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Chiara Donnini [*] and Marialaura Pesce [**]

Abstract

We propose a notion of *local* fairness by requiring absence of envy only among agents who are *related* each other. We identify conditions under which fairness is ensured in the entire society by imposing absence of envy just locally. Our analysis is conducted in atomless economies as well as in mixed markets.

JEL Classification: D51, D63

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1 Introduction

In this paper we suggest a notion of *local* fairness by obstructing envy only among *linked* people. We provide sufficient conditions under which *local* absence of envy is enough to ensure *global* fairness.
Our starting point is the notion of strict fairness introduced by Zhou (1992), according to which beyond efficiency it is required that each individual weakly prefers her own bundle to the average bundle of any coalitions. This notion is more demanding than the classical one due to Foley (1967) and in large economies it characterizes the set of equal-income Walrasian allocations. Based on the strict envy-freeness, each agent needs to know the consumption bundles all others receive and to consider the average bundles of all possible coalitions. This can be considered a strong requirement especially in large economies or in situations in which agents' knowledge is limited for some reasons (Cato (2010), Abebe et al. (2017), Beynier et al. (2018) among others, see also the concluding remarks of Zhou (1992)). In addition, empirical works suggest that individuals often exhibit myopia in the spatial sense, meaning that they are interested only on their immediate neighbours. Moreover, even a particularly inquiring person who would like to compare her bundle with the bundle of anybody else, probably cannot do it, because of her incomplete knowledge about the others. In many situations it is more reasonable to allow an agent to focus her comparisons only on those individuals she can relate herself, rather than on the entire population. For instance, a full professor of a faculty tends to compare himself/herself (salary, tasks, CV, etc.) with colleagues with similar position and to neglect the rest of the staff, like researchers or office workers. A person applying for an available job focuses on the CV of other candidates, who are his/her direct competitors, leaving out the rest of people looking for a different occupation. Similar applications, in which the object of potential envy is just the person with whom one is directly connected, include inheritance and divorce settlements as well as international border agreements.
In this paper we suggest a concept of relationship among people and the consequent definition of *local* strict envy-freeness, according to which only related individuals must not envy each other. It is natural to think that members of a society can be categorized in several ways, for instance, by phisical features (height, weight, gender), by occupation, spoken languages, citizenship and so on. Formally, we consider a countable (finite or infinite) covering of the set of agents T, that is a family $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$ of possible coalitions whose union gives back T. Each group C_i of $\mathcal{R}$ can be viewed as a country, a community, a group of co-workers or people with similar interests, or more generally any possible category of people on the basis of the attributes used to classify them. Clearly, a partition of T is a particular case of covering, but we do not impose elements of $\mathcal{R}$ to be pairwise disjoint. Therefore, each individual belongs to at least one group but possibly even to

several categories. This is the case, for instance, of people with dual citizenship, if any C_i of $\mathcal{R}$ is viewed as a country, or bilinguals if we gather people with respect to spoken languages. If any C_i of $\mathcal{R}$ is the editorial board of a certain journal, it is common that a professor is member of more than one editiorial board.

Given a covering $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$ of T, we impose absence of envy within each group C_i of $\mathcal{R}$, disregarding the possibility for an individual to envy a group of people not similarly situated. Essentially we reduce the number of coalitions that each agent must look at and we obtain a *local* version of strict envy-freeness that we call $\mathcal{R}$-strict envy-freeness. Any *global* strictly fair allocation is clearly *local* $\mathcal{R}$-strictly envy-free for any covering $\mathcal{R}$ of T. The converse is not true in general. Our main goal is to identify a class of coverings of T for which fairness is ensured in the entire society by imposing absence of envy just locally. To this end, we construct a one-to-one correspondence between a covering $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$ of T and a network, in which the nodes are the elements C_i of $\mathcal{R}$ and an undirected edge connects two nodes if and only if the two groups are not disjoint. We define a covering $\mathcal{R}$ connected if it corresponds to a network with no isolated nodes nor isolated subgraphs. We show that for any connected covering $\mathcal{R}$, any strictly fair allocation is $\mathcal{R}$-strictly fair and viceversa. This means that if in the society there is no isolated group, it is enough to obstruct envy locally to get fairness globally. We obtain this equivalence first in atomless economies and then in the more general framework of mixed markets, in which the space of agents may exhibit atoms.

In the case of atomless economies, we show that given any connected covering $\mathcal{R}$, equal-income Walrasian equilibria are the only $\mathcal{R}$-strictly fair allocations. Then, from the equivalence due to Zhou (1992) in atomless economies, we derive that the notions of global and local fairness coincide. For this, efficiency plays a crucial role. Indeed, we show that if we consider only absence of envy and we leave out the efficiency, then the equivalence fails even with connected coverings. The intuitive reason lies on the fact that efficiency involves the entire society, whereas *local* envy-freeness only a piece of it. Thus, in a sense, efficiency expands local into global fairness.

The same arguments cannot be applied to mixed markets for two main reasons. First, the Lyapuvon convexity theorem does not hold; and second, the equivalence between the set of equal-income Walrasian equilibria and *global* strictly fair allocations fails. Recently in Donnini and Pesce (2018), we provide two sufficient conditions on the space of agents, under which such equivalence is restored. The first imposes the presence of infinitely countably many atoms with the same utility function; the second requires for each atom the presence of an atomless fringe, that is a coalition of negiglible traders with the same tastes of the atom. One might guess that these two conditions are sufficient also for the notion of *local* fairness. On the contrary, we show that even if the covering is connected and the hypotheses of

Donnini and Pesce (2018) are satisfied, an $\mathcal{R}$-strictly fair allocation may not be *globally* strictly envy-free. We then show that, by further strenghtening the two assumptions on the space of agents, we can restore the equivalence even in the presence of atoms. Precisely, we assume that, given a connected covering $\mathcal{R} = \{C_i\}_{i \in I \subseteq I\!N}$ of T, if there are infinitely many atoms with the same utility function, then they must belong to a unique coalition C_i of $\mathcal{R}$. Alternatevely, if an element C_i of $\mathcal{R}$ contains an atom A, it must contain a non-negligible piece of A's atomless fringe.
Cato (2010) introduces a notion of local strict fairness based on a very similar intuition and he shows in atomless economies the equivalence with the global strict fairness. We prove that his equivalence result follows from ours, being his notion a particular case of $\mathcal{R}$-strict fairness.
The paper is organized as follows: in sections 2 and 3 we respectively introduce the notions and we get the equivalence theorem in atomless economies; in section 4 we make a comparison with the related literature and finally in section 5 we analysis the more general case of mixed markets.

