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Abstract

We consider a private ownership production economy with consumption and production externalities. Each household is characterized by a consumption set described by a possibility function, an endowment of commodities, and preferences described by a utility function. Each firm is owned by the household and it is characterized by technology desribed by a transformation function. Describing equlibria in terms of first order conditions and market clearing conditions, and using a homotopy approach, we prove the non-emptiness and compactness of the set of competitive equilibria with consumptions and prices strictly positive.

JEL Classification: C62, D50, D51, D62

Keywords: Externalities, Private ownership economy, Competitive equilibrium, Homotopy approach.

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Table of contents

1 Introduction

We consider a private ownership economy with consumption and production externalities. In a differentiable framework, our purpose is to prove the non-emptiness and compactness of the set of competitive equilibria with consumptions and prices strictly positive. In line with the classical contributions of Arrow and Hahn (1971) and Laffont and Laroque (1972), the paper considers a general equilibrium model of a private ownership economy where the choices of all households and firms affect individual consumption sets, individual preferences and production technologies. The importance of consumption and production externalities on individual preferences and production technologies has been widely recognized in literature. Externalities may also affect individual consumption sets and do not directly affect preferences. For instance, (*i*) in the case of internet or electricity, the congestion due to the global consumption limits the physically possible individual consumption; (*ii*) an increase in the production of transport services decreases the minimal threshold of consumption of fuel; (*iii*) an increase of polluting production makes worse the individual health, and consequently it increases the survival threshold of consumption of medicines.

In our model, each firm is characterized by a technology described by an inequality on a differentiable function called the transformation function. Each household is characterized by a consumption set, preferences and an initial endowment of commodities. Following del Mercato (2006), each consumption set is described by an inequality on a differentiable function called the possibility function. Individual preferences are represented by a utility function. Firms are owned by households. In the spirit of Laffont and Laroque (1972) the associated concept of competitive equilibrium is nothing else than an equilibrium *à la Nash*, the resulting allocation being feasible with the initial resources of agents.

Our main result states that for all initial endowments which satisfy classical survival conditions, the set of competitive equilibria with consumptions and prices strictly positive is non-empty and compact. Following the seminal work by Smale (1974), and the recent contributions made by del Mercato (2006) and del Mercato and Platino (2017), we prove our result using an homotopy argument and the topological *degree modulo 2*.[1] As shown by del Mercato and Platino (2017), due the fact that the production sets are not required to be convex, one needs to provide a "price-wise homotopy" which makes the proof of our result non-trivial.

We now compare our contribution with previous works. The existence re-

[1] The reader can find a survey on the theory of modulo 2 and related concepts in Milnor (1965), Geanakoplos and Shafer (1990) and in Villanacci et al. (2002).

sults by Arrow and Hahn (1971), Laffont and Laroque (1972), Bonnisseau and Médecin (2001) and Mandel (2008) are more general than ours since in these works individual consumption sets and firms technologies are represented by correspondences.[2] We are interested in a model where one can perform comparative static analysis, and therefore Pareto improving policies from a differentiable viewpoint. So, at the cost of loosing generality, we choose to use an inequality on differentiable functions, instead of more general correspondences, to describe individual consumption sets and firms technologies.

del Mercato (2006), Balasko (2015), Ericson and Kung (2015), and del Mercato and Platino (2017) use a differentiable approach to general equilibrium analysis. In Ericson and Kung (2015), individual preferences and production technology also depend on the price system. To get existence result, the authors perturb all fundamentals of the economy. In del Mercato (2006) a general model of pure exchange economies with externalities on consumption sets and preferences is studied. In Balasko (2015), wealth dependent preferences are taking into account. del Mercato and Platino (2017) consider a private ownership economies with externalities and standard consumption sets.

The paper is organized as follows. In Section 2, we present the model and the assumptions. In Section 3, the concept of competitive equilibrium is adapted to our economy. Then, we focus on the equilibrium function which is built on first order conditions associated with households and and firms maximization problems. In Section 4, we present our main result, that is Theorem 11. In Section 5 and 6, we prove Theorem 11 by constructing the test economy and the required homotopy. All the propositions are proved in Section 7. In Appendix, the reder can find the definition and the fundamental properties of the topological *degree modulo 2*.

2 The model and the assumptions

There is a finite number C of physical commodities or goods labeled by the superscript $c \in \mathcal{C} := \{1, \dots, C\}$. The commodity space is $\mathbb{R}^C$. There is a finite number H of households or consumers labeled by the subscript $h \in \mathcal{H} := \{1, \dots, H\}$. Each household h is characterized by an endowment of commodities, a possibility function and preferences described by a utility function. There is a finite number J of firms labeled by the subscript $j \in \mathcal{J} := \{1, \dots, J\}$. Each firm j is owned by the households and it is characterized by a technology described by a transformation function. Individual

[2] In Mandel (2008), each consumption set coincides with the positive orthant of the commodity space, so concerning the consumption side our model is more general since it also allows externalities on consumption sets.

utility, possibility and transformation functions are affected by the consumption choices of all households and the production activities of all firms which represent the *externalities* created on individual agents (households and firms) by all the other agents. The notations are summarized below.

- $y_j := (y_j^1, .., y_j^c, .., y_j^C)$ is the production plan of firm j. As usual, the output components are positive and the input components are negative; $y_{-j} := (y_z)_{z \neq j}$ denotes the production plan of firms other than j and $y := (y_j)_{j \in \mathcal{J}}$ denotes the production of all the firms.
- x_h^c is the consumption of commodity c by household h; $x_h := (x_h^1, .., x_h^c, .., x_h^C)$ denotes household h's consumption; $x_{-h} := (x_k)_{k \neq h}$ denotes the consumption of households other than h and $x := (x_h)_{h \in \mathcal{H}}$ denotes the consumption of all the households.
- The technology of firm j is described by an inequality on a trasformation function t_j, which depends on the production and consumption activities of all other agents. So, given y_{-j} and x, the production set of the firm j is described by the following set,

$$Y_j(y_{-j}, x) := \Big\{ y_j \in \mathbb{R}^C \ : \ t_j(y_j, y_{-j}, x) \leq 0 \Big\}$$

