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Abstract

In the literature on financial contagion, the possibility to deal only with imprecise information about the overall interbank exposures and the implications in the analysis of the stability of the financial system seems to be a relevant problem. In particular, previous literature has shown that fuzzy data arise naturally in this framework and turn to be sufficiently friendly to handle from the computational point of view. The present paper generalizes the well known *fictitious default algorithm* to the fuzzy setting, providing an existence result for the corresponding fuzzy fixed points, the convergence of the algorithm to fixed points, an implementation of the algorithm in MATLAB and numerical simulations.

Keywords: Financial networks, fuzzy financial data, fictitious default, fixed point.

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1 Introduction

The recent theoretical and empirical literature on financial contagion has investigated the relationships between the interbank exposure network and the financial stability of the banking system (see for instance Glasserman and Young (2016) or Hurd (2016) for recent surveys): the financial network has been recognized to be a source of financial crisis as shocks, which initially affect only few institutions, propagate through the entire banking system producing a *contagion cascade*. Aim of the present paper is to study the issue of financial contagion under uncertainty in the particular case in which financial obligations are represented by fuzzy numbers.

As pointed out in De Marco et al. (2018), only few papers have explored the issue of the lack of precise information about the overall interbank exposures and the implications in the analysis of the stability of the banking system. Nevertheless, this seems to be a relevant problem as banks' balance sheets are made public only few times each year. Furfine (2003) gives a first attempt to study the system's stability according to more realistic interbank exposure data based on daily observations along a two months period. As the recovery of precise and exhaustive interbank exposure data turns to be problematic, Furfine (2003) deduces interbank exposures by looking at the transaction data in the Federal Reserve's large-value system (Fedwire). Banks are then classified into four groups according to the volume of funds traded and the exposure of a bank from one group in another bank from another group is expressed by the minimum, the maximum and the average value of transactions between the two groups observed in the sample period. In De Marco et al. (2018), it has been shown that Furfine's data can be regarded as triangular fuzzy numbers where the minimum and the maximum are clearly the infimum and the supremum of the support of the fuzzy number and the average is simply the maximum point for the membership function. This interpretation has been the main motivation of the analysis of financial contagion in a fuzzy environment as it shows that fuzzy data appear in a natural way when we look at financial networks. Moreover, Furfine's work provides a data set of fuzzy interbank exposures that can be used to run simulations.

Many papers have studied the dynamics of contagion in the classical crisp case: Eisenberg and Noe (2001) is recognized as a pioneering contribution; it shows that the existence of payment vectors that *clear* the obligations in the market can be solved by using fixed points arguments. Then, Eisenberg and Noe (2001) develops an algorithm (known as the *fictitious default algorithm*) that converges to a *clearing payment vector* and, at the same time, gives information about the systemic risk faced by the banks in the network[3]. The present paper is a direct generalization of the Eisenberg and Noe (2001) analysis: interbank exposures and every other quantity in banks' balance sheets are allowed to be fuzzy numbers. As a consequence, we characterize fuzzy clearing vectors as fixed points of the fuzzy mappings that, in turn, define the fuzzy version of the fictitious default algorithm. So we provide an analogous of the fixed point theorem in Eisenberg and Noe (2001) for the fuzzy environment and show that the corresponding fuzzy fictitious default algorithm converges to a fixed point. In order to support our theoretical analysis we propose also some numerical results. The fuzzy fictitious default algorithm has been written and fully implemented in MATLAB, and tested numerically on a real financial date set. Finally, the present paper differs significatively from De Marco et al. (2018) which does not focus on the original fictitious default algorithm but rather investigates a variation based on the *zero recovery assumption*, as studied in Gai and Kapadia (2010) for

[3]Gai and Kapadia (2010), Gai et al. (2011), Lee (2013), Glasserman and Young (2015), among others, provide some further significative contributions from slightly different perspectives.

the crisp case, whose underlying idea is the following: since in the immediate aftermath of a default, the recovery rate and the timing of recovery are highly uncertain then it is likely to assume the pessimistic scenario in which each bank in the network is supposed to loose all its interbank deposits on a failed bank, regardless of the real net worth of such failed bank.

The paper is organized as follows: Section 2 recalls basic notions from fuzzy set theory. Section 3 introduces the basic elements of the Eisenberg and Noe (2001) model. Section 4 introduces the fuzzy financial network, the fuzzy fixed point problem and the fuzzy fictitious default algorithm. Section 5 provides the existence of fixed point while, the convergence of the fuzzy fictitious default algorithm is studied in Section 6. The implementation of the algorithm and results of the simulation are described in the final Section 7.

2 Fuzzy Numbers

In this section we recall some key notions and results from the theory of fuzzy numbers that are required in our model (see, for example, Buckley and Eslami (2002), Klir (1995), Zadeh (1995) and Zimmermann (2001) for extensive surveys and references).

Given a universal set X, a fuzzy subset A of X is a function, called *membership function*, which associates with each point in X a real number in the interval $[0,1]$.
Following Goetschel and Voxman (1986) (see also Chai and Zhang (2016)) a *fuzzy number* n is a particular fuzzy subset of $\mathbb{R}$, with membership function denoted by μ_n, satisfying the following conditions:

1. μ_n is normal, i.e., there is a real number x_0 such that $\mu_n(x_0) = 1$;

2. μ_n is compactly supported, i.e., the closure of the support of n is bounded (we remind that $supp(n) = \{x \in \mathbb{R} \mid \mu_n(x) > 0\}$ and we denote by $\overline{supp}(n)$ its closure);

3. μ_n is quasi-concave, i.e., $x \leqslant y \leqslant z$ implies $\min\{\mu_n(x), \mu_n(z)\} \leqslant \mu_n(y)$ for all $x, y, z \in \mathbb{R}$;

4. μ_n is upper semi-continuous, i.e., for each $\alpha \in]0,1]$, the α-cut, $\{x \in \mathbb{R} \mid \mu_n(x) \geqslant \alpha\}$, is closed.

Defining

$$n[\alpha] = \begin{cases} \{x \in \mathbb{R} \mid \mu_n(x) \geqslant \alpha\}, & \text{if } 0 < \alpha \leqslant 1; \\ \\ \overline{supp}(n), & \text{if } \alpha = 0; \end{cases} \tag{1}$$

it is easy to show that n is a fuzzy number if and only if

1. $n[\alpha]$ is a closed and bounded interval for each α in $[0,1]$;

2. $n[1] \neq \emptyset$.

Using this characterization, a fuzzy number, n, can be determined by the endpoints of the intervals $n[\alpha]$, therefore we can identify n with the parameterized representation $\{(\underline{n}[\alpha], \overline{n}[\alpha]) \mid \alpha \in [0,1]\}$, where $\underline{n}[\alpha]$ and $\overline{n}[\alpha]$ denote respectively the left hand endpoint and the right hand endpoint of $n[\alpha]$.[4]

[4]In the remainder of the paper we will use equivalently $\underline{n}$ or $\underline{n}[0]$ for $\inf supp(n)$ and $\overline{n}$ or $\overline{n}[0]$ for $\sup supp(n)$. Moreover, with abuse of notation, for every α in $[0,1]$, we will call $n[\alpha]$ α-cut.

A fuzzy number n is said to have *single peak* if the core is a singleton, i.e., $co(n) = \{\hat{n}\}$. We remind that, by definition, the core of a fuzzy number n is $co(n) = \{x \in \mathbb{R} \,|\, \mu_n(x) = 1\}$ and it coincides with $n[1]$. We denote by $\mathcal{F}$ the set of fuzzy numbers having single peak[5], with $\mathcal{F}_+ = \{n \in \mathcal{F} : n[0] \subseteq [0, +\infty[\}$ and with $\mathcal{F}^s$ the s-Cartesian product $\mathcal{F}^s = \{(n_1, n_2, \ldots, n_s) \,|\, n_k \in \mathcal{F}$ for every $k \in \{1, 2, \ldots, s\}\}$.

