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University of Naples Federico II

University of Salerno

Bocconi University, Milan

CSEF - Centre for Studies in Economics and Finance
DEPARTMENT OF ECONOMICS – UNIVERSITY OF NAPLES
80126 NAPLES - ITALY
Tel. and fax +39 081 675372 – e-mail: csef@unina.it
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Christian Di Pietro[*], Maria Gabriella Graziano[**] and Vincenzo Platino[**]

Abstract

We consider a pure exchange economy with externalities. We adopt a cooperative approach to equilibrium analysis, allowing each individual to cooperate with others and to form coalitions. Individual preferences are affected by the consumption of all other agents in the economy, and the consumption set of each agent is affected by the coalition to which he/she belongs. Following Montesano (2002), we introduce a measure of social loss with respect to the γ-core and α-core of the economy which completely characterizes the corresponding core allocations.

JEL Classification: C71, D11, D62, D64

Keywords: exchange economy, interdependent preferences, core, social loss

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* Università di Napoli Parthenope. Address: Dipartimento di Studi Aziendali ed Economici, Via Amm. F. Acton, 80133 Napoli, Italy. e-mail: christian.dipietro@uniparthenope.it

** Università di Napoli Federico II and CSEF. Address: Dipartimento di Scienze Economiche e Statistiche, Via Cintia, Complesso Monte S. Angelo, 80126 Napoli, Italy. e-mails: mgrazian@unina.it, vincenzo.platino@unina.it

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1 Introduction

This paper introduces the concept of *measure of social loss* associated to the core of a pure exchange economy with externalities. The idea of measuring the social loss associated to an allocation which is not efficient, interpreted as the amount of resources wasted with respect to a Pareto optimal distribution, dates back to Debreu (1951), Luenberger (1992, 1994) and Montesano (1997). We consider a measure of social loss relative to core allocations and use this measure to characterize the core under the assumption that coalitions of agents have an *optimistic* or a *pessimistic* attitude with respect to the behavior of outsiders. Based on these two perspectives we analyze the distinct notions of γ-*core* and α-*core*, and in both cases, allocations in the core are characterized as zero points of the social loss functions. Our analysis rests on the assumption that an optimal use of resources is possible provided that the agents are able to redistribute these resources to their benefit. Consequently, our results rest on the identification of suitable monotonicity assumptions to make such redistribution possible in the presence of externalities.

In line with Dufwenberg et al. (2011), the paper analyzes a pure exchange economy in which externalities are modeled through *other-regarding preferences* (ORP): agents do not care only about their own consumption, they care also about the consumption of other agents. In other words, we do not assume that agents are purely self-interested, but that they may also be concerned about issues such as others' well-being and social wealth.

It is a well-known fact that an agent's payoff is not independent of others' behavior (or more generally others' consumption). However, classical economic models assume that agents ignore others' well-being, and it is only recently that the role played by altruism or envy in motivating agents' choices has been highlighted. In particular, in contrast to the classical view that agents are guided purely by self-interest even in the presence of external effects, experimental economics suggests that more realistic predictions of economic behavior can be built by taking into account individual concern for others. A strand of work related to experimental analysis examines others' concerns based on social preferences and deviations from self-interested behavior modeled in various ways. People can deviate from pure self-interest in response to a friendly action (see Levine (1998), where altruistic preferences are used in ultimatum bargaining and public goods contribution games, Fehr and Schmidt (1999) and Sobel (2005), who define intrinsic reciprocity through preferences). Also, deviation might be guided by fairness behavior regardless of other agents' actions (Fehr and Schmidt (1999), Dufwenberg et al. (2011)). Although the importance of externalities has been acknowledged and studied widely, only a few theoretical papers analyze their impact on welfare within a general equilibrium approach. Most results are based on specific models. Therefore, the question naturally arises do these basic theoretical results persist if we assume ORP [1].

The main classical theory issues analized include competitive equilibria, cooperative solutions such as core solutions, welfare theorems and their special forms of core-equivalence theorems. The theoretical research on ORP, which started with the basic contributions of Arrow and Hahn (1971), Laffont and Laroque (1972) and Laffont (1988), puts more emphasis on these aspects: 1. the existence of competitive equilibria; 2. the failure of welfare theorems and subsequent identification of the conditions under which the optimality of equilibria is restored[2]. Little attention has been paid to the study of alternative cooperative solutions concepts such as the core, fair allocation, and stable sets despite the fact that cooperative solutions sometimes can provide successful alternatives to non-cooperative solutions[3].

Our paper analyzes the core of a market with externalities, and contributes to filling the gap in work on cooperative solutions. The analysis of the core employs an approach based on the idea of saving resources which is motivated by the fact that many core notions in the presence of externalities are defined similarly to the case where the focus is on the problem of "saving resources" (such as, e.g., environmental models). Moreover, it is hoped that our findings might provide an alternative view of the core existence problem since in the *resource core* approach, core allocations are reconsidered as zero points of suitable loss functions.

In the case of cooperative notions we observe that in line with the generally accepted definition, the core is based on a dominance relation between allocations which in turn, is built on a blocking mechanism due to coalitions. In models with externalities, it can be difficult to define the blocking procedure. This is because a blocking coalition S, i.e. a coalition which proposes a deviation from the status quo allocation, must take account of the possible reactions of the counter coalition $N \setminus S$ in response to its deviation. This reaction cannot be ignored by coalition S, since it directly affects the utility of its members. Consequently, different core notions are possible depending on the attitude of a blocking coalition S with respect to the reactions of the outsiders

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[1] Contributions such as Dufwenberg et al. (2011) have enhanced this line of investigation.

[2] See del Mercato (2006b) and Bonnisseau and del Mercato (2010) which prove the existence of equilibrium in models where externalities also imply consumption constraints, Balasko (2015) for a differentiable approach to the existence problem, He and Yannelis (2016) for existence results without continuity of preferences, and for welfare analysis see Borglin (1973) and Dufwenberg et al. (2011) and the references they include.

[3] See e.g. the results for cooperative solutions in Dufwenberg et al. (2011), Velez (2016), Graziano et al. (2017).

(cf. Graziano et al. (2017), Hervés-Beloso and Moreno-García (2018)).

The main notion on which our paper is based assumes that a coalition S is able to redistribute its resources among its members and that this is possible also for the counter-coalition $N \setminus S$. We focus on a classification of the blocking mechanism based on optimistic and pessimistic behaviors of coalitions with respect to outsiders' reactions. Also, we assume several possibilities for the resources available for redistribution in both S and $N \setminus S$.

This leads to the idea of γ-dominance and γ-core which is close to the notion analyzed in Dufwenberg et al. (2011). In this case, the solution assumes that the deviating coalition S does not accept the proposed allocation, and that this is distributed to the members of the complementary coalition. So the members of S redistribute their initial endowment in order to block, while agents outside the coalition S passively accept the deviation of S and stick to their initial status quo allocation. We interpret the behavior of coalition S as extremely optimistic, since the coalition assumes that the outsiders do not react and the coalition is willing to deviate as soon as this status ensures a better outcome for its members. In the γ-blocking mechanism just described, the coalition S is allowed to redistribute its initial resources while $N \setminus S$ potentially is able to redistribute the resources received under the status quo allocation. Hence, the final distribution of resources may not be feasible for the market [4].

Following Yannelis (1991a), in the second part of the paper we focus on the so called α-dominance and the corresponding α-core. In this case, the resources available for redistribution in S and $N \setminus S$ are the initial endowments. A deviating coalition S takes into account the possible reactions of the members of coalition $N \setminus S$, and all redistributions of their initial resources. We interpret the behavior of S as completely pessimistic (or cautious) in the sense that the coalition is willing to change its initial position only if all the redistributions by outsiders are favorable to all of its members. In contrast to γ-dominance, in the case of α-dominance the final distribution of resources is feasible for the market, whatever the reaction of the complementary coalition.

To analyze the γ-core and the α-core we study a problem typically investigated for Pareto efficiency. The usual notion of efficiency of an allocation requires that there is no other feasible allocation which is weakly preferred by each agent and is strictly preferred by at least one agent. Alternative definitions emphasize the optimal use of resources in the treatment of efficiency in the sense of no waste of resources. We focus on the notion of efficient allocation which requires that the utility levels achieved under the allocation

[4] This type of deviation can be considered closed in spirit of the Strong Nash equilibrium concept proposed by Aumann (1959): holding the outsiders' allocation fixed, a coalition can improve itself by a reallocation of resources such that each of its members is better off.

cannot be achieved through an alternative allocation which also allows resources saving (compare Allais (1943)). The two definitions are equivalent in selfish economies under standard regularity conditions on preferences[5] but in the presence of externalities, their equivalence may fail. This is because of the level of spitefulness among agents. It is well-known that in a *hateful society*, the second welfare theorem may fail and the full resource utilization (i.e. total consumption equal to total endowment) under a Pareto optimal allocation is not guaranteed (compare Example 1 in Dufwenberg et al. (2011)).

The duality between minimization of resources and maximization of preferences is key to introducing a measure of social loss expressed in terms of resources and due to inefficiency (see Montesano (1997)). In Montesano (2002), a measure of social loss is proposed with respect to the core. To capture a measure of social loss which does not rely on utilities, Montesano (2002) introduces the idea of *resources-core*: for an allocation not in the resources-core, there exists at least one coalition whose members can improve upon the given allocation by saving resources. The amount of resources that can be saved allows to define a measure of social loss associated to the given allocation. In turn, this measure can be used to provide a complete characterization of the core. The result holds true in a standard (selfish) pure exchange economy with regular, continuous and monotonic preferences.

Defining measures of social loss in our framework is not trivial. The idea of resources-core is based on the assumption that coalitions dislike resources waste and therefore builds on a monotonicity requirement. As recalled above, this simple idea could fail due to externalities effects. On the other hand, assumptions such as *social monotonicity* (see Borglin (1973) and Dufwenberg et al. (2011)) can be useful to reduce the degree of spitefulness of a society and to restore the second welfare theorem. This type of condition ensures that any increase in the resources available to the society can be used to make everyone better off and is satisfied in many specific models of ORP. We follow this idea in the study of a resources-core when coalitions are extremely optimistic i.e. under the γ-blocking mechanism. We present our results under the more restrictive condition of *social group monotonicity* which takes account of the effect of redistributions within each coalition not just the grand coalition. This assumption is naturally implied by the structure of the problem and ensures equivalence between the γ-resources core and the γ-preferences core. When coalitions are extremely pessimistic i.e. in the case of α-core, in order to deal with the high degree of variability in the reactions of outsiders, we assume a special condition of separability for preferences with respect to coalitions. This assumption of *social group separability*, implies that the preference of an agent i over the consumption of a coalition S to which he belongs, does not depend on the consumption of the outsiders. Moreover, it implies standard separability

[5] E.g. the equivalence holds under the assumptions of continuity and monotonicity.

with respect to agent i's own consumption. Under social group monotonicity and social group separability, the α-resources core and the α-preferences core are shown to be equivalent. The equivalence between allocations undominated in terms of resources and allocations which in turn are undominated in terms of preferences allows the introduction in both cases, of suitable measures of social loss associated to the core.

To conclude, we note that the γ-dominance and the α-dominance relations studied in the paper reduce to standard dominance in the absence of externalities. These two dominance relations are not directly comparable since the resources available for redistribution in the counter coalition $N \setminus S$ are different under the two approaches. Note also that the γ-dominance can be modified maintaining the assumption about the resources redistributed in S and $N \setminus S$ but assuming a pessimistic attitude of the blocking coalition S, i.e. allowing the outsiders to redistribute the status quo resources. Similarly, the α-dominance relation can be adapted to an optimistic attitude of S, by assuming that the outsiders stick to their initial consumption [6]. If this is the case, we end up with four different dominance relations which can be ranked from the more optimistic to the more pessimistic in terms of expected reactions. We focus on the extreme cases in order to highlight the main difficulties and the nature of assumptions necessary in the optimistic and pessimist case. However, similar results can be provided for the other core notions.

The paper is organized as follows. Section 2 introduces the model and the model assumptions. Sections 3 and 4 focus respectively on the study of the preferences-core and resources-core for the γ-dominance and the α-dominance relations. Sections 5 and 6 represent the main body of the paper: in Section 5 the notions of resource core are analyzed together with their equivalence with the preferences core; in Section 6, the γ-core and the α-core are characterized in terms of loss mappings. Section 7 provides some further remarks and conclusions. The Appendix provides some basic definitions and facts about Euclidean spaces and the proofs of technical results.

2 The model and basic assumptions

There is a finite number l of commodities. The commodity space is $\mathbb{R}^l$ [7]. There is a finite number of individuals (agents or traders) denoted by the subscript

[6] This dominance relation was introduced and studied by Chander and Tulkens (1997). A taxonomy of dominance relations based on the perspective of the blocking coalition toward the reaction of the counter-coalition is proposed in Graziano et al. (2017).

[7] For basic notations see Section 8.

$i \in N := \{1, \ldots, n\}$. The consumption of individual i is $x_i := (x_i^1, \ldots, x_i^l)$, and let $x := (x_i)_{i \in N}$ be a vector of consumption bundles. A coalition is any nonempty subset S of the set of agents N. The consumption set of agent i depends on the coalition which i joins. Formally, denoting by $\mathcal{P}_i(N)$ the set of all subsets S of N such that $i \in S$, $X_i : \mathcal{P}_i(N) \rightrightarrows \mathbb{R}_+^l$ is the correspondence which associates to each coalition S the consumption set $X_i(S) \subseteq \mathbb{R}_+^l$ available to agent i when i joins coalition S.[8] This way of modeling consumption sets is sufficiently general to recover the standard case in which $X_i(S)$ is the positive cone for each agent and each coalition. It also covers the asymmetric information framework, in which an exogenous rule regulates the information sharing of the individuals in a coalition (see Section 7).

The individual preferences of an agent are affected by the consumption of all the other agents in the economy. Formally, the preferences of individual i are described by a binary relation $\succsim_i$ over $\mathbb{R}_+^{l \cdot n}$. The situation in which each agent's preference depends only on his own consumption will be referred to as the *selfish* case. The initial endowment of individual i is $\omega_i := (\omega_i^1, \ldots, \omega_i^l)$, and let $\omega := (\omega_i)_{i \in N} \in \mathbb{R}_+^{l \cdot n}$ be the vector of all initial endowments. Thus, the economy with externalities under consideration is formalized by the list of elements summarized below:

$$E := \langle N, ((X_i(S)_{S \in \mathcal{P}_i(N)})_{i \in N}, (\succsim_i, \omega_i)_{i \in N} \rangle.$$

For every coalition $S \subseteq N$, and for any vector $z \in \mathbb{R}^{l \cdot n}$, we define $z_S := (z_i)_{i \in S}$ and $z_{N \setminus S} := (z_i)_{i \in N \setminus S}$. Given z_S and $z_{N \setminus S}$, without loss of generality, we denote z by $(z_S, z_{N \setminus S})$, and let $z(S) := \sum_{i \in S} z_i$.

An assignment for a coalition is a distribution of commodity bundles among its members which satisfies the consumption set and the physical feasibilities. An assignment for the grand coalition is an allocation. Formally,

Definition 1 (Assignment and Allocation) *Given a coalition $S \subseteq N$, an assignment for S is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^{l \cdot n}$ such that: i) $x_i \in X_i(S)$, for every $i \in S$ (consumption set feasibility); ii) $x(S) \leq \omega(S)$ (physical feasibility). An assignment for N is an allocation. $\mathcal{F}(\omega)$ will denote the set of allocations.*

We make the following basic assumptions about the consumption set.

Assumption 2 *For every $i \in N$, and for every coalition $S \subseteq N$, with $i \in S$,*

[8] A related approach is employed in del Mercato (2006a) to study the non emptiness of the core for a pure exchange economy without externalities, and in Graziano et al. (2017) to study the existence and the uniqueness of stable sets for exchange economies with interdependent preferences.

1. *$X_i(S)$ is a closed convex cone in $\mathbb{R}^l_+$;*

2. *$\omega_i \in X_i(S)$;*

3. *$\omega(S)$ belongs to the relative interior of $\sum_{i \in S} X_i(S)$;*

4. *$X_i(S) \subseteq X_i(N)$;*[9]

5. *$\dim \bigcap_{S \subseteq N} \langle \sum_{i \in S} X_i(S) \rangle \geq 1.$*[10]

Point 1 of Assumption 2 is standard. Point 2 states that the initial endowment of each agent belongs to each consumption set meaning that it is always available whatever the coalition S the agent joins. Point 3 is an interior assumption about the aggregate initial endowment. Point 4 requires that each agent who joins the grand coalition has at least the same consumption opportunities as in each coalition to which he/she belongs to. Finally, Point 5 imposes a technical assumption on the dimension of the director space associated to the affine hull of the aggregate consumption set. It ensures that there exists a common direction along which each coalition can slightly modify its assignments. It is meaningful only if restrictions are imposed on consumption sets, and it allows us to fix a reference bundle for the construction of a social loss mapping (see Section 6). Notice that clearly Assumption 2 is satisfied if the consumption set is the positive cone of the commodity space for each agent and each coalition.

We make the following basic assumptions about preference relations.

Assumption 3 *For every individual $i \in N$, $\succsim_i$ are complete, transitive and continuous over $\prod_{i \in N} X_i(N)$.*

Notice that we do not require any convexity assumption on preferences. Moreover, although we do not make use of utilities in the paper, the assumptions stated for preferences ensure that each agents' preference relation $\succsim_i$ can be represented by a utility function u_i defined over assignments for the coalition N.

Finally, note that a well-known example of preference under externalities is

[9] This assumption implies that any assignment can be completed in order to generate an allocation. Notice that we are requiring positive effects in terms of availability of consumption bundles with respect to the grand coalition, but not with respect to a larger coalition, i.e. it is not necessarily true that for every agent i, and for every coalitions $S, T \in \mathcal{P}_i(N)$, $S \subseteq T$, implies $X_i(S) \subseteq X_i(T)$.

[10] According to point 1 of Assumption 2, the director space associated to the affine hull of the set $\sum_{i \in S} X_i(S)$, i.e., $\langle \text{aff} \sum_{i \in S} X_i(S) \rangle$, coincides with the set $\text{aff} \sum_{i \in S} X_i(S)$. So, with little abuse of notation we denote it simply by $\langle \sum_{i \in S} X_i(S) \rangle$.

represented by the so called *separable preference*, i.e. a preference relation $\succsim_i$ for which $(x_i, x_{N\setminus i}) \succsim_i (x_i^{'}, x_{N\setminus i})$ for some $x_{N\setminus i}$ implies that $(x_i, x_{N\setminus i}^{'}) \succsim_i (x_i^{'}, x_{N\setminus i}^{'})$, for each $x_{N\setminus i}^{'} \in \mathbb{R}_+^{l\cdot(n-1)}$. This type of preference relation is analyzed in more detail in subsection 5.2. We note only that under separability of $\succsim_i$, it is possible to introduce a well-defined preference relation $\succsim_i^{(i)}$ over $\mathbb{R}_+^l$ i.e. over the consumption vectors, sometimes called *internal preference* of trader i. By definition $x_i \succsim_i^{(i)} y_i$, if and only if $(x_i, x_{N\setminus i}) \succsim_i (x_i^{'}, x_{N\setminus i})$, for some $x_{N\setminus i}$.

3 Preferences-core under externalities

According to the usual definition, the core is based on a certain dominance relation between allocations which is implemented by a coalition. In models with externalities, difficulties arise in defining such dominance. Indeed, a coalition S implementing a blocking procedure observes different possibilities for the reactions from the counter coalition $N \setminus S$.

We consider two notions of dominance based on the optimistic/pessimistic behavior of the blocking coalition S with respect to the reactions of the outsiders. For each of the corresponding dominance relations we have a core notion.

A first type of dominance can be introduced in line with Ichiishi (1981) and Dufwenberg et al. (2011). In order to distinguish this dominance relation from that with respect to resources we refer to it as the γ-dominance with respect to preferences. Below, we give the formal definition of γ-dominance in preferences and the corresponding notion of the γ-preferences core.

Definition 4 (γ-dominance with respect to preferences) *Let y and x be two allocations for the economy E. We say that y γ-dominates x in preferences, denoted by $y \succ_p^{\gamma} x$, if there exists a coalition $S \subseteq N$ such that:*

i) y is a assignment for S;

ii) for every $i \in S$, $(y_S, x_{N\setminus S}) \succ_i x$ holds true.