  where the transformation function t_j is a function from $\mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$ to $\mathbb{R}$. So, t_j describes the way firm j's technology is affected by the actions of the other agents. Denote $t := (t_j)_{j \in \mathcal{J}}$.
- As in general equilibrium models à la Arrow–Debreu, each household h has to choose a consumption in his consumption set X_h. Analogously to the production side, each consumption set X_h is described in terms of an inequality on a function χ_h.[3] We call χ_h the *possibility function* of households h. The main innovation of this paper comes from the dependency of the consumption set on the consumptions of the other households and the production activities of firms. So, given x_{-h} and y the consumption set of household h is given by

$$X_h(x_{-h}, y) := \Big\{ x_h \in \mathbb{R}^C_{++} \ : \ \chi_h(x_h, x_{-h}, y) \geq 0 \Big\}$$

  where the possibility function χ_h is a function from $\mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$ to $\mathbb{R}$. Thus, χ_h describes the way in which the set of all consumption alternatives which are *a priori possible* for household h is affected by the actions of the other agents. Denote $\chi := (\chi_h)_{h \in \mathcal{H}}$.
- Each household $h \in \mathcal{H}$ has preferences described by a utility function u_h from $\mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$ to $\mathbb{R}$, and $u_h(x_h, x_{-h}, y)$ is the utility level of household h associated with (x_h, x_{-h}, y). So, u_h describes the way household h's preferences are affected by the consumption and the production of the other agents. Denote $u := (u_h)_{h \in \mathcal{H}}$.

[3] In same spirit, see Smale (1974), and Bonnisseau and del Mercato (2010).

- $s_{jh} \in [0,1]$ is the share of firm j owned by household h; $s_h := (s_{jh})_{j\in\mathcal{J}} \in [0,1]^J$ denotes the vector of the shares of all firms owed by household h; $s := (s_h)_{h\in\mathcal{H}} \in [0,1]^{JH}$. As usual, $\sum_{h\in\mathcal{H}} s_{jh} = 1$. We denote by S the set of all the shares.
- $e_h := (e_h^1, .., e_h^c, .., e_h^C) \in \mathbb{R}^C_{++}$ denotes household h's endowment; $e := (e_h)_{h\in\mathcal{H}}$ and $r := \sum\limits_{h\in\mathcal{H}} e_h$.
- $E := ((\chi, u, e, s), t)$ is a private ownership economy with externalities.
- p^c is the price of one unit of commodity c; $p := (p^1, .., p^c, .., p^C) \in \mathbb{R}^C_{++}$.
- Given $w = (w^1, .., w^c, .., w^C) \in \mathbb{R}^C$, we denote $w^\backslash := (w^1, .., w^c, .., w^{C-1}) \in \mathbb{R}^{C-1}$.

We make the following assumptions on the transformation functions.

Assumption 1 *For all $j \in \mathcal{J}$,*

(1) The function t_j is continuous in its domain. For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, the function $t_j(\cdot, y_{-j}, x)$ is differentiable and $D_{y_j} t_j(\cdot,\cdot,\cdot)$ is continuous on $\mathbb{R}^{C(J)} \times \mathbb{R}^{CH}_{++}$.

(2) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, $t_j(0, y_{-j}, x) = 0$.

(3) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, the function $t_j(\cdot, y_{-j}, x)$ is differentiably strictly quasi-convex, i.e., it is C^2 function and for all $y'_j \in \mathbb{R}^C$, $D^2_{y_j} t_j(y'_j, y_{-j}, x)$ is positive definite on $\ker D_{y_j} t_j(y'_j, y_{-j}, x)$.

(4) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, $D_{y_j} t_j(y'_j, y_{-j}, x) \gg 0$, for all $y'_j \in \mathbb{R}^C$.

Fixing the externalities, the assumptions on t_j are standard in "smooth" general equilibrium models. In particular, from Points 1 and 4 of Assumption 1, the production set is a C^1 manifold with boundary of dimension C and its boundary is a C^1 manifold of dimension $C-1$. We point out that we do not require the production set to be convex with respect to the externalities.

For any given externality $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, $Y(x, y)$ denotes the set of all the production allocations that are consistent with (x, y), that is

$$Y(x,y) := \{y' \in \mathbb{R}^{CJ} \colon t_j(y'_j, y_{-j}, x) \leq 0, \ \forall\, j \in \mathcal{J}\} \tag{1}$$

The assumption below is analogous to Assumption UB (Uniform Boundedness) in Bonnisseau and Médecin (2001), and Assumption P(3) in Mandel (2008).[4]

Assumption 2 (Uniform Boundedness) *There exists a bounded set $C(r) \subseteq$*

[4] As it is pointed out by del Mercato and Platino (2017), Assumption 2 is weaker than the condition provided by Arrow and Hahn (1971).

$\mathbb{R}^{CJ}$ *such that for every* $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$,

$$Y(x,y) \cap \left\{ y' \in \mathbb{R}^{CJ} : \sum_{j \in \mathcal{J}} y'_j + r \gg 0 \right\} \subseteq C(r)$$

The following lemma is an immediate consequence of Assumption 2.

Lemma 3 *There exists a bounded set* $K(r) \subseteq \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ *such that for every* $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, *the following set is included in* $K(r)$.

$$A(x, y; r) := \left\{ (x', y') \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} : y' \in Y(x, y) \text{ and } \sum_{h \in \mathcal{H}} x'_h - \sum_{j \in \mathcal{J}} y'_j \leq r \right\}$$

From Assumption 2, the set of feasible allocations $A(x, y; r)$ is *uniformly bounded* with respect to the externalities. Lemma 3, is nedeed to prove the compactness of the homotopy, once externalities move along the homotopy arc (see Step 2.1 of the proof of Proposition 14 in Section 7).

We make the following assumptions on the utilities functions.

Assumption 4

(1) *The function* u_h *is continuous on its domain. For every* $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$, *the function* $u_h(\cdot, x_{-h}, y)$ *is differentiable and* $D_{x_h} u_h(\cdot, \cdot, \cdot)$ *is continuous on* $\mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$.

(2) *For every* $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, *the function* $u_h(\cdot, x_{-h}, y)$ *is differentiably strictly increasing, i.e.* $D_{x_h} u_h(x'_h, x_{-h}, y) \gg 0$ *for all* $x'_h \in \mathbb{R}^{C}_{++}$.

(3) *For every* $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, *the function* $u_h(\cdot, x_{-h}, y)$ *it is differentiably strictly quasi-concave, i.e. it is* C^2 *and for all* $x'_h \in \mathbb{R}^{C}_{++}$, $D^2_{x_h} u_h(x'_h, x_{-h}, y)$ *is negative definite on* $\operatorname{Ker} D_{x_h} u_h(x'_h, x_{-h}, y)$.