Remark 2.1: Using the parameterized representation of fuzzy numbers, we have that, given $n, m, l \in \mathcal{F}$ and $c \in \mathbb{R}$,

1. $n + m - l = \{(\underline{n}[\alpha] + \underline{m}[\alpha] - \bar{l}[\alpha], \overline{n}[\alpha] + \overline{m}[\alpha] - \underline{l}[\alpha]) \,|\, \alpha \in [0, 1]\}$
2. $n{\cdot}m = \{(a[\alpha], b[\alpha]) \,|\, \alpha \in [0, 1]\}$, where $a[\alpha] = \min\{\underline{n}[\alpha]\underline{m}[\alpha], \underline{n}[\alpha]\overline{m}[\alpha], \overline{n}[\alpha]\underline{m}[\alpha], \overline{n}[\alpha]\overline{m}[\alpha]\}$ and $b[\alpha] = \max\{\underline{n}[\alpha]\underline{m}[\alpha], \underline{n}[\alpha]\overline{m}[\alpha], \overline{n}[\alpha]\underline{m}[\alpha], \overline{n}[\alpha]\overline{m}[\alpha]\}$
3. $\dfrac{n}{m} = \left\{ \left(\dfrac{\underline{n}[\alpha]}{\overline{m}[\alpha]}, \dfrac{\overline{n}[\alpha]}{\underline{m}[\alpha]} \right) \,|\, \alpha \in [0, 1] \right\}$, if, for each α in $[0, 1]$, 0 does not belong to $m[\alpha]$,
4. $cn = \begin{cases} \{(c\underline{n}[\alpha], c\overline{n}[\alpha]) \,|\, \alpha \in [0, 1]\}, & \text{if } c > 0; \\ \{(c\overline{n}[\alpha], c\underline{n}[\alpha]) \,|\, \alpha \in [0, 1]\}, & \text{if } c < 0; \\ 0, & \text{if } c = 0. \end{cases}$

Note that given $n, m, l \in \mathcal{F}$ and $a \in \mathbb{R}$, $n + m - l$, $n \cdot m$, $\dfrac{n}{m}$ and an are still in $\mathcal{F}$.

Given two fuzzy numbers n and m in $\mathcal{F}$, we denote by p and q respectively their maximum and minimum, i.e., $p = MAX\{n, m\}$ and $q = MIN\{n, m\}$, where the membership functions of p and q are defined as follows (see for instance Buckley and Eslami (2002)):

$$\mu_p(z) = \sup \{\min\{\mu_m(x), \mu_n(y)\} \,|\, \max\{x, y\} = z\}$$

and

$$\mu_q(z) = \sup \{\min\{\mu_m(x), \mu_n(y)\} \,|\, \min\{x, y\} = z\}$$

Moreover, as proved in Hong and Kim (2006) (see also Chiu and Wang (2002)), it can be shown that each α-cut is given by

$$p[\alpha] = [\max\{\underline{n}[\alpha], \underline{m}[\alpha]\}, \max\{\overline{n}[\alpha], \overline{m}[\alpha]\}] \quad \text{or} \quad \underline{p}[\alpha] = \max\{\underline{n}[\alpha], \underline{m}[\alpha]\},\ \overline{p}[\alpha] = \max\{\overline{n}[\alpha], \overline{m}[\alpha]\}$$

and

$$q[\alpha] = [\min\{\underline{n}[\alpha], \underline{m}[\alpha]\}, \min\{\overline{n}[\alpha], \overline{m}[\alpha]\}] \quad \text{or} \quad \underline{q}[\alpha] = \min\{\underline{n}[\alpha], \underline{m}[\alpha]\},\ \overline{q}[\alpha] = \min\{\overline{n}[\alpha], \overline{m}[\alpha]\};$$

[5] We remark that the set $\mathbb{R}$ of real numbers is canonically embedded in the set of fuzzy numbers, identifying each real number r with the "crisp" fuzzy number with membership equal to

$$\mu_n(x) = \begin{cases} 1, & \text{if } x = r; \\ 0, & \text{otherwise;} \end{cases}$$

so, in particular, each real number can be seen as a fuzzy number having single peak.

so that p and q have the following representations

$$p = \{(\max\{\underline{n}[\alpha], \underline{m}[\alpha]\}, \max\{\overline{n}[\alpha], \overline{m}[\alpha]\}) \mid \alpha \in [0,1]\}$$

and

$$q = \{(\min\{\underline{n}[\alpha], \underline{m}[\alpha]\}, \min\{\overline{n}[\alpha], \overline{m}[\alpha]\}) \mid \alpha \in [0,1]\}.$$

It immediately follows that

PROPOSITION 2.2: *Given $n, m, l \in \mathcal{F}$ and the fuzzy number r defined by*

$$r = MIN\{n, MAX\{m, l\}\},$$

then, for every $\alpha \in [0,1]$, the α-cut of r is

$$r[\alpha] = [\underline{r}[\alpha], \overline{r}[\alpha]] \quad \text{with } \underline{r}[\alpha] = \min\{\underline{n}[\alpha], \max\{\underline{m}[\alpha], \underline{l}[\alpha]\}\},\ \overline{r}[\alpha] = \min\{\overline{n}[\alpha], \max\{\overline{m}[\alpha], \overline{l}[\alpha]\}\}.$$

Moreover $r \in \mathcal{F}$.

Proof. Consider $p = MAX\{m, l\}$. Then, using the parameterized representation,

$$p = \{(\max\{\underline{m}[\alpha], \underline{l}[\alpha]\}, \max\{\overline{m}[\alpha], \overline{l}[\alpha]\}) \mid \alpha \in [0,1]\}$$

and

$$r = MIN\{n, p\} = \{(\min\{\underline{n}[\alpha], \underline{p}[\alpha]\}, \min\{\overline{n}[\alpha], \overline{p}[\alpha]\}) \mid \alpha \in [0,1]\} =$$

$$\{(\min \underline{n}[\alpha], \{\max\{\underline{m}[\alpha], \underline{l}[\alpha]\}\}, \min\{\overline{n}[\alpha], \max\{\overline{m}[\alpha], \overline{l}[\alpha]\}\}) \mid \alpha \in [0,1]\}.$$

Hence, for every $\alpha \in [0,1]$, the α-cut of r is

$$r[\alpha] = \left[\min\{\underline{n}[\alpha], \max\{\underline{m}[\alpha], \underline{l}[\alpha]\}\}, \min\{\overline{n}[\alpha], \max\{\overline{m}[\alpha], \overline{l}[\alpha]\}\}\right].$$

Moreover, since $n[1]$, $m[1]$ and $l[1]$ are singleton, then r have a single peak, that is $r \in \mathcal{F}$. □

Finally, we say that a fuzzy number n is *positive*, whenever $\underline{n}[0] = \inf supp(n) > 0$ (or, equivalently, $supp(n) \subseteq \mathbb{R}_+$), while n is *negative* whenever $\overline{n}[0] = \sup supp(n) < 0$ (or, equivalently, if $supp(n) \subseteq \mathbb{R}_-$). We can remark that there are fuzzy numbers that are not positive neither negative. With abuse of notation, in the following, we will indicate that n is positive (negative) with $n > 0$ ($n < 0$).

3 Networks of Banks and Financial Contagion

Banks and financial Networks

Firstly, we introduce the market that is substantially the same of Eisenberg and Noe (2001). We consider a market composed by a set of banks $I = \{1, 2, \ldots, s\}$. Each bank is characterized by its balance sheet consisting on assets and liabilities, including, respectively, bank's claims and obligations to non-financial and financial entities.

The bank's assets are:

i) Outside assets c_i representing the aggregate claims of bank i on nonfinancial entities;

ii) In-network assets p_{ki}, for each $k \neq i$. Each number p_{ki} is the claim of bank i on bank k, that is, it consists in a payment obligation of bank k to bank i; in other words, p_{ki} represents the aggregate exposure of bank i in the bank k.

The bank's liabilities include:

i) Obligations b_i to nonfinancial entities;

ii) Obligations p_{ik}, for each $k \neq i$, to the bank k.

In the literature, the matrix $(p_{ik})_{i,k=1}^s$ is the adjacency matrix of a directed network, called *financial network*. Each node is a bank, and a directed edge runs from node i to node k if bank i has a payment obligation $p_{ik} > 0$ to node k. In this case, we say that bank i is directly connected to bank k.