The set of allocations which can be γ-improved by no coalition with respect to preference relations is called γ-preferences core. It is denoted by $\mathcal{C}_p^{\gamma}(E)$.

The γ-dominance concept is based on the following mechanism: a coalition S deviates and does not accept the status quo allocation x. The blocking coalition S has an optimistic attitude. In particular, S considers that the members of the counter coalition $N \setminus S$ do not react but stick to $x_{N\setminus S}$. Notice that the final redistribution $(y_S, x_{N\setminus S})$ is not necessarily physically feasible for the grand coalition and thus might not be an allocation. The Strong Nash equilibrium concept defined by Aumann (1959) for non-cooperative games, makes the same assumption about the behavior of non-coalition members.

In the spirit of Yannelis (1991a) we introduce a different type of dominance, namely the α-dominance with respect to preferences and the corresponding core notion.

Definition 5 (α-dominance with respect to preferences) *Let y and x be two allocations for the economy E. We say that y α-dominates x with respect to preferences, denoted by $y \succ_p^\alpha x$, if there exists a coalition $S \subseteq N$ such that:*

i) y is an assignment for S;

ii) for every $i \in S$ and for every $z_{N \setminus S}$ with $z_i \in X_i(N)$ and $z(N \setminus S) \leq \omega(N \setminus S)$, $(y_S, z_{N\setminus S}) \succ_i x$ holds true.

The set of allocations which can be α-improved by no coalition with respect to preference relations is called α-preferences core. It is denoted by $\mathcal{C}_p^\alpha(E)$.

Hence, in the α-dominance, a coalition S deviates and does not accept the proposed allocation x. The blocking coalition S has a pessimistic attitude. Indeed, S takes account of all possible redistributions of the initial endowments among the outsiders, and S is willing to deviate from the status quo allocation only if these reactions ensure a better outcome for its members [11]. Notice that the final redistribution $(y_S, z_{N\setminus S})$ is an allocation and, in particular, for every $i \in N \setminus S$, $z_i \in X_i(N)$. Moreover, allocations which cannot be dominated by the grand coalition are the same in the γ-dominance and in the α-dominance. These allocations form the set of (weakly) Pareto optimal allocations of the economy E.

Each of the previous core notions can be empty. Yannelis (1991a) has proved that for a pure exchange economy with only two agents, the α-preferences core is non-empty. Holly (1994) provides an examples with an empty α-core for an economy with three or more agents. The example shows that when there is interdependency in the utility functions, the core is empty also in the simple case of linear utilities. A similar idea can be used to show that the γ-preferences core with more than two agents can be empty (see (Dufwenberg et al., 2011, Example 4)). So the problem of the existence of the core is central in models with externalities.

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[11] An equivalent formulation of condition *ii*) in Definition 5 is the following:

ii) for every $i \in S$, for every allocation z with $z_S = y_S$, $(y_S, z_{N\setminus S}) \succ_i x$ holds true.

In both formulations, we do not impose the very demanding requirement of the physical feasibility of consumption for the members of the counter coalition.

Table 1 below summarizes the dominance relations introduced so far, the resources supposed to be available for the counter coalition under each of these relations, and the expected reaction. Notice that cross comparison between the γ and α dominance relations is not possible since the resources available for a redistribution reported in the first column, are different in the two approaches.

Resources of $N \setminus S$	View point of S	Expected reaction	Dominance
$x(N \setminus S)$	optimistic	$x_{N \setminus S}$	γ
$\omega(N \setminus S)$	pessimistic	any redistribution	α

We conclude this section by observing that since we assume no cooperation between S and $N \setminus S$, the γ and the α dominance relations reduce to the standard blocking mechanism in the case of economic models without externalities [12] .

4 Resources-core under externalities

Measures of social loss proposed in economics are related to a Paretian definition of efficiency. According to Allais (1943), an allocation x is efficient if there is no other allocation that is weakly preferred to the status quo x, and which allows individuals to save resources. The duality between the classical Pareto efficiency notion given in terms of preferences and the notion formulated in terms of minimization of resources by Allais, has been widely studied (see Debreu (1951), Luenberger (1992) and Montesano (1997)). Employing the dual notion allows to define a measure of social loss due to inefficiency in terms of quantity of resources. This analysis of efficiency was extended to the case of the core of economy without externalities by Montesano (2002), who introduced the concept of resources core. The duality between the maximization of preferences and the minimization of resources is key to transforming equilibrium notions which pertain (invisible) preferences in equilibrium notions which in turn pertain to (concrete) resources. So, this concept becomes particularly relevant in the presence of externalities since in several models used to study the core, resources play a central role [13] .

In this section, we define the resources-core of an exchange economy with externalities in order to analyze a similar duality. For each notion of the core

[12] Scarf (1971) provides proof that the α-core of an exchange economy is non-empty. In the α-dominance it is assumed that a coalition S is able to block if it can redistribute its resources to the whole society so that its members are better off for each redistribution of resources to the society by $N \setminus S$. Hence, a kind of cooperation between the two coalitions is assumed.

[13] E.g. this applies to the case of environmental models.

introduced in Section 3, we provide a specular version in terms of resources. For an allocation that is not in the resources-core, there exists at least one coalition whose members can improve upon the given allocation by saving resources. Below, we formally introduce the notion of γ-dominance and α-dominance with respect to use of resources and the corresponding notions of resources core.

Definition 6 (γ-dominance with respect to resources) *Let y and x be two allocations for the economy E. We say that y γ-dominates x with respect to resources, denoted by $y \succ_r^\gamma x$, if there exists a coalition $S \subseteq N$ such that:*

i) y is an assignment for S with $y(S) < \omega(S)$;

ii) for all $i \in S$, $(y_S, x_{N\setminus S}) \succsim_i x$ holds true.

The set of allocations which can be γ-improved by no coalition with respect to resources is called γ-resources core. It is denoted by $\mathcal{C}_r^\gamma(E)$.

A coalition S γ-improves an allocation x with respect to the use of resources, if there exists y which is weakly preferred to x assuming that the outsiders consume their status quo allocation and which allows the members of S to preserve resources. In fact, if the allocation x is blocked by y in the spirit of Definition 6, the positive quantity $\omega(S) - y(S)$, can be saved without damage the members of S. Hence a loss in terms of resources emerges for the coalition S under the status quo allocation x.

In Definition 6 coalitions have an optimistic attitude so that the consumption of outsiders is considered fixed, and they are not concerned about market clearing. In line with the α-dominance introduced in Section 3, we assume now that coalitions are pessimistic, and therefore they evaluate any variation in the consumption of the outsiders. Moreover, the final distribution for the society is required to be feasible.

Definition 7 (α-dominance with respect to resources) *Let y and x be two allocations for the economy E. We say that y α-dominates x with respect to resources, denoted by $y \succ_r^\alpha x$, if there exists a coalition $S \subseteq N$ such that:*

i) y is an assignment for S with $y(S) < \omega(S)$;

ii) for every $i \in S$ and for every $z_{N\setminus S}$ with $z_i \in X_i(N)$ and $z(N \setminus S) \leq \omega(N \setminus S)$, $(y_S, z_{N\setminus S}) \succsim_i x$ holds true.

The set of allocations which can be α-improved by no coalition with respect to the use of resources is called α-resources core. It is denoted by $\mathcal{C}_r^\alpha(E)$.

A coalition S α-improves a given allocation with respect to (the use of) resources if there exists an assignment y which is weakly preferred to x by the members of S whatever the redistribution of initial resources outside S, and which allows S to save resources. Again, allocations which cannot be dominated by the grand coalition in resources are the same in both the γ and the α-dominance. These allocations form the set of (weakly) Pareto optimal allocations with respect to resources.

5 Preferences and resources core equivalence

In this section, the equivalence between the preferences core and the resources core is proved for the two dominance relations introduced so far. This requires some additional assumptions. The new assumptions vary depending on the optimistic/pessimistic attitude of the coalitions. So, the main results are presented in separate subsections.

5.1 γ-core equivalence

The first part is devoted to proving that the γ-preferences core and the γ-resources core coincide. Proposition 8 shows that under our basic assumptions, the γ-resources core is a subset of the γ-preferences core. The transitivity of preferences is not invoked in order to obtain this first inclusion.

Proposition 8 *Under the basic assumptions, the inclusion $\mathcal{C}_r^\gamma(E) \subseteq \mathcal{C}_p^\gamma(E)$ holds true.*

Proof.
Let $x \in \mathcal{C}_r^\gamma(E)$ and suppose by contradiction that $x \notin \mathcal{C}_p^\gamma(E)$. Then there exists a coalition $S \subseteq N$ and an assignment x' for S such that $(x'_S, x_{N\setminus S}) \succ_i x$ for all $i \in S$.

If $(x' - \omega)(S) < 0$, a contradiction follows, so, we can assume that $x'(S) = \omega(S)$. By continuity of preferences, there exists a positive δ such that, if $z_i \in X_i(S)$ for all $i \in S$ and $\|(z_S, x_{N\setminus S}) - (x'_S, x_{N\setminus S})\| < \delta$ then $(z_S, x_{N\setminus S}) \succ_i (x_S, x_{N\setminus S})$, for every $i \in S$.

Since $x'(S) = \omega(S)$, by Point 3 of Assumption 2, $x'(S)$ belongs to the relative interior of $\sum_{i\in S} X_i(S)$ and, consequently, there exists an agent $j \in S$ such that $x'_j > 0$.

Choose $\varepsilon > 0$ such that $0 < (1-\varepsilon)\|x'_j\| < \delta$. Define x'' by choosing $x''_i = x'_i$, for $i \in S \setminus \{j\}$ and $x''_j = \varepsilon x'_j$.[14] Notice that x'' is an assignment for S since x' is an assignment for S and $X_j(S)$ is a cone by Point 1 of Assumption 2. Then $\|(x''_S, x_{N\setminus S}) - (x'_S, x_{N\setminus S})\| = (1-\varepsilon)\|x'_j\| < \delta$ and consequently $(x''_S, x_{N\setminus S}) \succ_i x$, for every $i \in S$. Since $x''(S) < x'(S)$, we have a contradiction. ■

In order to prove the inclusion of the preferences core in the resources core, we need the additional assumption of monotonicity. The fact that the notion of resources core is based on the idea that waste of resources is not desirable for coalitions, makes the introduction of monotonicity natural. In particular, we introduce an assumption of group monotonicity in the spirit of Borglin (1973) and Dufwenberg et al. (2011). This generalizes the so called Social Monotonicity (SM) assumption, which refers only to the grand coalition and is usually adopted to reduce the degree of spitefulness and to ensure that the Second Welfare Theorem holds true if the preferences are separable.[15] Under (SM), one can prove that the set of (weakly) Pareto efficient allocations coincides with the set of (weakly) Pareto efficient allocations defined in terms of resources[16] .