2 Local fairness notion

In this section we introduce and analyze a notion of local strict fairness in atomless exchange economies with a continuum of agents and ℓ private divisible goods. The space of agents is represented by a complete atomless measure space (T, Σ, μ), with $\mu(T) = 1$. As usual, a coalition is a measurable subset of T with positive measure. The commodity space is $I\!R^{\ell}_{++}$ and each agent t is endowed with an utility function $u_t : I\!R^{\ell}_{++} \to I\!R$ and an initial endowment $\omega(t)$. We assume that $\omega : T \to I\!R^{\ell}_{++}$ is integrable and we denote by $\boldsymbol{\omega}$ the total initial endowment in the economy, i.e., $\boldsymbol{\omega} = \int_T \omega(t)\, d\mu$. For any t in T, u_t is strictly increasing, differentiable on $I\!R^{\ell}_{++}$ and satisfies the usual measurability condition[1].
An *allocation* is an integrable function $x : T \to I\!R^{\ell}_{++}$ which is said to be *feasible* if $\int_T x(t)\, d\mu = \boldsymbol{\omega}$. A *Walrasian equilibrium allocation* is a feasible allocation x for which there exists a price vector $p \in I\!R^{\ell}_{++}$ such that, for almost every agent t in T, $x(t)$ maximizes u_t on the budget set $B_t(p) = \{y \in I\!R^{\ell}_{++} : p \cdot y \leq p \cdot \omega(t)\}$. If $p \cdot \omega(t)$ is agent-independent, x is called *equal-income* Walrasian equilibrium allocation[2]. We denote by $\mathcal{W}_{ei}(\mathcal{E})$ the set of equal-income Walrasian equilibrium allocations.
A feasible allocation x is *Pareto optimal* (or *efficient*) if there is no alternative feasible allocation y such that $u_t(y(t)) > u_t(x(t))$ for almost all t in T. An efficient allocation is *fair* if it is *envy-free*, meaning that any agent prefers

[1]The mapping $(t, x) \mapsto u_t(x)$ is $\Sigma \otimes \mathcal{B}(\mathbb{R}^{\ell}_{++})$−measurable, where $\mathcal{B}(\mathbb{R}^{\ell}_{++})$ is the σ-field of Borel subsets of $\mathbb{R}^{\ell}_{++}$.

[2]A special case is when the total initial endowment $\boldsymbol{\omega}$ is equally divided among agents and x is called *equal-division* Walrasian equilibrium allocation.

her own bundle rather than the bundle of any other, i.e., for almost every agents $t, s \in T$, $u_t(x(t)) \geqslant u_t(x(s))$ (see Foley (1967)). It is known that any equal-income Walrasian equilibrium allocation is fair, but the converse may be not true. Zhou (1992) introduces the stronger notion of strict fairness, according to which, beyond efficiency, each agent compares her own bundle with the average bundle of any possible coalition.

Definition 2.1. *Given an allocation x, an agent t* envies *a coalition S at x if $u_t(\bar{x}(S)) > u_t(x(t))$, where $\bar{x}(S) = \frac{1}{\mu(S)} \int_S x(t) d\mu \in \mathbb{R}^\ell_{++}$. An allocation x is* strictly envy-free *if almost every agent does not envy any coalition at x; it is* strictly fair *if it is strictly envy-free and efficient. We denote by $SF(\mathcal{E})$ the set of strictly fair allocations.*

Zhou (1992) shows that equal-income Walrasian equilibria are the only strictly fair allocations, i.e., $\mathcal{W}_{ei}(\mathcal{E}) = SF(\mathcal{E})$.
According to the above definition, an envious trader t envies the possibility to join a certain coalition S, since she prefers what she would have in average being a member of S rather than what she gets alone. Notice that, as for the fairness notion of Foley (1967), utility comparisons are made by a single individual t who must know only her tastes and the entire allocation, that is the achievement of everybody in the society. This can be considered a strong requirement especially in large economies or in situations in which agents' knowledge is limited for some reasons.
Motivated by this, we introduce a definition of *local* fairness by imposing absence of envy only among people that are "related" or "connected" in some way. Our first goal is then to formalize the concept of connection among individuals. We assume that the society is made up of different groups, so that two individuals are related if they are members of the same coalition. Formally, we consider a countable (finite or infinite) covering of the set of agents T, that is a family $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$ of possible coalitions whose union gives back T, i.e., $C_i \in \Sigma$, $\mu(C_i) > 0$ for any $i \in I$ and $\bigcup_{i \in I} C_i = T$.
We introduce the following definition of local strict fairness that, given a covering $\mathcal{R}$ of T, obstructs agents to envy only people within the same class C_i of $\mathcal{R}$.

Definition 2.2. *Let $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$ be a covering of T. An allocation x is said to be $\mathcal{R}$-*strictly envy-free *if for any C_i in $\mathcal{R}$ and for almost all t in C_i, there does not exist a coalition $S \subseteq C_i$ such that $u_t(\bar{x}(S)) > u_t(x(t))$. The allocation x is $\mathcal{R}$-*strictly fair *if it is both efficient and $\mathcal{R}$-strictly envy-free. We denote by $\mathcal{R}SF(\mathcal{E})$ the set of $\mathcal{R}$-strictly fair allocations.*

Given a covering $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$, the notion of $\mathcal{R}$-strict fairness imposes absence of envy in each C_i but allows an individual to envy a group of people not similarly situated; hence it does not rule out *(global)* envy. Therefore, $SF(\mathcal{E}) \subseteq \mathcal{R}SF(\mathcal{E})$, while the reverse may be not true as shown below.

