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The Core of Economies with Collective Goods and a Social Division of Labor

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Abstract

This paper considers the Core of a competitive market economy with private commodities as well as (non-Samuelsonian) collective goods that are provided through an endogenous social division of labour. Our approach is founded on the hypothesis that every agent is a "consumer-producer", producing private commodities as well as consuming collective and private goods. We develop the σ-core concept, assuming that collective goods are scalable with community size.

We show that the σ-core can be founded on deviations of coalitions of arbitrary size, extending the seminal insights of Vind and Schmeidler for pure exchange economies. Our analysis also shows that self-organisation in a social division of labour can be incorporated into the Edgeworthian barter process directly. This is formulated as an equivalence of the σ-core and a structured σ-core concept based on blocking coalitions that use internal divisions of labour. Furthermore, Grodal's theorem is extended, allowing applications of metrics that express productive similarities between agents making up blocking coalitions.

Finally, we consider the equivalence of the σ-core and the set of cost share equilibrium allocations.

JEL Classification: D41, D51

Keywords: Collective goods; Consumer-producer; Social division of labour; Competitive equilibrium; σ-Core; Core equivalence.

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Table of contents

1 Introduction: Collective goods, social divisions of labor and the core

One of the oldest ideas in social philosophy and economics is that wealth is created through a functional social division of labor (Plato, 380 BCE; Smith, 1776; Mandeville, 1714; Babbage, 1835). A social division of labor is founded on the interplay between two fundamental principles, namely that there are *increasing returns to specialisation* and that full exploitation of these returns is possible through the principle of *gains from trade*, allowing individuals to specialise in certain productive tasks and to resolve the generated commodity bundles through competitive trade.

Using the notion of a *consumer-producer* to represent economic agents—seminally developed by Yang (1988, 2001) and Yang and Ng (1993)—a fully specified general equilibrium theory of an economy with an endogenous social division of labor has been developed in Gilles (2019b). Subsequently, Gilles, Pesce, and Diamantaras (2019) extended this theory to economies with (non-Samuelsonian) collective goods.[1] These contributions focus on general equilibrium concepts and the welfare theorems. In our contribution, we consider the Edgeworthian representation of competitive trade processes through the definition of an appropriate Core concept.

The development of an appropriate notion to represent Edgeworthian barter processes incorporating the provision of collective goods has been an arduous and long undertaking. Initially, the literature focussed mainly on blocking coalitions to provide an indivisible quantity of the collective good, instituting a relatively large Core, and investigated the link with Lindahl (1919)'s equilibrium concept based on individualised taxation (Foley, 1970; Ruys, 1972).[2] This line of research was extended to non-Samuelsonian collective goods by Mas-Colell (1980), for which Mas-Colell and Silvestre (1989) investigated the relationship of the core and linear cost-share equilibria.

An appropriate Core concept for economies with non-Samuelsonian collective goods has been explored further in Gilles and Diamantaras (1998), who introduced the hypothesis that collective goods are scalable and provided locally at proportionally lower costs. The corresponding notion of a σ-core for this class of economies is founded on a scaling measure σ for the coalitional provision of collective goods. The σ-core concept was further explored in Graziano and Romaniello (2012) and Basile, Graziano, and Pesce (2016).

In this paper we generalise this σ-core concept further by replacing the scaling measure with a multi-dimensional cost contribution measure. This allows the introduction of highly non-linear scaling of the costs of coalitional provision of collective goods. This significantly extends the scope of the theory, incorporating a much larger class of scalable collective good configurations.

Extensions of the Schmeidler-Vind-Grodal Theorems. For the introduced class of general scalable collective goods we explore the properties of the σ-core. First, we investigate the size of

[1] Samuelson (1954) introduced "public goods" as quantifiable bundles of non-rivalrous goods that are provided non-exclusively to the economy by an authority. Mas-Colell (1980) seminally considered non-Samuelsonian collective goods that are represented as abstract elements of an unstructured set of configurations. Non-Samuelsonian collective goods can represent any collective feature or configuration such as market systems (Gilles and Diamantaras, 2003), and allocations of land and knowledge (Gilles, Pesce, and Diamantaras, 2019).

[2] Alternative notions to Lindahl's equilibrium were also proposed and investigated, mainly founded in a reconsideration of the properties of collective goods such as developed by Buchanan (1965) and Foley (1967).

blocking coalitions by devising conditions under which the theorems of Schmeidler (1972) and Vind (1972) hold in the framework developed here. Gilles (2018a) showed that both of these results extend unconditionally to continuum economies with an endogenous social division of labour. Here, in the context of collective good provision through a social division of labor, we show that the extension of Schmeidler's result that any non-σ-core allocation can be blocked by an arbitrarily small coalition, applies in our framework. However, for Vind's result that any non-core allocation can be blocked by an arbitrarily large coalition, we have that it only holds for a certain class of allocations. Such allocations represent the property that collective good configurations are essentially interchangeable. We denote these as "Vind allocations".[3]

We also investigate the extension of Grodal (1972)'s theorem that non-Core allocations can be blocked by arbitrarily small coalitions of neighboring agents. We show that this result extends to our framework without onerous additional conditions. This allows applications for certain well-selected metrics. The obvious applications in our framework concern the structuring of blocking coalitions through internal social divisions of labor.

Blocking through social divisions of labor. Recently, Gilles (2018a) introduced the idea that in economies of consumer-producers, coalitions can use internal social divisions of labor in Edgeworthian re-trade. Hence, blocking coalitions can be assumed to be organised internally through a fully developed social division of labor, founded on the principle that almost every member of the coalition is fully specialised in the production of a single commodity. Hence, every member assumes a *profession*. This represents the idea that, as in coalition production economies (Hildenbrand, 1968, 1974), coalitions are assumed to be able to create wealth through an appropriate fine-tuning of the specialisation of its constituting members.

Using the formalisation of *Increasing Returns to Specialisation* (IRSpec) as introduced in Gilles (2019b), we are able to formalise how blocking coalitions can be structured through fully developed internal social divisions of labor. Indeed, under IRSpec, any non-σ-core allocation can be improved upon by a coalition with an internal social division of labor. This insight can be enhanced by relating it to Grodal (1972)'s theorem, showing that non-σ-core allocations can be blocked by arbitrarily small coalitions with an internal social division of labor founded on agents with very similar productive abilities.

Core equivalence. Finally, we investigate the equivalence of the devised notion of the σ-core with an appropriately formulated concept of linear cost share equilibrium. We focus on equilibria that are founded on cost distributions of the total provision cost of the selected collective good configuration. Our notion is, therefore, more general than the linear cost share equilibrium concepts explored by Mas-Colell (1980); Mas-Colell and Silvestre (1989); Gilles and Diamantaras (1998) and Graziano and Romaniello (2012).

We show that our general notion of the σ-core is equivalent to this generalised notion of cost share equilibrium. In particular, the set of cost share equilibrium allocations for some cost distribu-

[3] In the case that the set of collective good configurations is trivial this notion becomes vacuous and the result of Gilles (2018a) is recovered.

tion φ is for arbitrary continuum economies a subset of the $\int \varphi d\mu$-core, where μ is the measure on the space of consumer-producers. Conversely, under certain standard assumptions for continuum economies, the σ-core is part of the set of cost share equilibrium allocations based on the Radon-Nikodym derivative $\partial\sigma/\partial\mu$ of the contribution measure σ. In particular, these required hypotheses include continuity and monotonicity of individual preferences, integrably boundedness of productive abilities, Mas-Colell (1980)'s notion of essentiality and Gale (1957)'s irreducibility condition. These results extend the equivalence results for σ-cores in the literature.

In summary, our paper establishes a wide range of properties and results regarding the generalised σ-core concept for continuum economies with collective goods and an endogenous social division of labor. These insights that Edgeworthian re-trade through social divisions of labor are a viable framework for considering the provision of collective goods.

Outline of the paper: In Section 2 of this paper we develop the model of an economy with (non-Samuelsonian) collective goods in which private commodities are provided through an endogenous social division of labour founded on individualised production sets. We also conceptualise the appropriate notion of the σ-core to model scalability of collective good provision and state the main hypotheses for the analysis of this framework.

Section 3 extends the theorems of Schmeidler (1972) and Vind (1972) on the size of blocking coalitions. We show that Schmeidler's theorem holds without additional assumptions, while Vind's result only holds for particular allocations. We also investigate coalitional blocking based on internal social divisions of labor, denoted as the *structured* σ-core. Structured blocking is linked to Grodal (1972)'s theorem in the sense that non-core allocations can be blocked by coalitions that are internally structured by a social division of labor founded on groups of very similar fully specialised agents.

Finally, we conclude the paper in Section 4 that investigates the equivalence of the σ-core and the set of cost share equilibria, extending the results of Mas-Colell (1980), Gilles and Diamantaras (1998) and Graziano and Romaniello (2012).

2 Collective goods in an economy with a social division of labor

We assume throughout that there are $\ell \geqslant 1$ marketable commodities. The *commodity space* is represented as the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$ and the *consumption space* is its nonnegative orthant $\mathbb{R}^\ell_+$. The commodity space represents all bundles of tradable goods in this economy. In particular, for $k = 1, \ldots, \ell$ we denote by $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ the k-th unit bundle in $\mathbb{R}^\ell_+$ and by $e = (1, \ldots, 1)$ the bundle consisting of one unit of each good.[4]

We assume that collective goods are modelled completely abstractly. Thus, we let $\mathcal{Z}$ be some unstructured set of *collective good configurations.* $\mathcal{Z}$ is some abstract provision space as considered in Mas-Colell (1980) and Diamantaras and Gilles (1996).[5] Throughout we assume that all collective

[4]Throughout, we employ the vector inequality notation that $x \geqslant x'$ if $x_k \geqslant x'_k$ for all commodities $k = 1, \ldots, \ell$; $x > x'$ if $x \geqslant x'$ and $x \neq x'$; and $x \gg x'$ if $x_k > x'_k$ for all commodities $k = 1, \ldots, \ell$.

[5]Clearly, the collective goods introduced here generalise Samuelson's quantifiable notion of a public good (Samuelson, 1954). As pointed out by Mas-Colell (1980) and Diamantaras and Gilles (1996), these *non-Samuelsonian* collective goods

goods can be provided at any local or coalitional level.

This is supported through the assumption that all ℓ commodities as well as the collective goods in $\mathcal{Z}$ are produced and allocated through a social division of labor (Gilles, 2018b, 2019a,b; Gilles, Pesce, and Diamantaras, 2019). A social division of labor is modelled through the conception of a consumer-producer representation of every economic agent (Yang, 1988, 2001), thus supporting the perfect scalability of this social division of labor and, therefore, the provision of private commodities as well as collective goods in the economy. This is developed initially before addressing how the social division of labour is formed.

Consumer-producers. Using standard formulations, we introduce a complete probability space (A, Σ, μ) of economic agents. Here, A denotes the set of all *economic agents*; $\Sigma \subset 2^A$ a σ-algebra of *coalitions*; and $\mu \colon \Sigma \to [0,1]$ a complete probability measure on (A, Σ).

Using the framework set out in Gilles, Pesce, and Diamantaras (2019), every agent $a \in A$ is modelled as a *consumer-producer*, endowed with consumptive *as well as* productive abilities represented, using a pair $(u_a, \mathcal{P}_a)$, where $u_a \colon \mathbb{R}^\ell_+ \times \mathcal{Z} \to \mathbb{R}$ is a utility function representing a's consumptive preferences[6] and $\mathcal{P}_a \colon \mathcal{Z} \twoheadrightarrow \mathbb{R}^\ell$ is a's production correspondence that assigns to every configuration of the collective good $z \in \mathcal{Z}$ a production set $\mathcal{P}_a(z) \subset R^\ell$ consisting of input-output bundles that agent a can produce.[7]

Clearly, all ℓ commodities are in principle provided through a distributed system described by the production correspondence $\mathcal{P}$—supported through a perfectly scalable social division of labor. Hence, all collective goods in $\mathcal{Z}$ can be provided through any coalition in the economy, representing different provision "levels" of these collective goods to all agents in the economy. Prime examples of such coalitionally provided collective goods are education, infrastructure and judicial systems. All of these collective good configurations are provided at the global, regional as well as local levels of the economy.

Throughout, we assume that collective goods are provided through the expenditure of private goods only. This can be represented by a cost function $c \colon \mathcal{Z} \to \mathbb{R}^\ell_+$.