Assumption 9 (Social Group Monotonicity (SGM)) *For any coalition $S \subseteq N$, any vector $x \in \prod_{i\in N} X_i(N)$ and $z > x(S)$, if $z \in \sum_{i\in S} X_i(S)$, then there exist vectors $x'_i \in X_i(S)$, $i \in S$, with $x'(S) = z$, and $(x'_S, x_{N\setminus S}) \succ_i (x_S, x_{N\setminus S})$, for all $i \in S$.*[17]

The (SGM) condition states that any increase in the resources available to the coalition S can be redistributed to make every member of S better off. (SGM) fails in the presence of hateful agents and generalizes the (SM) condition. Hence, it ensures that the second welfare theorem holds true when preferences are separable (see Dufwenberg et al. (2011)).

Under the (SGM) assumption, it can be shown that the resources core coincides with the preferences core under coalitions with an optimistic attitude, i.e. in the γ-dominance.

Theorem 10 (γ-resource core equivalence) *Under the basic assumptions and (SGM) assumption, the equality $\mathcal{C}^\gamma_p(E) = \mathcal{C}^\gamma_r(E)$ holds true.*

[14] We arbitrarily define the other elements x''_i for all $i \in N \setminus S$.
[15] See Dufwenberg et al. (2011) for details.
[16] However, the (SM) assumption is not enough to guarantee the equivalence between the set of weakly Pareto efficient allocations and the set of strongly Pareto efficient allocations when both notions are given in terms of preferences.
[17] By Point 2 of Assumption 2 and Assumption 9, for any allocation x such that $x(S) < \omega(S)$, we can find a S-feasible assignment that γ-dominates the allocation x.

Proof.
Proposition 8 shows that it is necessary only to prove the inclusion $\mathcal{C}_p^\gamma(E) \subseteq \mathcal{C}_r^\gamma(E)$.
Let $x \in \mathcal{C}_p^\gamma(E)$ and suppose by contradiction that there exists a coalition $S \subseteq N$ and the vectors $x'_i \in X_i(S)$, $i \in S$, such that $(x' - \omega)(S) < 0$ and $(x'_S, x_{N\setminus S}) \succsim_i (x_S, x_{N\setminus S})$, for all $i \in S$. Notice that: $\omega(S) > x'(S)$. Moreover, Point 2 of Assumption 2 implies that $\omega(S) \in \sum_{i\in S} X_i(S)$ and Point 4 of Assumption 2 implies that $(x'_S, x_{N\setminus S}) \in \prod_{i\in S} X_i(N)$. Consequently, by Assumption 9, there exists vectors $x''_i \in X_i(S)$, $i \in S$, with $x''(S) = \omega(S)$, such that $(x''_S, x_{N\setminus S}) \succ_i (x'_S, x_{N\setminus S})$ for all $i \in S$. Finally, by transitivity, we obtain $(x''_S, x_{N\setminus S}) \succ_i (x_S, x_{N\setminus S})$, for all $i \in S$, which implies a contradiction. ∎

The equality $\mathcal{C}_p^\gamma(E) = \mathcal{C}_r^\gamma(E)$ proved by Theorem 10 states that if an allocation x cannot be improved by a group of agents in terms of preferences, then in the given no-worst-than-set it does not determine a loss of resources for each coalition. Under utility representation for preferences, Theorem 10 establishes the equivalence between these classes of allocations: allocations which give to each (optimistic) coalition a utility vector which is undominated by any other utility vector; and allocations which generate for each (optimistic) coalition a utility vector that cannot be reached through any other feasible allocation requiring fewer resources.

Finally, according to the assumptions in Theorem 10, we can denote the γ-Core of the economy E simply by $\mathcal{C}^\gamma(E)$.

5.2 α-core equivalence

In this section the equivalence between the α-preferences core and the α-resources core is proved. Compared to Section 5.1, coalitions have a pessimistic attitude and take into account any possible redistribution of resources among outsiders. Moreover, they care about market clearing. Proposition 11 shows that under our basic assumptions, the α-resources core is a subset of the α-preferences core. Again, the transitivity of the preferences is not needed to prove this first inclusion.

Proposition 11 *Under our basic assumptions, $\mathcal{C}_r^\alpha(E) \subseteq \mathcal{C}_p^\alpha(E)$ holds true.*

Proof.
Let $x \in \mathcal{C}_r^\alpha(E)$ and suppose by contradiction that there exists a coalition $S \subseteq N$ and an assignment x' such that for every agent $i \in S$, $(x'_S, y_{N\setminus S}) \succ_i (x_S, x_{N\setminus S})$ holds true for all $y_{N\setminus S} \in \prod_{i\in N\setminus S} X_i(N)$ with $(y - \omega)(N \setminus S) \leq 0$. We can suppose that $x'(S) = \omega(S)$. Define the non empty sets

$$\mathrm{K}_S(x') := \{(x'_S, y_{N\setminus S}) \mid y_{N\setminus S} \in \prod_{i\in N\setminus S} X_i(N) \text{ and } (y-\omega)(N\setminus S) \leq 0\}$$

and

$$\mathrm{NBT}_S(x) \coloneqq \bigcup_{i\in S}\{z \in \prod_{i\in N} X_i(N) | \; x \succsim_i z\,\}$$

and their distance

$$\Lambda_S(\mathrm{K}_S(x'), \mathrm{NBT}_S(x)) \coloneqq \inf_{q\in \mathrm{K}_S(x'),\, z\in \mathrm{NBT}(x)} \|z-q\|.$$

Since $\mathrm{K}_S(x')$ is compact, $\mathrm{NBT}_S(x)$ is a closed subset of $\prod_{i\in N} X_i(N)$ and $\mathrm{K}(x') \cap \mathrm{NBT}_S(x) = \emptyset$,[18] the distance is strictly positive. Denote the distance $\Lambda_S(\mathrm{K}_S(x'), \mathrm{NBT}_S(x))$ simply by δ.

For every element $(x'_S, y_{N\setminus S}) \in \mathrm{K}_S(x')$ consider the open ball $B((x'_S, y_{N\setminus S}); \delta)$ centered in $(x'_S, y_{N\setminus S})$ and with ray $\delta > 0$. Then, for any $z \in V_\delta(y) \coloneqq B((x'_S, y_{N\setminus S}); \delta) \cap \prod_{i\in N} X_i(N)$, we must have $z \succ_i x$.

For any y such that $(x'_S, y_{N\setminus S}) \in \mathrm{K}_S(x')$, define $z(y)$ as $z_S(y) = \varepsilon x'_S$ with $\varepsilon > 0$ such that $0 < (1-\varepsilon)\|x'_S\| < \delta$, and $z_{N\setminus S}(y) = y_{N\setminus S}$. By Points 1 and 4 of Assumption 2, every vector $z(y)$ belongs to the corresponding neighborhood $V_\delta(y)$. Therefore for every agent $i \in S$, it is the case that, $(\epsilon x', y_{N\setminus S}) \succ_i (x_S, x_{N\setminus S})$ for every $y_{N\setminus S} \in \prod_{i\in N\setminus S} X_i(N)$ with $(y-\omega)(N \setminus S) \leq 0$. By construction, $\epsilon x'(S) < \omega(S)$, and a contradiction to the fact that $x \in \mathcal{C}_r^\alpha(E)$. ■

We now have to prove the inclusion of the preferences core in the resources core. In order to deal with the several possible reactions by the outsiders, we associate the (SGM) condition to the special case of preferences which are separable with respect to coalitions. The formal definition is an extension of the separability introduced at the end of Section 2 (see Borglin (1973) and Dufwenberg et al. (2011) for standard separability).

Assumption 12 (Social Group Separability (SGS)) *For any coalition S and $i \in S$, the preference relations $\succsim_i$ are S-separable: for all x_S and x'_S in $\prod_{i\in S} X_i(S)$, if there exists $x_{N\setminus S} \in \prod_{i\in N\setminus S} X_i(N)$ such that $(x'_S, x_{N\setminus S}) \succsim_i (x_S, x_{N\setminus S})$ (resp. $(x'_S, x_{N\setminus S}) \succ_i (x_S, x_{N\setminus S})$) then $(x'_S, x'_{N\setminus S}) \succsim_i (x_S, x'_{N\setminus S})$ (resp. $(x'_S, x'_{N\setminus S}) \succ_i (x_S, x'_{N\setminus S})$) for all $x'_{N\setminus S} \in \prod_{i\in N\setminus S} X_i(N)$.*

[18] If $\mathrm{K}_S(x') \cap \mathrm{NBT}_S(x) \neq \emptyset$, then there exists an $y'_{N\setminus S} \in \prod_{i\in N\setminus S} X_i(N)$ with $(y' - \omega)(N\setminus S) \leq 0$ such that the vector $(x'_S, y'_{N\setminus S})$ belongs to $\mathrm{NBT}_S(x)$. This contradicts the fact that x' α-dominates in preferences x, i.e, $x' \succ_i^\alpha y$.

Condition (SGS) states that if a member of coalition S likes the S-assignment x'_S better than the S-assignment x_S when the outsiders consume $x_{N\setminus S}$, then the coalition member will also prefer x'_S to x_S if each of them is joined with any other consumption by the outsiders. Consequently, the preference of i for the consumption of a coalition S to which i belongs, does not depend on the choice of others outside S. Notice that in each comparison the consumption of the counter coalition $N \setminus S$ is held constant. Hence, the (SGS) on its own is not enough to identify the γ and α dominances.

We note also that in the standard separable assumption, each agent has a preference relation which is separable only with respect to the coalition formed by the agent alone. Consequently, the (SGS) implies the standard separability of preferences. Under (SGS), each preference $\succsim_i$ induces a well-defined (internal) preference of i in the coalition S to which i belongs which is defined as follows: $x_S \succsim_i^{(S)} y_S$, if and only if $(x_S, x_{N\setminus S}) \succsim_i (x'_S, x_{N\setminus S})$, for some $x_{N\setminus S}$ [19].

Below we give an example of a utility function which satisfies (SGS) and was inspired by classical Edgeworth well-being externalities (see Dufwenberg et al. (2011)). In this example, agent i cares about his own internal utility and the sum of the internal utilities of the other agents. Hence, it is the internal well-being of the others that enters the utility of the agent i. Under this type of preference, it is easy to verify that (SGS) is satisfied.

Example 13 *Each agent $i \in N$ has an (internal) utility function u_i that depends only on his own consumption x_i, and the interdependent utility function U_i aggregates these individual utilities for each agent according to the formula:*

$$U_i(x) \coloneqq u_i(x_i) + \frac{\beta_i}{n-1} \sum_{j \neq i} u_j(x_j)$$

If β_i is positive, then agent i is altruistic or benevolent. On the contrary, the case of a negative β_i this denotes an envious or spiteful agent.

More generally, the example could be formulated assuming that each agent $i \in N$ has an internal utility function $u_i^{(S)}$ which depends on his own consumption x_i in the coalition S. In this case, the interdependent utility function U_i aggregates individual internal utilities for each i and a weighted average of the internal utilities of other agents. Again, the coefficient could be used to express the altruism/spitefulness of i with respect to a coalition S to which he does not belong [20].

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[19] According to Ichiishi (1981), the dependence of a preference relation $\succsim_i^{(S)}$ on S reflects the fact that agent i enjoys an environment specific to coalition S.

[20] Core and cooperative solutions defined using utility functions depending on coalitions are studied in Ichiishi (1981) and del Mercato (2006a).

Notice that in Example 13 if agents are altruistic (i.e. their coefficient β_i are assumed to be non negative), the (SGM) assumption also is satisfied. Under (SGM) and (SGS) assumptions, the preferences and the resources core in the α-dominance coincide.

Theorem 14 (α-resources core equivalence) *Under the basic assumptions, assumptions (SGM) and (SGS), $\mathcal{C}_p^\alpha(E) = \mathcal{C}_r^\alpha(E)$ holds true.*

Proof.
By Proposition 11, we have only to prove the inclusion $\mathcal{C}_p^\alpha(E) \subseteq \mathcal{C}_r^\alpha(E)$.
Let $x \in \mathcal{C}_p^\alpha(E)$ and suppose by contradiction that there exists a coalition $S \subseteq N$ and an allocation x' such that x' α-dominates x with respect to the use of resources. Then $(x' - \omega)(S) < 0$ and for all $i \in S$ $(x'_S, y_{N\setminus S}) \succsim_i (x_S, x_{N\setminus S})$, for all $y_{N\setminus S} \in \Pi_{i\in N\setminus S} X_i(N)$ such that $y(N \setminus S) \leq \omega(N \setminus S)$.

According to (SGM), since $\omega(S) \in \sum_{i\in S} X_i(S)$ and $\omega(S) > x'(S)$ there exists x'' such that $x''(S) = \omega(S)$ and $(x''_S, y_{N\setminus S}) \succ_i (x'_S, y_{N\setminus S})$. By (SGS) it is also true that $(x''_S, y'_{N\setminus S}) \succ_i (x'_S, y'_{N\setminus S})$ for every $y'_{N\setminus S}$ and for all $i \in S$. Consequently, the transitivity of $\succsim_i$ produces $(x''_S, y'_{N\setminus S}) \succ_i (x_S, x_{N\setminus S})$ for all $i \in S$ and for all $y'_{N\setminus S}$ such that $y'(N \setminus S) \leq \omega(N \setminus S)$. This contradicts the fact that $x \in \mathcal{C}_p^\alpha(E)$. ■

Under (SGM) and (SGS) Theorem 14 establishes an equivalence between allocations which cannot be dominated in terms of preferences and allocations which in turn cannot be dominated in terms of resources if coalitions are assumed to be pessimistic. In particular, under the utility representation for preferences, the theorem ensures that a feasible allocation x which cannot be improved by a group of agents in terms of preferences, guarantees a level of utilities to each coalition such that the coalition do not suffer loss of resources.

Finally, under the assumptions of Theorem 14, we can denote the α-Core of the economy E simply by $\mathcal{C}^\alpha(E)$.

6 Loss Mapping

In the previous section, we characterized the usual notions of γ-core and α-core based on maximization of preferences in terms of resources. In this section, first we define a *measure of social loss* as the amount of resources which can be saved under a given allocation. Then we use this measure to characterize the core. In subsection 6.1 we provide a complete characterization of the γ-core, and in subsection 6.2 we characterize the α-core.

In the remainder of this section, we assume that all the basic assumptions and the (SGM) condition hold true. Following Montesano (2002), we define a measure of loss for every coalition S. Given an allocation x, we define the set of resources which gives to coalition S the possibility to reach a redistribution that is weakly preferred to x by all the members of S, fixing the consumption of the outsiders at $x_{N\setminus S}$. This set is denoted by $\mathcal{R}_S^\gamma(x)$. Formally,

$$\mathcal{R}_S^\gamma(x) := \{z \in \mathbb{R}^l |\ \forall i \in S\ \exists\ x_i' \in X_i(S) : x'(S) = z \text{ and } (x_S', x_{N\setminus S}) \succsim_i x\}.$$

The set $\mathcal{R}_S^\gamma(x)$ is closed in $\sum_{i\in S} X_i(S)$ [21] .
If the vector z is an element of $\mathcal{R}_S^\gamma(x)$, then based on Assumption 9 and the transitivity property of $\succsim_i$, it can easily be shown that all vectors $z' \in \sum_{i\in S} X_i(S)$ greater than z still belong to $\mathcal{R}_S^\gamma(x)$. As a consequence, if $\omega(S) \notin \mathcal{R}_S^\gamma(x)$, then S is not a blocking coalition since S cannot implement a blocking procedure. However, if $\omega(S) \in \mathcal{R}_S^\gamma(x)$, the possibility that the allocation x is blocked by S cannot be excluded. Finally, notice that $\omega(N) \in \mathcal{R}_N^\gamma(x)$ also holds true. [22]

By considering the differences between the initial resources and elements in the set $\mathcal{R}_S^\gamma(x)$, we can define the set $\Psi_S(x)$ of resources that can be saved by coalition S while still allowing S to achieve a resources allocation for its members that is at least as good as x, keeping consumption of the counter coalition $N \setminus S$ fixed at $x_{N\setminus S}$. Formally,

$$\Psi_S^\gamma(x) := \left\{z \in \mathbb{R}^l |\ \omega(S) - z \in \mathcal{R}_S^\gamma(x) \cap \mathcal{A}_S(\omega)\right\}$$

where $\mathcal{A}_S(\omega) := \left\{y \in \mathbb{R}_+^l : y - \omega(S) \leq 0\right\}$.

When x_S is not an assignment for S, the vector $x(S)$ may not be in $\mathcal{R}_S^\gamma(x)$. This implies the possibility for $\mathcal{R}_S^\gamma(x) \cap \mathcal{A}_S(\omega)$ to be empty, and consequently implies emptiness of $\Psi_S^\gamma(x)$. [23] The next result gives a necessary and sufficient condition under which the set $\Psi_S^\gamma(x)$ is not empty

[21] This follows from the fact that the consumption sets are closed and preferences are continuous.
[22] If $x(N) = \omega(N)$ the result is trivial. If $x(N) < \omega(N)$, it is enough to invoke Point 3 of Assumption 2 and (SGM) for this statement to hold.
[23] In order to ensure that $x(S)$ belongs to $\mathcal{R}_S^\gamma(x)$, it should be assumed that $X_i(N) \subseteq X_i(S)$ for every $i \in S$. This assumption, together with Point 4 in Assumption 2 implies that the consumption set of every agent i does not depend on the coalition i.e. $X_i(S) = X_i$, for every S and for every $i \in S$.

Lemma 15 *$\Psi_S^\gamma(x) \neq \emptyset$ if and only if $\omega(S) \in \mathcal{R}_S^\gamma(x)$.*

Proof.
Trivially if $\omega(S) \in \mathcal{R}_S^\gamma(x)$ then $0 \in \Psi_S^\gamma(x)$ and so $\Psi_S^\gamma(x) \neq \emptyset$.
Conversely, if $\Psi_S^\gamma(x) \neq \emptyset$, then there exists $z \in \mathbb{R}_+^l$ and a redistribution x' with $x'_S \in \prod_{i\in S} X_i(S)$ such that: $\omega(S) - z = x'(S) \in \mathcal{R}_S^\gamma(x)$ and $\omega(S) \geq x'(S)$. If $\omega(S) = x'(S)$, then the implication is proved. Consider now the case where $\omega(S) > x'(S)$. Point 4 in Assumption 2 implies that $(x'_S, x_{N\setminus S}) \in \prod_{i\in S} X_i(N)$. Furthermore, according to Point 2 in Assumption 2, $\omega(S) \in \sum_{i\in S} X_i(S)$ holds true. Consequently, by Assumption 9, there exists $x'' \in \prod_{i\in S} X_i(S)$ such that $x''(S) = \omega(S)$, and $(x''_S, x_{N\setminus S}) \succ_i (x'_S, x_{N\setminus S})$ for all $i \in S$. Finally, $x'(S) \in \mathcal{R}_S^\gamma(x)$ and transitivity imply that $(x''_S, x_{N\setminus S}) \succ_i (x_S, x_{N\setminus S})$ for all $i \in S$. Thus $\omega(S) \in \mathcal{R}_S^\gamma(x)$. ■

Let us fix a vector $g \in \mathbb{R}_+^l$ with $g \neq 0$. We will call g the *reference bundle*. Below, we introduce the *loss mapping* as a function measuring the maximum amount of resources that can be saved by a coalition S with respect to an allocation x in the direction of the reference bundle g. [24]

Formally, the loss mapping $\mathcal{L}_{S,g}^\gamma : \mathcal{F}(\omega) \to \mathbb{R}$ is defined as follows

$$\mathcal{L}_{S,g}^\gamma(x) := \begin{cases} \max\{\lambda_S \colon \lambda_S \cdot g \in \Psi_S^\gamma(x)\} & \text{if } \Psi_S^\gamma(x) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Notice that, if $\Psi_S^\gamma(x) \neq \emptyset$ then the maximum is well-defined (i.e. finite value), since $\Psi_S^\gamma(x)$ is compact. [25] Furthermore, $\mathcal{L}_{S,g}^\gamma(x) \geq 0$ since $\Psi_S^\gamma(x) \subseteq \mathbb{R}_+^l$ and $g \in \mathbb{R}_+^l$ with $g \neq 0$.

We observe that in the related literature on Pareto optimal allocations in terms of resources, the reference bundle g can be chosen arbitrarily. For example, in a classical setting without externalities, Debreu (1951) chooses $g = \omega(N)$ while Allais (1943) and Groves (1979) use $g = (1, 0 \ldots, 0)$. In our framework, some preliminary remarks are needed here, given, in particular, the dependence of consumption sets on coalitions. For instance, Lemma 16 below shows that not all vectors can be taken as a reference bundle. Indeed, for vectors $g \in \mathbb{R}_+^l$ outside the director space generated by the aggregate consumption set of coalition S, the loss mapping is always equal to zero.

Lemma 16 *If $g \notin \langle \sum_{i\in S} X_i(S) \rangle$, then $\mathcal{L}_{S,g}^\gamma(x) = 0$ for any $x \in \mathcal{F}(\omega)$.*

[24] Equivalently, it measures the loss, in terms of g procured to coalition S by an allocation x.
[25] The set $\Psi_S^\gamma(x)$ is compact, since it is a translation of the intersection $\mathcal{A}_S(\omega) \cap \mathcal{R}_S^\gamma(x)$, where $\mathcal{A}_S(\omega)$ is compact and $\mathcal{R}_S^\gamma(x)$ is closed.

Proof.
See Section 8.

For vectors belonging to the director space, the loss mappings generally are different. However, if there exists g such that $\mathcal{L}^{\gamma}_{S,g}(x)$ is equal to zero, then all the other loss mappings are equal to zero as proved in Proposition 17.

Proposition 17 *For a given allocation x, if $\mathcal{L}^{\gamma}_{S,\widehat{g}}(x) \neq 0$ for a vector $\widehat{g} \in \langle\sum_{i\in S} X_i(S)\rangle$ with $\widehat{g} \neq 0$, then $\mathcal{L}^{\gamma}_{S,g}(x) \neq 0$ for every $g \in \langle\sum_{i\in S} X_i(S)\rangle$ with $g \neq 0$.*

Proof.
See Section 8.

Moving now from the loss (in terms of g) procured to each coalition S by an allocation x, introduce the measure of social loss with respect to x as the *social loss mapping* $\mathcal{L}^{\gamma}_{g} \colon \mathcal{F}(\omega) \to \mathbb{R}$ which can be defined as

$$\mathcal{L}^{\gamma}_{g}(x) \coloneqq \max_{S\subseteq N} \mathcal{L}^{\gamma}_{S,g}(x)$$

The social loss mapping $\mathcal{L}^{\gamma}_{g}(x)$ is well-defined because for every coalition S, the loss mapping $\mathcal{L}^{\gamma}_{S,g}$ is well-defined. $\mathcal{L}^{\gamma}_{g}(x)$ is the maximal loss procured for a coalition by the allocation x. Theorem 18 shows that the maximal loss vanishes if and only if the allocation belongs to the γ-core. Consequently, we obtain a full characterization of the core in terms of loss mappings.

Theorem 18 *For any non null reference bundle $g \in \bigcap\limits_{S\subseteq N} \langle\sum\limits_{i\in S} X_i(S)\rangle$ it is true that $\mathcal{L}^{\gamma}_{g}(x) = 0$ if and only if $x \in \mathcal{C}^{\gamma}(E)$.*

Proof.
We start by proving that if $x \in \mathcal{C}^{\gamma}(E)$ then $\mathcal{L}^{\gamma}_{g}(x) = 0$. Suppose by contradiction that $\mathcal{L}^{\gamma}_{g}(x) > 0$. Then there exists a coalition S such that $\Psi^{\gamma}_{S}(x) \setminus \{0\} \neq \emptyset$. Consequently, we can find a vector $x' \in \prod_{i\in N} X_i(N)$ such that the consumption bundle $(x'_S, x_{N\setminus S})$ is better than $(x_S, x_{N\setminus S})$, for every $i \in S$. Since it is also true that $(x' - \omega)(S) < 0$, a contradiction with $x \in \mathcal{C}^{\gamma}(E)$ is obtained.

Let us show now that $\mathcal{L}^{\gamma}_{g}(x) = 0$ implies $x \in \mathcal{C}^{\gamma}(E)$. By contradiction, suppose that $x \notin \mathcal{C}^{\gamma}(E)$. So, there exists a coalition $S \subseteq N$ and an allocation x' such that $(x'_S, x_{N\setminus S}) \succsim_i (x_S, x_{N\setminus S})$ for every $i \in S$ and $(x' - \omega)(S) < 0$. Therefore, the set $\Psi^{\gamma}_{S}(x) \setminus \{0\}$ is nonempty. Thus, $\mathcal{L}^{\gamma}_{S,g}(x) > 0$ for any $g \in \langle\sum_{i\in S} X_i(S)\rangle$ and consequently for all $g \in \bigcap_{S\subseteq N}\langle\sum_{i\in S} X_i(S)\rangle$ we obtain $\mathcal{L}^{\gamma}_{g}(x) > 0$ which contradicts $\mathcal{L}^{\gamma}_{g}(x) = 0$. ■

We conclude this section by observing that Theorem 18 allows the problem of existence of γ-core allocations to be managed by looking at the zeros of the social loss mapping, considered as a function defined over allocations.

6.2 Characterization of the α-core

This section analyzes the α-core equivalence in terms of loss mappings. In what follows we assume that the basic assumptions and the (SGM) and (SGS) conditions are satisfied.

As in Section 6.1, we first define the set of resources that give coalition S the possibility to reach an allocation at least as good as x. However, here coalitions are assumed to be pessimistic. To simplify the notation, we introduce the set

$$\mathrm{Y}_S(\omega) := \{y_{N\setminus S} \in \prod_{i\in N\setminus S} X_i(N)\colon\ y(N\setminus S) \leq \omega(N\setminus S)\}$$

and then $\mathcal{R}^\alpha_S(x)$ as the set of aggregate vectors z which have at least one redistribution x'_S which is preferred with respect to the status quo x for any reaction of the counter coalition $N \setminus S$. Formally, we have

$\mathcal{R}^\alpha_S(x) := \{z \in \mathbb{R}^l | \forall\, i \in S\, \exists\, x'_i \in X_i(S)\colon x'(S) = z,\, (x'_S, y_{N\setminus S}) \succsim_i x,\, \forall\, y_{N\setminus S} \in \mathrm{Y}_S(\omega)\}$.

Lemma 19 *For every allocation x, the set $\mathcal{R}^\alpha_S(x)$ is closed in $\mathbb{R}^l_+$.*

Proof.
Consider a sequence $(z^\nu)_{\nu\in\mathbb{N}} \subseteq \mathcal{R}^\alpha_S(x)$ converging to some $z^\infty \in \mathrm{cl}\,\mathcal{R}^\alpha_S(x)$. We must prove that $z^\infty \in \mathcal{R}^\alpha_S(x)$. Since $z^\nu \in \mathcal{R}^\alpha_S(x)$ for every $\nu \in \mathbb{N}$, there exists $x^\nu_i \in \prod_{i\in S} X_i(S)$ such that $x^\nu(S) = z^\nu$ and $(x^\nu_S, y_{N\setminus S}) \succsim_i x$ for all $i \in S$ and $y_{N\setminus S} \in \mathrm{Y}_S(\omega)$. Notice that, for every $i \in S$, $0 \leq x^\nu_i \leq z^\nu$ and z^ν converges to z, hence $(x^\nu_i)_{\nu\in\mathbb{N}}$ is a bounded sequence. Therefore, there exists for each $i \in S$ a subsequence $(x^k_i)_{k\in\mathbb{N}} \subseteq (x^\nu_i)_{\nu\in\mathbb{N}}$ converging to $x^\infty_i \in X_i(S)$. Since we have finitely many agents, we can assume that the subsequence involves the same indexes $k \in \mathbb{N}$ for each agent $i \in S$ by considering new subsequences if necessary.

Hence, we find a sequence $x^k \in \prod_{i\in S} X_i(S)$ converging to some $x^\infty \in \prod_{i\in S} X_i(S)$. Notice that,

$$z^\infty = \lim_{k\to\infty} z^k = \lim_{k\to\infty} \sum_{i\in S} x^k_i = \sum_{i\in S} \lim_{k\to\infty} x^k_i = \sum_{i\in S} x^\infty_i$$

Thus, we have proved that $x^\infty(S) = z$.

Fix $y_{N\setminus S} \in Y_S(N)$. Since $z^\nu \in \mathcal{R}^\alpha_S(x)$ for every $\nu \in \mathbb{N}$, then it must be the case that $(x^\nu_S, y_{N\setminus S}) \succsim_i x$ for every $i \in S$. In particular, $(x^k_S, y_{N\setminus S}) \succsim_i x$ for every $i \in S$ holds true for every k. Taking the limit, by the continuity of the preference relations, we obtain $(x^\infty_S, y_{N\setminus S}) \succsim_i x$ for every $i \in S$. Finally, since $(x^\infty_S, y_{N\setminus S}) \succsim_i x$, $i \in S$ holds for every $y_{N\setminus S} \in Y_S(\omega)$, we obtain $z^\infty \in \mathcal{R}^\alpha_S(x)$. ■

According to the (SGM) and (SGS) assumptions, if $z \in \mathcal{R}^\alpha_S(x)$, every vector $z' \in \sum_{i\in S} X_i(S)$ with $z' > z$ also belongs to $\mathcal{R}^\alpha_S(x)$.[26] The set of resources which can be saved by coalition S with respect to x is denoted by $\Psi^\alpha_S(x)$ i.e.

$$\Psi^\alpha_S(x) := \left\{ z \in \mathbb{R}^l \,|\, \omega(S) - z \in \mathcal{R}^\alpha_S(x) \cap \mathcal{A}_S(\omega) \right\}.$$

Lemma 20 *$\Psi^\alpha_S(x) \neq \emptyset$ if and only if $\omega(S) \in \mathcal{R}^\alpha_S(x)$.*

Proof.
Trivially, if $\omega(S) \in \mathcal{R}^\alpha_S(x)$ then $0 \in \Psi^\alpha_S(x)$.
Conversely, if $\Psi^\alpha_S(x) \neq \emptyset$, then there exists a vector $z \in \mathbb{R}^l_+$ such that $\omega(S) - z \in \mathcal{R}^\alpha_S(x)$. The definition of $\mathcal{R}^\alpha_S(x)$ implies that there is a redistribution $x'_S \in \prod_{i\in S} X_i(S)$ of $\omega(S) - z$ with $(x'_S, y_{N\setminus S}) \succsim_i x$ for every agent $i \in S$ and for every $y_{N\setminus S} \in Y_S(\omega)$. If $\omega(S) = x'(S)$, then $z = 0$ and the conclusion follows. So $\omega(S) > x'(S)$ and by Point 4 in Assumption 2, $(x'_S, y_{N\setminus S}) \in \prod_{i\in N} X_i(N)$. Since by Points 2 and 4 in Assumption 2, $\omega(S) \in \sum_{i\in S} X_i(N)$, then Assumption 9 implies that there is a redistribution $x''_i \in X_i(S)$, $i \in S$ with $\omega(S) = x''(S)$ and $(x''_S, y_{N\setminus S}) \succ_i (x'_S, y_{N\setminus S})$, for every member i of coalition S. Finally the transitivity property of the preferences implies $(x''_S, y_{N\setminus S}) \succ_i (x_S, x_{N\setminus S})$. Note that by (SGS) this conclusion holds true for every vector $y_{N\setminus S} \in Y_S(\omega)$, then we have that $\omega(S) \in \mathcal{R}^\alpha_S(x)$. ■

Given a reference bundle $g \in \mathbb{R}^l_+$ with $g \neq 0$, we introduce for any coalition $S \subseteq N$, the loss mapping $\mathcal{L}^\alpha_{S,g} : \mathcal{F}(\omega) \to \mathbb{R}$ associated to the α-dominance relation. The function is defined as

$$\mathcal{L}^\alpha_{S,g}(x) := \begin{cases} \max\{\lambda_S \colon \lambda_S g \in \Psi^\alpha_S(x)\} & \text{if } \Psi^\alpha_S(x) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Notice that, if $\Psi^\alpha_S(x) \neq \emptyset$, then the loss mapping is well-defined, since $\Psi^\gamma_S(x)$ is compact. Furthermore, $\mathcal{L}^\alpha_{S,g}(x) \geq 0$.

[26] Indeed, $z \in \mathcal{R}^\alpha_S(x)$ and Point 4 in Assumption 2 imply that there exists a redistribution $\widetilde{x}$ of z such that $(\widetilde{x}_S, y_{N\setminus S}) \in \prod_{i\in N} X_i(N)$ for every vector $y_{N\setminus S} \in \mathrm{Y}_S(\omega)$. Fixing $y_{N\setminus S} \in \mathrm{Y}_S(\omega)$, Assumption 9 implies that there exists a redistribution $x'_S \in \prod_{i\in S} X_i(S)$ of z', such that $(x', y_{N\setminus S}) \succ_i (\widetilde{x}_S, y_{N\setminus S})$, $i \in S$. By transitivity we obtain $(x'_S, y_{N\setminus S}) \succ_i x$ for every $i \in S$. Finally, noting that the (SGS) condition ensures that $(x'_S, y_{N\setminus S}) \succ_i x$, $i \in S$ holds true for every $y_{N\setminus S} \in \mathrm{Y}_S(\omega)$, we conclude that $z' \in \mathcal{R}^\alpha_S(x)$.

As in the case of γ-dominance, also under the α-dominance relation we need preliminary results. The proof of the next Lemma is not presented here.

Lemma 21 *If $g \notin \langle \sum_{i \in S} X_i(S) \rangle$, then $\mathcal{L}^{\alpha}_{S,g}(x) = 0$ for any $x \in \mathcal{F}(\omega)$.*

For every reference bundle g in the director space associated to the aggregate consumption set of coalition S, the zero points of the loss mapping are g-independent.

Proposition 22 *For any allocation x, $\mathcal{L}_{S,\widehat{g}}(x) \neq 0$ for a vector $\widehat{g} \in \langle \sum_{i \in S} X_i(S) \rangle$ with $\widehat{g} \neq 0$, then $\mathcal{L}_{S,g}(x) \neq 0$ for every $g \in \langle \sum_{i \in S} X_i(S) \rangle$ with $g \neq 0$.*

Proof.
See Section 8.

The *social loss mapping* $\mathcal{L}^{\alpha}_{g} \colon \mathcal{F}(\omega) \to \mathbb{R}$ is defined by

$$\mathcal{L}^{\alpha}_{g}(x) \coloneqq \max_{S \subseteq N} \mathcal{L}^{\alpha}_{S,g}(x).$$

$\mathcal{L}^{\alpha}_{g}(x)$ is well-defined since for every coalition S, the mapping $\mathcal{L}^{\alpha}_{S,g}$ is well-defined, and N is a finite set. Our final result is the core characterization under externalities and with a pessimistic attitude in the coalitions.

Theorem 23 *For all non null reference bundles $g \in \bigcap_{S \subseteq N} \langle \sum_{i \in S} X_i(S) \rangle$ it is true that $\mathcal{L}^{\alpha}_{g}(x) = 0$ if and only if $x \in \mathcal{C}^{\gamma}(E)$.*

Proof. First prove that if $x \in \mathcal{C}^{\alpha}(E)$ then $\mathcal{L}^{\alpha}_{g}(x) = 0$. Suppose by contradiction that $\mathcal{L}^{\alpha}_{g}(x) > 0$. Then a coalition S exists such that $\Psi^{\alpha}_{S}(x) \setminus \{0\} \neq \emptyset$. Then, we can find a redistribution $x' \in \prod_{i \in N} X_i(N)$ of $\omega(S) - \mathcal{L}^{\alpha}_{S}(x)g$, such that, the allocations $(x'_S, y_{N \setminus S})$, for every $y_{N \setminus S} \in Y_S(\omega)$, are weakly preferred to $(x_S, x_{N \setminus S})$ for every agent i of S, and $(x' - \omega)(S) \leq 0$. By Point 4 in Assumption 2 and Assumption 9, there exists a redistribution $x'' \in \prod_{i \in N} X_i(S)$ of $\omega(S)$ such that $(x'', y_{N \setminus S}) \succ_i (x', y_{N \setminus S})$. Finally the transitivity of the preferences and (SGS) contradicts the fact that $x \in \mathcal{C}^{\alpha}(E)$.

$\mathcal{L}^{\alpha}_{g}(x) = 0$ implies $x \in \mathcal{C}^{\alpha}(E)$. By contradiction, suppose that $x \notin \mathcal{C}^{\alpha}(E)$. Then there exists a coalition $S \subseteq N$ and an assignment $x' \in \prod_{i \in S} X_i(S)$ such that $(x'_S, y_{N \setminus S}) \succsim_i (x_S, y_{N \setminus S})$ for every $i \in S$ and $y_{N \setminus S} \in Y_S(\omega)$. Therefore, the set $\Psi^{\alpha}_{S}(x) \setminus \{0\}$ is nonempty. Thus, $\mathcal{L}^{\alpha}_{S,g}(x) > 0$ for any $g \in \langle \sum_{i \in S} X_i(S) \rangle$ and consequently for all $g \in \bigcap_{S \subseteq N} \langle \sum_{i \in S} X_i(S) \rangle$, the social loss mapping satisfies $\mathcal{L}^{\alpha}_{g}(x) > 0$. ■

7 Conclusions

We provided a characterization of the core in markets with externalities using the duality between preferences maximization and resources minimization. The characterization is formulated in terms of social loss mappings. Our results extend the previous findings for efficient allocations and the core of a pure exchange economy without externalities. In the case of a pure exchange economy with externalities, several core notions are possible. We focused in this paper on two notions which embody the idea of optimistic/pessimistic behavior of coalitions: the γ-core and the α-core. For these notions, we proved that suitable monotonicity and separability, the (SGM) and (SGS) conditions, are enough to establish the duality. The (SGM) condition allows us to study the γ-core and, therefore, the case of optimistic behavior. It is linked to the results for the second welfare theorem. Employing the (SGS) condition to deal with the α-core and with pessimistic behavior of coalitions is a novelty in this framework. Our results provide an alternative approach to the existence of core allocations in the presence of externalities. Future research should investigate conditions related to the social loss mappings to ensure the existence of their zero points.

For completeness, we conclude by showing how our model includes the case of economies with asymmetric information. In these models the economy takes place over two time periods, $t = 0$ and $t = 1$. At time $t = 0$ agents subscribe contracts which may be contingent on the realized state of nature at $t = 1$. Uncertainty is described by a a measurable space $(\Omega, \mathscr{F})$, where Ω is the set of possible states and $\mathscr{F}$ is the algebra of all possible events. Each trader i has partial information about the true state of nature and this information is described by a measurable partition Π_i of Ω. In a model with an information sharing rule, the private information can change if the trader joins a coalition S, according to a function which associates a new partition $\Gamma_i(S)$ to each member $i \in S$. In the blocking mechanism, traders can choose only consumption plans they are able to distinguish, meaning therefore plans that are measurable with respect to their private information $\Gamma_i(S)$ [27]. This makes the consumption set $X_i(S)$ depending on S.

8 Appendix

In this section we recall the topological properties of Euclidean spaces and some basic concepts in affine spaces. We need to state some preliminary definitions before starting, first the one of *affine hull* associated to a generic subset

[27] See Yannelis (1991b).

A of $\mathbb{R}^m$.

Definition 24 *We denote by* aff A *the affine hull of* A*, which is the smallest affine space that contains* A*.* [28]

The vector space underlying aff A will be denoted by $\langle \text{aff } A\rangle$. It is isomorphic to $\mathbb{R}^r$ for some $r \leq m$.

Definition 25 *A point* $a \in A$ *belongs to the relative interior of* A *if and only if there exists a open ball* B_a *in* $\mathbb{R}^m$ *centered in* a *such that* $B_a \cap \text{aff } A \subseteq A$*.*

Lemma 26 *Let* A *be a convex set and assume that* aff A *has dimension* r*. A point* x *is in the relative interior of* A *iff for every* $v \in \langle \text{aff } A\rangle$ *there exists* $\alpha > 0$ *such that* $x + \alpha v \in A$*.*

Proof. Without loss of generality we can consider $x = \underline{0}$. The underlying vector space of aff A can be identified with $\mathbb{R}^r$ via isomorphism. Let e_i denote the i-unitary vector of $\mathbb{R}^r$.
According to the convexity assumption, for any i there exists α_i, $\beta_i > 0$ such that $\underline{0} + \alpha_i e_i$ and $\underline{0} + \beta_i(-e_i)$ belong to A. By letting $\alpha = \min_i\{\alpha_i, \beta_i\}$ we obtain the assertion. ■

We recall also that a subset X of $\mathbb{R}^m$ is a cone if $\lambda x \in X$ for every $\lambda > 0$. Clearly, for a closed cone X and $x \in X$, the half-ray $\{\lambda \cdot x \,:\, \lambda \geq 0\}$ is included in X.

To conclude, we recall that for any $x \in \mathbb{R}^m$, the l_p-norm of x is defined as

$$\|x\|_p := \left(\sum_{i=1}^{m} x_i^p\right)^{\frac{1}{p}} \quad 1 \leq p < +\infty. \tag{1}$$

All the topologies defined by l_p-norms are equivalent in the sense that they have the same family of open sets. In particular, in the sequel we make use of the l_1-norms of an x vector, which reduces to the sum of x-components.

We conclude the section with the proofs of the technical results.

Proof of Lemma 16.

If $\Psi_S^\gamma(x) = \emptyset$, we obtain the statement by the definition of the loss function. Suppose otherwise that $\Psi_S^\gamma(x) \neq \emptyset$. Notice that $\Psi_S^\gamma(x) \subseteq \langle \sum_{i\in S} X_i(S)\rangle$ [29] and

[28] Formally, $A = \left\{\sum_{i=1}^{k} \lambda_i x_i \mid k \in \mathbb{N}, \quad x_i \in A, \quad \lambda_i \in \mathbb{R}, \quad \sum_{i=1}^{k} \lambda_i = 1\right\}$.
[29] For a vector $z \in \Psi_S^\gamma(x)$, there exists η such that $\eta \in \mathcal{R}_S^\gamma(x) \cap \mathcal{A}_S(\omega) \subseteq \langle \sum_{i\in S} X_i(S))\rangle$ and $\eta = \omega(S) - z$. By Point 3 of Assumption 2, $\omega(S)$ belongs to

consequently, $\mathcal{L}^{\gamma}_{S,g}(x)\cdot g$ belongs to the set $\langle\sum_{i\in S} X_i(S)\rangle$ too. Use M_g to denote the vector space of dimension 1 generated by g. Since $\mathcal{L}^{\gamma}_{S,g}(x)g \in M_g$, and $g \notin \langle\sum_{i\in S} X_i(S)\rangle$, we obtain $M_g \cap \langle\sum_{i\in S} X_i(S)\rangle = \{0\}$ and consequently, $\mathcal{L}^{\gamma}_{S,g}(x) = 0$. ■

Proof of Proposition 17.

First notice that $\mathcal{L}^{\gamma}_{S,g}(x) = \lambda_g \neq 0$ if and only if $\omega(S) - \lambda_g g \in \mathcal{R}^{\gamma}_S(x) \cap \mathcal{A}_S(\omega)$. Therefore, we can provide an equivalent formulation of the statement:

if $\omega(S) - \widehat{\lambda}\widehat{g} \in \mathcal{R}^{\gamma}_S(x)$ for some vector $\widehat{g} \in \langle\sum_{i\in S} X_i(S)\rangle$, $\widehat{g} > 0$, $\widehat{\lambda} > 0$, then for any $g \in \langle\sum_{i\in S} X_i(S)\rangle$, $g > 0$ there exists $\lambda_g > 0$ such that $\omega(S) - \lambda_g g \in \mathcal{R}^{\gamma}_S(x)$. [30]

By Assumption 9, the transitivity property of preference relations and by the fact that $\omega(S) - \widehat{\lambda}\widehat{g} \in \mathcal{R}^{\gamma}_S(x)$, it follows that $\omega(S) \in \mathcal{R}^{\gamma}_S(x)$. Precisely, for all $i \in S$ one can find $x'_i \in X_i(S)$ such that $\omega(S) = x'(S)$ and $(x'_S, x_{N\setminus S}) \succ_i (x_S, x_{N\setminus S})$, for each $i \in S$. Therefore we have that $\omega(S) \in \mathcal{R}^{\gamma}_S(x)$ and we have found a vector $(x'_S, x_{N\setminus S})$ which is strictly preferred to $(x_S, x_{N\setminus S})$ by the members of coalition S.

The rest of proof involves the following three steps:

Step 1 (Neighborhood of bundles better than the status quo allocation x) By continuity property of preference relations, there exists an open ball centered in $(x'_S, x_{N\setminus S})$, $B((x'_S, x_{N\setminus S}); \delta_1)$, such that for all vectors z which belong to $B((x'_S, x_{N\setminus S}); \delta_1) \cap \prod_{i\in N} X_i(N)$ we get $z \succ_i x$, for each $i \in S$.
In particular, for any $(x''_S, x_{N\setminus S}) \in B((x'_S, x_{N\setminus S}); \delta_1) \cap \left(\prod_{i\in S} X_i(S) \times \{x_{N\setminus S}\}\right)$ it is true that $x''(S) \in \mathcal{R}^{\gamma}_S(x)$. [31]

Step 2 (Neighborhood of aggregate resources) By Point 3 in the Assumption 2, $\omega(S)$ belongs to the relative interior of $\sum_{i\in S} X_i(S)$. Thus there is an open ball $B(\omega(S); \delta_2) \subseteq \mathbb{R}^l$ such that

$$B(\omega(S); \delta_2) \cap \left\langle \sum_{i\in S} X_i(S) \right\rangle \subseteq \sum_{i\in S} X_i(S)$$

As consequence, if $||v|| < \delta_2$ and $v \in \langle\sum_{i\in S} X_i(S)\rangle$ then $\omega(S) + v$ belongs to $\sum_{i\in S} X_i(S)$. As a consequence, there exists $z_i \in X_i(S)$, $i \in S$, such that

$\langle\sum_{i\in S} X_i(S)\rangle$. Therefore, $z = \omega(S) - \eta \in \langle\sum_{i\in S} X_i(S)\rangle$ since $\langle\sum_{i\in S} X_i(S)\rangle$ is a vector space.

[30] Whenever we get $\lambda g > 0$ trivially we have $\omega(S) - \lambda g \in \mathcal{A}_S(\omega)$.

[31] Recall that $x'_S \in \prod_{i\in S} X_i(S)$.

$z(S) = \omega(S) + v$.

Step 3 (Relation between the two neighborhoods) Let $\delta \coloneqq \min\{\delta_1, \delta_2\}$ and take $z \in B(\omega(S); \delta) \cap \langle \sum_{i \in S} X_i(S) \rangle$. Then there exists $(x''_S, x_{N \setminus S}) \in \prod_{i \in S} X_i(S) \times \{x_{N \setminus S}\}$ such that $x''(S) = z$ and

$$||(x''_S, x_{N \setminus S}) - (x'_S, x_{N \setminus S})|| = ||(x''_S - x'_S, 0)|| \leq ||x''(S) - x'(S)|| < \delta$$

where the inequality between the two norms holds true since the consumption bundles have positive components. So $(x''_S, x_{N \setminus S}) \succ_i x$ for all $i \in S$, and consequently $z \in \mathcal{R}^{\gamma}_S(x)$.

Now we complete the proof showing that, for any $g \in \langle \sum_{i \in S} X_i(S) \rangle$, there exists a $\lambda_g > 0$ such that $\omega(S) - \lambda_g g \in B(\omega(S); \delta) \cap \langle \sum_{i \in S} X_i(S) \rangle$. By Step 2, it is enough to take λ_g with $0 < \lambda_g < \delta ||g||^{-1}$, to obtain the desired result. Indeed, $||\lambda_g g|| < \delta$ and Step 3 allow us to find a consumption bundle $(x''_S, x_{N \setminus S})$ which belongs to $\prod_{i \in S} X_i(S) \times \{x_{N \setminus S}\}$, which is better than the status quo allocation and such that $\omega(S) - \lambda_g g = x''(S)$. Hence, $\omega(S) - \lambda_g g \in \mathcal{R}^{\gamma}_S(x)$. ■

Proof of Proposition 22.

The statement of Proposition 22 is equivalent to the following requirement:

if $\omega(S) - \widehat{\lambda}\widehat{g} \in \mathcal{R}^{\alpha}_S(x)$ for a vector $\widehat{g} \in \langle \sum_{i \in S} X_i(S) \rangle$ with $\widehat{g} > 0$ and $\widehat{\lambda} > 0$ then for any $g \in \langle \sum_{i \in S} X_i(S) \rangle$ there exists $\lambda > 0$ such that $\omega(S) - \lambda g \in \mathcal{R}^{\alpha}_S(x)$.

If $\omega(S) - \widehat{\lambda}\widehat{g} \in \mathcal{R}^{\alpha}_S(x)$, then we get $\omega(S) \in \mathcal{R}^{\alpha}_S(x)$, and we find vectors $x'_i \in X_i(S)$ such that $\omega(S) = x'(S)$ and $(x'_S, y_{N \setminus S}) \succ_i (x_S, x_{N \setminus S})$, for all $y_{N \setminus S} \in Y_S(\omega)$ and for all $i \in S$. By continuity of the preference relations, we obtain an open set $B(x'_S; \delta_1)$ such that $z \in B(x'_S; \delta_1) \cap \prod_{i \in N} X_i(N)$ implies that $z \succ_i x$, for each $i \in S$.

Since $\omega(S)$ is in the relative interior of $\sum_{i \in S} X_i(S)$, we find an open ball $B(\omega(S); \delta_2)$ such that for any vector v in $B(\omega(S); \delta_2) \cap \langle \sum_{i \in S} X_i(S) \rangle$ there are $z_i \in X_i(S)$ for all $i \in S$ and $z(S) = v$.

As in Proposition 17, with $\delta = \min\{\delta_1, \delta_2\}$ we obtain the conclusion. ■

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