Example 2.3. Consider an economy whose consumption set is $I\!R^2_{++}$, $T = (0,1)$ and the total initial endowment is $\boldsymbol{\omega} = (1,1)$. Agents' utility functions are given by

$$u_t(x,y) = \begin{cases} xy & \text{if } t \in (0, \frac{1}{2}] \\ x^2 y & \text{if } t \in (\frac{1}{2}, 1). \end{cases}$$

Consider the covering $\mathcal{R} = \{C_1, C_2\}$, where $C_1 = \big(0, \frac{1}{2}\big]$ and $C_2 = \big(\frac{1}{2}, 1\big)$. The *efficient* allocation

$$(x(t), y(t)) = \begin{cases} \big(1, \frac{4}{3}\big) & \text{if } t \in C_1 \\ \big(1, \frac{2}{3}\big) & \text{if } t \in C_2. \end{cases}$$

is $\mathcal{R}$*-strictly envy-free*, because for any $i \in I := \{1,2\}$, for every $t \in C_i$ and $S \subseteq C_i$, $u_t(x(t), y(t)) = u_t(\bar{x}(S), \bar{y}(S))$, but it is *not (globally) strictly fair* since, for instance, every t in C_2 envies any subcoalition of C_1. △

Therefore, in atomless economies, this chain of relationships $\mathcal{W}_{ei}(\mathcal{E}) = SF(\mathcal{E}) \subsetneqq \mathcal{R}SF(\mathcal{E})$ holds, which guarantees the existence of a (*local*) $\mathcal{R}$-strictly fair allocation whenever the set of equal-income Walrasian equilibrium allocations is non empty.

Remark 2.4. In Donnini and Pesce (2018), we observe that any strictly fair allocation is individually rational. The above example shows that this is not true with the *local* $\mathcal{R}$-strict fairness notion. Indeed, if we assume in Example 2.3 that the total initial endowment is equally shared among agents, then $u_t(x(t), y(t)) = \frac{2}{3} < 1 = u_t(\omega(t))$ for any $t \in C_2$.

3 Equivalence Theorem and Remarks

We have already shown that an $\mathcal{R}$-strictly fair allocation does not exclude that someone envies a group of people not similarly situated. In this section, we identify a class of coverings for which local and global fairness notions coincide. Borrowing the terminology used in the social network literature, we can interpret the elements of a certain covering $\mathcal{R} = \{C_i\}_{i \in I \subseteq I\!N}$ as nodes of a network. An undirected edge connects two nodes C_i and C_j if and only if $\mu(C_i \cap C_j) > 0$. In this case, C_i and C_j are said to be connected. Each covering $\mathcal{R}$ designs a network and viceversa. A partition of T, for instance, corresponds to an edgeless graph, that is a graph with isolated nodes. Viceversa, a network with isolated nodes defines a partition of T. Our goal is then to find a class of coverings, or equivalently conditions on the topology of the corresponding networks, ensuring global fairness by imposing absence of envy only among neighbors, that is only within each node. To this end, the following definitions are needed.

Definition 3.1. *Given a covering $\mathcal{R}$ of T, a* path *is a sequence of elements of $\mathcal{R}$ that are connected to each other. A covering $\mathcal{R}$ is said to be* connected *if for every pair of its elements there exists a path linking up them.*

A connected covering $\mathcal{R}$ designs a network with no isolated nodes nor isolated subgraphs and viceversa. Clearly a partition of T is not connected, whereas examples of connected coverings are fully connected networks, cyclic networks as well as star networks.

We now show that for every connected covering $\mathcal{R}$ of T, equal-income Walrasian equilibria are the only $\mathcal{R}$-strictly fair allocations. This, together with the equivalence proved by Zhou (1992), implies that once the society is structured in such a way that no group is isolated, it is enough to avoid envy locally to ensure fairness globally.

Theorem 3.2. *For every connected covering $\mathcal{R}$ of T, $\mathcal{R}SF(\mathcal{E}) = \mathcal{W}_{ei}(\mathcal{E})$.*

The proof of Theorem 3.2 needs Lemma 3.3 of Zhou (1992), stated below, which consists into two conclusions. The former is an average version of the Lyapunov convexity theorem since it concerns the set of average integrals of an integrable function on an atomless measure space. The latter points out that the bundle of almost every agent can be approximated by a sequence of average integrals.

Lemma 3.3 in Zhou (1992): *Let h be an integrable function from an atomless measure space (T, Σ, μ) to $I\!R^\ell$, in which $0 < \mu(T) < \infty$. Denote by H the set of all average integrals of h on measurable sets of positive measure:*

$$H = \{x \in I\!R^\ell : \; x = \bar{h}(S) \, \text{for some} \, S \in \Sigma, \, \text{with} \, \mu(S) > 0\}.$$

Then, (i) H is convex; and (ii) $h(t) \in cl(H)$ almost everywhere in T, where $cl(H)$ is the closure of H.

This lemma together with the connectedness of the covering $\mathcal{R}$ is crucial to our result, whose proof is organized in the following steps. Take an arbitrary connected covering $\mathcal{R} = \{C_i\}_{i \in I \subseteq I\!N}$ and an arbitrary $\mathcal{R}$-strictly fair allocation x. Since x is efficient, for the second welfare theorem there exists a supporting price p. Absence of envy within each element C_i of $\mathcal{R}$ and Lemma 3.3. in Zhou (1992) imply that the value of x is constant on any C_i. Being $\mathcal{R}$ connected, the value of x is actually constant over T. Feasibility of x concludes the proof, since it ensures that (x, p) is an equal-income Walrasian equilibrium.