For every consumer-producer $a \in A$ and collective good configuration $z \in \mathcal{Z}$, we assume that a typical production bundle $y \in \mathcal{P}_a(z)$ can be written as $y = y^+ - y^-$ where $y^+, y^- \in \mathbb{R}^\ell_+$. Here, y^+ denotes the outputs of a's production process, while y^- denotes the tradable inputs required for producing y^+. We explicitly assume that the collective good is not used as an input in any production process, although the collective good is allowed to produce widespread externalities, as reflected in the dependence of the production sets on $z \in \mathcal{Z}$.[8] With regard to the productive

can represent discrete configurations of public projects such as infrastructural design, types of plants and works of art used in public parks. Furthermore, $z \in \mathcal{Z}$ can be subject to saturation in consumption such as road and air transport systems.

[6]We refer to Gilles, Pesce, and Diamantaras (2019) for a more generalised development of consumptive preferences in the context of an economy with an endogenous social division of labor and collective goods. In that paper, the preferences are just assumed to be non-satiated, following the framework developed in Hildenbrand (1969).

[7]Here, we employ the notation that a point-to-set correspondence represented as $\mathcal{F} \colon A \twoheadrightarrow \mathbb{R}^\ell$ can be defined equivalently as a mapping $\mathcal{F} \colon A \to 2^{\mathbb{R}^\ell}$.

[8]We also assume that there can be non-tradable inputs in this production process—such as a's labour time and land resources—that are not explicitly modelled. We allow the possibility that all outputs are generated using non-tradable inputs only.

abilities of a consumer-producer we introduce a regularity requirement.

Definition 2.1 *Given an agent $a \in A$ and a collective good configuration $z \in \mathcal{Z}$, the production set $\mathcal{P}_a(z) \subset \mathbb{R}^\ell$ is said to be **regular** if $\mathcal{P}_a(z)$ is a closed set that is bounded from above, $0 \in \mathcal{P}_a(z)$ and it is comprehensive, i.e.,*

$$\mathcal{P}_a(z) - \mathbb{R}^\ell_+ = \{ y - t \mid y \in \mathcal{P}_a(z) \text{ and } t \geqslant 0 \} \subset \mathcal{P}_a(z). \tag{1}$$

Definition 2.1 imposes that we only consider production sets that satisfy certain standard regularity properties. In particular, a regular production set satisfies the properties that one has the ability to cease production altogether and the assumption of free disposal in production. Both properties are used throughout the literature.

Furthermore, it is natural to assume that individual consumer-producers can only manage bounded production processes and are not able to grow their operations arbitrarily, imposing that there is an upper bound on the individual's production set.

Note that regularity does not include convexity, allowing production to exhibit non-convexities, in particular increasing returns to scale and specialisation, introducing constructions of production sets developed in the existing literature on economies with an endogenous social division of labour (Yang, 1988, 2001; Sun, Yang, and Zhou, 2004; Gilles, 2019b).

Defining an economy. We introduce an economy as a continuum of consumer-producers endowed with productive abilities to generate regular commodities as well as collective goods, introduced above.

Definition 2.2 *An **economy** with ℓ (private) commodities is a list $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, c, u, \mathcal{P} \rangle$ where*

(i) *(A, Σ, μ) is a complete probability space of consumer-producers, where every $a \in A$ is represented by the pair $(u_a, \mathcal{P}_a)$;*

(ii) *$\mathcal{Z}$ is an unstructured set of collective good configurations;*

(iii) *$c \colon \mathcal{Z} \to \mathbb{R}^\ell_+$ is a cost function that assigns to every collective good configuration $z \in \mathcal{Z}$ a bundle of commodities $c(z) \in \mathbb{R}^\ell_+$ that is used in its provision;*

(iv) *For every agent $a \in A$, the function $u_a \colon \mathbb{R}^\ell_+ \times \mathcal{Z} \to \mathbb{R}$ represents a's preferences over consumption configurations $(x, z) \in \mathbb{R}^\ell_+ \times \mathcal{Z}$, defining a utility function representation of consumptive preferences $u \colon A \times \mathbb{R}^\ell_+ \times \mathcal{Z} \to \mathbb{R}$.*
We assume that, for every consumption configuration $(x, z) \in \mathbb{R}^\ell_+ \times \mathcal{Z}$, the function $u(\cdot, x, z) \colon A \to \mathbb{R}$ is measurable on the probability space (A, Σ, μ);

(v) *For every agent $a \in A$, the correspondence $\mathcal{P}_a \colon \mathcal{Z} \to 2^{\mathbb{R}^\ell}$ represents a's productive abilities. We assume that the resulting production correspondence $\mathcal{P} \colon A \times \mathcal{Z} \to 2^{\mathbb{R}^\ell}$ is such that*

- *for every $a \in A$ and every $z \in \mathcal{Z}$ the production set $\mathcal{P}_a(z) \subset \mathbb{R}^\ell$ is regular according to Definition 2.1;*

- *for every $z \in \mathcal{Z}$ there exists an integrable function $g\colon A \longrightarrow \mathbb{R}^\ell_+$ such that $g(a) \in \mathcal{P}_a(z)$ for all $a \in A$ and $\int g d\mu \geqslant c(z) \in \mathbb{R}^\ell_+$.*

An economy $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ is a ***continuum economy*** *if the complete probability space of consumer-producers (A, Σ, μ) is atomless.*

For certain results the following additional assumptions are needed. Concerning the utility function u assigned to any consumer-producer we assume the following regularity properties that are used throughout the literature on collective good economies:

Assumption 2.3 *Consider an economy $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$.*

(a) *For every $a \in A$, the utility function $u_a(\cdot, z)\colon \mathbb{R}^\ell_+ \to \mathbb{R}$ is* **continuous** *on the consumption space $\mathbb{R}^\ell_+$,* **strictly quasi-concave** *and* **strictly monotone** *in the sense that for all consumption bundles $x, x' \in \mathbb{R}^\ell_+$ with $x > x'$ it holds that $u_a(x, z) > u_a(x', z)$.*

(b) ***Essentiality condition*** *– For almost all $a \in A$ and for all collective good configurations $z, z' \in \mathcal{Z}$ it holds that $u_a(0, z) \leqslant u_a(x, z')$ for every consumption bundle $x \in \mathbb{R}^\ell_+$.*

Assumption 2.3(a) imposes that we only consider "regular" preferences in this economy. The imposed properties of continuity and monotonicity are standard assumptions in the literature on general equilibrium in economies with public and/or collective goods.

Assumption 2.3(b) imposes a weak form of the essentiality condition introduced by Diamantaras and Gilles (1996, Definition 4.2). It states that zero consumption of private commodities cannot be compensated by any collective good configurations. In that regard, the consumption of private commodity bundles is primal and essential.

With regard to the productive abilities of a consumer-producer we introduce the following hypotheses.

Assumption 2.4 *For each $z \in \mathcal{Z}$, the production correspondence $\mathcal{P}(\cdot, z)\colon A \longrightarrow 2^{\mathbb{R}^\ell}$ has a measurable graph on the probability space (A, Σ, μ) such that the correspondence $\mathcal{P}$ is* ***integrably bounded from above****.*[9]

The used formulation of a consumer-producer has been developed in Gilles (2019b) and extends the original approach in such economies with consumer-producers developed in Yang (2001); Sun, Yang, and Zhou (2004) and Diamantaras and Gilles (2004), in which all production is achieved through the use of non-tradable inputs only. Yang's original formulation can be represented within our framework by imposing that for every $a \in A$ and $z \in \mathcal{Z}$ it holds that $\mathcal{P}(a, z) = \overline{\mathcal{P}}(a, z) - \mathbb{R}^\ell_+$, where $\overline{\mathcal{P}}(a, z) \subset \mathbb{R}^\ell_+$, i.e., essentially the production of tradable outputs is based on the usage of non-tradable, privately owned inputs only, to which free disposal is applied.

Definition 2.5 *An* ***allocation*** *in the economy $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ is a triple (f, g, z) where*

(i) *$z \in \mathcal{Z}$;*

[9] Hence, we impose that there exists some integrable function $\overline{g}\colon A \longrightarrow \mathbb{R}^\ell$ such that for almost every agent $a \in A$ and every production plan $y \in \mathcal{P}_a(z)$ and $y \leqslant \overline{g}(a)$.

(ii) $g : A \longrightarrow \mathbb{R}^\ell$ *is an integrable function such that* $g(a) \in \mathcal{P}_a(z)$ *for all* $a \in A$;

(iii) $f : A \longrightarrow \mathbb{R}^\ell_+$ *is an integrable function.*

An allocation (f, g, z) *is* ***feasible*** *if* $\int f \, d\mu + c(z) = \int g \, d\mu$.

Following Gilles, Pesce, and Diamantaras (2019), the basic Pareto efficiency notion in the context of an economy with collective goods that are provided through an endogenous social division of labor is defined as follows.

Definition 2.6 *A feasible allocation* (f, g, z) *in the economy* $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ *is* ***Pareto optimal*** *if there does not exist an alternative feasible allocation* (f', g', z') *such that* $u_a(f'(a), z') \geqslant u_a(f(a), z)$ *for almost all* $a \in A$ *and* $u_a(f'(a), z') > u_a(f(a), z)$ *for almost all* $a \in S$*, for some coalition* $S \in \Sigma$ *with* $\mu(S) > 0$.

These notions complete the standard framework for the development of Core concepts that represent outcomes of Edgeworthian barter processes in an economy. Here, these barter processes refer to the scaled provision of collective goods in the context of coalitions of economic agents forming communities in the economy.

Introducing an appropriate Core concept. In their contribution, Gilles and Diamantaras (1998) introduced a Core concept for economies with collective goods that are assumed to be perfectly scalable with the size of the community that is provided.[10] This inception is particularly relevant for economies in which these collective goods are provided through an endogenous social division of labor. Such provision mechanisms implicitly introduce scalable provision in the context of communities in the economy.

The next Core concept generalises the σ-core concept introduced in Gilles and Diamantaras (1998). This notion is represented through perfectly scalable contributions in collective good provision, conceptualised as a contribution measure.

Definition 2.7 *A function* $\sigma \colon \Sigma \times \mathcal{Z} \to \mathbb{R}^\ell_+$ *is a* ***contribution measure*** *if for each* $z \in \mathcal{Z}$, $\sigma(\cdot, z)$ *is a finite* ℓ*-dimensional vector measure on* Σ *such that*

(i) σ *is absolutely continuous with regard to* μ *in the sense that for every* $S \in \Sigma$, $\mu(S) = 0$ *implies that* $\sigma(S, z) = \mathbf{0}$ *and*

(ii) $\sigma(A, z) = c(z)$ *for any* $z \in \mathcal{Z}$.

A contribution measure indicates how collective good configurations are scaled over coalitions or communities of different sizes in the economy. Here the configuration $z \in \mathcal{Z}$ denotes a provision level that fully determines the satisfaction of the economic agents in any community. Now $\sigma(S, z) \in \mathbb{R}^\ell_+$ is the total quantities of all ℓ commodities that are required to provide the collective goods at level z in community S.

[10] Hence, a collective good configuration $z \in \mathcal{Z}$ has to be interpreted as a qualitative design that can be scaled with the size of the economy. Standard examples are government provided public goods such as education, defense, and infrastructure as well as non-Samuelsonian collective goods such as market institutions, law enforcement, including the judicial system, and monetary systems.

We emphasise that a contribution σ can be highly non-linear in the sense that smaller coalitions require disproportionate contributions of private goods to provide a certain collective good configuration in the represented community. This is explored below in Example 2.9.

The next formalisation captures the idea that private commodities as well as collective goods are provided and traded within communities $S \in \Sigma$ with $\mu(S) > 0$ according to the scale introduced by the contribution measure σ. These barters will continue until no further improvements in the various communities can be established. These outcomes are denoted as σ-core allocations.

Definition 2.8 *Let $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ be an economy with ℓ commodities. Given a contribution measure σ, a coalition $S \in \Sigma$* ***is able to improve upon*** *an allocation (f, g, z) via an alternative allocation (f', g', z') if*

(i) $\mu(S) > 0$;

(ii) $u_a(f'(a), z') > u_a(f(a), z)$ *for almost all* $a \in S$;

(iii) $\int_S f' d\mu + \sigma(S, z') = \int_S g' d\mu$.