(4) *For every* $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$ *and for every* $u \in \operatorname{Im} u_h(\cdot, x_{-h}, y)$,

$$\operatorname{cl}_{\mathbb{R}^C} \{ x_h \in \mathbb{R}^{C}_{++} : u_h(x_h, x_{-h}, y) \geq u \} \subseteq \mathbb{R}^{C}_{++}$$

Fixing the externalities, the assumptions on u_h are standard in "smooth" general equilibrium models. We remark that in Point 1 and Point 4 of Assumption 4 we consider consumption x_{-h} in the closure of a $\mathbb{R}^{C(H-1)}$, just to look at the limit of a behavior (see Step 2.2 of the proof of Proposition 14 in Section 7).

We make the following assumptions on the possibility functions.

Assumption 5 *For all* $h \in \mathcal{H}$,

(1) χ_h *is continuous in its domain. For every* $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, *the*

function $\chi_h(\cdot, x_{-h}, y)$ *is differentiable and* $D_{x_h}\chi_h(\cdot,\cdot,\cdot)$ *is continuous on* $\mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$.

(2) For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$*, the function* $\chi_h(\cdot, x_{-h}, y)$ *is quasi-concave.*

(3) There exists $\overline{x}_h \in \mathbb{R}^C_{++}$ *such that* $\chi_h(\overline{x}_h, x_{-h}, y) \geq 0$ *for every* $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$.

(4) For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$ *and for all* $x'_h \in \mathbb{R}^C_{++}$*,*
(a) $D_{x_h}\chi_h(x'_h, x_{-h}, y) \neq 0$ *; (b)* $D_{x_h}\chi_h(x'_h, x_{-h}, y) \notin -\mathbb{R}^C_{++}$.

The above assumptions are adapted from del Mercato (2006). Notice that by Points 1 and 4(a) of Assumption 5, the consumption set is a C^1 manifold with boundary of dimension C and it boundary is a C^1 manifold of dimension $C-1$. We only point out that del Mercato (2006) requires a *global desiderability* assumption in order to get positive prices at equilibrium. We do not require this assumption because of Point 4 of Assumption 1 (see Steps 1.3 and 2.3 of the proof of Proposition 14 in Section 7). In Assumption 5, we consider consumption bundles x_{-h} in the closure of $\mathbb{R}^C_{++}$, just to look at limit of a behavior (see step 2.2 of the proof of Proposition 14 in Section 7).

Remark 6 *As in del Mercato and Platino (2017), one may restrict the domain of the utility and the possibility functions by considering strictly positive consumption bundles* x_{-h}*, but in addition, one needs to require the existence of a continuous extension of* χ_h *on* $\mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$*, and one of the two assumptions provided by del Mercato and Platino (2017) at page 84, namely Assumption 7 and 8.*

We now define, the set of endowments which satisfy the *Survival Assumption* for given possibility functions.

Definition 7 *Define the set* $\Omega := \prod_{h \in \mathcal{H}} \Omega_h \subseteq \mathbb{R}^{CH}_{++}$ *where*

$$\Omega_h := \left\{ x'_h \in \mathbb{R}^C_{++} \;:\; \chi_h(x'_h, x_{-h}, y) \geq 0,\ \forall\ (x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ} \right\} + \mathbb{R}^C_{++}$$

From Point 3 of Assumption 5, Ω is nonempty and it is open by definition. From Points 3 and 4(a) of Assumption 5, the Survival Assumption is satisfied on the set Ω since for all $e \in \Omega$ the following property holds true.

$$\forall (x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ},\ \exists \widehat{x}_h \in \mathbb{R}^C_{++} : \chi_h(\widehat{x}_h, x_{-h}, y) > 0 \text{ and } \widehat{x}_h \ll e_h \quad (2)$$

As a direct consequence of Points 1 and 2 of Assumptions 5 and (2) we get the following proposition. The continuous selection functions given by Proposition 8 will play a fundamental role in the construction of the continuous homotopy used to show our main result (see Theorem 11). Specifically, we use Proposition

8 to define the homotopies given in Section 6.[5]

Proposition 8 *For all $h \in \mathcal{H}$, there exists a continuous selection function $\widehat{x}_h : \mathbb{R}_{+}^{C(H-1)} \times \mathbb{R}^{CJ} \times \Omega_h \to \mathbb{R}_{++}^{C}$ such that for each $(x_{-h}, y, e_h) \in \mathbb{R}_{+}^{C(H-1)} \times \mathbb{R}^{CJ} \times \Omega_h$, $\chi_h(\widehat{x}_h(x_{-h}, y, e_h), x_{-h}, y) > 0$ and $\widehat{x}_h(x_{-h}, y, e_h) \ll e_h$.*

Remark 9 *From now on, we only consider economies E for which the initial endowments $e = (e_h)_{h\in\mathcal{H}}$ belong to the set Ω.*

3 Competitive equilibrium and equilibrium function

In this section, we provide the notion of competitive equilibrium associated with our economy and the equilibrium function. Without loss of generality, commodity C is the *numeraire good*. So, given $p^{\backslash} \in \mathbb{R}_{++}^{C-1}$ with innocuous abuse of notation, we denote $p := (p^{\backslash}, 1) \in \mathbb{R}_{++}^{C}$.

Definition 10 $(x^*, y^*, p^{*\backslash}) \in \mathbb{R}_{++}^{CH} \times \mathbb{R}^{CJ} \times \mathbb{R}_{++}^{C-1}$ *is a competitive equilibrium for the economy E if*

(1) for all $j \in \mathcal{J}$, y_j^* solves the following problem

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} \; p^* \cdot y_j \\ &\text{subject to } \; t_j(y_j, y_{-j}^*, x^*) \leq 0 \end{aligned} \tag{3}$$

(2) For all $h \in \mathcal{H}$, x_h^* solves the following problem

$$\begin{aligned} &\max_{x_h \in \mathbb{R}_{++}^C} \; u_h(x_h, x_{-h}^*, y^*) \\ &\text{subject to } \; \chi_h(x_h, x_{-h}^*, y^*) \geq 0 \\ &\qquad p^* \cdot x_h \leq p^* \cdot \left(e_h + \sum_{j\in\mathcal{J}} s_{jh} y_j^* \right) \end{aligned} \tag{4}$$