Proof of Theorem 3.2: Fix an arbitrary connected covering $\mathcal{R} = \{C_i\}_{i \in I \subseteq I\!N}$ of T. Since $\mathcal{W}_{ei}(\mathcal{E}) = SF(\mathcal{E}) \subseteq \mathcal{R}SF(\mathcal{E})$, it is sufficient to prove that

$\mathcal{R}SF(\mathcal{E}) \subseteq \mathcal{W}_{ei}(\mathcal{E})$. Let x an arbitrary $\mathcal{R}$-strictly fair allocation. Since x is efficient, the second welfare theorem ensures the existence of a price vector p supporting x. In order to get that x is an equal-income Walrasian equilibrium allocation, we need to show that $p \cdot x(t) = p \cdot \boldsymbol{\omega}$ for almost all $t \in T$.

Let C_i be an arbitrary element of $\mathcal{R}$, then the set C_i^p of agents in C_i for whom $x(t)$ does not maximize u_t on $\{y \in I\!R^{\ell}_{++} : p \cdot y = p \cdot x(t)\}$ has null measure. On the other hand, since x is $\mathcal{R}$- strictly envy-free, even the set C_i^e of agents in C_i envying some subcoalition of C_i has null measure. We now show that $p \cdot x(t) = p \cdot \bar{x}(C_i)$ for almost all $t \in \hat{C}_i := C_i \setminus (C_i^p \cup C_i^e)$. To this end, let $\bar{C}_i = \{t \in \hat{C}_i : p \cdot x(t) < p \cdot \bar{x}(C_i)\}$ and assume to the contrary that $\mu(\bar{C}_i) > 0$.
Consider the set $X_i := \{y \in \mathbb{R}^{\ell}_{++} : y = \bar{x}(S) \text{ for some } S \subseteq C_i \text{ with } \mu(S) > 0\}$ and its subset $Y_i := \{y \in \mathbb{R}^{\ell}_{++} : y = \bar{x}(S) \text{ for some } S \subseteq \bar{C}_i \text{ with } \mu(S) > 0\}$. Applying Lemma 3.3 (ii) of Zhou (1992) to $(\bar{C}_i, \Sigma_{|\bar{C}_i}, \mu_{|\bar{C}_i})$, we get the existence of an agent s in $\bar{C}_i$ such that $x(s) \in cl(Y_i)$.

Since $s \in \bar{C}_i \subseteq \hat{C}_i = C_i \setminus (C_i^p \cup C_i^e)$, in particular $s \notin C_i^p$, then $x(s)$ maximizes the differentiable function $u_s(\cdot)$ on the set $\{y \in I\!R^{\ell}_{++} : p \cdot y = p \cdot x(s)\}$ and $p \cdot x(s) < p \cdot \bar{x}(C_i)$. This implies the existence of $\alpha \in (0,1)$ for which

$$u_s(x(s)) < u_s(x(s) + \alpha(\bar{x}(C_i) - x(s))). \tag{1}$$

Since $x(s) \in cl(Y_i)$ and $Y_i \subseteq X_i$, there exists a sequence $(y_k)_{k \in \mathbb{N}}$ in X_i converging to $x(s)$. Clearly $\bar{x}(C_i)$ belongs to X_i which is convex because of Lemma 3.3 (i) of Zhou (1992). Then, $y_k + \alpha(\bar{x}(C_i) - y_k) \in X_i$, for every k. Since $s \in \bar{C}_i \subseteq \hat{C}_i = C_i \setminus (C_i^p \cup C_i^e)$, in particular $s \notin C_i^e$, which means, by the $\mathcal{R}$-strict envy-freeness of x, that $u_s(x(s)) \geqslant u_s(y_k + \alpha(\bar{x}(C_i) - y_k))$ for every k. Taking the limit of the sequence, we get $u_s(x(s)) \geqslant u_s(x(s) + \alpha(\bar{x}(C_i) - x(s)))$, which contradicts (1). Hence $\mu(\bar{C}_i) = 0$, that is $p \cdot x(t) \geq p \cdot \bar{x}(C_i)$ for almost all $t \in C_i$. Since $\int_{C_i} p \cdot x(t) d\mu = p \cdot \bar{x}(C_i)\mu(C_i)$, it follows that $p \cdot x(t) = p \cdot \bar{x}(C_i)$ for almost all $t \in C_i$. Therefore, being C_i taken arbitrarily in $\mathcal{R}$, for all $i \in I$ and for almost all $t \in C_i$, $x(t)$ maximizes u_t on $\{y \in \mathbb{R}^{\ell}_{++} : p \cdot y = p \cdot \bar{x}(C_i)\}$. Moreover, since $\mathcal{R}$ is connected, for any $i \in I$ there exists $j \in I \setminus \{i\}$ for which $\mu(C_i \cap C_j) > 0$ and hence $p \cdot \bar{x}(C_i) = p \cdot \bar{x}(C_j)$. This means that $p \cdot \bar{x}(C_i)$ is constant over I and so is $p \cdot x(t)$ over T. Feasibility of x concludes the proof as it ensures that $p \cdot x(t) = p \cdot \boldsymbol{\omega}$ for almost all $t \in T$. □

Corollary 3.3. *$\mathcal{R}SF(\mathcal{E}) = SF(\mathcal{E})$ for every connected covering $\mathcal{R}$ of T.*

Remark 3.4. Theorem 3.2 also implies a further characterization of the core and the set of coalitional fair allocation (see Varian (1974), Gabszewicz (1975) among others), because they both coincide with the set of Walrasian equilibrium allocations.

Remark 3.5. According to Definition 2.2, given a covering $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$ with $\mu(C_i \cap C_j) > 0$ for some $i, j \in I$; any agent in $C_i \cap C_j$ must compare her own bundle separately with the average bundle of any subcoalitions of C_i and of C_j, and not of all the coalitions in the union $C_i \cup C_j$. This restriction reduces the number of bundle-comparisons and then induces a weaker notion of fairness since more allocations pass the envy-freeness test. However, this constraint is irrelevant under the hypotheses of Theorem 3.2 as it implies the equivalence between *local* and *global* strict fairness (see Corollary 3.3).

Remark 3.6. We want to highlight the role of the hypothesis that the covering is connected in the equivalence theorem. In Example 2.3 the society is structured as a partition, because agents have been categorized on the basis of their type, so that two individuals belong to the same group if they have the same initial endowment and the same utility function. This is a natural way to split an atomless economy with a finite number of types. In Example 2.3, the equivalence $\mathcal{R}SF(\mathcal{E}) = SF(\mathcal{E})$ fails. This does not contradict Theorem 3.2 because the covering $\mathcal{R}$ is a partition and hence it is not connected. The same happens if in Example 2.3, we consider a covering corresponding to a network with isolated subgraphs; for instance, by considering the covering $\mathcal{R} = \{C_1, C_2, C_3, C_4\}$, with $C_1 = \left(0, \frac{3}{8}\right)$, $C_2 = \left(\frac{1}{8}, \frac{1}{2}\right]$, $C_3 = \left(\frac{1}{2}, \frac{7}{8}\right)$ and $C_4 = \left(\frac{5}{8}, 1\right)$. Note that the pairs $\{C_1, C_2\}$ and $\{C_3, C_4\}$ are connected, because $\mu(C_1 \cap C_2) > 0$ and $\mu(C_3 \cap C_4) > 0$, whereas all the other possible pairs are not connected. Thus $\mathcal{R}$ is not connected and still the *efficient* allocation

$$(x(t), y(t)) = \begin{cases} \left(1, \frac{4}{3}\right) & \text{if } t \in C_1 \cup C_2 \\ \left(1, \frac{2}{3}\right) & \text{if } t \in C_3 \cup C_4. \end{cases}$$

is $\mathcal{R}$*-strictly envy-free*, but it is *not (globally) strictly fair.*

The connection of the covering $\mathcal{R}$ is therefore crucial. However, it is the unique condition on the topology of the network needed for the equivalence theorem. Indeed, other features as *the weight of the edge connecting two nodes* C_i and C_j, definable as the non negative number $\mu(C_i \cap C_j)$, or *the degree of a node* C_i, definable as the number of elements C_j of $\mathcal{R}$ to which it is connected are irrelevant for our analysis.

Remark 3.7. In what follows, we illustrate an example in which, given a connected covering $\mathcal{R}$ of T, there is an $\mathcal{R}$-strictly envy-free allocation x which is not (globally) strictly envy-free. This does not contradict Theorem 3.2 because the allocation x is not efficient.

Example 3.8. Consider an atomless economy with $T = (0, 1)$ and two goods. The total initial endowment is $\boldsymbol{\omega} = \left(\frac{11}{8}, \frac{7}{6}\right)$ and agents' utility function are defined as follows

$$u_t(x,y) = \begin{cases} x^2y & \text{if } t \in \left(0, \frac{1}{3}\right] \\ x^3y & \text{if } t \in \left(\frac{1}{3}, \frac{2}{3}\right] \\ xy^2 & \text{if } t \in \left(\frac{2}{3}, 1\right). \end{cases}$$

Consider the covering $\mathcal{R} = \{C_1, C_2\}$ with $C_1 = \left(0, \frac{1}{3}\right] \cup \left(\frac{2}{3}, 1\right)$ and $C_2 = \left(\frac{1}{3}, 1\right)$, which is connected since $C_1 \cap C_2 = \left(\frac{2}{3}, 1\right)$. The allocation

$$(x(t), y(t)) = \begin{cases} \left(\frac{13}{8}, 1\right) & \text{if } t \in \left(0, \frac{1}{3}\right] \subseteq C_1 \\ \left(\frac{3}{2}, 1\right) & \text{if } t \in \left(\frac{1}{3}, \frac{2}{3}\right] \subseteq C_2 \\ \left(1, \frac{3}{2}\right) & \text{if } t \in \left(\frac{2}{3}, 1\right) \subseteq C_1 \cap C_2 \end{cases}$$

is feasible and, it can be shown that, for any $i = 1, 2$, $t \in C_i$ and $S \subseteq C_i$, $u_t(x(t), y(t)) \geqslant u_t(\bar{x}(S), \bar{y}(S))$. Hence x is $\mathcal{R}$-strictly envy-free. On the other hand, x is not (globally) strictly envy-free given that, for instance, any individual in $\left(\frac{1}{3}, \frac{2}{3}\right]$ envies any subcoalition of $\left(0, \frac{1}{3}\right]$. △

Notice that the above allocation (x, y) is not efficient, as it is Pareto blocked by the alternative feasible allocation

$$(x^*(t), y^*(t)) = \begin{cases} \left(\frac{53}{36}, \frac{11}{9}\right) & \text{if } t \in \left(0, \frac{1}{3}\right] \\ \left(\frac{815}{504}, \frac{4}{5}\right) & \text{if } t \in \left(\frac{1}{3}, \frac{2}{3}\right] \\ \left(\frac{29}{28}, \frac{133}{90}\right) & \text{if } t \in \left(\frac{2}{3}, 1\right). \end{cases}$$

Thus, if we focus only on absence of envy ignoring the efficiency, even though the covering is connected, local absence of envy cannot be expanded into the global one. Example 3.8 underlines the role of efficiency for our equivalence theorem. Intuitively this is because efficiency involves the entire society, contrary to the notion of $\mathcal{R}$-strict envy-freeness which instead considers only a piece of it. Roughly speaking, once there is no envy in each element of the covering and there is no isolated group in the society, efficiency is the glue that attaches each piece of the covering guaranteeing global fairness.

Remark 3.9. With similar arguments used in Lemma 3.5 of Donnini and Pesce (2018), it is possible to prove that under a $\mathcal{R}$-strictly envy-free allocation, members of the same group and with the same strictly quasi-concave utility function get the same bundle. In other words, we get a local version of the equal treatment property, because "*equals are treated equally*" at least within the same coalition. We now show that this property fails globally under a $\mathcal{R}$-strictly envy-free allocation even though $\mathcal{R}$ is connected, because there might exist two agents with the same tastes, belonging to two distinct elements of $\mathcal{R}$ receiving different bundles. To this end, consider a revised version of Example 3.8 with an atomless economy in which $T = (0, 1)$ and with two goods. The total initial endowment is $\boldsymbol{\omega} = (\frac{11}{8}, \frac{7}{6})$ and agents' utility function are given by

$$u_t(x, y) = \begin{cases} x^2 y & \text{if } t \in \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right) \\ \\ xy^2 & \text{if } t \in \left[\frac{1}{3}, \frac{2}{3}\right] \end{cases}$$

Consider the connected covering $\mathcal{R} = \{C_1, C_2\}$ with $C_1 = \left(0, \frac{2}{3}\right]$ and $C_2 = \left[\frac{1}{3}, 1\right)$. Then, it can be proved that the allocation

$$(x(t), y(t)) = \begin{cases} \left(\frac{13}{8}, 1\right) & \text{if } t \in \left(0, \frac{1}{3}\right) \subseteq C_1 \\ \\ \left(1, \frac{3}{2}\right) & \text{if } t \in \left[\frac{1}{3}, \frac{2}{3}\right] \subseteq C_1 \cap C_2 \\ \\ \left(\frac{3}{2}, 1\right) & \text{if } t \in \left(\frac{2}{3}, 1\right) \subseteq C_2. \end{cases}$$

is $\mathcal{R}$-strictly envy-free but agents in $\left(0, \frac{1}{3}\right)$ and in $\left(\frac{2}{3}, 1\right)$, nevertheless with the same utility, get different bundles being members of different groups. According to Lemma 3.5 of Donnini and Pesce (2018), the allocation $(x(t), y(t))$ cannot be *globally* strictly fair. Indeed, agents in $\left(\frac{2}{3}, 1\right)$ envies any subcoalition in $\left(0, \frac{1}{3}\right)$.

4 A comparison with the related literature

The idea of local fairness is not new and has been studied in several contexts, in particular in social networks (see, for instance, Abebe et al. (2017)) and in house allocation problems (see, for instance, Beynier et al. (2018)). A similar idea can be also found in the matching literature. For example, in marriage matching the society is divided into two disjoint sets, men and women. A man can only envy another man, that is a person within his same group and similarly for the women. As far as we know, theoretical models of economies with divisible goods have tended to focus on global envy. An exception is Cato (2010), who introduces a local version of strict fairness, called $\varepsilon-$strict fairness, with a very similar intuition, but which, as we show below, is a particular case of our notion of $\mathcal{R}$-strict fairness.

In Cato (2010) the set of agents is the interval $T = (0, 1)$ and for any fixed $\varepsilon \in (0, 1)$, and any trader $t \in T$, the set of t's neighbours is the interval $I_t(\varepsilon) = (t - \varepsilon, t + \varepsilon) \cap T$. According to the notion of ε-strict fairness, each individual has to compare her own bundle with the average bundle of any coalitions composed by her neighbours. Formally, an allocation x is *ε-strictly envy-free*, if for almost every agent t in T, $u_t(x(t)) \geqslant u_t(\bar{x}(S))$, with S subcoalition of $I_t(\varepsilon)$. An allocation x is *ε-strictly fair* if it is ε-strictly envy-free and efficient. We now show that our notion of local strict envy-freeness generalizes the one of Cato (2010).

Proposition 4.1. *For any $\varepsilon \in (0,1)$ there exists a connected covering $\mathcal{R}_\varepsilon$ of T such that any ε-strictly envy-free allocation is also $\mathcal{R}_\varepsilon$-strictly envy-free. The reverse is not true.*

Proof: Given an arbitrary ε in $(0,1)$, define the finite family $\mathcal{R}_\varepsilon = \{C_1, \dots, C_n\}$ with $C_i := \left(\frac{i-1}{2}\varepsilon, \frac{i+1}{2}\varepsilon\right]$ for $i \in \{1, \dots, n-1\}$ and $C_n := \left(\frac{n-1}{2}\varepsilon, 1\right)$, where $n = \left\lfloor \frac{2-\varepsilon}{\varepsilon} \right\rfloor + 1$.[3] Note that $\mathcal{R}_\varepsilon$ is a covering of T for which, for any $i \in \{1, ..., n-1\}$, $\mu(C_i) = \varepsilon$ and $\mu(C_n) < \varepsilon$ and it is connected. Therefore, for any $i \in I := \{1, ..., n\}$ and for any t in C_i, each subcoalition S of C_i is made by neighbours of t, that is $S \subseteq (t - \varepsilon, t + \varepsilon) \cap T = I_t(\varepsilon)$. This implies that any ε- strictly envy-free allocation is $\mathcal{R}_\varepsilon$-strictly envy-free, and it concludes the first part of the proposition. To show that the reverse is not true, consider the economy described in Example 3.8 and notice that the $\mathcal{R}$-strictly envy-free allocation (x, y) is not ε-strictly envy-free for any $\varepsilon \in (0,1)$. Indeed, given any $\varepsilon \in (0,1)$, any individual t in $\left(\frac{1}{3}, \frac{1}{3} + \varepsilon\right) \cap \left(\frac{1}{3}, \frac{2}{3}\right]$ strictly envies any coalition $S \subseteq \left(0, \frac{1}{3}\right] \cap I_\varepsilon(t)$. □

Remark 4.2. Cato (2010) shows that efficiency expands the local notion into the global one, in the sense that ε-strict envy-freeness does not guarantee absence of envy globally, but when combined with the efficiency it does. In the light of Proposition 4.1, the equivalence obtained by Cato (2010) follows from Theorem 3.2.