*A feasible allocation (f, g, z) is a σ-***core** *allocation if there is no coalition $S \in \Sigma$ that can improve upon it. We denote by C_σ as the set of all σ-core allocations, also referred to as the σ-core of $\mathbb{E}$.*

We remark that, for any contribution measure σ, evidently every σ-core allocation in an economy $\mathbb{E}$ is Pareto optimal.[11]

To illustrate the σ-core notion introduced here, the next example shows that there exist non-trivial economies with a non-empty σ-core where σ describes a highly non-linear scaling in the provision of collective goods.

Example 2.9 Consider an economy $\mathbb{E}$ with $A = [0, 1]$ being the standard continuum endowed with the σ-algebra of Lebesgue measurable coalitions Σ and the Lebesgue measure $\lambda \colon \Sigma \to [0, 1]$. Furthermore, we let $\ell = 2$ and consider $\mathcal{Z} = [0, 1]$ with provision costs given by $c(z) = (z, 0)$.
We complete the description of the economy by describing productive abilities and consumptive desires, where for every agent $a \in A$ and collective good configuration $z \in \mathcal{Z}$:

$$\mathcal{P}_a(z) = \{ (2, 0), (0, z + 1) \} - \mathbb{R}^2_+$$

and for commodity bundle $(x, y) \in \mathbb{R}^2_+$ and collective good configuration $z \in \mathcal{Z}$:

$$u_a(x, y, z) = \frac{4(az + 1)^4 xy}{(2(az + 1)^2 - z(z + 1))^2} .$$

In this economy, we consider the non-trivial contribution measure σ on Σ that is defined by the density function $\varphi \colon A \times \mathcal{Z} \to \mathbb{R}^2_+$ with $\sigma(S, z) = \int_S \varphi(\cdot, z)\, d\lambda$, where

$$\varphi(a, z) = \left(\frac{z(z + 1)}{(az + 1)^2}, 0 \right) .$$

[11] We remark that in the definition of Pareto optimality we use a weak improvement formulation, while in the definition of the σ-core the coalitional improvement is formulated as a strong notion. We also note that, under the hypotheses stated in Assumption 2.3, weak and strong improvement are equivalent.

Note that $\int_A \varphi(\cdot, z)d\lambda = c(z)$, thereby showing that σ is indeed a contribution measure.

We now claim that the contribution measure σ is not a *linear* contribution measure in the sense that there is a probability measure $\tilde{\sigma} \colon \Sigma \longrightarrow [0,1]$ such that $\sigma(\cdot, z) = \tilde{\sigma}(\cdot) \cdot c(z)$.
Indeed, consider a coalition $S = [0, \bar{a}]$, with $\bar{a} \in A$, then

$$\sigma(S, z) = \frac{\bar{a}(z+1)z}{\bar{a}z+1} \quad \text{cannot be represented as} \quad \tilde{\sigma}(S) \cdot c(z)$$

since in that case $\tilde{\sigma}(S) = \frac{\bar{a}(z+1)}{\bar{a}z+1}$, which depends on z.

Next, we argue that the allocation (f^*, g^*, z^*) with

$$z^* = 1;\ f^*(a) = \left(\frac{(a+1)^2 - 1}{(a+1)^2}, \frac{(a+1)^2 - 1}{(a+1)^2} \right) \text{ and } g^*(a) = \begin{cases} (2,0) & \text{for } 0 \leqslant a \leqslant \frac{3}{4} \\ (0, z^*+1) = (0,2) & \text{for } \frac{3}{4} < a \leqslant 1 \end{cases}$$

constitutes a σ-core allocation.
Note that $c(z^*) = (1,0)$ and $\int f^* \, d\lambda = \left(\frac{1}{2}, \frac{1}{2}\right)$, which is supported by the production plan g^*. ♦

3 Extending the Schmeidler-Vind-Grodal Theorems

One of the classical questions in mathematical economics has been what type of blocking coalitions there are. One particular concern is about the size of blocking coalitions. This has been pursued in contributions by Schmeidler (1972); Vind (1972); Grodal (1972); Hervés-Beloso, Moreno-García, and Yannelis (2005); Greenberg, Weber, and Yamazaki (2007); Evren and Hüsseinov (2008) and Gilles (2018a) for economies with private commodities only. We show below that these results can naturally be extended to the setting of an economy with an endogenous social division of labour and collective good provision. These insights build on the results established by Gilles (2018a), which, in turn, extended the theorems from Schmeidler (1972) and Vind (1972) to the realm of continuum economies with an endogenous social division of labor.

The second question concerns the nature or composition of blocking coalitions. Gilles (2018a) showed that in a continuum economy with an endogenous social division of labor and private goods only, non-core allocations can be blocked by coalitions with internal divisions of labor based on full specialisation of its constituting members. Indeed, in the context of our notion of an economy, the social division of labour is perfectly fluid in a continuum of consumer-producers. As such, this framework incorporates the property that, therefore, production is completely scalable through the endogenous adjustment of the social division of labour. This property carries over to blocking coalitions in the sense that improving on a proposed allocation can be done through a set of completely specialised coalitions of consumer-producers. This analysis is pursued in the second part of this section.

Furthermore, in her seminal contribution to Core theory, Grodal (1972) investigated blocking coalitions that are composed of small coalitions within a social characteristics space that is endowed with a complete metric. She showed that every non-core allocation can be blocked by a coalition that is composed of ℓ arbitrarily small coalitions of neighboring agents within that social characteristics space. We extend this result to our setting, linking it to the organisation of a blocking coalition through an internal social division of labor.

3.1 The size of blocking coalitions

Our first discussion concerns the size of blocking coalitions as first addressed by Schmeidler (1972) and Vind (1972) for continuum exchange economies. Here, we extend these theorems by considering the size of blocking coalitions in the case that perfectly scalable collective goods can be provided in communities in an economy with an endogenous social division of labor. Scalability of the collective goods—as described by a contribution measure—works in tandem with the natural scalability of production of private economies in an economy with an endogenous social division of labor.

Extension of Schmeidler's Theorem: The theorem stated in Schmeidler (1972) addresses the effectiveness of blocking by arbitrarily small coalitions. He showed that any non-core allocation in a continuum exchange economy can be blocked by an arbitrarily small coalition. We show here that this result straightforwardly extends to our framework.

Theorem 3.1 *Consider a continuum economy* $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ *and some contribution measure* $\sigma \colon \Sigma \times \mathcal{Z} \to \mathbb{R}^{\ell}_{+}$*. Furthermore, let* $(f, g, z) \notin C_{\sigma}(\mathbb{E})$ *be a feasible non-core allocation such that there exists some non-negligible coalition* $S \in \Sigma$ *which can improve upon* (f, g, z)*. Then for every* $0 < \delta \leqslant \mu(S)$ *there exists a coalition* $S_{\delta} \in \Sigma$ *with* $\mu(S_{\delta}) = \delta$ *that can improve upon* (f, g, z)*.*

Proof. Suppose that S with $\mu(S) > 0$ improves upon (f, g, z) with an alternative allocation (f', g', z'). Hence, $u_a(f'(a), z') > u_a(f(a), z)$ for all $a \in S$ and $\int_S f' \, d\mu + \sigma(S, z') = \int_S g' \, d\mu$.
A proof can easily be constructed on the argument introduced in Schmeidler (1972). Indeed, introduce a multi-dimensional measure $\nu \colon \Sigma \to \mathbb{R}^{\ell+1}$ on (A, Σ, μ) restricted to S by

$$\nu(T) = \left(\int_T (f' - g') \, d\mu + \sigma(T, z') \, , \, \mu(T) \right) \in \mathbb{R}^{\ell+1} \quad \text{for any } T \subset S \tag{2}$$

Now by Lyapunov's Convexity Theorem (Hildenbrand, 1974, Theorem 3, page 62), it follows that ν results in a convex image.
Obviously $\nu(\emptyset) = (0, \dots, 0, 0)$ and $\nu(S) = (0, \dots, 0, \mu(S))$. Let $0 < \delta \leqslant \mu(S)$. Then there has to exist some $S_{\delta} \subset S$ such that $\nu(S_{\delta}) = (0, \dots, 0, \delta)$. Clearly the coalition S_{δ} now improves upon (f, g, z) through (f', g', z') such that $\mu(S_{\delta}) = \delta$, showing the assertion. ■

Extension of Vind's Theorem: The insight of Vind (1972) was that any non-core allocation can be blocked by an arbitrarily large coalition in a continuum exchange economy. This result does not translate easily to the setting of an economy with collective goods. The next example illustrates the difficulties of extending Vind's Theorem to our setting. We devise an economy with collective goods in which there exists a non-core allocation for which there exists no blocking coalition of a particular, larger size.

Example 3.2 Consider a continuum economy with two private commodities, a basic good and an intermediary input for providing any of two collective good configuration. These two collective good configurations are given by $\mathcal{Z} = \{\alpha, \beta\}$ with $c(\alpha) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $c(\beta) = \left(\frac{3}{4}, \frac{3}{4}\right)$.
For all consumer-producers, the production sets are given by $\mathcal{P}_a(k) = \{(1,1),(0,0)\} - \mathbb{R}^2_+$ for any $k \in \mathcal{Z}$. Agents only consume the basic good and, therefore, the agents' utility functions are defined by $u_a(h_1, h_2, \alpha) = u_a(h_1, h_2, \beta) = h_1$ for each $a \in A$. Consider the contribution measure equals for each $S \in \Sigma$ to

$$\sigma(S, \alpha) = \frac{\mu(S)}{2} e \tag{3}$$

$$\sigma(S, \beta) = \frac{3e}{4} \left[0.4\mu(S \cap [0, \tfrac{1}{2}]) + 2.4\mu(S \cap (\tfrac{1}{2}, \tfrac{3}{4})) + 0.8\mu(S \cap (\tfrac{3}{4}, 1])\right] \tag{4}$$

Note that for any $k \in \mathcal{Z}$, $\sigma(A, k) = c(k)$.

Consider the allocation (f, g, α) with $f(a) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $g(a) = (1, 1)$ for all $a \in A$.

- (f, g, α) is feasible. Indeed,

$$\int_A f d\mu + c(\alpha) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) + \left(\tfrac{1}{2}, \tfrac{1}{2}\right) = (1, 1) = \int_A g d\mu.$$

- (f, g, α) does not belong to the σ-core, since it is blocked by the coalition $T = \left[0, \frac{1}{2}\right]$ via the allocation (f', g, β) with $f'(a) = (0.7, 0.7)$.
  Indeed, it results that

$$u_a(f'(a), \beta) = 0.7 > 0.5 = u_a(f(a), \alpha), \quad \text{for each } a \in T$$

  and

$$\int_T f' d\mu + \sigma(T, \beta) = \tfrac{1}{2}(0.7, 0.7) + \tfrac{0.4}{2}\left(\tfrac{3}{4}, \tfrac{3}{4}\right) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \int_T g d\mu$$

- (f, g, α) cannot be blocked by a coalition of measure $\frac{7}{8}$.
  Indeed, assume by contradiction that there exists a coalition S which σ-blocks (f, g, α) by means of a suitable allocation (f', g', k) with $k \in \mathcal{Z} = \{\alpha, \beta\}$ and such that $\mu(S) = \frac{7}{8}$.
  This implies particularly that

  (a) $u_a(f'(a), k) = f'_1(a) > \frac{1}{2} = f_1(a) = u_a(f(a), \alpha)$, for each $a \in S$, and
  (b) $\int_S f' \, d\mu + \sigma(S, k) = \int_S g \, d\mu$.