The market clearing condition

The key element of the model in Eisenberg and Noe (2001) is given by a vector $\eta = (\eta_1, \dots, \eta_s)$, where η_k is the proportion of debt that bank k is able to repay to its creditors in the financial networks.
Therefore, the *asset side* of i's balance sheet is given by

$$c_i + \sum_{k \neq i} \eta_k p_{ki},$$

while the *liability side* by

$$b_i + \sum_{k \neq i} p_{ik}.$$

The node's *net worth* is

$$w_i = c_i + \sum_{k \neq i} \eta_k p_{ki} - b_i - \sum_{k \neq i} p_{ik}.$$

This latter equation implies that the net worth of each bank i depends on the proportion of the exposures that the bank receives back from the others. On the other hand, the proportion of debt that bank i it is able to give back to the others is a function of its net worth: $\eta_i = 1$ if bank i is able to honor all of its obligations, $\eta_i = 0$ if bank i is unable to honor each of its obligations and $\eta_i \in]0, 1[$ in case bank i is only able to partially repay its obligations. In the model introduced by Eisenberg and Noe (2001) each η_i is the maximal proportion of debt that the bank i is able to repay to the others, given the repayments of the other banks η_k with $k \neq i$. More precisely, each η_i is implicitly computed as the solution ξ_i of the *market clearing* equation

$$c_i + \sum_{k \neq i} \eta_k p_{ki} = b_i + \sum_{k \neq i} \xi_i p_{ik}. \tag{2}$$

in case

$$\xi_i = \frac{1}{\sum_{k \neq i} p_{ik}} \left(c_i + \sum_{k \neq i} \eta_k p_{ki} - b_i \right) \in]0, 1[,$$

while, $\eta_i = 1$ in case $\xi_i \geqslant 1$ and $\eta_i = 0$ in case $\xi_i \leqslant 0$.
Hence,

$$\eta_i = \Phi_i\left((\eta_k)_{k\neq i}\right) = \min\left\{1, \max\left\{\frac{1}{\sum\limits_{k\neq i} p_{ik}}\left(c_i + \sum_{k\neq i}\eta_k p_{ki} - b_i\right), 0\right\}\right\}. \qquad (3)$$

Therefore, denote with $\Phi : [0,1]^s \to 0,1]^s$ the function defined by

$$\Phi(\eta) = \left(\Phi_1\left((\eta_k)_{k\neq 1}\right), \ldots, \Phi_s\left((\eta_k)_{k\neq s}\right)\right) \qquad \forall \eta = (\eta_1, ..., \eta_s) \in [0,1]^s \qquad (4)$$

then a vector of market clearing proportions is a fixed point of the function Φ. If some of the components of this vector is different from 1 then it means that the corresponding bank has failed and the financial system is unstable.

Eisenberg and Noe (2001) introduce a fictitious default algorithm in order to study the dynamics of contagion. Starting from any initial condition $\eta^0 = (\eta_1^0, \eta_2^0, ..., \eta_s^0)$, the contagion is regulated by the (so called) *fictitious default algorithm*:

$$\eta^{\nu+1} = \Phi(\eta^\nu) \quad \nu = 0, 1, \ldots$$

They show that:

i) There exist at least a fixed point for Φ.

ii) A sequence $(\eta^\nu)_{\nu\in\mathbb{N}}$ regulated by the fictitious default algorithm converges to a fixed point.

In the next sections, these results will be extended to the fuzzy setting.

4 Fuzzy Contagion

In this section we consider the generalization of the model presented in the previous one to the fuzzy setting. For each i and j, the quantities c_i, b_i, p_{ij} and p_{ji} belong to $\mathcal{F}_+$ and, for technical reasons, we assume that

$$\sigma_i = \inf supp\left(\sum_{k\neq i} p_{ik}\right) > 0.$$

In the present framework, the proportion of debt that bank k is able to repay to its creditors in the financial networks is allowed to be a fuzzy number η_k such that $\eta_k[0] \subseteq [0,1]$. Therefore, denote with $\mathcal{N}$ the subset of fuzzy numbers contained in $\mathcal{F}$ having the closure of the support entirely contained in $[0,1]$, i.e.

$$\mathcal{N} = \{n \in \mathcal{F} \mid n[0] \subseteq [0,1]\},$$

then η_k must belong to $\mathcal{N}$. Moreover, we fix fuzzy numbers z and u in $\mathcal{N}$ such that $z[1] = \{0\}$ and $u[1] = \{1\}$ and call them respectively fuzzy zero and fuzzy unity. It immediately follows from Proposition 2.2 that

Corollary 4.1: *Let l be any fuzzy number in $\mathcal{F}$ then the fuzzy number $r = MIN\{u, MAX\{l, z\}\}$ belongs to $\mathcal{N}$.*

Proof. From Proposition 2.2, r has single peak. Moreover

$$r[0] = \left[\min\{\underline{u}[0], \max\{\underline{z}[0], \underline{l}[0]\}\}, \min\{\overline{u}[0], \max\{\overline{z}[0], \overline{l}[0]\}\}\right].$$

Now, by construction $\underline{z}[0], \underline{u}[0] \in [0,1]$; so, whatever is $l \in \mathcal{F}$, it follows that $\max\{\underline{z}[0], \underline{l}[0]\} \geqslant 0$, then $\underline{r} = \min\{\underline{u}[0], \max\{\underline{z}[0], \underline{l}[0]\}\} \in [0,1]$. It follows that $\max\{\overline{z}[0], \overline{l}[0]\} \geqslant 0$; as $\overline{u}[0] \in [0,1]$, then $\overline{r} = \min\{\overline{u}[0], \max\{\overline{z}[0], \overline{l}[0]\}\} \in [0,1]$. Hence $r[0] \subseteq [0,1]$ and $r \in \mathcal{N}$. □

Given the set of fuzzy numbers $\mathcal{N}$, define the s-Cartesian product $\mathcal{N}^s$ of $\mathcal{N}$ as:

$$\mathcal{N}^s = \{(n_1, n_2, \ldots, n_s) \,|\, n_k \in \mathcal{N} \text{ for every } k \in \{1, 2, \ldots, s\}\}.$$

For each $\eta = (\eta_1, \ldots \eta_s) \in \mathcal{N}^s$, with an abuse of notation denote

$$\Phi_i\left((\eta_k)_{k\neq i}\right) = MIN\left\{u, MAX\left\{\frac{1}{\sum_{k\neq i} p_{ik}}\left(c_i + \sum_{k\neq i}\eta_k p_{ki} - b_i\right), z\right\}\right\} \quad \forall i \in \{1, \ldots, s\}. \tag{5}$$

We get:

Lemma 4.2: *Let $\eta \in \mathcal{N}^s$ and $\alpha \in [0,1]$ then, for every $i \in \{1, \ldots, s\}$, the α-cut of the fuzzy number $\Phi_i\left((\eta_k)_{k\neq i}\right)$, defined in (5), is*

$$\Phi_i\left((\eta_k)_{k\neq i}\right)[\alpha] = \left[\underline{\Phi}_i\left((\eta_k)_{k\neq i}\right)[\alpha], \overline{\Phi}_i\left((\eta_k)_{k\neq i}\right)[\alpha]\right]$$

where

$$\underline{\Phi}_i\left((\eta_k)_{k\neq i}\right)[\alpha] = \min\left\{\underline{u}[\alpha], \max\left\{\frac{1}{\sum_{k\neq i}\overline{p}_{ik}[\alpha]}\left(\underline{c}_i[\alpha] + \sum_{k\neq i}\underline{\eta}_k[\alpha]\underline{p}_{ki}[\alpha] - \overline{b}_i[\alpha]\right), \underline{z}[\alpha]\right\}\right\}$$

and

$$\overline{\Phi}_i\left((\eta_k)_{k\neq i}\right)[\alpha] = \min\left\{\overline{u}[\alpha], \max\left\{\frac{1}{\sum_{k\neq i}\underline{p}_{ik}[\alpha]}\left(\overline{c}_i[\alpha] + \sum_{k\neq i}\overline{\eta}_k[\alpha]\overline{p}_{ki}[\alpha] - \underline{b}_i[\alpha]\right), \overline{z}[\alpha]\right\}\right\}$$

Proof. Firstly, note that the fuzzy number

$$\xi_i := \frac{1}{\sum_{k\neq i} p_{ik}}\left(c_i + \sum_{k\neq i}\eta_k p_{ki} - b_i\right)$$

is well defined as $\sigma_i > 0$. Moreover, from the algebra of fuzzy numbers reported in Remark 2.1, it follows that ξ_i belongs to $\mathcal{F}$.