(3) $(x^*, y^*) \in \mathbb{R}_{++}^{CH} \times \in \mathbb{R}^{CJ}$ satisfies market clearing conditions, that is

$$\sum_{h\in\mathcal{H}} x_h^* = \sum_{h\in\mathcal{H}} e_h + \sum_{j\in\mathcal{J}} y_j^* \tag{5}$$

Let $\Xi := (\mathbb{R}_{++}^C \times \mathbb{R}_{++} \times \mathbb{R})^H \times (\mathbb{R}^C \times \mathbb{R}_{++})^J \times \mathbb{R}_{++}^{C-1}$ be the set of endogenous variables with generic element $\xi := (x, \lambda, \mu, y, \alpha, p^{\backslash}) := ((x_h, \lambda_h, \mu_h)_{h\in\mathcal{H}}, (y_j, \alpha_j)_{j\in\mathcal{J}}, p^{\backslash})$, where λ_h and μ_h denote respectively the Lagrange multiplier associated with

[5] The proof of Proposition 8 is based on Points 1 and 2 of Assumption 5, and on the Micheal's Selection Theorem. See del Mercato (2006) for details.

the household h's budget constraint and consumption set, and α_j denotes the Lagrange multiplier associated with the firm j's technological constraint. Competitive equilibria for an economy E can be described using the *equilibrium function* $F : \Xi \to \mathbb{R}^{\dim \Xi}$,

$$F(\xi) := \left(\left(F^{h.1}(\xi), F^{h.2}(\xi), F^{h.3}(\xi) \right)_{h \in \mathcal{H}} \left(F^{j.1}(\xi), F^{j.2}(\xi) \right)_{j \in \mathcal{J}}, F^{M}(\xi) \right)$$

where $F^{h.1}(\xi) := D_{x_h} u_h(x_h, x_{-h}, y) - \lambda_h p + \mu_h D_{x_h} \chi_h(x_h, x_{-h}, y)$, $F^{h.2}(\xi) := -p \cdot (x_h - e_h - \sum_{j \in \mathcal{J}} s_{jh} y_j)$, $F^{h.3}(\xi) := \min\{\mu_h, \chi_h(x_h, x_{-h}, y)\}$, $F^{j.1}(\xi) := p - \alpha_j D_{y_j} t_j(y_j, y_{-j}, x)$, $F^{j.2}(\xi) := -t_j(y_j, y_{-j}, x)$, and $F^{M}(\xi) := \sum_{h \in \mathcal{H}} x_h^{\setminus} - \sum_{j \in \mathcal{J}} y_j^{\setminus} - \sum_{h \in \mathcal{H}} e_h^{\setminus}$. An element $\xi^* \in \Xi$ is an *extended equilibrium* for the economy E if and only if $F(\xi^*) = 0$. With innocuous abuse of terminology, we call an extended equilibrium simply an equilibrium.

4 Compactness and non-emptiness of the set of competitive equilibria

In this section, we state the main result of the paper, and we provide the idea of its proof.

Theorem 11 (Existence and compactness) *The set of competitive equilibria with strictly positive consumption and prices associated with an economy $E = ((\chi, u, e, s), t)$ with $e \in \Omega$, is compact and non-empty.*

In order to prove Theorem 11, we use an homotopy approach following the seminal paper by Smale (1974). The following theorem is a consequence of the homotopy invariance of the topological degree. Following recent contributions by del Mercato (2006), Bonnisseau and del Mercato (2008) and del Mercato and Platino (2017), our homotopy approach is based on the *degree modulo* 2, hereafter "$\deg_2$".[6]

Theorem 12 (Homotopy Theorem) *Let M and N be C^2 manifolds of the same dimension contained in euclidean spaces. Let $y \in N$ and $f, g : M \to N$ be two functions such that f is continuous, g is C^1 in an open neighborhood of $g^{-1}(y)$, y is a regular value for g and $\# g^{-1}(y)$ is odd. Let L be a continuous homotopy from g to f such that $L^{-1}(y)$ is compact. Then,*
(1) $g^{-1}(y)$ is compact and $\deg_2(g, y) = 1$,
(2) $f^{-1}(y)$ is compact and $\deg_2(f, y) = 1$.

The equilibrium function F defined in Section 3 plays the role of the function f in Theorem 12. We use Theorem 12 to prove the compactness of $F^{-1}(0)$ and

[6] See the Appendix for a brief review of the topological degree modulo 2.

$\deg_2(F,0)=1$. As a consequence of the *non-trivial property* of the topological degree, the set $F^{-1}(0)$ is nonempty.

In order to construct the required homotopy and the function that will play the role of the function g, we proceed as follows. First, we fix the externalities and we consider a Pareto optimal allocation of a standard production economy without externalities. Second, using the Second Theorem of Welfare Economics, we construct an appropriate private ownership economy $\widetilde{E}$ that has a unique regular equilibrium. The economy $\widetilde{E}$ is called "test economy" and it is an economy à la Arrow–Debreu *without externalities at all.* Third, we construct the equilibrium function G associated with the test economy playing the role of the function g. This is analysed in detail in Section 5. Finally, in Section 6 we provide the required homotopy H from G to F playing the role of L.

5 The test economy and its properties

Fix the externalities at $(\overline{x},\overline{y})$ and define $\overline{u}_h(x_h) := u_h(x_h,\overline{x}_{-h},\overline{y})$ for every household h and $\overline{t}_j(y_j) := t_j(y_j,\overline{y}_{-j},\overline{x})$ for every firm j. Consider the production economy à la Arrow–Debreu, $\overline{E} := ((\mathbb{R}^C_{++},\overline{u}),\overline{t},r)$ with $r := \sum_{h\in\mathcal{H}} e_h$. Notice that, the consumption set of each household coincides with the strictly positive orthant of the commodity space. Since there are no externalities at all, the notions of feasibility and Pareto optimality are standard. Under Assumptions 1, 2 and 4, there exists a Pareto optimal allocation $(\widetilde{x},\widetilde{y}) \in \mathbb{R}^{CH}_{++}\times\mathbb{R}^{CJ}$ of the economy $\overline{E}$, and there exist Lagrange multipliers $(\widetilde{\theta},\widetilde{\gamma},\widetilde{\beta}) = ((\widetilde{\theta}_h)_{h\neq 1},\widetilde{\gamma},(\widetilde{\beta}_j)_{j\in\mathcal{J}}) \in \mathbb{R}^{H-1}_{++}\times\mathbb{R}^C_{++}\times\mathbb{R}^J_{++}$ such that $(\widetilde{x},\widetilde{y},\widetilde{\theta},\widetilde{\gamma},\widetilde{\beta})$ is the unique solution to the following system.[7]

$$\begin{cases} (1)\ D_{x_1}\overline{u}_1(x_1)=\gamma\ (2)\ \forall\ h\neq 1,\ \theta_h D_{x_h}\overline{u}_h(x_h)=\gamma\ (3)\ \forall\ h\neq 1,\ \overline{u}_h(x_h)=\overline{u}_h(\widetilde{x}_h) \\ (4)\ \forall\ j\in\mathcal{J},\ \gamma=\beta_j D_{y_j}\overline{t}_j(y_j)\ (5)\ \forall\ j\in\mathcal{J},\ -\overline{t}_j(y_j)=0\ (6)\ \sum\limits_{h\in\mathcal{H}} x_h-\sum\limits_{j\in\mathcal{J}} y_j=r \end{cases} \tag{6}$$

It is well known that the Pareto optimal allocation, $(\widetilde{x},\widetilde{y})$ is an *equilibrium relative to some price system* $\widetilde{p}$. From system above, one easily deduces a supporting price $\widetilde{p}$, a redistribution of the initial endowments $\widetilde{e} := (\widetilde{e}_h)_{h\mathcal{H}}$ and the equilibrium equations satisfied by $(\widetilde{x},\widetilde{y})$ for appropriate Lagrange multipliers. Define

$$\widetilde{e}_h := \widetilde{x}_h - \sum_{j\in\mathcal{J}} s_{jh}\widetilde{y}_j \tag{7}$$

and the **test economy** $\widetilde{E} := ((\mathbb{R}^C_{++},\overline{u},\widetilde{e},s),\overline{t})$. Notice that the economy $\widetilde{E}$ is

[7] For a formal proof, see for instance del Mercato and Platino (2017).

a private ownership economy à la Arrow–Debreu with no externalities at all. Consider the function $G : \Xi \to \mathbb{R}^{\dim \Xi}$,

$$G(\xi) := \left(\left(G^{h.1}(\xi), G^{h.2}(\xi), G^{h.3}(\xi) \right)_{h \in \mathcal{H}}, \left(G^{j.1}(\xi), G^{j.2}(\xi) \right)_{j \in \mathcal{J}}, G^{M}(\xi) \right) \tag{8}$$

where $G^{h.1}(\xi) := D_{x_h} u_h(x_h, \overline{x}_{-h}, \overline{y}) - \lambda_h p$, $G^{h.2}(\xi) := -p \cdot (x_h - \widetilde{e}_h - \sum_{j \in \mathcal{J}} s_{jh} y_j)$, $G^{h.3}(\xi) := \min\{\mu_h, \chi_h(\widehat{x}_h(x_{-h}, y, e_h), x_{-h}, y)\}$, $G^{j.1}(\xi) := p - \alpha_j D_{y_j} t_j(y_j, \overline{y}_{-j}, \overline{x})$, $G^{j.2}(\xi) := -t_j(y_j, \overline{y}_{-j}, \overline{x})$, and $G^{M}(\xi) := \sum_{h \in \mathcal{H}} x_h^{\setminus} - \sum_{j \in \mathcal{J}} y_j^{\setminus} - \sum_{h \in \mathcal{H}} \widetilde{e}_h^{\setminus}$.

We remark that the continuous function $\widehat{x}_h$ is given by Proposition 8 and $G^{h.3}(\widetilde{\xi}) = \widetilde{\mu}_h = 0$ since $\chi_h(\widehat{x}_h(x_{-h}, y, e_h), x_{-h}, y) > 0$. As a consequence, the function G is nothing else than the equilibrium function associated with the test economy $\widetilde{E}$. Finally define the vector $\widetilde{\xi} := (\widetilde{x}, \widetilde{\lambda}, \widetilde{\mu}, \widetilde{y}, \widetilde{\alpha}, \widetilde{p}^{\setminus}) \in \Xi$ with $\widetilde{\lambda}_1 := \widetilde{\gamma}^C$, $\widetilde{\lambda}_h := \frac{\widetilde{\gamma}^C}{\widetilde{\theta}_h}$ for all $h \neq 1$, $\widetilde{\mu}_h = 0$ for all $h \in \mathcal{H}$, $\widetilde{\alpha}_j := \frac{\widetilde{\beta}_j}{\widetilde{\gamma}^C}$ and $\widetilde{p}^{\setminus} := \frac{\widetilde{\gamma}^{\setminus}}{\widetilde{\gamma}^C}$. From system (6), one easily deduces $G(\widetilde{\xi}) = 0$.

The next proposition shows that $\widetilde{\xi}$ is the unique equilibrium for the economy $\widetilde{E}$. Furthermore, it is a regular equilibrium.

Proposition 13 *$G^{-1}(0) = \{\widetilde{\xi}\}$, G is C^1 in an open neighborhood of $\widetilde{\xi}$ and 0 is a regular value for G.*

6 The homotopy and its properties

The basic idea is to homotopize endowments and externalities by a segment in the two economies $\widetilde{E}$ and E. Due to the fact that the production sets are not required to be convex, the individual budget set may be empty along the homotopy arc.[8] To overcome this difficulty, we define the homotopy H by using the two homotopies Φ and Γ defined below. In the homotopy Φ, we only homotopize the initial endowments. In the homotopy Γ we homotopize the externalities in utility and transformation functions, and the consumption choices in possibility functions. In order to simplify the notation, we define the following convex combinations,

$$e_h(\tau) := \tau e_h + (1-\tau)\widetilde{e}_h, \quad x(\tau) := \tau x + (1-\tau)\overline{x}, \quad y(\tau) := \tau y + (1-\tau)\overline{y}$$

and the homotopies, $\Gamma, \Phi : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$ defined by

$$\Phi(\xi, \tau) := \left(\left(\Phi^{h.1}(\xi, \tau), \Phi^{h.2}(\xi, \tau), \Phi^{h.3}(\xi, \tau) \right)_{h \in \mathcal{H}}, \left(\Phi^{j.1}(\xi, \tau), \Phi^{j.2}(\xi, \tau) \right)_{j \in \mathcal{J}}, \Phi^{M}(\xi, \tau) \right)$$

[8] See del Mercato and Platino (2017) for details.

where $\Phi^{h.1}(\xi,\tau) := D_{x_h}u_h(x_h,\overline{x}_{-h},\overline{y}) - \lambda_h p$, $\Phi^{h.2}(\xi,\tau) := -p\cdot[x_h - e_h(\tau) - \sum_{j\in\mathcal{J}} s_{jh}y_j]$,$\Phi^{h.3}(\xi,\tau) := \min\{\mu_h, \chi_h(\widehat{x}_h(x_{-h},y,e_h),x_{-h},y)\}$,$\Phi^{j.1}(\xi,\tau) := p - \alpha_j D_{y_j}t_j(y_j,\overline{y}_{-j},\overline{x})$,$\Phi^{j.2}(\xi,\tau) := -t_j(y_j,\overline{y}_{-j},\overline{x})$,$\Phi^{M}(\xi,\tau) := \sum_{h\in\mathcal{H}} x_h^{\backslash} - \sum_{j\in\mathcal{J}} y_j^{\backslash} - \sum_{h\in\mathcal{H}} e_h(\tau)^{\backslash}$.

$$\Gamma(\xi,\tau) := \left(\left(\Gamma^{h.1}(\xi,\tau), \Gamma^{h.2}(\xi,\tau), \Gamma^{h.3}(\xi,\tau)\right)_{h\in\mathcal{H}}, \left(\Gamma^{j.2}(\xi,\tau), \Gamma^{j.2}(\xi,\tau)\right)_{j\in\mathcal{J}}, \Gamma^{M}(\xi,\tau)\right)$$

where $\Gamma^{h.1}(\xi,\tau) := D_{x_h}u_h(x_h,x_{-h}(\tau),y(\tau)) - \lambda_h p + \tau\mu_h D_{x_h}\chi_h(\tau x_h + (1-\tau)\widehat{x}_h(x_{-h},y,e_h),x_{-h},y)$, $\Gamma^{h.2}(\xi,\tau) := -p\cdot[x_h - e_h - \sum_{j\in\mathcal{J}} s_{jh}y_j]$, $\Gamma^{h.3}(\xi,\tau) := \min\{\mu_h, \chi_h(\tau x_h + (1-\tau)\widehat{x}_h(x_{-h},y,e_h),x_{-h},y)\}$, $\Gamma^{j.1}(\xi,\tau) := p - \alpha_j D_{y_j}t_j(y_j, y_{-j}(\tau), x(\tau))$, $\Gamma^{j.2}(\xi,\tau) := -t_j(y_j,y_{-j}(\tau),x(\tau))$, $\Gamma^{M}(\xi,\tau) := \sum_{h\in\mathcal{H}} x_h^{\backslash} - \sum_{j\in\mathcal{J}} y_j^{\backslash} - \sum_{h\in\mathcal{H}} e_h^{\backslash}$.

We remind that, the continuous function $\widehat{x}_h$ is given by Proposition 8.

Define the homotopy $H : \Xi\times[0,1]\to\mathbb{R}^{\dim\Xi}$,

$$H(\xi,\psi) := \begin{cases} \Phi(\xi,2\psi) \text{ if } 0\le\psi\le\frac{1}{2} \\ \Gamma(\xi,2\psi-1) \text{ if } \frac{1}{2}\le\psi\le 1 \end{cases}$$

Observe that H is a continuous function. Indeed, Φ and Γ are continuous because they are composed by continuous functions (see Point 1 of Assumptions 1, 4 and 5, and Proposition 8). Moreover, $H\left(\xi,\frac{1}{2}\right)$ is well defined since $\Phi(\xi,1)=\Gamma(\xi,0)$. Finally, observe that $H(\xi,0)=\Phi(\xi,0)=G(\xi)$ and $H(\xi,1)=\Gamma(\xi,1)=F(\xi)$.

Proposition 14 *For each $e\in\Omega$, $H^{-1}(0)$ is compact.*

7 Proofs

Proof of Proposition 13. The proof is standard and it is available upon request from the author. We just point out that G is C^1 in an open neighborhood of $G^{-1}(0)=\widetilde{\xi}$. By the continuity of χ_h and $\widehat{x}$, the function $g_h : \xi\in\Xi\to g_h(\xi) := (\chi_h(\widehat{x}_h(x_{-h},y,e_h),x_{-h},y)-\mu_h)\in\mathbb{R}$ is continuous. For all $h\in\mathcal{H}$, $g_h(\widetilde{\xi})>0$ since $\chi_h(\widehat{x}_h(\widetilde{x}_{-h},\widetilde{y},e_h),\widetilde{x}_{-h},\widetilde{y})>0$ and $\widetilde{\mu}_h=0$. Thus, in some open neighborhood $\mathcal{I}(\widetilde{\xi})\subseteq\Xi$ of $\widetilde{\xi}$ we get $g_h(\xi)>0$ for all $h\in\mathcal{H}$. Therefore, in the open neighborhood $\mathcal{I}(\widetilde{\xi})$, the component $G^{h.3}(\xi)=\mu_h$ for all $h\in\mathcal{H}$ while the components $G^{h.1}(\xi), G^{h.2}(\xi), G^{j.1}(\xi), G^{j.2}(\xi)$ and $G^{M}(\xi)$ are given by (8). So, $G(\xi)$ is obviously a C^1 function in $\mathcal{I}(\widetilde{\xi})$. ■

Proof of Proposition 14. Observe that $H^{-1}(0)=\Phi^{-1}(0)\cup\Gamma^{-1}(0)$. Since the union of a finite number of compact sets is compact, it is enough to show that $\Phi^{-1}(0)$ and $\Gamma^{-1}(0)$ are compact.

Claim 1. $\Phi^{-1}(0)$ *is compact.*

We prove that, up to a subsequence, every sequence $(\xi^\nu, \tau^\nu)_{\nu\in\mathbb{N}} \subseteq \Phi^{-1}(0)$ converges to an element of $\Phi^{-1}(0)$, where $\xi^\nu := (x^\nu, \lambda^\nu, \mu^\nu, y^\nu, \alpha^\nu, p^{\nu\setminus})_{\nu\in\mathbb{N}}$. First observe that, since $\{\tau^\nu : \nu \in \mathbb{N}\} \subseteq [0,1]$, up to a subsequence, $(\tau^\nu)_{\nu\in\mathbb{N}}$ converges to some $\tau^* \in [0,1]$. From Steps 1.1, 1.2, 1.3 and 1.4 below, we have that up to a subsequence, $(\xi^\nu)_{\nu\in\mathbb{N}}$ converges to some $\xi^* := (x^*, \lambda^*, \mu^*, y^*, \alpha^*, p^{*\setminus}) \in \Xi$. Since the homotopy Φ is continuous, taking the limit, we get the desired result, that is $(\xi^*, \tau^*) \in \Phi^{-1}(0)$.

Step 1.1. *Up to a subsequence,* $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ *converges to some* $(x^*, y^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$. We show that for $r = \sum_{h\in\mathcal{H}} e_h$, the sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ belongs to the bounded set $K(r)$ given in Lemma 3. By $\Phi^{j.2}(\xi^\nu, \tau^\nu) = 0$, for every j and for any ν, the sequence $(y^\nu)_{\nu\in\mathbb{N}}$ is included in the set $Y(\overline{x}, \overline{y})$ given by (1). Summing $\Phi^{h.2}(\xi^\nu, \tau^\nu) = 0$ over h, by $\Phi^M(\xi^\nu, \tau^\nu) = 0$ one gets $\sum_{h\in\mathcal{H}} x^\nu_h - \sum_{j\in\mathcal{J}} y^\nu_j = \sum_{h\in\mathcal{H}} e_h(\tau^\nu)$ for every $\nu \in \mathbb{N}$. By the definition of $\widetilde{e}_h$ given in (7), system (6) and Proposition 13, one easily gets $\sum_{h\in\mathcal{H}} e_h(\tau^\nu) = r$. Therefore, $(x^\nu, y^\nu) \in A(\overline{x}, \overline{y}; r) \subseteq K(r)$. Consequently, the sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ belongs to the compact set $\operatorname{cl} K(r)$ which is included in $\mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$. Therefore, up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to some $(x^*, y^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$.

Step 1.2. *The consumption allocation* x^* *is strictly positive, i.e.* $x^*_h \gg 0$ *for every* $h \in \mathcal{H}$. The argument is similar to the one used in Step 2.1 of Claim 1. It suffices to replace:

(1) the problem (9) with the following problem

$$\max_{x_h\in\mathbb{R}^C_{++}} u_h(x_h, \overline{x}_{-h}, \overline{y})$$
$$\text{subject to } p^\nu \cdot x_h \leq p^\nu \cdot [\tau^\nu e_h + (1-\tau^\nu)\widetilde{x}_h] + p^\nu \cdot \sum_{j\in\mathcal{J}} s_{jh}(y^\nu_j - (1-\tau^\nu)\widetilde{y}_j)$$

according to $\Phi^{h.1}(\xi^\nu, \tau^\nu) = \Phi^{h.2}(\xi^\nu, \tau^\nu) = 0$;

(2) the point $\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h)$ given by Proposition 8, with the bundle $\widehat{e}_h(\tau^\nu) := \tau^\nu e_h + (1-\tau^\nu)\widetilde{x}_h \in \mathbb{R}^C_{++}$.

(3) the problem (10) with the following problem

$$\max_{y_j\in\mathbb{R}^C} p^\nu \cdot y_j$$
$$\text{subject to } t_j(y_j, \overline{y}_{-j}, \overline{x}) \leq 0$$

according to $\Phi^{j.1}(\xi^\nu, \tau^\nu) = \Phi^{j.2}(\xi^\nu, \tau^\nu) = 0$.

Next, as in Step 2.2 of Claim 2 one easily shows that x^*_h belongs to the closure of the upper counter set of $e(\tau^*)$, which is included in $\mathbb{R}^C_{++}$ by Point 4 of Assumption 4. Thus, $x^*_h \in \mathbb{R}^{CH}_{++}$.

Step 1.3. *Up to a subsequence,* $(\alpha^\nu, p^{\nu\,\setminus})_{\nu\in\mathbb{N}}$ *converges to some* $(\alpha^*, p^{*\,\setminus}) \in \mathbb{R}^J_{++} \times \mathbb{R}^{C-1}_{++}$. Using Points 1 and 3 of Assumption 1, the proof is similar to the one of Step 1.3 in Claim 1. Thus, $p^{*\,\setminus} \in \mathbb{R}^{C-1}_{++}$.

Step 1.4. *Up to a subsequence,* $(\lambda^\nu, \mu^\nu)_{\nu\in\mathbb{N}}$ *converges to some* $(\lambda^*, \mu^*) \in \mathbb{R}^H_{++} \times \mathbb{R}^H_{+}$. By $\Phi^{h.3}(\xi^\nu, \tau^\nu) = 0$ and Proposition 8, we have $\mu^\nu_h = 0$ for every $\nu \in \mathbb{N}$. Taking the limit, we get $\mu^*_h = 0$.

For any household h, fix a commodity $c(h)$. By $\Phi^{h.1}(\xi^\nu, \tau^\nu) = 0$, for every $\nu \in \mathbb{N}$ we have $\lambda^\nu_h = \frac{D_{x_h^{c(h)}} u_h(x^\nu_h, \overline{x}_{-h}, \overline{y})}{p^{\nu c(h)}}$. Taking the limit and using the continuity of Du_h (see Point 1 of Assumption 4) we have $\lambda^*_h = \frac{D_{x_h^{c(h)}} u_h(x^*_h, \overline{x}_{-h}, \overline{y})}{p^{*c(h)}}$ which is strictly positive since fixing the externalities the function u_h is strictly increasing (see Point 2 of Assumption 4).

Claim 2. $\Gamma^{-1}(0)$ *is compact.*

Let $(\xi^\nu, \tau^\nu)_{\nu\in\mathbb{N}}$ be a sequences in $\Gamma^{-1}(0)$. As in Claim 1, $(\tau^\nu)_{\nu\in\mathbb{N}}$ converges to $\tau^* \in [0,1]$. From Seps 2.1, 2.2, 2.3 and 2.4 below, we have that, up to a subsequence, $(\xi^\nu)_{\nu\in\mathbb{N}}$ converges to an element $\xi^* := (x^*, \lambda^*, \mu^*, y^*, \alpha^*, p^{*\setminus}) \in \Xi$. Since Γ is a continuous function, taking limit one gets $(\xi^*, \tau^*) \in \Gamma^{-1}(0)$.

Step 2.1. *Up to a subsequence,* $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ *converges to some* $(x^*, y^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$. We show that, for $r = \sum_{h\in\mathcal{H}} e_h$, the sequence $(\xi^\nu, \tau^\nu)_{\nu\in\mathbb{N}}$ is included in the bounded set $K(r)$ given by Lemma 3. By $\Gamma^{j.2}(\xi^\nu, \tau^\nu) = 0$, for every j we get [9]

$$t_j(y^\nu_j, y^\nu_{-j}(\tau^\nu), x^\nu(\tau^\nu)) = 0, \quad \forall \nu \in \mathbb{N}$$

Thus, for every $\nu \in \mathbb{N}$, the production plan y^ν_j belongs to the set $Y(x^\nu(\tau^\nu), y^\nu(\tau^\nu))$ given by (1). Summing $\Gamma^{h.2}(\xi^\nu, \tau^\nu) = 0$ over h, by $\Gamma^M(\xi^\nu, \tau^\nu) = 0$ we get $\sum\limits_{h\in\mathcal{H}} x^\nu_h - \sum\limits_{j\in\mathcal{J}} y^\nu_j = \sum\limits_{h\in\mathcal{H}} e_h$ for all $\nu \in \mathbb{N}$. Therefore, for every $\nu \in \mathbb{N}$, (x^ν, y^ν) belongs to the set $A(x^\nu(\tau^\nu), y^\nu(\tau^\nu))$ which is included in $K(r)$. Consequently, the sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ belongs to $\operatorname{cl} K(r)$ which is a compact set. So, up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to some $(x^*, y^*) \in \operatorname{cl} K(r) \subseteq \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$, and thus $(x^*, y^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$.

Step 2.2. *The consumption allocation x^* is strictly positive, i.e. $x^*_h \gg 0$ for every $h \in \mathcal{H}$.* The proof is based on Point 4 of Assumption 4. By $\Gamma^{h.1}(\xi^\nu, \tau^\nu) = \Gamma^{h.2}(\xi^\nu, \tau^\nu) = \Gamma^{h.3}(\xi^\nu, \tau^\nu) = 0$ and the KKT sufficient conditions, x^ν_h solves the

[9] For every ν, we use the notation $x^\nu(\tau^\nu) := \tau^\nu x^\nu + (1-\tau^\nu)\overline{x}$ and $y^\nu(\tau^\nu) := \tau^\nu y^\nu + (1-\tau^\nu)\overline{y}$.

following problem for every $\nu \in \mathbb{N}$

$$\begin{aligned} \max_{x_h \in \mathbb{R}^C_{++}} \; & u_h(x_h, x^\nu_{-h}(\tau^\nu), y^\nu(\tau^\nu)) \\ \text{subject to } \; & \chi_h(\tau^\nu x_h + (1-\tau^\nu)\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h), x^\nu_{-h}, y^\nu) \geq 0 \\ & p^\nu \cdot x_h \leq p^\nu \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} y^\nu_j) \end{aligned} \tag{9}$$

We show now that $\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h)$ belongs to the constraint set of the problem above. We first claim that $\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h)$ belongs to the budget set of agent h. By $\Gamma^{j.1}(\xi^\nu, \tau^\nu) = \Gamma^{j.2}(\xi^\nu, \tau^\nu) = 0$ and KKT sufficient conditions, y^ν_j solves

$$\begin{aligned} & \max_{y_j \in \mathbb{R}^C} p^\nu \cdot y_j \\ & \text{subject to } t_j(y_j, y^\nu_{-j}(\tau^\nu), x^\nu(\tau^\nu)) \leq 0 \end{aligned} \tag{10}$$

By Point 2 of Assumption 1, 0 belongs to the production set of firm j. Therefore, $p^\nu \cdot y^\nu_j \geq 0$ and $\sum_{j \in \mathcal{J}} s_{jh} p^\nu \cdot y^\nu_j \geq 0$. As a consequence, the initial endowment e_h belongs to the budget set of agent h. By Proposition 8, $\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h) \ll e_h$ for any ν, and consequently, $p^\nu \cdot \widehat{x}_h(x^\nu_{-h}, y^\nu, e_h) < p^\nu \cdot e_h$, which completes the proof of the claim. Finally, $\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h)$ belongs to the possibility set of the agent by Proposition 8.

We claim now that x^*_h belongs to the closure of some upper contour set. For every $\nu \in \mathbb{N}$, $u_h(x^\nu_h, x^\nu_{-h}(\tau^\nu), y^\nu(\tau^\nu)) \geq u_h(\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h), x^\nu_{-h}(\tau^\nu), y^\nu(\tau^\nu))$.

By Point 2 of Assumption 4, for every $\varepsilon > 0$ we have $u_h(x^\nu_h + \varepsilon \mathbf{1}, x^\nu_{-h}(\tau^\nu), y^\nu(\tau^\nu)) > u_h(\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h), x^\nu_{-h}(\tau^\nu), y^\nu(\tau^\nu))$, where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^C_{++}$. So, taking the limit for $\nu \to +\infty$ and using the continuity of u_h and $\widehat{x}_h$ (see Point 1 of Assumption 4 and Proposition 8), we get $u_h(x^*_h + \varepsilon \mathbf{1}, x^*_{-h}(\tau^*), y^*(\tau^*)) \geq u_h(\widehat{x}_h(x^*_{-h}, y^*, e_h), x^*_{-h}(\tau^*), y^*(\tau^*))$ for every $\varepsilon > 0$. That is, x^*_h belongs to the closure of the upper counter set of $\widehat{x}_h(x^*_{-h}, y^*, e_h)$, which is included in $\mathbb{R}^C_{++}$ by Point 4 of Assumption 4. Thus, $x^*_h \in \mathbb{R}^{CH}_{++}$. One should notice that, since $\tau^* \in [0,1]$, $x^*_{-h}(\tau^*)$ and x^*_{-h} are not necessarily strictly positive. For that reason, in Point 4 of Assumption 4 and in Point 2 Assumption 5 we consider x_{-h} in $\mathbb{R}^{C(H-1)}_{+}$.

Step 2.3. *Up to a subsequence,* $(\alpha^\nu, p^{\nu \setminus})_{\nu \in \mathbb{N}}$ *converges to some* $(\alpha^*, p^{* \setminus}) \in \mathbb{R}^J_{++} \times \mathbb{R}^{C-1}_{++}$. By $\Gamma^{j.1}(\xi^\nu, \tau^\nu) = 0$, considering commodity C for every $\nu \in \mathbb{N}$, we get $\alpha^\nu_j = \frac{1}{D_{y^C_j} t_j(y^\nu_j, y^\nu_{-j}(\tau^\nu), x^\nu(\tau^\nu))}$. