C}(x) = [4x, 4x + 1/5] \quad \forall x \in [0, 1/5].$$

This set valued map is obviously convex and has convex images. Nevertheless it is not quasi-convex nor strongly quasi-convex. In fact for $x = 1/2(0) + 1/2(1/5) = 1/10$, it follows that $\mathcal{C}(1/10) = [4/10, 6/10]$ while $\mathcal{C}(0) \cup \mathcal{C}(1/5) = [0, 1/5] \cup [4/5, 1]$.

PROPOSITION (3.9): *Let $\mathcal{C} : X \rightsquigarrow \mathcal{T}$ be a set-valued map with convex images, then $\mathcal{C}$ is quasi-concave if and only if for every convex $A \subseteq \mathcal{T}$, with $A \neq \emptyset$, the set*

$$\overline{L}_A = \{x \mid A \subseteq \mathcal{C}(x)\} \tag{33}$$

is a convex subset of X.

Proof of Proposition 3.9. Assume first that for every subset convex $A \subseteq \mathcal{T}$, $\underline{L}_A$ is a convex subset of X. Given x' and x'', it immediately follows that $x', x'' \in \overline{L}_{co(\mathcal{C}(x')\cap\mathcal{C}(x''))}$ which is a convex set. Then, for every $t \in]0,1[$, $tx' + (1-t)x'' \in \overline{L}_{co(\mathcal{C}(x')\cap\mathcal{C}(x''))}$ implying that

$$\mathcal{C}(x') \cap \mathcal{C}(x'') \subseteq co(\mathcal{C}(x') \cap \mathcal{C}(x'')) \subseteq \mathcal{C}(tx' + (1-t)x''),$$

so $\mathcal{C}$ is quasi-concave.

Conversely, assume that $\mathcal{C}$ is quasi-concave. Given a convex subset $A \subseteq \mathcal{T}$, let $x', x'' \in \overline{L}_A$. By definition, $A \subseteq \mathcal{C}(x')$ and $A \subseteq \mathcal{C}(x'')$, then it follows that

$$A \subseteq \mathcal{C}(x') \cap \mathcal{C}(x'') \subseteq \mathcal{C}(tx' + (1-t)x'').$$

So $tx' + (1-t)x'' \in \overline{L}_A$ an $\overline{L}_A$ is a convex subset of X. □

PROPOSITION (3.10): *Given the set valued map $\mathcal{C} : X \rightsquigarrow \mathcal{T}$, then*

i) If $\mathcal{C}$ is strongly quasi-concave then $\mathcal{C}$ is quasi-concave.

ii) If $\mathcal{C}$ is concave then $\mathcal{C}$ is quasi-concave

Proof of Proposition 3.10. i) Since $\mathcal{C}(x') \subseteq \mathcal{C}(tx' + (1-t)x'')$ or $\mathcal{C}(x'') \subseteq \mathcal{C}(tx' + (1-t)x'')$ then $\mathcal{C}(x') \cap \mathcal{C}(x'') \subseteq \mathcal{C}(tx' + (1-t)x'')$. The assertion follows.
ii) Since $\mathcal{C}(x') \cap \mathcal{C}(x'') \subseteq t\mathcal{C}(x') + (1-t)\mathcal{C}(x'')$ then the assertion immediately follows. □

The next counter examples show that there are no links between strong quasi-concavity and concavity.

EXAMPLE 9.3: Here it is shown that the a strongly quasi-concave set-valued map is not necessarily concave. Consider the set valued map $\mathcal{C} : [0,1] \rightsquigarrow [0,1]$ defined as follows

$$\mathcal{C}(x) = [1 - x^2, 1] \quad \forall x \in [0,1].$$

This set valued map is obviously strongly quasi-concave (hence quasi-concave) as $x' < x'' \implies \mathcal{C}(x') \subset \mathcal{C}(x'')$. Moreover $\mathcal{C}$ has convex images. Nevertheless it is not concave. In fact for $x = 1/2 * 0 + 1/2 * 1 = 1/2$, it follows that

$$1/2\mathcal{C}(0) + 1/2\mathcal{C}(1) = 1/2\{1\} + 1/2[0,1] + = [1/2, 1] \not\subseteq \mathcal{C}(1/2) = [3/4, 1].$$

EXAMPLE 9.4: Here it is shown that the a concave set-valued map is not necessarily strongly quasi-concave. Consider the set valued map $\mathcal{C} : [0,1] \rightsquigarrow [0,1]$ defined as follows

$$\mathcal{C}(x) = [0, 1 - (x - 1/2)^2] \quad \forall x \in [0,1].$$

This set valued map is obviously concave and has convex images. Nevertheless it is not strongly quasi-concave. In fact for $x = 1/2(0) + 1/2(1) = 1/2$, it follows that $\mathcal{C}(1/2) = [0,1]$ while $\mathcal{C}(0) = \mathcal{C}(1) = [0, 3/4]$.

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