Since, for each k and i, $\eta_k[0] \subseteq [0,1]$ and $p_{ki}[0] \subseteq [0,+\infty[$, then, for each α in $[0,1]$, it follows that

$$\eta_k[\alpha]p_{ki}[\alpha] = [\underline{\eta}_k[\alpha]\underline{p}_{ki}[\alpha], \overline{\eta}_k[\alpha]\overline{p}_{ki}[\alpha]].$$

Therefore, from Remark 2.1, it follows that $\xi_i[\alpha] = \left[\underline{\xi}_i[\alpha], \overline{\xi}_i[\alpha]\right]$, where

$$\underline{\xi}_i[\alpha] = \frac{1}{\sum_{k\neq i} \overline{p}_{ik}[\alpha]} \left(\underline{c}_i[\alpha] + \sum_{k\neq i} \underline{\eta}_k[\alpha]\underline{p}_{ki}[\alpha] - \overline{b}_i[\alpha] \right)$$

and

$$\overline{\xi}_i[\alpha] = \frac{1}{\sum_{k\neq i} \underline{p}_{ik}[\alpha]} \left(\overline{c}_i[\alpha] + \sum_{k\neq i} \overline{\eta}_k[\alpha]\overline{p}_{ki}[\alpha] - \underline{b}_i[\alpha] \right)$$

for each α in $[0,1]$.
Since $\Phi_i\left((\eta_k)_{k\neq i}\right) = MIN\{u, MAX\{\xi_i, z\}\}$ then the assertion follows from Proposition 2.2. □

Problem Statement

With an abuse of notation, let $\Phi : \mathcal{N}^s \to \mathcal{N}^s$ be the function defined by

$$\Phi(\eta) = \left(\Phi_1\left((\eta_k)_{k\neq 1}\right), \ldots, \Phi_n\left((\eta_k)_{k\neq s}\right)\right) \qquad \forall \eta \in \mathcal{N}^s \tag{6}$$

then

DEFINITION 4.3: A vector $\eta = (\eta_1, \ldots, \eta_s) \in \mathcal{N}^s$ is a vector of *market clearing proportions* if it is a fixed point of the map Φ, that is, $\eta = \Phi(\eta)$.

The problems that we study in the next sections are

i) Existence of fixed points of Φ.

ii) Convergence of the *fuzzy fictitious default* algorithm to a vector of market clearing proportions: given a sequence $(\eta^h)_{h\in\mathbb{N}} \subset \mathcal{N}^s$ governed by the rule

$$\eta^{h+1} = \Phi(\eta^h) \quad \text{for every } h \in \mathbb{N},$$

the question is to study whether $(\eta^h)_{h\in\mathbb{N}}$ converges to a fixed point of Φ.

5 Existence of market clearing proportions

In this section we prove the existence of fixed points for the function Φ. To this purpose, we firstly recall the Tarski fixed point Theorem (see Tarski (1955)).

5.1 Tarski fixed point theorem

Recall that

DEFINITION 5.1: Given a set A and a binary relation $\succsim$ on A then the pair $(A, \succsim)$ is said to be a lattice if $\succsim$ is a partial order, that is a reflexive, antisymmetric and transitive binary relation, and for every $a, b \in A$ there exist a least upper bound (supremum) $a \vee b$ and a greatest lower bound (infimum) $a \wedge b$.

A lattice is said to be complete if for every subset $B \subseteq A$ there exist a least upper bound $\vee B$ and a least lower bound $\wedge B$

and

DEFINITION 5.2: A function $f : A \to A$ is said to be increasing with respect to the partial order $\succsim$ if, for every $a, b \in A$,

$$a \succsim b \implies f(a) \succsim f(b).$$

Then

THEOREM 5.3 (Tarski (1955)): *If $(A, \succsim)$ is a complete lattice and $f : A \to A$ is increasing with respect to the partial order $\succsim$ then the set of fixed points P of f is not empty and $(P, \succsim)$ is a complete lattice.*

5.2 The lattice structure of $\mathcal{N}^s$

DEFINITION 5.4: Let $\succsim_R$ be the binary relation on $\mathcal{N}$ defined by

$$n \succsim_R m \iff \begin{cases} i) & \underline{n}[\alpha] \geqslant \underline{m}[\alpha], \\ ii) & \overline{n}[\alpha] \geqslant \overline{m}[\alpha]. \end{cases} \forall \alpha \in [0,1]$$

We say that n is *R-related to* m if $n \succsim_R m$.

DEFINITION 5.5: Let $\succsim_{NW}$ be the binary relation on $\mathcal{N}^s$ defined by

$$(n_1, \dots, n_s) \succsim_{NW} (m_1, \dots, m_s) \iff n_k \succsim_R m_k \quad \forall k \in \{1, \dots, s\}$$

We say that $(n_1, \dots, n_s)$ is *NW-related to* $(m_1, \dots, m_s)$ if $(n_1, \dots, n_s) \succsim_{NW} (m_1, \dots, m_s)$.

Then, it follows that

PROPOSITION 5.6: *The pair $(\mathcal{N}^s, \succsim_{NW})$ is a complete lattice.*

Proof. The binary relation $\succsim_R$ is a partial order as it is reflexive ($\forall n \in \mathcal{N},\ n \succsim_R n$), antisymmetric ($\forall n, m \in \mathcal{N}$, $n \succsim_R m$ and $m \succsim_R n$ imply that $n = m$) and transitive ($\forall n, m, l \in \mathcal{N}$, $n \succsim_R m$ and $m \succsim_R l$ imply $n \succsim_R l$). Then, it immediately follows that $\succsim_{NW}$ is a partial order as well.

Consider a subset $B \subseteq \mathcal{N}^s$ and let $w = (w_1, \dots, w_s)$ be defined as follows: for every $i \in \{1, \dots, s\}$, w_i is a fuzzy number whose representation is

$$w_i = \{(\underline{w}_i[\alpha], \overline{w}_i[\alpha]) \mid \alpha \in [0,1]\}$$

where

$$\underline{w}_i[\alpha] = \inf_{b \in B} \underline{b}_i[\alpha], \quad \overline{w}_i[\alpha] = \inf_{b \in B} \overline{b}_i[\alpha] \quad \forall \alpha \in [0,1].$$

Since $b \in \mathcal{N}^s$, then each b_i has a single peak; so

$$\underline{w}_i[1] = \inf_{b \in B} \underline{b}_i[1] = \inf_{b \in B} \overline{b}_i[1] = \overline{w}_i[1]$$

and w_i has a single peak. Moreover $b \in \mathcal{N}^s$ implies that, for every i, $b_i[0] \in [0,1]$. Hence $w_i[0] \in [0,1]$ as well, so $w_i \in \mathcal{N}$. Now, consider $w = (w_1, \dots, w_s)$; by construction; it follows that $b \succsim_{NW} w$ for every $b \in B$, so w is a lower bound for B in $\mathcal{N}^s$; moreover, let $n \in \mathcal{N}^s$ be such that $b \succsim_{NW} n$ for every $b \in B$, then it follows that for every $\alpha \in [0,1]$

$$\underline{n}_i[\alpha] \leqslant \inf_{b \in B} \underline{b}_i[\alpha] = \underline{w}_i[\alpha] \quad \text{and} \quad \overline{n}_i[\alpha] \leqslant \inf_{b \in B} \overline{b}_i[\alpha] = \overline{w}_i[\alpha]$$

so $w \succsim_{NW} n$ and w is the greatest lower bound for B, that is $w = \wedge B$.

Let $y = (y_1, \dots, y_s)$, where each y_i is a fuzzy number whose representation is

$$y_i = \{(\underline{y_i}[\alpha], \overline{y_i}[\alpha]) \mid \alpha \in [0,1]\}$$

with

$$\underline{y_i}[\alpha] = \sup_{b \in B} \underline{b}_i[\alpha], \quad \overline{y_i}[\alpha] = \sup_{b \in B} \overline{b}_i[\alpha] \quad \forall \alpha \in [0,1].$$

Then, following the same steps of the previous part, we get that y is the least upper bound for B in $\mathcal{N}$, that is $y = \vee B$, and $(\mathcal{N}^s, \succsim_{NW})$ is a complete lattice. □

5.3 Existence of fixed points of Φ

Definition 5.7: We say that a function $F : \mathcal{N}^s \to \mathcal{N}^s$ is *NW-increasing* if and only if

$$(n_1, \dots, n_s) \succsim_{NW} (m_1, \dots, m_s) \implies F(n_1, \dots, n_s) \succsim_{NW} F(m_1, \dots, m_s). \tag{7}$$

Proposition 5.8: *The function $\Phi : \mathcal{N}^s \to \mathcal{N}^s$ defined in (6) is NW-increasing.*

Proof. Let η' and η'' be vectors in $\mathcal{N}^s$ such that $\eta' \succsim_{NW} \eta''$ then, for every $k = 1, \dots, s$ it follows that $\eta'_k \succsim_R \eta''_k$. Then for each α in $[0,1]$

i) $\underline{\eta}'_k[\alpha] \geqslant \underline{\eta}''_k[\alpha]$,

ii) $\overline{\eta}'_k[\alpha] \geqslant \overline{\eta}''_k[\alpha]$.