5 Local fairness in mixed markets

In this section we extend our analysis to mixed markets, that are large economies with atoms. We assume, as usual, the set of agents T to be partitioned into two sets T_0 and $T_1 = T \setminus T_0$. T_0 is the atomless part representing the set of negligible (small) agents, while T_1 is the union of at most countably many atoms, representative of non negligible (large) agents. Since any atom is treated as a single trader, T_1 can be seen as $T_1 = \{A_1, A_2, ..., A_n, ...\}$ and the notation $A_i \in T_1$ can be used instead of $A_i \subseteq T_1$. Two agents are said to be identical or of the same type if they have the same initial endowment and the same utility function. Zhou (1992) shows that in mixed markets, under a strictly fair allocation, small traders always get less income than any large traders. He also illustrates an economy with two identical atoms in which a strictly fair allocation is not supported by an equilibrium price. In Donnini and Pesce (2018) we show that even in an economy with an arbitrary

[3]We denote by $\left\lfloor \frac{2-\varepsilon}{\varepsilon} \right\rfloor$ the integer part of $\frac{2-\varepsilon}{\varepsilon}$.

finite number of atoms of the same type, the equivalence between the set of equal-income Walrasian equilibria and strictly fair allocations fails. In other words, the hypotheses used by Shitovitz (1973) to prove the Core-Walras equivalence theorem, that is *all atoms must be identical and at least two*, is not enough for the strict fairness. In Donnini and Pesce (2018) we provide two stronger assumptions on the space of agents in order to nullify the market power of the atoms and restore the equivalence for mixed markets. The former requires the existence of countable infinitely many identical atoms and we implicitly impose a condition on the measure of the atoms. The latter needs, regardless the number and the relations among large traders, that for each atom there is a coalition of negligible agents of the same type of the atom (atomless fringe)[4].
The next two examples show that the previous two assumptions, sufficient for the (global) strict fairness, are not enough for the (local) $\mathcal{R}$-strict fairness even though the covering $\mathcal{R}$ is connected.

Example 5.1. Consider a mixed economy whose consumption set is $\mathbb{R}^2_{++}$, $T = T_0 \cup T_1$ with $T_0 = [0, \frac{1}{4}]$ and $T_1 = \{A_n\}_{n\in\mathbb{N}}$ with $\mu(A_n) = \frac{3}{2^{n+2}}$. Notice that $\mu(T_1) = \frac{3}{4}$. The total initial endowment is $\boldsymbol{\omega} = (1, 1)$. Agents' utility function is given by

$$u_t(x, y) = \begin{cases} xy & \text{if } t \in T_0 \\ x^2 y & \text{if } t \in T_1. \end{cases}$$

Consider the connected covering $\mathcal{R} = \{C_1, C_2\}$ of T, where $C_1 = T_0 \cup \{A_1, A_2\}$ and $C_2 = T_1$.

We now show that the following feasible allocation

$$(x(t), y(t)) = \begin{cases} \left(\frac{4+2\sqrt{19}}{15}, \frac{2+2\sqrt{19}}{9}\right) & \text{if } t \in T_0 \\ \\ \left(\frac{26-2\sqrt{19}}{15}, \frac{16-2\sqrt{19}}{9}\right) & \text{if } t \in T_1. \end{cases}$$

is *$\mathcal{R}$-strictly fair*, but it is not *globally strictly fair* and a fortiori it is not an equal-income Walrasian equilibrium allocation.

FACT 1. (x, y) is efficient.
Assume to the contrary that for some feasible allocation $(\tilde{x}, \tilde{y})$, $u_t(x(t), y(t)) < u_t(\tilde{x}(t), \tilde{y}(t))$, for almost every t in T. As observed in Remark 3.1 of Donnini and Pesce (2018) (see also Lemma in García-Cutrín and Hervés-Beloso

[4]Similar assumptions are used, for different contexts, in Gabszewicz and Mertens (1971), Shitovitz (1992), Greenberg and Shitovitz (1986) and Basile et al. (2016).

(1993) and Lemma 7.1 in Basile et al. (2017)) the allocation defined by

$$(a(t), b(t)) = \begin{cases} (a, b) = \frac{1}{\mu(T_0)} \int_{T_0} (\tilde{x}(t), \tilde{y}(t))\, d\mu & \text{if } t \in T_0 \\ (2-a, 2-b) = \frac{1}{\mu(T_1)} \int_{T_1} (\tilde{x}(t), \tilde{y}(t))\, d\mu & \text{if } t \in T_1. \end{cases}$$

still improves upon (x, y). Therefore,

$$\begin{cases} ab > \frac{4+2\sqrt{19}}{15} \frac{2+2\sqrt{19}}{9} \\ (2-a)^2(2-b) > \left(\frac{26-2\sqrt{19}}{15}\right)^2 \frac{16-2\sqrt{19}}{9}. \end{cases}$$

By the first inequality, $b > \frac{4}{45}(7+\sqrt{19})$, which implies in the second condition $5(2-a)^2(90a-4(7+\sqrt{19})) > 16(111-22\sqrt{19})a$, that, after algebraic computation, leads to $225a^3 - 10(97+\sqrt{19})a^2 + 4(73-54\sqrt{19})a - 40(7+\sqrt{19}) > 0$, which has no solution in $(0, 2)$. This means that (x, y) is efficient.

FACT 2. (x, y) is $\mathcal{R}$-strictly envy-free.
For every t in C_2 and $S \subseteq C_2$, $(x(t), y(t)) = (\bar{x}(S), \bar{y}(S))$, therefore (x, y) is envy-free in C_2. For every coalition S in C_1 let $\alpha = \frac{\mu(S\cap T_0)}{\mu(S)}$, and $1 - \alpha = \frac{\mu(S\cap T_1)}{\mu(S)}$. Then, the average bundle of (x, y) over S is

$$(\bar{x}(S), \bar{y}(S)) = \left(\frac{2}{15}(\alpha(2\sqrt{19}-11) + 13 - \sqrt{19}), \frac{2}{9}(\alpha(2\sqrt{19}-7) + 8 - \sqrt{19})\right).$$

For every t in T_0, $u_t(x(t), y(t)) \geqslant u_t(\bar{x}(S), \bar{y}(S))$ is equivalent to

$$(2+\sqrt{19})(1+\sqrt{19}) \geqslant [\alpha(2\sqrt{19}-11) + 13 - \sqrt{19}][\alpha(2\sqrt{19}-7) + 8 - \sqrt{19}],$$

which, after algebraic computation, leads to $(12\sqrt{19}-51)(\alpha-1)(3\alpha-2) \geqslant 0$. This inequality is always satisfied since for every S in C_1, $\alpha \in [0, \frac{2}{3}] \cup \{1\}$.
For every t in T_1, $u_t(x(t), y(t)) \geqslant u_t(\bar{x}(S), \bar{y}(S))$ is equivalent to

$$(13-\sqrt{19})^2(8-\sqrt{19}) \geqslant [\alpha(2\sqrt{19}-11) + 13 - \sqrt{19}]^2[\alpha(2\sqrt{19}-7) + 8 - \sqrt{19}],$$

that is, by algebraic computation, $9\alpha[(78\sqrt{19}-339)\alpha^2 + (862-199\sqrt{19})\alpha + 168\sqrt{19} - 734] \leqslant 0$. This inequality holds, in particular, for any α in $[0, 1]$. Therefore there is no envy neither in C_1, and hence (x, y) is $\mathcal{R}$-strictly envy-free.

FACT 3. (x, y) is not (globally) strictly fair.
For every t in T_0, consider the coalition $S = [0, \frac{3}{16}] \cup \{A_3\}$. Then

$$(\bar{x}(S), \bar{y}(S)) = \left(\frac{2}{45}(17+\sqrt{19}), \frac{2}{27}(10+\sqrt{19})\right)$$

and $u_t(\bar{x}(S), \bar{y}(S)) = 3u_t(x(t), y(t)) > u_t(x(t), y(t))$. This means that any small trader $t \in T_0$ envies the coalition S. $\triangle$

Notice that in the economy described in Example 5.1 there are infinitely countably many identical atoms and the considered covering is connected. With slight modifications we now show that even though the economy has for each atom an atomless fringe of negligible traders of the same type of the atom, there might exist a connected covering $\mathcal{R}$ and a (local) $\mathcal{R}$-strictly fair allocation which is not an equal-income Walrasian equilibrium.

Example 5.2. Consider a mixed economy whose consumption set is $\mathbb{R}^2_{++}$, $T = T_0 \cup T_1$ with $T = [0, \frac{1}{2}]$ and $T_1 = \{A_1, A_2\}$ with $\mu(A_n) = \frac{1}{4}$. The total initial endowment is $\boldsymbol{\omega} = (1, 1)$. Agents' utility function is given by

$$u_t(x, y) = \begin{cases} xy & \text{if } t \in [0, \frac{1}{4}] \\ x^2 y & \text{if } t \in (\frac{1}{4}, \frac{1}{2}] \text{ and } t \in T_1. \end{cases}$$

Consider the connected covering $\mathcal{R} = \{C_1, C_2\}$ where $C_1 = [0, \frac{1}{4}] \cup T_1$ and $C_2 = (\frac{1}{4}, \frac{1}{2}] \cup T_1$. With similar computation done in the previous example, it can be shown that the following feasible allocation

$$(x(t), y(t)) = \begin{cases} \left(\frac{4+2\sqrt{19}}{15}, \frac{2+2\sqrt{19}}{9}\right) & \text{if } t \in T_0 \\ \\ \left(\frac{26-2\sqrt{19}}{15}, \frac{16-2\sqrt{19}}{9}\right) & \text{if } t \in T_1. \end{cases}$$

is *$\mathcal{R}$-strictly fair*, but it is not *(globally) strictly fair* and a fortiori it is not an equal-income Walrasian equilibrium allocation. $\triangle$

Examples 5.1 and 5.2 make clear that under a (local) $\mathcal{R}$-strictly fair allocation, the market power of large traders should be *locally* crumbled. We can then restore the equivalences $SF(\mathcal{E}) = \mathcal{R}SF(\mathcal{E})$ in mixed markets by further strengthening the two sufficient conditions of Donnini and Pesce (2018). The former can be reformulated by requiring that if there are infinitey many identical large traders, they must belong to the same set C_i of the covering $\mathcal{R}$. The second can be strengthened by imposing that if a certain set C_i of $\mathcal{R}$ contains an atom A, then it must contain a piece of its atomless fringe.

Theorem 5.3. *Assume that for almost every t in T, u_t is strictly quasi-concave on $\mathbb{R}^\ell_{++}$. Given a connected covering $\mathcal{R} = \{C_1, ..., C_n\}$ of T, if one of the following statements holds*

1. *T_1 consists in countable infinitely many identical atoms and $\mu(C_j \cap T_1) > 0$, then $T_1 \subseteq C_j$,*

2. *for each atom A there exists a coalition S_A of negligible agents identical to A and $A \subseteq C_j$, then $\mu(S_A \cap C_j) > 0$,*

then the equal-division Walrasian allocations are the only $\mathcal{R}$-strictly fair allocations, and a fortiori, the only global strictly fair allocations.

Proof. Apply the arguments of the proof of Theorem 3.6 in Donnini and Pesce (2018) in any set C_i of $\mathcal{R}$. □

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