  Denote by $S_1 = S \cap \left[0, \frac{1}{2}\right]$, $S_2 = S \cap \left(\frac{1}{2}, \frac{3}{4}\right]$ and $S_3 = S \cap \left(\frac{3}{4}, 1\right]$. Then, from $\sum_{i=1}^3 \mu(S_i) = \mu(S) = \frac{7}{8}$, $\mu(S_1) \leqslant \frac{1}{2}$ and $\mu(S_i) \leqslant \frac{1}{4}$ for $i = 2, 3$, it follows that $\mu(S_1) \geqslant \frac{3}{8}$ and $\mu(S_i) \geqslant \frac{1}{8}$ for $i = 2, 3$.
  If $k = \beta$, from (a) and (b) it follows that

$$\tfrac{1}{2}\mu(S) + 0.4\tfrac{3}{4}\mu(S_1) + 2.4\tfrac{3}{4}\mu(S_2) + 0.8\tfrac{3}{4}\mu(S_3) < \int_S f'_1 d\mu + \sigma(S, \beta) = \int_S g'_1 d\mu \leqslant \mu(S).$$

Hence, $0.4\frac{3}{4}\mu(S_1) + 2.4\frac{3}{4}\mu(S_2) + 0.8\frac{3}{4}\mu(S_3) < \frac{1}{2}\mu(S) = \frac{1}{2}[\mu(S_1) + \mu(S_2) + \mu(S_3)]$ implying that $0.2\mu(S_1) - 1.3\mu(S_2) - 0.1\mu(S_3) > 0$. Therefore,

$$\tfrac{1}{2} \geqslant \mu(S_1) > \tfrac{1}{0.2}[1.3\mu(S_2) + 0.1\mu(S_3)] \geqslant \tfrac{1}{0.2}[1.3\tfrac{1}{8} + 0.1\tfrac{1}{8}] = \tfrac{1.4}{0.2}\tfrac{1}{8} = \tfrac{7}{8},$$

which is a contradiction.

If $k = \alpha$, from (a) and (b) the following contradiction arises

$$\mu(S) = \frac{1}{2}\mu(S) + \frac{1}{2}\mu(S) < \int_S f_1' d\mu + \sigma(S,\alpha) = \int_S g_1' d\mu \leqslant \mu(S).$$

Note that, by assumption, the feasible allocation (f, g, α) does not satisfy the Definition 3.3 because, given the alternative collective good configuration β, there does not exist any feasible allocation (f', g', β) such that $u_a(f'(a), \beta) \geqslant u_a(f(a), \alpha)$ for almost all $a \in A$. Indeed, assume to the contrary that there exists (f', g', β) feasible such that $u_a(f'(a), \beta) \geqslant u_a(f(a), \alpha)$ for almost all $a \in A$. Then, $f_1'(a) \geqslant f_1(a) = \frac{1}{2}$ for almost all $a \in A$ and hence $\int_A f_1' d\mu \geqslant \frac{1}{2}$. On the other hand, feasibility implies that $\int_A f_1' d\mu + \frac{3}{4} = \int_A g_1' d\mu \leqslant 1$, that is $\int_A f_1' d\mu \leqslant \frac{1}{4}$. Thus, the following contradiction arises

$$\tfrac{1}{2} \leqslant \int_A f_1' d\mu \leqslant \tfrac{1}{4}.$$

This contradiction shows indeed that (f, g, α) cannot be blocked by a coalition of measure $\frac{7}{8}$. ♦

The example shows that Vind's result can only be extended for particular allocations and under certain regularity conditions. We introduce a particular class of allocations to which we can extend Vind's theorem. In particular, the following property is needed, which becomes trivial in the context of an economy with a single collective good configuration, i.e., such that $\#\mathcal{Z} = 1$.

Definition 3.3 *Let (f, g, z) be a feasible allocation in an economy $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$. We say that (f, g, z) is a* ***Vind allocation*** *if for any collective good configuration $z' \in \mathcal{Z}$, there exists a feasible allocation (f', g', z') such that $u_a(f'(a), z') \geqslant u_a(f(a), z)$ for almost all $a \in A$.*

Definition 3.3 introduces a rather strong requirement that essentially imposes the interchangeability of collective good configurations. Indeed, it in effect requires that *all* collective good configurations are socially optimal in the sense that they are part of a Pareto optimal allocation. This is subject of the next proposition.

On the other hand, it should also be clear that all feasible allocations are Vind allocations if the economy only has private goods in the sense that the set of collective good configurations is a singleton, i.e., such that $\#\mathcal{Z} = 1$.

Proposition 3.4 *Consider an economy $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ such that there exists a Pareto optimal allocation (f^*, g^*, z^*) that is a Vind allocation in the sense of Definition 3.3. Then for every collective good configuration $z \in \mathcal{Z}$ there exists a Pareto optimal allocation (f, g, z) such that $u_a(f(a), z) = u_a(f^*(a), z^*)$ for almost all $a \in A$.*

Proof. Let $z \in \mathcal{Z}$. Applying the property stated in Definition 3.3 to the Pareto optimal allocation (f^*, g^*, z^*) and collective good configuration $z \in \mathcal{Z}$, there exists a pair (f, g) such that (f, g, z) is feasible and $u_a(f(a), z) \geqslant u_a(f^*(a), z^*)$ for almost all $a \in A$. Pareto optimality of (f^*, g^*, z^*) implies that (f, g, z) is Pareto optimal as well. Moreover, since (f^*, g^*, z^*) is Pareto optimal, it has to hold that in fact $u_a(f(a), z) = u_a(f^*(a), z^*)$ for almost all $a \in A$, showing the assertion. ■

Proposition 3.4 asserts that, if there exists a Pareto optimal Vind allocations, all collective good configurations are equivalent in their Pareto ranking. Hence, all collective good configurations are non-trivial and are part of Pareto optimal allocations. Thus, the existence of a Vind allocation as constructed in Definition 3.3 therefore excludes collective good configurations that are Pareto inferior in the broadest sense.

Next we state our main result that generalises the theorem of Vind (1972) to our setting.

Theorem 3.5 *Consider a continuum economy $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ and some contribution measure $\sigma \colon \Sigma \times \mathcal{Z} \to \mathbb{R}^{\ell}_{+}$. Furthermore, let $(f, g, z) \notin C_{\sigma}(\mathbb{E})$ be a feasible non-core allocation such that there exists some non-negligible coalition $S \in \Sigma$ which can improve upon (f, g, z).*
If (f, g, z) is a Vind allocation in the sense of Definition 3.3 and $u_a(\cdot, z)$ satisfies Assumption 2.3 for almost all $a \in A$ and all $z \in \mathcal{Z}$, then for every $\delta \in (\mu(S), 1)$ there exists a coalition $S_{\delta} \in \Sigma$ with $\mu(S_{\delta}) = \delta$ that can improve upon (f, g, z).

Proof. Take $\mu(S) < \delta < 1$. Since the Vind allocation (f, g, z) can be improved upon by S, there exists a triple (f', g', z') such that

(1) $u_a(f'(a), z') > u_a(f(a), z)$ for almost all $a \in S$,

(2) $\displaystyle\int_S f' d\mu + \sigma(S, z') = \int_S g' d\mu.$

By the essentiality condition (Assumption 2.3(b)) it now holds that $\int_S f' \, d\mu > 0$.
By continuity, there now exist $\varepsilon \in (0, 1)$ and $S' \subseteq S$ with $\mu(S') > 0$ such that $u_a(\varepsilon f'(a), z') > u_a(f(a), z)$ for almost every $a \in S'$. Define $\tilde{f}(a) = \varepsilon f'(a)$ if $a \in S'$ and $\tilde{f}(a) = f'(a)$ if $a \in S \setminus S'$. Now, $\int_S f' \, d\mu > 0$ implies that $\int_S \tilde{f} \, d\mu > 0$. Therefore, there exists $y \in \mathbb{R}^{\ell}_{+} \setminus \{0\}$ such that

(1) $u_a(\tilde{f}(a), z') > u_a(f(a), z)$ for almost all $a \in S$,

(2) $\displaystyle\int_S \tilde{f} d\mu + \sigma(S, z') = \int_S g' d\mu - y.$

From Definition 3.3 for (f, g, z), there exists an allocation (f'', g'', z') such that

(1) $u_a(f''(a), z') \geqslant u_a(f(a), z)$ for almost all $a \in A$,

(2) $\displaystyle\int_A f'' d\mu + c(z') = \int_A g'' d\mu.$

Now by Lyapunov's Convexity Theorem applied to

$$\nu(T) = \left(\int_T (f'' - g'') \, d\mu + \sigma(T, z') \, , \, \mu(T) \right) \in \mathbb{R}^{\ell+1} \quad \text{with } A \setminus S \tag{5}$$

for any $\varepsilon \in (0,1)$ there exists $B \subseteq A \setminus S$ such that $\nu(B) = (1-\varepsilon)\nu(A \setminus S)$.

Note that, since $\int_S \mathcal{P}(\cdot, z')\, d\mu$ is a convex set, it follows that

$$\varepsilon \int_S g'\, d\mu + (1-\varepsilon) \int_S g''\, d\mu \in \int_S \mathcal{P}(\cdot, z')\, d\mu.$$

Hence, there exists some integrable selection $\hat{g}$ of $\mathcal{P}(\cdot, z')$ such that

$$\int_S \hat{g}\, d\mu = \varepsilon \int_S g'\, d\mu + (1-\varepsilon) \int_S g''\, d\mu \in \int_S \mathcal{P}(\cdot, z')\, d\mu. \tag{6}$$

Now define $(\bar{f}, \bar{g}, z')$ with

$$\bar{f}(a) = \begin{cases} \varepsilon \tilde{f}(a) + (1-\varepsilon) f''(a) & \text{for } a \in S \\ f''(a) + \frac{\varepsilon y}{\mu(B)} & \text{for } a \in B \end{cases} \quad \text{and} \quad \bar{g}(a) = \begin{cases} \hat{g}(a) & \text{for } a \in S \\ g''(a) & \text{for } a \in B \end{cases}$$

Then, since for all $a \in A$ the function $u_a(\cdot, z')$ is strictly quasi-concave and strictly monotone, it follows that $u_a(\bar{f}(a), z') > u_a(f(a), z)$ for almost all $a \in S \cup B$. Moreover, $\bar{g}(a) \in \mathcal{P}(a, z')$ for almost all $a \in S \cup B$ and

$$\begin{aligned}
\int_{S \cup B} \bar{f}\, d\mu + \sigma(S \cup B, z') = \\
&= \varepsilon \int_S \tilde{f}\, d\mu + (1-\varepsilon) \int_S f''\, d\mu + \int_B f''\, d\mu + \varepsilon y + \sigma(S, z') + \sigma(B, z') = \\
&= \varepsilon \int_S g'\, d\mu + (1-\varepsilon)\sigma(S, z') + (1-\varepsilon) \int_S f''\, d\mu + \\
&\qquad + (1-\varepsilon) \int_{A \setminus S} f''\, d\mu + (1-\varepsilon)\sigma(A \setminus S, z') = \\
&= (1-\varepsilon) \int_A f''\, d\mu + (1-\varepsilon)c(z') + \varepsilon \int_S g'\, d\mu = \\
&= (1-\varepsilon) \int_A g''\, d\mu + \varepsilon \int_S g'\, d\mu = (1-\varepsilon) \int_S g''\, d\mu + (1-\varepsilon) \int_{A \setminus S} g''\, d\mu + \varepsilon \int_S g'\, d\mu = \\
&= \int_S \hat{g}\, d\mu + \int_B g''\, d\mu = \int_{S \cup B} \bar{g}\, d\mu.
\end{aligned}$$

Then, (f, g, z) is improved upon by $S \cup B$ with $(\bar{f}, \bar{g}, z')$, showing the assertion once $\varepsilon = \frac{1-\delta}{\mu(A \setminus S)}$ since, in this case, $\mu(S \cup B) = \delta$. ■

3.2 Structured σ-cores

Gilles (2018a) developed a formalisation of the intuition that in an economy with an endogenous, fluid social division of labor, blocking coalitions can be organised internally through a social division of labor. A social division of labor emerges through the specialisation of consumer-producers in the production of a single output. This is captured through the auxiliary notion of a "full specialisation" production plan.

Definition 3.6 *Let $k \in \{1, \ldots, \ell\}$ be some commodity. A production plan $y^k \in \mathbb{R}^\ell$ is a* ***full special-***

isation production plan *for commodity* k *if there exists some positive output quantity* $Q^k > 0$ *and some input vector* $t^k \in \mathbb{R}^\ell_+$, *with* $t^k_k = 0$, *such that* $y^k = Q^k e_k - t^k$, *where* e_k *is the* k*-th unit vector in the* ℓ*-dimensional Euclidean space* $\mathbb{R}^\ell$.

A full specialisation production plan can be interpreted as a mathematical representation of a "profession". Indeed, at y^k an agent would be fully specialised in the production of commodity k. At this production plan, the agent only generates quantities of commodity k as an output, while all other commodities are only used as inputs into the production of that particular output.

The existence of full specialisation production plans in the production set of almost every consumer-producer in the economy is a pre-requisite for that economy to be organisable through a social division of labor. In such a social division of labor every agent is fully specialised in her production. This has been explored in Gilles (2019b).

The next definition builds on this notion by introducing the property that an agent is maximally productive in a fully specialised state. The next property states that there are ℓ full specialisation production plans that form the corner points of that agent's production set. This fully represents the seminal ideas of Smith (1776) on productive specialisation—as set out in his discussion of the pin factory—and the notion of "infra-marginal analysis" developed in Yang (2001).

Definition 3.7 (Gilles, 2019b)
A regular production set $\mathcal{P} \subset \mathbb{R}^\ell$ ***satisfies Increasing Returns to Specialisation*** *(IRSpec) if there exists a full specialisation production plan for each of the* ℓ *commodities, represented by the set* $Q = \{y^1, \ldots, y^\ell\}$ *with* $y^k = Q^k e_k - t^k$ *for every* $k \in \{1, \ldots, \ell\}$, *such that* [12]

$$Q \subset \mathcal{P} \subset \operatorname{Conv} Q - \mathbb{R}^\ell_+ \tag{7}$$

An economy $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ ***exhibits IRSpec*** *if for every agent* $a \in A$ *and every collective good configuration* $z \in \mathcal{Z}$ *the production set* $\mathcal{P}_a(z)$ *satisfies Increasing Returns to Specialisation. We denote the set of corresponding full specialisation production plans by* $Q_a(z) \subset \mathcal{P}_a(z)$.

In an economy that exhibits IRSpec, every agent can specialise fully in the production of any of the ℓ commodities that is a corner point of her production set. Hence, these full specialisation plans represent that agent's most productive state. It can be shown that income over $\mathcal{P}_a(z)$ is maximised in a full specialisation production plan selected from $Q_a(z)$—known as the *specialisation theorem* (Gilles, 2019b, Theorem 2).

Naturally, in an economy that exhibits IRSpec, agents should be guided into a full specialisation production plan. This selection forms a representation of a social division of labor in which each agent assumes a certain profession.

Definition 3.8 *Let* $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ *be an economy that exhibits IRSpec and let* $S \in \Sigma$ *be a coalition with* $\mu(S) > 0$. *Let* $z \in \mathcal{Z}$ *be any collective good configuration. Then an assignment of production plans* $g \colon S \to \mathbb{R}^\ell$ ***structures*** S *at* z *if* $g(a) \in Q_a(z)$ *for almost every* $a \in S$.

[12] Here, we use the notational convention that $\operatorname{Conv} S = \{\lambda x + (1 - \lambda) y \mid x, y \in S \text{ and } 0 \leqslant \lambda \leqslant 1\}$ denotes the convex hull of the set S.

This definition introduces the notion of a social division of labor in the Edgeworthian trade processes underlying the Core allocations. Indeed, if a production plan structures a coalition, it essentially imposes a proper social division of labor on a blocking coalition by assigning professions to each of them.

Formally, if the production plan g structures S at $z \in \mathcal{Z}$, then for any commodity $k \in \{1, \dots, \ell\}$ we determine the set of agents that are fully specialised in the production of k as

$$S_k(z) = \{a \in S : g_k(a) > 0\};$$
$$K = \{k \in \{1, \dots, \ell\} : \mu(S_k(z)) > 0\}.$$

Now the collection $(S_k(z))_{k \in K}$ forms a partition of S, which represents the internal social division of labor within coalition S corresponding to the production plan assignment g.

Definition 3.9 *Let* $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ *be an economy that exhibits IRSpec.*
Given a contribution measure $\sigma \colon \Sigma \times \mathcal{Z} \to \mathbb{R}^\ell_+$*, a feasible allocation* (f, g, z) *is a* ***structured*** σ***-core allocation*** *if there does not exist a coalition* $S \in \Sigma$ *with* $\mu(S) > 0$ *and an alternative allocation* (f', g', z') *such that*

(i) $u_a(f'(a), z') > u_a(f(a), z)$ *for almost all* $a \in S$*;*

(ii) $\int_S f' d\mu + \sigma(S, z') = \int_S g' d\mu$*;*

(iii) g' *structures* S*.*

We denote by $C^S_\sigma(\mathbb{E})$ *the structured* σ*-core of the economy* $\mathbb{E}$*.*

A coalition that organises itself through an internal social division of labour based on full specialisation production plans only can be interpreted as an alliance between ℓ different professional guilds, which members specialise in the production of only one good. The structured core now collects exactly those allocations that cannot be blocked through such alliances. It means that trade only occurs between fully specialised economic agents in the prevailing social division of labour. This imposes restrictions on blocking, typically enlarging the core considerably, i.e., for a given contribution measure σ it typically holds that $C_\sigma(\mathbb{E}) \subsetneq C^S_\sigma(\mathbb{E})$.

The next theorem asserts that there is an equivalence between standard σ-core and structured σ-core allocations in continuum economies in which the production technologies satisfy the increasing returns to specialisation (IRSpec) property, subject to standard regularity conditions on the preferences of the economic agents.

Theorem 3.10 (Structured Core Equivalence)
Let $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ *be a continuum economy that exhibits IRSpec and such that for all* $a \in A$ *and* $z \in \mathcal{Z}$*,* $\mathcal{P}(a, z)$ *satisfies Assumption 2.4. Furthermore, let* $\sigma \colon \Sigma \times \mathcal{Z} \to \mathbb{R}^\ell_+$ *be a contribution measure.*
Then, if all utility functions u_a*,* $a \in A$*, are strictly monotone, every non-*σ*-core allocation can be improved upon by a non-negligible coalition* $S \in \Sigma$ *with an internal social division of labor, i.e.,*

$$C^S_\sigma(\mathbb{E}) = C_\sigma(\mathbb{E}). \tag{8}$$

Proof of Theorem 3.10: Let $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ be some continuum economy, in which (A, Σ, μ) is a complete atomless probability space for which the assumptions introduced in Assumption 2.4 hold and all utility functions u_a, $a \in A$, are strictly monotone.
As stated in the assertion, we also suppose that all agents in $\mathbb{E}$ have productive abilities that are subject to Increasing Returns to Specialisation (IRSpec), i.e., for all $z \in \mathcal{Z}$ and for almost every agent $a \in A$: $Q(a,z) \subset \mathcal{P}_a(z) \subset \operatorname{Conv} Q(a,z) - \mathbb{R}^\ell_+$, where

$$Q(a,z) = \left\{ y^1(a,z), \dots, y^\ell(a,z) \right\} \tag{9}$$

is the set of full specialisation production plans for a at z.
We introduce some auxiliary notation. In particular, we define for every $a \in A$ and $z \in \mathcal{Z}$:

$$\overline{Q}(a,z) = \operatorname{Conv} Q(a,z) \tag{10}$$

Now we can prove the following assertion:

Claim 3.11 *For any $z \in \mathcal{Z}$, the correspondences $Q(\cdot, z) \colon A \to 2^{\mathbb{R}^\ell}$ and $\overline{Q}(\cdot, z) \colon A \to 2^{\mathbb{R}^\ell}$ have a measurable graph.*

Proof. For any $z \in \mathcal{Z}$, first, as assumed in Definition 2.2 and Assumption 2.4, the correspondence $\mathcal{P}(\cdot, z) \colon A \to 2^{\mathbb{R}^\ell}$ has a measurable graph and is closed-valued. Let $H(a,z)$ be the hyperplane containing each $y(a,z) \in Q(a,z)$. Then $\overline{Q}(\cdot, z) \subseteq H(a,z)$. Let $\mathbf{n}$ be the normal vector to $H(a,z)$, with $\mathbf{n} \gg 0$, and for each $k = \{1, \dots, \ell\}$, $\bar{n}_k = \mathbf{n} + n_k e_k = (n_1, \dots, n_{k-1}, 2n_k, n_{k+1}, \dots, n_\ell) \gg 0$. From the IRSpec property it follows that for every $k \in \{1, \dots, \ell\}$ the maximisation problem

$$\max \phi_k(x) = \bar{n}_k \cdot x \quad \text{such that } x \in \mathcal{P}_a(z)$$

has a unique solution given by $\phi_k(y^k(a,z))$, with $y^k(a,z) \in Q(a,z)$.
Proposition 3 in Hildenbrand (1974, page 60) now implies that $a \mapsto y^k(a,z)$ is a measurable function on the complete probability space (A, Σ, μ). This, in turn, implies that

$$a \mapsto Q(a,z) = \bigcup_{k=1}^{\ell} \left\{ y^k(a,z) \right\}$$

has a measurable graph on (A, Σ, μ).
Finally, the Corollary of Proposition 3 in Hildenbrand (1974, page 60) implies that the correspondence $\overline{Q}(\cdot, z) \colon A \to 2^{\mathbb{R}^\ell}$ that assigns to every $a \in A$ the convex hull $\operatorname{Conv} Q(a,z)$ of the finite set $Q(a,z)$ has a measurable graph, showing Claim 3.11. ■

Returning to the proof of Theorem 3.10, let $(f, g, z) \notin C_\sigma(\mathbb{E})$ be some non-σ-core allocation in $\mathbb{E}$. Hence, there exist some $S \in \Sigma$, with $\mu(S) > 0$, and a coalitional allocation (f', g', z') with $f' \colon S \to \mathbb{R}^\ell_+$ and $g' \colon S \to \mathbb{R}^\ell$ such that

(i) $u_a(f'(a), z') > u_a(f(a), z)$ for every $a \in S$;

(ii) $g'(a) \in \mathcal{P}_a(z')$ for every $a \in S$, and;

(iii) $\int_S f' \, d\mu + \sigma(S, z') = \int_S g' \, d\mu$.

Next, we may define for each $z \in \mathcal{Z}$

$$Q_S(z) = \int_S Q(a, z) \, d\mu(a) \tag{11}$$

We note that, since Q has a measurable graph by Claim 3.11 and the space (A, Σ, μ) is atomless, by Theorem 4 in Hildenbrand (1974, page 64), it follows that

$$Q_S(z) = \int_S \operatorname{Conv} Q(a, z) \, d\mu(a) = \int_S \overline{Q}(a, z) \, d\mu(a) \neq \varnothing \tag{12}$$

is a closed and convex set.[13]

Therefore, since $\int_S g' d\mu \in Q_S(z') - \mathbb{R}^\ell_+$, there exists some integrable selection $g'' \colon S \longrightarrow \mathbb{R}^\ell$ with $g''(\cdot) \in Q(\cdot, z')$ such that

$$\int_S g'' \, d\mu \geqslant \int_S g' \, d\mu.$$

We conclude that S improves upon (f, g, z) through (f', g'', z'). Indeed, $u_a(f'(a), z') > u_a(f(a), z)$ for all $a \in S$, $g''(a) \in Q(a, z')$ for all a, and

$$\int_S f' \, d\mu + \sigma(S, z') \leqslant \int_S g'' \, d\mu,$$

which, by strict monotonicity, can be assumed to be

$$\int_S f' \, d\mu + \sigma(S, z') = \int_S g'' \, d\mu.$$

Hence, $(f, g, z) \notin C^S_\sigma(\mathbb{E})$, showing the assertion of Theorem 3.10. ■

3.3 An extension of Grodal's Theorem

Grodal (1972) showed that non-core allocations can be blocked by coalitions that are composed of at most ℓ groups of neighboring agents. In our setting this result can be reformulated to incorporate the internal structuring of a blocking coalition through a social division of labor. Indeed, such a social division of labor structures a coalition into exactly ℓ professional groups of fully specialised agents. It can be shown—following Grodal's seminal insight—that each of these professional groups has an arbitrarily small diameter for any (pseudo-) metric on the space of consumer-producers A.

Theorem 3.12 *Let $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ be some continuum economy. Next, let (f, g, z) be blocked by a coalition S via the triple (f', g', z') such that $f' \gg 0$ on S and g' structures S.*

For any $k = 1, \dots, \ell$, let d_k be a measurable pseudo-metric defined on $S_k(z')$ so that $S_k(z')$ is separable

[13] Nonemptiness of Q_S follows from the integrably boundedness of the correspondence $\mathcal{P}$ and, therefore, of Q. This implies that all measurable selections in Q are integrable.

in the corresponding topology.

If for almost all $a \in A$, $u_a(\cdot, z')$ is monotone, then for any $\varepsilon \in (0,1)$ and $k = 1,\dots,\ell$ there exists $U_k \subseteq S_k(z')$ such that $\mu(U_k) \leqslant \varepsilon$, $\mathrm{diam}_k(U_k) \leqslant \varepsilon$ and $U = \bigcup_{k=1}^{\ell} U_k$ blocks (f, g, z), where diam_k denotes the diameter operator for pseudo-metric d_k on $S_k(z')$.

Proof. Let (f, g, z) be blocked by a coalition S via the triple (f', g', z') such that g' structures S. For any $k = 1, \dots, \ell$,

$$S_k(z') = \{a \in S \mid g'_k(a) > 0\}.$$

Let $0 < \varepsilon < 1$ be arbitrarily taken. Without loss of generality by Theorem 3.1 we may assume that $\mu(S) \leqslant \varepsilon$.

For each $k = 1, \dots, \ell$, let $\{s_i^k \mid i \in \mathbb{N}\} \subseteq S_k(z')$ be a dense subset in $S_k(z')$ for the pseudo-metric d_k. Then, $S_k(z') = \bigcup_{i\in\mathbb{N}} B^k\left(s_i^k, \frac{\varepsilon}{2}\right)$. Now define,

$$\begin{aligned} F_1^k &= S_k(z') \cap B^k\left(s_1^k; \frac{\varepsilon}{2}\right) \\ F_i^k &= S_k(z') \cap B^k\left(s_i^k; \frac{\varepsilon}{2}\right) \setminus \bigcup_{j=1}^{i-1} F_j^k \quad \text{for all } i > 1 \end{aligned}$$

Let $N_0^k = \{i \in \mathbb{N} \mid \mu(F_i^k) > 0\}$. Then, $(F_i^k)_{i \in N_0^k}$ is a family of disjoint subcoalitions of $S_k(z')$ such that $\mu\left(\bigcup_{i\in N_0^k} F_i^k\right) = \mu(S_k(z'))$.

For every $i \in N_0^k$, define

$$\begin{aligned} x_i^k &:= \int_{F_i^k} f' - g' d\mu + \sigma(F_i^k, z'), \text{ and} \\ C &:= \mathrm{conv}\,\{x_i^k \mid k = 1, \dots, \ell; \text{ and } i \in N_0^k\}. \end{aligned}$$

Since $\int_S f' - g' d\mu + \sigma(S, z') = 0$, it follows that

$$\sum_{k=1}^{\ell} \sum_{i\in N_0^k} x_i^k = 0 \tag{13}$$

Let H be the smallest affine space containing C. From (13) it follows that H is a subspace. We claim now that $0 \in \mathrm{int}_H C$. Indeed, otherwise there exists $p \in H$ with $p \neq 0$ such that $p \cdot x_i^k \geqslant 0$ for all $k = 1, \dots, \ell$ and $i \in N_0^k$. From (13), we have that for all $k = 1, \dots, \ell$ and $i \in N_0^k$,

$$x_i^k = -\sum_{j\in N_0^k\setminus\{i\}} x_j^k - \sum_{k'\neq k} \sum_{i\in N_0^{k'}} x_i^{k'}.$$

Then,

$$p \cdot x_i^k = -\sum_{j\in N_0^k\setminus\{i\}} p \cdot x_j^k - \sum_{k'\neq k} \sum_{i\in N_0^{k'}} p \cdot x_i^{k'} \leqslant 0,$$

implying that $p \cdot x_i^k = 0$ for all $k = 1, \ldots, \ell$ and $i \in N_0^k$. Therefore, $H' = \{x \in H : p \cdot x = 0\}$ is a smaller affine space than H which contains C and this contradicts the definition of H.

Let $\dim H = m \leqslant \ell$. By Caratheodory's theorem, there exist $t_1, t_2, \ldots, t_{m+1} \in C$ such that $\sum_{r=1}^{m+1} \lambda_r t_r = 0$, with $\lambda_r \in [0,1]$ for all r and $\sum_{r=1}^{m+1} \lambda_r = 1$. For any $r = 1, \ldots, m+1$, $t_r = x_{i_r}^{k_r}$ where $i_r \in N_0^{k_r}$.
Now $T = \operatorname{conv}\{t_r \mid r = 1, \ldots, m+1\}$ is a simplex. Since $0 \in T$, there is a boundary point v of T so that $v \leqslant 0$. Thus, $v = \sum_{r=1}^{m} \alpha_r t_r$, with $\alpha \in [0,1]$ for any r, and $\sum_{r=1}^{m} \alpha_r = 1$.
For any $r = 1, \ldots, m$, consider the measure $\nu_r : \Sigma_{|F_{i_r}^{k_r}} \to \mathbb{R}^{\ell+1}$ such that

$$\nu_r(B) = \left(\mu(B), \int_B f' - g' d\mu + \sigma(B, z')\right).$$

By Lyapunov's theorem there exists $U_r \subseteq F_{i_r}^{k_r}$ such that $\nu_r(U_r) = \alpha_r \nu_r(F_{i_r}^{k_r})$, that is

$$\mu(U_r) = \alpha_r \mu(F_{i_r}^{k_r}) \qquad \text{and}$$
$$\int_{U_r} f' - g' d\mu + \sigma(U_r, z') = \alpha_r \int_{F_{i_r}^{k_r}} f' - g' d\mu + \sigma(F_{i_r}^{k_r}, z').$$

Now, define $U = \bigcup_{r=1}^{m} U_r$. Then

$$\mu(U) = \sum_{r=1}^{m} \mu(U_r) = \sum_{r=1}^{m} \alpha_r \mu(F_{i_r}^{k_r}) \leqslant \varepsilon \sum_{r=1}^{m} \alpha_r = \varepsilon,$$

and, a fortiori, $\mu(U_r) \leqslant \varepsilon$ for any $r = 1, \ldots, m$.
Furthermore, since for any r, $U_r \subseteq F_{i_r}^{k_r} \subseteq B^{k_r}\left(s_{i_r}^{k_r}; \frac{\varepsilon}{2}\right)$, it follows that $\operatorname{diam}_k U_r \leqslant \varepsilon$.
Since $U \subseteq S$, it can be concluded that $u_a(f'(a), z') > u_a(f(a), z)$ for almost all $a \in U$. Moreover,

$$\begin{aligned}
\int_U f' - g' d\mu + \sigma(U, z') &= \sum_{r=1}^{m} \left[\int_{U_r} f' - g' d\mu + \sigma(U_r, z')\right] \\
&= \sum_{r=1}^{m} \alpha_r \left[\int_{F_{i_r}^{k_r}} f' - g' d\mu + \sigma(F_{i_r}^{k_r}, z')\right] \\
&= \sum_{r=1}^{m} \alpha_r x_{i_r}^{k_r} = \sum_{r=1}^{m} \alpha_r t_r = v \leqslant 0.
\end{aligned}$$

By monotonicity of u_a, $\tilde{f}(a) = f'(a) - \frac{v}{\mu(U)}$ is such that

$$u_a(\tilde{f}(a), z') \geqslant u_a(f'(a), z') > u_a(f(a), z) \qquad \text{for almost all } a \in U,$$

and

$$\int_U \tilde{f} - g' d\mu + \sigma(U, z') = \sum_{r=1}^{m} \left[\int_{U_r} f' - g' d\mu + \sigma(U_r, z') \right] - \frac{v}{\mu(U)} \sum_{r=1}^{m} \mu(U_r)$$

$$= \sum_{r=1}^{m} \left[\alpha_r \int_{F_{i_r}^{k_r}} f' - g' d\mu + \sigma(F_{i_r}^{k_r}, z') \right] - \frac{v}{\mu(U)} \sum_{r=1}^{m} \alpha_r \mu(F_{i_r}^{k_r})$$

$$= \sum_{r=1}^{m} \alpha_r t_r - \frac{v}{\mu(U)} \mu(U)$$

$$= v - v = 0.$$

Hence, we conclude that $(\tilde{f}, g', z')$ blocks (f, g, z) via U. Furthermore, since $f'(a) \gg 0$ for almost all $a \in S$, by definition $\tilde{f}(a) \gg 0$ for almost all $a \in U \subseteq S$. Hence $\sum_{r=1}^{m} \int_{U_r} g' d\mu = \int_U g' d\mu \gg 0$. This implies that for all $k = 1, \ldots, \ell$ there exists $r = 1 \ldots, m$ such that $k_r = k$. Hence, $m = \ell$, showing the assertion of Theorem 3.12. ■

Theorem 3.12 can be applied to a variety of settings in the context of economies with collective goods that are provided through a social division of labor. This variety is established through the appropriate selection of the pseudo-metrics $(d_1, \ldots, d_\ell)$ to reflect a certain specification of social neighborhood. In the setting of our model there a few natural conceptions of these pseudo-metrics.

Assuming that the economy exhibits IRSpec and the set of collective good configurations $\mathcal{Z}$ is finite, for any agent $a \in A$ we consider the set of full specialisation production plans $Q_a(z)$ for any collective good configuration $z \in \mathcal{Z}$. This allows us to introduce metrics based on these full specialisation production sets:

Productive similarity – Let $a, b \in A$ be two consumer-producers in $\mathbb{E}$. For every private good $k \in \{1, \ldots, \ell\}$ we can introduce the metric δ by

$$\delta_k(a, b) = \frac{1}{\#\mathcal{Z}} \sum_{z \in \mathcal{Z}} \left\| y^k(a, z) - y^k(b, z) \right\| \tag{14}$$

where $y^k(a, z) \in Q_a(z)$ is the k-th full specialisation production plan for agent a under collective good configuration $z \in \mathcal{Z}$. Due to the finiteness of $\mathcal{Z}$, the definition of δ_k constitutes a proper measurable metric on (A, Σ).

Using the assertion of Theorem 3.12 for $d_k = \delta_k$, the application of the metric δ imposes that a blocking coalition consists of professional groups of consumer-producers who are similarly effective in the production of a particular private good. Thus, the blocking coalition is a union of very similar specialised producers.

More generally, using the assertion of Theorem 3.12 for $d_k = \delta = \sum_{k=1}^{\ell} \delta_k$ for all $k = 1, \ldots, \ell$ imposes that blocking coalitions consist of consumer-producers that are virtually equivalent in their productive abilities when specialised.

Professional similarity – In a similar fashion as under productive similarity, we can construct metrics that focus on output levels of products under full specialisation only. Hence, for any two consumer-producers $a, b \in A$ and for every private good $k \in \{1, \ldots, \ell\}$ we can introduce the metric δ' by

$$\delta'_k(a, b) = \frac{1}{\# \mathcal{Z}} \sum_{z \in \mathcal{Z}} \left| y^k_k(a, z) - y^k_k(b, z) \right| \tag{15}$$

where $y^k_k(a, z) > 0$ is the output quantity of good k if agent a is fully specialised in the production of the k-th good under collective good configuration $z \in \mathcal{Z}$. Again, using the finiteness of $\mathcal{Z}$, this defines a proper metric on A that is measurable for Σ.

Using the assertion of Theorem 3.12 for $d_k = \delta'_k$, the application of the metric δ' introduces the requirement that blocking coalitions are formed as a union of fully specialised consumer-producers that achieve very similar output levels in their selected specialisation.

As before, using the assertion of Theorem 3.12 for $d_k = \delta' = \sum_{k=1}^{\ell} \delta'_k$ for all $k = 1, \ldots, \ell$ imposes that blocking coalitions consist of consumer-producers that produce very similar output levels when specialised. Note that the difference with the metric δ is that input quantities are not considered under δ', just output levels.

4 Cost sharing and σ-core equivalence

Core equivalence has been a rather elusive and difficult proposition in the context of economies with collective or public goods. We pursue here the equivalence of the σ-core and cost sharing equilibria as initially developed in Gilles and Diamantaras (1998) and further extended by Graziano and Romaniello (2012) and Basile, Graziano, and Pesce (2016). Here we consider the extension of cost sharing and σ-core equivalence to economies with collective goods, building on the results in Gilles and Diamantaras (1998) and Gilles (2018a).

The main conception of collective goods in the context of the σ-core is that they are assumed to be perfectly scalable and that they can be provided at any level in the economy according to the corresponding contribution measure σ. In an equilibrium all provision decisions are decentralised and are required to be coordinated in the case of collective good provision. This requires, in turn, the consideration of a cost distribution through which such coordination can be established. Cost distributions are the subject of the next definition.