Now, *i)* implies that $\underline{\eta}'_k[\alpha]\underline{p}_{ki}[\alpha] \geqslant \underline{\eta}''_k[\alpha]\underline{p}_{ki}[\alpha]$ and so

$$\sum_{k \neq i} \underline{\eta}'_k[\alpha]\underline{p}_{ki}[\alpha] \geqslant \sum_{k \neq i} \underline{\eta}''_k[\alpha]\underline{p}_{ki}[\alpha].$$

Therefore

$$\underline{\xi}'_i[\alpha] = \frac{1}{\sum_{k \neq i} \overline{p}_{ik}[\alpha]} \left(\underline{c}_i[\alpha] + \sum_{k \neq i} \underline{\eta}'_k[\alpha]\underline{p}_{ki}[\alpha] - \overline{b}_i[\alpha] \right) \geqslant$$

$$\frac{1}{\sum_{k \neq i} \overline{p}_{ik}[\alpha]} \left(\underline{c}_i[\alpha] + \sum_{k \neq i} \underline{\eta}''_k[\alpha]\underline{p}_{ki}[\alpha] - \overline{b}_i[\alpha] \right) = \underline{\xi}''_i[\alpha].$$

Moreover
$$\underline{\xi}_i'[\alpha] \geqslant \underline{\xi}_i''[\alpha] \implies \max\{\underline{z}[\alpha], \underline{\xi}_i'[\alpha]\} \geqslant \max\{\underline{z}[\alpha], \underline{\xi}_i''[\alpha]\}.$$
Finally, it follows that
$$\underline{\Phi}_i\left((\eta_k')_{k\neq i}\right)[\alpha] = \min\{\underline{u}[\alpha], \max\{\underline{z}[\alpha], \underline{\xi}_i'[\alpha]\} \geqslant \min\{\underline{u}[\alpha], \max\{\underline{z}[\alpha], \underline{\xi}_i''[\alpha]\} = \underline{\Phi}_i\left((\eta_k'')_{k\neq i}\right)[\alpha]$$

In similar way, ii) implies that
$$\overline{\Phi}_i\left((\eta_k')_{k\neq i}\right)[\alpha] = \min\{\overline{u}[\alpha], \max\{\overline{z}[\alpha], \overline{\xi}_i'[\alpha]\}\} \geqslant \min\{\overline{u}[\alpha], \max\{\overline{z}[\alpha], \overline{\xi}_i''[\alpha]\}\} = \overline{\Phi}_i\left((\eta_k'')_{k\neq i}\right)[\alpha].$$
Therefore
$$\underline{\Phi}_i\left((\eta_k')_{k\neq i}\right)[\alpha] \geqslant \underline{\Phi}_i\left((\eta_k'')_{k\neq i}\right)[\alpha], \quad \overline{\Phi}_i\left((\eta_k')_{k\neq i}\right)[\alpha] \geqslant \overline{\Phi}_i\left((\eta_k'')_{k\neq i}\right)[\alpha],$$
but α is arbitrary, then it follows that
$$\Phi_i\left((\eta_k')_{k\neq i}\right) \succsim_R \Phi_i\left((\eta_k'')_{k\neq i}\right).$$
Since the previous relation holds for every $i \in \{1, 2, \ldots, s\}$, it finally follows that
$$\Phi(\eta') \succsim_{NW} \Phi(\eta'').$$
□

Finally, we get

THEOREM 5.9: *The function $\Phi : \mathcal{N}^s \to \mathcal{N}^s$ defined in (6) has at least a fixed point.*

Proof. From Proposition 5.6, $(\mathcal{N}^s, \succsim_{NW})$ is a complete lattice. From the previous Proposition 5.8, Φ is NW-increasing. Then, a direct application of Theorem 5.3. gives the assertion. □

6 Convergence of the fuzzy fictitious default algorithm

In this section we study the convergence of sequences $(\eta^h)_{h\in\mathbb{N}} \in \mathcal{N}^s$, with $\eta^h = (\eta_1^h, \eta_2^h, \ldots, \eta_s^h)$, governed by the rule
$$\eta^{h+1} = \Phi(\eta^h)$$
starting from an initial condition $\eta^0 = (\eta_1^0, \eta_2^0, \ldots, \eta_s^0) \in \mathcal{N}^s$. The vector η^0 represents the fuzzy vector of market clearing proportions before an exogenous shock. For numerical purposes, it could useful to fix $\eta^0 = (1, \ldots, 1)$. The exogenous shock is implicitly included in the capital $b_i - c_i$. Therefore it could be possible to study the financial contagion for every choice of capital $b_i - c_i$. The only assumption that we need is that the choice $b_i - c_i$ gives rise to a contagion cascade. In particular, the only assumption that we need is that $\eta^0 \succsim_{NW} \eta^1$ which perfect sense as market clearing proportions do not increase after a shock[6]. More precisely,

[6]From a theoretical point of view, this assumption may include the case that η^0 is a fixed point.

Definition 6.1: Given an initial fuzzy vector of market clearing proportions $\eta^0 = (\eta_1^0, \eta_2^0, \ldots, \eta_s^0) \in \mathcal{N}^s$ then the sequence $(\eta^h)_{h\in\mathbb{N}} \subset \mathcal{N}^s$ is a *contagion dynamics* if,

$$\eta^{h+1} = \Phi(\eta^h) \quad \text{for every } h = 0, 1, 2, \ldots \tag{8}$$

and

$$\eta^0 \succsim_{NW} \eta^1. \tag{9}$$

Proposition 6.2: *Let $\left(\eta^h\right)_{h\in\mathbb{N}} \subset \mathcal{N}^s$ be a contagion dynamics, that is, a sequence satisfying (8) and (9) in Definition 6.1. Then, $\left(\eta^h\right)_{h\in\mathbb{N}} \subset \mathcal{N}^s$ is NW-decreasing, that is*

$$\eta^h \succsim_{NW} \eta^{h+1} \quad \forall h = 0, 1, \ldots \tag{10}$$

Proof. We prove the property by induction. By construction, (9) holds, that is $\eta^0 \succsim_{NW} \eta^1$.

Fixed $h \in \mathbb{N}$, we show that

$$\eta^{h-1} \succsim_{NW} \eta^h \implies \eta^h \succsim_{NW} \eta^{h+1}.$$

Firstly, recall that

$$\underline{\xi}_i^h[\alpha] = \frac{1}{\sum\limits_{k\neq i} \overline{p}_{ik}[\alpha]} \left(\underline{c}_i[\alpha] + \sum_{k\neq i} \underline{\eta}_k^h[\alpha]\underline{p}_{ki}[\alpha] - \overline{b}_i[\alpha] \right)$$

and

$$\overline{\xi}_i^h[\alpha] = \frac{1}{\sum\limits_{k\neq i} \underline{p}_{ik}[\alpha]} \left(\overline{c}_i[\alpha] + \sum_{k\neq i} \overline{\eta}_k^h[\alpha]\overline{p}_{ki}[\alpha] - \underline{b}_i[\alpha] \right).$$

If $\eta^{h-1} \succsim_{NW} \eta^h$ then $\eta_i^{h-1} \succsim_R \eta_i^h$ for every $i = 1, 2, \ldots, s$; then, for every $i = 1, 2, \ldots, s$ and for every $\alpha \in [0, 1]$, it follows that

$$\underline{\eta}_i^{h-1}[\alpha] \geqslant \underline{\eta}_i^h[\alpha] \quad \text{and} \quad \overline{\eta}_i^{h-1}[\alpha] \geqslant \overline{\eta}_i^h[\alpha] \quad \forall i \in \{1, \ldots, s\}.$$

Hence,

$$\underline{\xi}_i^{h-1}[\alpha] \geqslant \underline{\xi}_i^h[\alpha] \quad \text{and} \quad \overline{\xi}_i^{h-1}[\alpha] \geqslant \overline{\xi}_i^h[\alpha] \quad \forall i \in \{1, \ldots, s\}, \quad \forall \alpha \in [0, 1].$$

It follows that

$$\underline{\eta}_i^h[\alpha] = \min\{\underline{u}[\alpha], \max\{\underline{z}[\alpha], \underline{\xi}_i^{h-1}[\alpha]\}\} \geqslant \min\{\underline{u}[\alpha], \max\{\underline{z}[\alpha], \underline{\xi}_i^h[\alpha]\}\} = \underline{\eta}_i^{h+1}[\alpha]$$

and

$$\overline{\eta}_i^h[\alpha] = \min\{\overline{u}[\alpha], \max\{\overline{z}[\alpha], \overline{\xi}_i^{h-1}[\alpha]\}\} \geqslant \min\{\overline{u}[\alpha], \max\{\overline{z}[\alpha], \overline{\xi}_i^{h-1}[\alpha]\}\} = \overline{\eta}_i^{h+1}[\alpha]$$

for every $i = 1, 2, \ldots, s$ and for every $\alpha \in [0, 1]$. Therefore,

$$\eta_i^h \succsim_R \eta_i^{h+1} \quad \forall i \in \{1, ..., s\}$$

and

$$\eta^h \succsim_{NW} \eta^{h+1}.$$

By induction, it follows that $\eta^h \succsim_{NW} \eta^{h+1}$ for every $h \in \mathbb{N}$ and the assertion follows.

$\square$

We give the following definition

Definition 6.3: Given a sequence $\left(n^h\right)_{h\in\mathbb{N}} \subset \mathcal{N}$ then it $\mathcal{N}$-converges to $n \in \mathcal{N}$ if, for every $\alpha \in [0,1]$, the sequence $\left(\underline{n}^h[\alpha]\right)_{h\in\mathbb{N}} \subset \mathbb{R}$ converges to $\underline{n}[\alpha]$ and $\left(\overline{n}^h[\alpha]\right)_{h\in\mathbb{N}} \subset \mathbb{R}$ converges to $\overline{n}[\alpha]$. Given a sequence $\left((n_1^h, n_2^h, \ldots, n_s^h)\right)_{h\in\mathbb{N}} \subset \mathcal{N}^s$ then it $\mathcal{N}^s$-converges to $(n_1, n_2, \ldots, n_s) \in \mathcal{N}^s$ if, for every $j \in \{1, 2, \ldots, s\}$ the sequence $\left(n_j^h\right)_{h\in\mathbb{N}}$ $\mathcal{N}$-converges to n_j.

Then,

Proposition 6.4: *Let $\left(\eta^h\right)_{h\in\mathbb{N}} \subset \mathcal{N}^s$ be a contagion dynamics, that is, a sequence satisfying (8) and (9) in Definition 6.1. Then, the sequence $\left(\eta^h\right)_{h\in\mathbb{N}}$ $\mathcal{N}^s$-converges to a fixed point for the function Φ.*

Proof. Let $\left(\eta^h\right)_{h\in\mathbb{N}} \subset \mathcal{N}^s$ be a contagion dynamics; from the previous proposition it follows that it is NW-decreasing, implying that for every α in $[0,1]$ and every $i = 1, ..., s$, $(\underline{\eta}_i^h[\alpha])_{h\in\mathbb{N}}$ and $(\overline{\eta}_i^h[\alpha])_{h\in\mathbb{N}}$ are non increasing sequences in $[0,1]$. So they converge respectively to $\underline{\eta}_i^*[\alpha]$ and $\overline{\eta}_i^*[\alpha]$ belonging to $[0,1]$. Now we show that the vector of fuzzy numbers $\eta^* = (\eta_1^*, \ldots, \eta_s^*)$, such that each η_i^* is the fuzzy number defined by

$$\eta_i^* = \left\{ \left[\underline{\eta}_i^*[\alpha], \overline{\eta}_i^*[\alpha]\right] \mid \alpha \in [0,1] \right\},$$

is a fixed point for Φ.

In fact, from the assumptions it follows that

$$\underline{\eta}_i^{h+1}[\alpha] = \underline{\Phi}_i\left(\left(\eta_k^h\right)_{k\neq i}\right)[\alpha] =$$

$$\min\left\{\underline{u}[\alpha], \max\left\{\frac{1}{\sum_{k\neq i}\overline{p}_{ik}[\alpha]}\left(\underline{c}_i[\alpha] + \sum_{k\neq i}\underline{\eta}_k^h[\alpha]\underline{p}_{ki}[\alpha] - \overline{b}_i[\alpha]\right), \underline{z}[\alpha]\right\}\right\}.$$

From the continuity of the $\min\{\cdot,\cdot\}$ and $\max\{\cdot,\cdot\}$ operators it follows that

$$\underline{\eta}_i^*[\alpha] = \lim_{h\to\infty}\underline{\eta}_i^{h+1}[\alpha] = \lim_{h\to\infty}\underline{\Phi}_i\left(\left(\eta_k^h\right)_{k\neq i}\right)[\alpha] =$$

$$\lim_{h\to\infty}\left[\min\left\{\underline{u}[\alpha], \max\left\{\frac{1}{\sum_{k\neq i}\overline{p}_{ik}[\alpha]}\left(\underline{c}_i[\alpha] + \sum_{k\neq i}\underline{\eta}_k^h[\alpha]\underline{p}_{ki}[\alpha] - \overline{b}_i[\alpha]\right), \underline{z}[\alpha]\right\}\right\}\right] =$$

$$\min\left\{\underline{u}[\alpha], \max\left\{\frac{1}{\sum_{k\neq i}\overline{p}_{ik}[\alpha]}\left(\underline{c}_i[\alpha] + \sum_{k\neq i}\underline{\eta}_k^*[\alpha]\underline{p}_{ki}[\alpha] - \overline{b}_i[\alpha]\right), \underline{z}[\alpha]\right\}\right\} = \underline{\Phi}_i\left((\eta_k^*)_{k\neq i}\right)[\alpha].$$

Hence

$$\underline{\eta}_i^*[\alpha] = \underline{\Phi}_i\left((\eta_k^*)_{k\neq i}\right)[\alpha].$$

Following the same steps, we get

$$\overline{\eta}_i^*[\alpha] = \overline{\Phi}_i\left((\eta_k^*)_{k\neq i}\right)[\alpha].$$

Since α is arbitrary, then

$$\eta_i^* = \Phi_i\left((\eta_k^*)_{k\neq i}\right)$$

The previous condition holds for every $i \in \{0, \dots, s\}$, then

$$\eta^* = \Phi\left(\eta^*\right)$$

and the assertion follows. □

7 Implementation and simulation

This section contains some numerical results which support our theoretical analysis. With this aim, the fuzzy fictitious default algorithm has been written, fully implemented in MATLAB, and tested numerically on a real financial date set.

The algorithm is *versatile* inasmuch the user, as input:

1. *choose* the fuzzy numbers z and u to be used as fuzzy zero and fuzzy unit;
2. *choose* which banks will be shocked;
3. *set* the intensity of the shock;
4. *choose* the maximum number of steps to be computed.

The algorithm

The fuzzy fictitious default algorithm starts with a "stability" initial condition under which each bank is able, with certainty, to honor all its debts. It means that the initial vector of market clearing proportions $\eta^0 = (\eta_1^0, ..., \eta_s^0) \in \mathcal{N}^s$ coincides with the "crisp" fuzzy vector $(1, ..., 1)$.

Then the following steps are accomplished.

- At the first step ($h = 1$) an exogenous shock hits some bank in the network, causing a fuzzy default of some of them. Hence the vector of fuzzy net worth $w^1 = (w_1^1, ..., w_s^1) \in \mathcal{F}^s$ and the vector of market clearing proportions $\eta^1 = (\eta_1^1, ..., \eta_s^1) = \Phi(\eta^0) \in \mathcal{N}^s$ are computed.
- At the second step ($h = 2$), starting from w^1 and η^1, the vectors $w^2 = (w_1^2, ..., w_s^2) \in \mathcal{F}^s$ and $\eta^2 = (\eta_1^2, ..., \eta_s^2) = \Phi(\eta^1) \in \mathcal{N}^s$ are computed. If $\eta^1 = \eta^2$, the exogenous default does not propagate and the process stops. Otherwise a new step is taken.
- At step h ($h > 2$), starting from $w^{h-1} = (w_1^{h-1}, ..., w_s^{h-1})$ and $\eta^{h-1} = (\eta_1^{h-1}, ..., \eta_s^{h-1})$ the vectors $w^h = (w_1^h, ..., w_s^h) \in \mathcal{F}^s$ and $\eta^h = (\eta_1^h, ..., \eta_s^h) = \Phi(\eta^{h-1}) \in \mathcal{N}^s$ are computed. If $\eta^h = \eta^{h-1}$ the fuzzy contagion stops, else the process continues until the maximum number of steps has been reached.

Details on computing approach
To extend the *fictitious default algorithm* to a *fuzzy framework* it has been necessary to implement an algebra over the set of fuzzy numbers and a method for computing $MIN/MAX\{n,m\}$ for n and m fuzzy numbers. It is known that, in many practical applications when handling with fuzzy numbers, it is necessary to have a permanent switch from a fuzzy representation to a numerical one. This transformation is usually carried out by the defuzzification process which, however, may cause loss of information. On the other hand, as observed in Section 1, a fuzzy number n can be identified with the representation $\{(\underline{n}[\alpha], \overline{n}[\alpha]) \,|\, \alpha \in [0,1]\}$, where $\overline{n}[\alpha]$ and $\underline{n}[\alpha]$ are the end points of the interval $n[\alpha]$. Hence, fuzzy numbers may be combined one to another without any defuzzification method, but making use of the *interval algebra* instruments, without loosing any information. For this reasons, in this paper, operations between fuzzy numbers are implemented on a computer by means of interval arithmetic on $n[\alpha]$. The intervals $n[\alpha]$ are univocally determined computing left/right side membership inverse. Thus, the algorithm makes use of the *Symbolic Package of Matlab* [7] in order to handle $n[\alpha]$ as a *symbolic object* described by two different functions of α: the α-*Lower bounds* and the α-*Upper bounds*, for each $\alpha \in [0,1]$. The implemented code has in input fuzzy triangular data assigned by three-dimensional vectors of real numbers[8]. In order to handle fuzzy numbers as symbolic objects to be combined by the interval algebra instruments, the sets $n[\alpha]$ of each input fuzzy number are determined computing left/right side membership *inverse*. Once the input data have been transformed into symbolic triangular functions of α, the process starts with step $h=1$, as described before. At each step h, the computation of the vector of the fuzzy market clearing proportions (5) involves also the computation of MIN/MAX between two fuzzy numbers which, as it is known, may produce a *piecewise function*[9]. Using the *Symbolic Package of Matlab*, piecewise functions are handled as *symbolic objects*, hence they are defined, combined and plotted *without any loop*. Thus, at the end of each step h, it is obtained a *symbolic vector* $\eta^h = (\Phi(\eta_1^{h-1}), \ldots, \Phi(\eta_s^{h-1}))$ composed by symbolic functions of α.
The algorithm for computing $p = MAX\{n,m\}$ and $q = MIN\{n,m\}$ has been implemented considering n as a triangular fuzzy number and m as fuzzy number having a single peak and it is structured in order to involve p and q in equation (5) using their parameterized representations $p = \{(\underline{p}[\alpha], \overline{p}[\alpha]) \,|\, \alpha \in [0,1]\}$, $q = \{(\underline{q}[\alpha], \overline{q}[\alpha]) \,|\, \alpha \in [0,1]\}$.

The MIN/MAX algorithm.

[7] *Symbolic Math Toolbox* enables you to perform symbolic computations by defining a special data type: the *symbolic object*. It provides functions for solving, plotting, and manipulating symbolic math equations.

[8] We remind that, given a triangular fuzzy number n, denoting by $\hat{n}$ the unique element contained in the core and by $\underline{n} = \underline{n}[0]$ and $\overline{n} = \overline{n}[0]$, n can be identified by $n = (\underline{n}, \hat{n}, \overline{n})$. For completeness we remind that the membership function of the triangular fuzzy number n is defined as follows

$$\mu_n(x) = \begin{cases} \dfrac{x-\underline{n}}{\hat{n}-\underline{n}}, & \text{if } \underline{n} \leqslant x \leqslant \hat{n}; \\[2ex] \dfrac{x-\overline{n}}{\hat{n}-\overline{n}}, & \text{if } \hat{n} < x \leqslant \overline{n}; \\[2ex] 0, & \text{otherwise.} \end{cases}$$

[9] That is a function defined by multiple sub functions, each one applying to a certain interval of the main function's domain.

Given a triangular fuzzy number n and a fuzzy number m having a single peak, let us describe how the algorithm computes $\underline{p}[\alpha]$ and $\underline{q}[\alpha]$, for $0 \leqslant \alpha \leqslant 1$ (the computation of $\overline{p}[\alpha]$ and $\overline{q}[\alpha]$ is equivalent).

- *Step 1.* Compute $\underline{n}[\alpha]$ and $\underline{m}[\alpha]$ for $0 \leq \alpha \leq 1$.
- *Step 2.* Solve equation $\underline{n}[\alpha] = \underline{m}[\alpha]$ (for α real in $[0, 1]$), to compute the intersections between *α-lowers bound* of n and m.

If *no intersection* exist:

- *Step 3.* the relation between $\underline{n}[\alpha]$ and $\underline{m}[\alpha]$, for each $0 \leqslant \alpha \leqslant 1$, is determined by the relation existing between $\underline{n}[0]$ and $\underline{m}[0]$ as follows:

$$\underline{n}[0] < \underline{m}[0] \Rightarrow \begin{cases} \underline{q}[\alpha] = \underline{n}[\alpha] \\ \\ \underline{p}[\alpha] = \underline{m}[\alpha] \end{cases}$$

$$\underline{m}[0] < \underline{n}[0] \Rightarrow \begin{cases} \underline{q}[\alpha] = \underline{m}[\alpha] \\ \\ \underline{p}[\alpha] = \underline{n}[\alpha] \end{cases}$$

The process stops with $\underline{p}[\alpha]$ and $\underline{q}[\alpha]$, for each $0 \leqslant \alpha \leqslant 1$, as output.

Else, if some *intersection exists*, the computation of $\underline{p}[\alpha]$ and $\underline{q}[\alpha]$ is constructed *piecewise* on a partition of the interval [0,1] opportunely generated at the following step.

- *Step 4.* The solutions of the equation $\underline{n}[\alpha] = \underline{m}[\alpha]$ (for α real in $[0, 1]$) are computed: $sol = [sol_1, sol_2, \cdots, sol_r]$, and a vector of nodes $nod = [nod_1, nod_2, \cdots, nod_k]$ s opportunely set in order to create a partition of the interval $[0, 1]$ by the elements of sol. Note that $r = k$ if and only if $\{0, 1\}$ are elements of sol.
- *Step 5.* Indicating with $\underline{p}_i[\alpha] = \{\underline{p}[\alpha],\ nod_i \leqslant \alpha \leqslant nod_{i+1}\}$ and $\underline{q}_i[\alpha] = \{\underline{q}[\alpha],\ nod_i \leqslant \alpha \leqslant nod_{i+1}\}$, $\underline{p}[\alpha]$ and $\underline{q}[\alpha]$ are constructed *piecewise* on the partition of $[0, 1]$ as follows.

  Two *indicators* $\boldsymbol{n}$, $\boldsymbol{m}$, initialized by:

$$\boldsymbol{n} = \underline{n}[nod_1], \boldsymbol{m} = \underline{m}[nod_1]$$

  are created. If $\boldsymbol{n} = \boldsymbol{m}$, nod_1 is substituted by a point in the interval $[nod_1, nod_2]$ in order to have $\mathbf{n} \neq \mathbf{m}$.

  for $i = 1 : k - 1$

$$\boldsymbol{n} < \boldsymbol{m} \Rightarrow \begin{cases} \underline{q}_i[\alpha] = \underline{n}[\alpha] \\ & \text{for } nod_i \leqslant \alpha \leqslant nod_{i+1}; \\ \underline{p}_i[\alpha] = \underline{m}[\alpha] \end{cases}$$

$$\boldsymbol{m} < \boldsymbol{n} \Rightarrow \begin{cases} \underline{q}_i[\alpha] = \underline{m}[\alpha] \\ & \text{for } nod_i \leqslant \alpha \leqslant nod_{i+1}; \\ \underline{p}_i[\alpha] = \underline{n}[\alpha] \end{cases}$$

New indicators $\boldsymbol{n}$, $\boldsymbol{m}$ are computed for the next step of the *for* loop.
endfor

- *Step 6.* $\underline{p}[\alpha]$, $\underline{q}[\alpha]$ are finally computed defining *symbolic Matlab piecewise functions* composed by sub functions $\underline{p}_i[\alpha]$ and $\underline{q}_i[\alpha]$ respectively, for $i = 1 : k - 1$.

Simulation results

The real financial data we refer as been derived by De Marco et al. (2018). Using Furfine (2003) data set, De Marco et al. (2018) construct a system of 719 commercial banks trading on Federwire System, classified into seven groups (called A, B, C, D4, D3, D2, D1) according to the volume of funds traded. The exposures of a bank from one group in another bank from another group are defined as triangular fuzzy numbers for which the infimum and the supremum of the support and the core coincide with the minimum, the maximum and the average value of the transactions between the two groups, taken over the sample period. (See De Marco et al. (2018) for a detailed description of the construction of the data set). Since the fuzzy fictitious default algorithm involves symbolic calculus, it has an *high computational cost.* For this reason it has been opportunely constructed a subset of 50 of the 719 banks. Then the algorithm was run on this data set, inflicting a fixed shock x on a different type of bank, for each simulation[10].

[10] Following the classical approach, the exogenous shock is assumed to be a vector $x = (x_1, \ldots, x_n)$ where each component, x_i, is a triangular fuzzy number representing the exogenous shock which affects the (ante-shock) capital of bank i, moreover it is characterized by the difference $b_i - c_i$.

	A	B	C	$D4$	$D3$	$D2$	$D1$
A $\underline{K}_A = 1000000000$ $\hat{K}_A = 3260000000$ $\overline{K}_A = 10000000000$	32464297 47235030 60805315	2147385 3653538 6143909	730707 2557476 4853986	9333 24864 51446	9333 24864 51446	9333 24864 51446	9333 24864 51446
B $\underline{K}_B = 1000000000$ $\hat{K}_B = 3260000000$ $\overline{K}_B = 10000000000$	17046189 24552768 45151178	3485197 5842173 10790707	878001 2379317 4135320	18889 43627 78433	18889 43627 78433	18889 43627 78433	18889 43627 78433
C $\underline{K}_C = 1000000000$ $\hat{K}_C = 3260000000$ $\overline{K}_C = 10000000000$	830422 1898107 4401237	106768 366063 955832	55361 257035 818558	4633 8197 16598	4633 8197 16598	4633 8197 16598	4633 8197 16598
$D4$ $\underline{K}_{D4} = 1000000000$ $\hat{K}_{D4} = 3260000000$ $\overline{K}_{D4} = 10000000000$	6514003 9512080 14240866	3440001 5168762 7031853	2060496 2841359 3863431	76552 100098 117728	76552 100098 117728	76552 100098 117728	76552 100098 117728
$D3$ $\underline{K}_{D3} = 100000000$ $\hat{K}_{D3} = 269230769,2$ $\overline{K}_{D3} = 1000000000$	1398712 2042471 3057854	738650 1109857 1509907	442437 610107 829571	16437 21493 25279	16437 21493 25279	16437 21493 25279	16437 21493 25279
$D2$ $\underline{K}_{D2} = 10000000$ $\hat{K}_{D2} = 23684210,53$ $\overline{K}_{D2} = 100000000$	719337 1050413 1572610	379877 570783 776523	227539 313769 426636	8453 11053 13000	8453 11053 13000	8453 11053 13000	8453 11053 13000
$D1$ $\underline{K}_{D1} = 1000000$ $\hat{K}_{D1} = 5284552,846$ $\overline{K}_{D1} = 10000000$	519521 758632 1135774	274355 412232 560822	164334 226611 308126	6105 7983 9389	6105 7983 9389	6105 7983 9389	6105 7983 9389

Table: Data set

With this data set, $h = 50$ has been set to be the maximum number of steps to be computed. We get that the fixed point is never reached. However, focusing on the first four digits, it is possible to deduce that, if the initial exogenous shock hints a bank in D1 causing its default, the equality $\eta^{h-1} = \eta^h$ is reached for $h = 2$, while, if the initial exogenous shock hints banks in any other group, the equality $\eta^{h-1}[0] = \eta^h[0]$ is reached for $h = 5$. Moreover, when banks in D1 fail, the contagion does not propagate in the financial network. When the exogenous shock causes a fuzzy default of banks on a group different to D1, the contagion propagates only within the small banks belonging to D1.
We get that each simulation gives $\eta_i = \eta_H$ for every i in a group of banks H; so the output can be summarized by a set of graphs representing the fuzzy numbers η_H in the plane (α, η), by the plots of the α-lower bounds, $\underline{\eta}_H[\alpha]$, and the α-upper bounds, $\overline{\eta}_H[\alpha]$, for $0 \leqslant \alpha \leqslant 1$.
The graphs obtained by simulating that the shock hits banks in group C are shown below[11].

[11]Since for computational reasons the input data have to be triangular, for the simulations, z and u are chosen to be the triangular fuzzy numbers defined respectively by $z = (0, 0.1, 0.2)$ and and $u = (0.9, 1, 1.1)$.

C-shocked

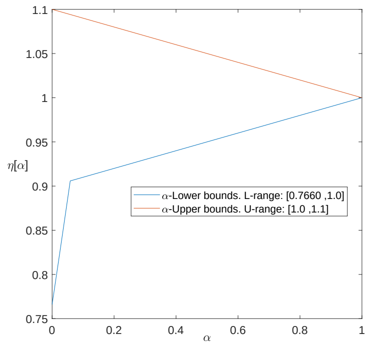

Figure 1: η_{D1}.

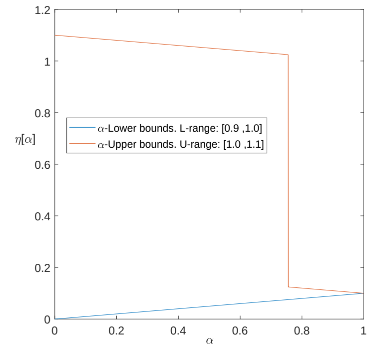

Figure 2: η_C.

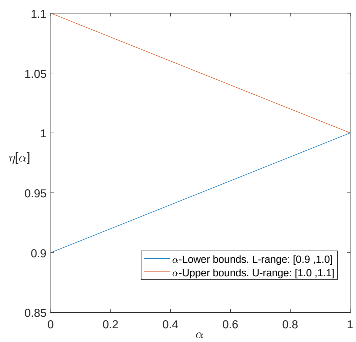

Figure 3: η_H for H different from C and D1.

References

Buckley, J. J. and E. Eslami (2002). An introduction to fuzzy logic and fuzzy sets. *Physica-Verlag*.

Chai, Y. and D. Zhang (2016). A representation of fuzzy numbers. *Fuzzy sets and systems 295*, 1–18.

Chiu, C.-H. and W.-J. Wang (2002). A simple computation of min and max operations for fuzzy numbers of fuzzy numbers. *Fuzzy sets and systems. 126*, 273–276.

De Marco, G., C. Donnini, F. Gioia, and F. Perla (2018). On the measure of contagion in fuzzy financial networks. *Applied Soft Computing 67*, 584–595.

Eisenberg, L. and T. H. Noe (2001). Systemic risk in financial systems. *Management Science 47*, 236–249.

Furfine, C. H. (2003). Quantifying the risk of contagion. *Journal of Money, Credit and Banking 45*, 111–128.

Gai, P., A. Haldane, and S. Kapadia (2011). Complexity, concentration and contagion. *Journal of Monetary Economics 58*, 453–470.

Gai, P. and S. Kapadia (2010). Contagion in financial networks. *Proceedings of the Royal Society Series A 466*, 2401–2423.

Glasserman, P. and H. P. Young (2015). How likely is contagion in financial networks?. *Journal of Banking and Finance 50*, 383–399.

Glasserman, P. and H. P. Young (2016). Contagion in financial networks. *Journal of Economic Literature 56*, 779–831.

Goetschel, R. J. and W. Voxman (1986). Elementary fuzzy calculus. *Fuzzy sets and systems 18*, 31–43.

Hong, D.-H. and K.-T. Kim (2006). n easy computation of min and max operations for fuzzy numbers of fuzzy numbers. *J. Appl. Math Computing 21*, 555–561.

Hurd, T. R. (2016). Contagion! systemic risk in financial networks. *Springer*.

Klir, G. J.and Yuan, B. (1995). Fuzzy sets and fuzzy logic, theory and applications. *Prentice Hall PTR, Upper Saddle River, New Jersey*.

Lee, S. (2013). Systemic liquidity shortages and interbank network structures. *Journal of Financial Stability 9*, 1–12.

Tarski, A. (1955). A lattice-theoretical fixpoint theorem and its application. *Pacific Journal of Mathematics 5*, 285–309.

Zadeh, L. A. (1995). Fuzzy sets. *Information and Control. 8*, 338–353.

Zimmermann, H. J. (2001). Fuzzy set theory and its application. *Kluver Academic Publishers, Boston/Dordrecht/London*.