Definition 4.1 *Let $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ be some economy. A **cost distribution** in $\mathbb{E}$ is a function $\varphi \colon A \times \mathcal{Z} \to \mathbb{R}^\ell_+$ such that for all $z \in \mathcal{Z}$, $\varphi(\cdot, z)$ is integrable and $\int \varphi(\cdot, z)\, d\mu = c(z)$.*

For any cost distribution we can now devise a cost sharing equilibrium in which all agents are required to contribute to the provision of the collective goods according to this cost distribution. In an equilibrium there emerge commodity prices at which all agents coordinate their decisions on a certain collective good configuration and there emerges a feasible allocation to support the provision of that collective good configuration.

We emphasise that in the cost share equilibrium concept the commodity prices are explicitly *conjectural* and vary according to which alternative collective good configuration is considered.

Conjectural price systems were introduced by Diamantaras and Gilles (1996) in the context of valuation equilibria.[14]

Definition 4.2 *Let $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ be some economy and let $\varphi\colon A \times \mathcal{Z} \to \mathbb{R}^{\ell}_{+}$ be a cost distribution in $\mathbb{E}$.*

*A feasible allocation (f, g, z) is a **cost share equilibrium allocation** for φ if there exists a (conjectural) price system $p\colon \mathcal{Z} \to \mathbb{R}^{\ell}_{+} \setminus \{0\}$ such that, almost everywhere on A,*

(i) $p(z) \cdot f(a) + p(z) \cdot \varphi(a, z) \leqslant p(z) \cdot g(a)$;

(ii) $(f(a), g(a), z)$ *maximizes* u_a *on*

$$B_a(p, \varphi) = \{ (x, y, z') \mid y \in \mathcal{P}_a(z') \text{ and } p(z') \cdot x + p(z') \cdot \varphi(a, z') \leqslant p(z') \cdot y \} \tag{16}$$

We denote by $CSE_{\varphi}(\mathbb{E})$ the set of cost share equilibria for cost distribution φ.

This particular formulation of cost share equilibrium concept is founded on the equilibrium concept introduced by Gilles (2019b) for economies with an endogenous social division of labor. Note that every agent $a \in A$ maximises her income from production as part of the utility maximisation over the budget set $B_a(p, \varphi)$. Indeed, agent a selects production plan $y \in \mathcal{P}_a(z)$ that maximises the generated income $p(z) \cdot y$ independently of the consumption plan x for that given maximal budgetary income.[15]

The cost share equilibrium concept was developed by Mas-Colell (1980) for economies with a single private commodity only and extended to the case of multiple private commodities by Diamantaras and Gilles (1996). In these contributions, the notion of a *linear* cost share equilibrium was considered, whenever $\varphi(a, z) = \phi(a) c(z)$ for all $a \in A$ and $z \in \mathcal{Z}$, where $\phi\colon A \to \mathbb{R}_{+}$ is a probability density function on (A, Σ, μ). A special case of this is the *egalitarian* cost share equilibrium for the egalitarian cost distribution $\varphi^{e}(a, z) = c(z)$ for all $a \in A$ and $z \in \mathcal{Z}$, being the linear cost share equilibrium for the uniform cost distribution.

There is a natural one-to-one correspondence between cost distributions and contribution measures. More precisely,

- Given a cost distribution $\varphi\colon A \times \mathcal{Z} \to \mathbb{R}^{\ell}_{+}$ there is a unique contribution measure $\sigma_{\varphi}\colon \Sigma \times \mathcal{Z} \to \mathbb{R}^{\ell}_{+}$ corresponding to φ defined for each $z \in \mathcal{Z}$ by

$$\sigma_{\varphi}(S, z) := \int_{S} \varphi(\cdot, z)\, d\mu.$$

- Conversely, given a contribution measure $\sigma = (\sigma_1, \ldots, \sigma_{\ell})\colon \Sigma \times \mathcal{Z} \to \mathbb{R}^{\ell}_{+}$, where each σ_k is absolutely continuous with respect to μ, there exists a unique cost distribution $\varphi^{\sigma} =$

[14] For a discussion of valuation equilibria in the setting of an economy with collective goods and an endogenous social division of labor, we refer also to Gilles, Pesce, and Diamantaras (2019).

[15] The dichotomy of the production and consumption decisions are investigated in depth in Diamantaras and Gilles (2004) and Gilles (2019a,b) for settings with and without transaction costs in economies with private commodities only that are produced through an endogenous social division of labor.

$(\varphi_1^\sigma, \dots, \varphi_\ell^\sigma) \colon A \times \mathcal{Z} \to \mathbb{R}_+^\ell$ corresponding to σ. This is constructed such that each φ_k^σ is defined for any $z \in \mathcal{Z}$ as the k-th partial Radon-Nikodym derivative $\partial \sigma_k / \partial \mu$ of σ_k with respect to μ.

We investigate the conditions under which there is an equivalence of the set of cost share equilibria for φ with the σ-core for appropriately selected cost distributions φ and contribution measures σ. In particular, we confirm that the binary relationship pointed out above indeed supports such an equivalence. We show this equivalence in two steps. First, we show that every cost share equilibrium allocation for φ is in the σ_φ-core.

Proposition 4.3 *Let $\varphi \colon A \times \mathcal{Z} \to \mathbb{R}_+^\ell$ be a cost distribution in the economy $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ and σ_φ be the corresponding contribution measure. Then it holds that $CSE_\varphi(\mathbb{E}) \subset C_{\sigma_\varphi}(\mathbb{E})$.*

Proof. Let (f, g, z) be a cost share equilibrium allocation with respect to the $p : \mathcal{Z} \to \mathbb{R}_+^\ell \setminus \{0\}$ and the contribution cost φ. Assume to the contrary that there are a coalition S, with $\mu(S) > 0$, and a triple (f', g', z') such that $f' : S \to \mathbb{R}_+^\ell$ integrable, $g' : S \to \mathbb{R}^\ell$ integrable with $g(a) \in \mathcal{P}_a(z')$ for all $a \in S$ and

(i) $u_a(f'(a), z') > u_a(f(a), z)$ for almost all $a \in S$;

(ii) $\int_S f' d\mu + \sigma_\varphi(S, z') = \int_S g' d\mu$.

From (i) it follows that $(f'(a), g'(a), z') \notin B_a(p, \varphi)$ for almost all $a \in S$, that is

$$p(z') \cdot f'(a) + p(z') \cdot \varphi(a, z') > p(z') \cdot g'(a) \text{ for almost all } a \in S,$$

and hence

$$p(z') \cdot \int_S f' d\mu + p(z') \cdot \sigma_\varphi(S, z') > p(z') \cdot \int_S g' d\mu,$$

which contradicts (ii) above. ■

We now prove the converse relationship and show that every σ-core allocation can be supported as a cost share equilibrium for the Radon-Nikodym derivative φ^σ. To this end, the following hypotheses are needed.

Assumption 4.4 *Let $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ be an economy. Then we impose the following properties:*

A.1 – Normality: *For every collective good configuration $z \in \mathcal{Z}$, the associated preferences represented by $u_a(\cdot, z)$ are continuous as well as monotone on $\mathbb{R}_+^\ell$, and u measurable in the sense that for any integrable allocation of consumption bundles $f \colon A \to \mathbb{R}_+^\ell$ and for all $z, z' \in \mathcal{Z}$:*

$$\{(a, x) \in A \times \mathbb{R}_+^\ell \mid u_a(x, z') > u_a(f(a), z)\} \in \Sigma \otimes \mathcal{B}(\mathbb{R}_+^\ell).$$

A.2 – Essentiality: *For all collective good configurations $z, z' \in \mathcal{Z}$ and any integrable allocation $f \colon A \to \mathbb{R}_+^\ell$, there exists an integrable allocation $f' \colon A \to \mathbb{R}_+^\ell$ such that for all $a \in A$, $u_a(f'(a), z') > u_a(f(a), z)$.*

A.3 – Irreducibility: *For any allocation (f, g, z), any collective good configuration $z' \in \mathcal{Z}$, and any measurable partitioning $\{A_1, A_2\}$ of the population A with $\mu(A_i) > 0$, for $i = 1, 2$, there exists a triple (f', g', z'), with $f' : A \to \mathbb{R}^\ell_+$ integrable and $g' : A_2 \to \mathbb{R}^\ell$ integrable with $g'(a) \in \mathcal{P}_a(z')$ for all $a \in A_2$ such that*

(i) $u_a(f'(a), z') > u_a(f(a), z)$ *for almost all* $a \in A_1$;

(ii) $\int_{A_1} f' d\mu + c(z') \leqslant \int_{A_2} g' d\mu - \int_{A_2} f' d\mu$.

A.4 – Boundedness of production: *For any $a \in A$ and $z \in \mathcal{Z}$, $\mathcal{P}_a(z)$ is regular, it satisfies Assumption 2.4 and there exists an integrable selection g of $\mathcal{P}(\cdot, z)$ such that $c(z) \ll \int_A g d\mu$.*

All of the hypotheses introduced above are well known from the literature on general equilibrium in economies with collective goods. Monotonicity in consumption of private commodities in its various forms (A.1) is a standard, normal hypothesis that guarantees that budgets are exhausted in equilibrium.

Essentiality (A.2) was introduced by Mas-Colell (1980) and imposes that private commodities have primacy over collective goods in the sense that any change in collective good provision can be compensated through the allocation of a different bundle of private commodities. Closely related to essentiality is the irreducibility hypothesis (A.3) that any coalition can compensate the other agents (outside the coalition) if they impose a different collective good configuration on the economy. This property was initially introduced into the literature on general equilibrium in economies with private goods only by Gale (1957), while Graziano (2007) extended its application to economies with collective goods.

Finally, boundedness of productive abilities (A.4) has been considered in the context of the social division of labor by Gilles (2019b). It is a critical property that guarantees that a meaningful social division of labor can emerge if the economy exhibits IRSpec.

The next result states that the reverse relationship between cost share equilibria and the σ-core indeed holds under the hypotheses introduced in Assumption 4.4 if the cost distribution is selected to be the Radon-Nikodym derivative of the contribution measure.

Theorem 4.5 *Let $E = \langle (A, \Sigma, \mu), \mathcal{Z}, u, \mathcal{P} \rangle$ be a continuum economy and let $\sigma : \Sigma \times \mathcal{Z} \to \mathbb{R}^\ell_+$ be a contribution measure. Now, let $\varphi^\sigma = \partial\sigma / \partial\mu$ be the corresponding Radon-Nikodym cost distribution. Then under Assumption 4.4,*

$$C_\sigma(\mathbb{E}) = CSE_{\varphi^\sigma}(\mathbb{E}). \tag{17}$$

Proof of Theorem 4.5: Let (f, g, z) be a σ-core allocation in $\mathbb{E}$. Define for each $a \in A$ and each collective good confiuguration $z' \in \mathcal{Z}$ the following sets:

$$\begin{aligned}
\Psi(a, z') &:= \{x \in \mathbb{R}^\ell_+ \ : \ u_a(x, z') > u_a(f(a), z)\} \\
F(a, z') &:= \{\Psi(a, z') - \mathcal{P}_a(z')\} \cup \{-\varphi^\sigma(a, z')\} \\
F(z') &:= \int F(\cdot, z') d\mu + c(z').
\end{aligned}$$

Note that $\Psi(a,z) \neq \emptyset$ due to hypothesis 4.4 (A.2). Furthermore, from assumption 4.4 ($A.1, A.2$ and $A.4$) it follows that $F(z')$ is well defined.
We proceed the proof of the assertion now through a series of claims.

Claim 4.6 *Let $z' \in \mathcal{Z}$ be an arbitrary collective good configuration. Then $0 \in F(z')$.*

Proof. Note that $-\varphi^\sigma(a,z') \in F(a,z')$ for almost all $a \in A$, thus $\int_A -\varphi^\sigma(\cdot,z')d\mu \in \int_A F(\cdot,z')d\mu$ and hence $0 = -c(z') + c(z') = -\sigma(A,z') + c(z') = -\int_A \varphi^\sigma(\cdot,z')d\mu + c(z') \in F(z')$. ■

Claim 4.7 *Let $z' \in \mathcal{Z}$ be an arbitrary collective good configuration. Then $F(z') \cap \operatorname{int}(\mathbb{R}^\ell_-) = \emptyset$.*

Proof. If to the contrary there is some $t \in F(z')$ with $t \ll 0$, then there exists an integrable selection $h(\cdot,z')$ of $F(a,z')$ such that $t = \int h(\cdot,z')d\mu + c(z') \ll 0$. Let $S = \{a \in A : h(a,z') \neq -\varphi^\sigma(a,z')\}$. S has positive measure. Indeed if, on the contrary, $\mu(S) = 0$, then

$$t = \int_A h(\cdot,z')d\mu + c(z') = -\int_A \varphi^\sigma(\cdot,z')d\mu + c(z') = 0,$$

which is a contradiction. Hence, $\mu(S) > 0$. For any $a \in S$, $h \in \Psi(a,z') - \mathcal{P}(a,z')$, then there exist two integrable selections f' and g' respectively of $\Psi(\cdot,z')$ and $\mathcal{P}(\cdot,z')$ such that $u_a(f'(a),z') > u_a(f(a),z)$ for almost all $a \in S$ and $\int_S h d\mu = \int_S f'd\mu - \int_S g'd\mu$. Furthermore,

$$\begin{aligned} t &= \int_A h d\mu + c(z') = \int_S f'd\mu - \int_S g'd\mu - \int_{A\setminus S} \varphi^\sigma(\cdot,z')d\mu + c(z') = \\ &= \int_S f'd\mu - \int_S g'd\mu - \sigma(A\setminus S,z') + \sigma(A,z') = \\ &= \int_S f'd\mu - \int_S g'd\mu + \sigma(S,z') \ll 0. \end{aligned}$$

This means that S σ-improves upon (f,g,z) via (f',g',z'), which is a contradiction. Hence, $F(z') \cap \operatorname{int}(\mathbb{R}^\ell_-) = \emptyset$. ■

Note that $F(z')$ is convex for any $z' \in \mathcal{Z}$. We can apply Minkowski's separating hyperplane theorem, which ensures for each $z' \in \mathcal{Z}$ the existence of $p(z') \in \mathbb{R}^\ell_+ \setminus \{0\}$ such that $p(z') \cdot t \geqslant 0$ for any $t \in F(z')$.

Claim 4.8 *For almost all $a \in A$, if there exists some (x',y',z') such that $u_a(x',z') > u_a(f(a),z)$ and $y' \in \mathcal{P}_a(z')$, then*

$$p(z') \cdot x' + p(z') \cdot \varphi^\sigma(a,z') \geqslant p(z') \cdot y'. \tag{18}$$

Proof. Since $p(z') \cdot t \geqslant 0$ for any $t \in F(z')$, $\inf p(z') \cdot F(z') \geqslant 0$, and from claim 4.6 it follows that

$$\inf p(z') \cdot F(z') = 0 \qquad \text{for all } z' \in \mathcal{Z}.$$

From Proposition 6 of (Hildenbrand, 1974, page 63), applied to $F(\cdot,z')$ it follows that

$$0 = \inf p(z') \cdot F(z') = \inf p(z') \cdot \int_A F(\cdot,z')d\mu + p(z') \cdot c(z') = \int_A \inf p(z') \cdot F(\cdot,z')d\mu + p(z') \cdot c(z'). \tag{19}$$

Since for all $a \in A$, $-\varphi^\sigma(a,z') \in F(a,z')$, $\inf p(z') \cdot F(a,z') + p(z') \cdot \varphi^\sigma(a,z') \leqslant 0$, and by (19),

$$\inf p(z') \cdot F(a,z') + p(z') \cdot \varphi^\sigma(a,z') = 0 \qquad \text{for almost all } a \in A,$$

that is

$$\inf p(z') \cdot F(a,z') = -p(z') \cdot \varphi^\sigma(a,z') \qquad \text{for almost all } a \in A. \tag{20}$$

Therefore, for almost all $a \in A$, if (x', y', z') is such that $x' - y' \in F(a,z')$, that is $u_a(x',z') > u_a(f(a),z)$, and $y' \in \mathcal{P}_a(z')$ then by (20)

$$p(z') \cdot (x' - y') \geqslant \inf p(z') \cdot F(a,z') = -p(z') \cdot \varphi^\sigma(a,z'),$$

implying that $p(z') \cdot x' + p(z') \cdot \varphi^\sigma(a,z') \geqslant p(z') \cdot y'$. ■

Claim 4.9 *For almost all $a \in A$, $p(z) \cdot f(a) + p(z) \cdot \varphi^\sigma(a,z) = p(z) \cdot g(a)$.*

Proof. From Claim 4.8 and the continuity of $u_a(\cdot,z)$ it follows that for all $a \in A$ $p(z) \cdot f(a) + p(z) \cdot \varphi^\sigma(a,z) \geqslant p(z) \cdot g(a)$. Assume to the contrary that for some coalition S, with $\mu(S) > 0$, $p(z) \cdot f(a) + p(z) \cdot \varphi^\sigma(a,z) > p(z) \cdot g(a)$ for almost all $a \in S$. Then,

$$p(z) \cdot \int_A f d\mu + p(z) \cdot \int_A \varphi^\sigma(\cdot,z) d\mu > p(z) \cdot \int_A g d\mu,$$

that is

$$p(z) \cdot \int_A f d\mu + p(z) \cdot c(z) > p(z) \cdot \int_A g d\mu,$$

which contradicts the feasibility of (f,g,z). ■

Claim 4.10 *For all $z' \in \mathcal{Z}$ and for almost all $a \in A$,*

$$p(z') \cdot \varphi^\sigma(a,z') < \sup p(z') \cdot \mathcal{P}_a(z'). \tag{21}$$

Proof. For each $z' \in \mathcal{Z}$ define the sets

$$\begin{aligned} A_1(z') &= \{a \in A \,:\, p(z') \cdot \varphi^\sigma(a,z') < \sup p(z') \cdot \mathcal{P}_a(z')\} \\ A_2(z') &= A \setminus A_1(z'). \end{aligned}$$

Notice that $\mu(A_1(z')) > 0$ for all $z' \in \mathcal{Z}$, otherwise if for some z', $\mu(A_1(z')) = 0$, then by A.4, there exists $g' : A \to \mathbb{R}^\ell$ such that $g'(\cdot) \in \mathcal{P}(\cdot, z')$ and

$$\int_A p(z') \cdot g' d\mu > p(z') \cdot c(z') = \int_A p(z') \cdot \varphi^\sigma(\cdot,z')\, d\mu \geqslant \int_A \sup p(z') \cdot \mathcal{P}(\cdot,z')\, d\mu,$$

which is a contradiction. We now show that $\mu(A_2(z')) = 0$ for all $z' \in \mathcal{Z}$. Assume to the contrary that for some z', $\mu(A_2(z')) > 0$. Then, by A.3 there exist $f' : A \to \mathbb{R}^\ell_+$ integrable and $g' : A_2(z) \to \mathbb{R}^\ell$

integrable such that

(i) $u_a(f'(a), z') > u_a(f(a), z)$ for almost all $a \in A_1(z')$;

(ii) $\int_{A_1(z')} f' d\mu + c(z') \leqslant \int_{A_2(z')} g' d\mu - \int_{A_2(z')} f' d\mu$.

From (i) and Claim 4.8 it follows that for almost all $a \in A_1(z')$ and all $\tilde{g}(a) \in \mathcal{P}(a, z')$,

$$p(z') \cdot f'(a) + p(z') \cdot \varphi^\sigma(a, z') \geqslant p(z') \cdot \tilde{g}(a),$$

and hence

$$p(z') \cdot f'(a) + p(z') \cdot \varphi^\sigma(a, z') \geqslant \sup p(z') \cdot \mathcal{P}_a(z').$$

Therefore,

$$\begin{aligned}
0 &\leqslant \int_{A_1(z')} p(z') \cdot \varphi^\sigma(\cdot, z') d\mu < \int_{A_1(z')} \sup p(z') \cdot \mathcal{P}(\cdot, z') d\mu \leqslant \\
&\leqslant \int_{A_1(z')} p(z') \cdot f' d\mu + \int_{A_1(z')} p(z') \cdot \varphi^\sigma(\cdot, z') d\mu = \\
&= \int_{A_1(z')} p(z') \cdot f' d\mu + p(z') \cdot c(z') - \int_{A_2(z')} p(z') \cdot \varphi^\sigma(\cdot, z') d\mu \leq \\
&\leqslant \int_{A_2(z')} p(z') \cdot g' d\mu - \int_{A_2(z')} p(z') \cdot f' d\mu - \int_{A_2(z')} \sup p(z') \cdot \mathcal{P}(\cdot, z') d\mu \leq \\
&\leqslant -\int_{A_2(z')} p(z') \cdot f' d\mu \leqslant 0,
\end{aligned}$$

which is a contradiction. This means that for any $z' \in \mathcal{Z}$, $\mu(A_2(z')) = 0$ and hence almost everywhere on A, $p(z') \cdot \varphi^\sigma(a, z') < \sup p(z') \cdot \mathcal{P}_a(z')$. ■

Claim 4.11 *For almost all $a \in A$, if there exists some (x', y', z') such that $u_a(x', z') > u_a(f(a), z)$ and $y' \in \mathcal{P}_a(z')$, then $p(z') \cdot x' + p(z') \cdot \varphi^\sigma(a, z') > p(z') \cdot y'$.*

Proof. Assume to the contrary the existence of a coalition S, with $\mu(S) > 0$, such that for all $a \in S$ there exists (x', y', z') such that $y' \in \mathcal{P}(a, z')$,

(i) $u_a(x', z') > u_a(f(a), z)$, and

(ii) $p(z') \cdot x' + p(z') \cdot \varphi^\sigma(a, z') = p(z') \cdot y'$.

From Claim 4.10, there exists $\tilde{y} \in \mathcal{P}(a, z')$ such that

$$p(z') \cdot \varphi^\sigma(a, z') < p(z') \cdot \tilde{y} < \sup p(z') \cdot \mathcal{P}(a, z').$$

Consider the triple $(x', \tilde{y}, z')$. By Claim 4.8 it follows that

$$p(z') \cdot \tilde{y} \leqslant p(z') \cdot x' + p(z') \cdot \varphi^\sigma(a, z') < p(z') \cdot x' + p(z') \cdot \tilde{y},$$

and hence $p(z') \cdot x' > 0$. From (i) above and the continuity of $u_a(\cdot, z')$, there exists $\epsilon \in (0,1)$ such that $u_a(\epsilon x', z') > u_a(f(a), z)$. Then, by claim 4.8, (ii) above it follows that

$$\begin{aligned} p(z') \cdot y' &\leqslant p(z') \cdot \epsilon x' + p(z') \cdot \varphi^\sigma(a, z') \\ &< p(z') \cdot x' + p(z') \cdot \varphi^\sigma(a, z') \\ &= p(z') \cdot y', \end{aligned}$$

which is impossible, showing the claim. ■

The series of claims above show that the assertion of Theorem 4.5 indeed holds. ■

The following example returns to Example 2.9 to discuss the cost share equilibrium corresponding to the core allocation identified there.

Example 4.12 Consider the economy $\mathbb{E}$ discussed in Example 2.9. There we considered the allocation (f^*, g^*, z^*) and argued that this particular allocation is a σ-core allocation corresponding to the cost distribution $\varphi = \varphi^\sigma$ with

$$\varphi(a, z) = \varphi^\sigma(a, z) = \left(\frac{z(z+1)}{(az+1)^2} \, , \, 0 \right).$$

This allocation can be supported as a φ-cost share equilibrium with conjectural price system $p(z) = (z+1, 2)$ for $z \in [0,1]$. Indeed, the generated income from both full specialisation production plans in $\mathcal{P}_a(z) = \{(2,0), (0, z+1)\} - \mathbb{R}^2_+$ is identical and independent of the agent, given by $I(a,z) = 2(z+1)$. Now for the cost share

$$p(z) \cdot \varphi(a, z) = \frac{z(z+1)^2}{(az+1)^2} \leqslant I(a, z) = 2(z+1),$$

the generated budget set is now given by

$$B_a = \{ (x, y, z) \mid (z+1)x + 2y + p(z) \cdot \varphi(a, z) \leqslant I(a, z) \}.$$

One can now compute that the optimising consumption bundle in the generated budget set for a fixed collective good configuration $z \in [0,1]$ is given by

$$f^*_z(a) = \frac{2(az+1)^2 - z(z+1)}{2(az+1)^2} \left(1 \, , \, \frac{z+1}{2} \right).$$

Now, we compute that $u_a(f^*_z(a), z) = \frac{z+1}{2}$, implying that the generated utility is optimal for $z^* = 1$. Noting that $f^*_1(a) = f^*(a)$ as stated in Example 2.9, we have shown that indeed the σ-core allocation (f^*, g^*, z^*) is supported as a φ^σ-cost share equilibrium. ♦

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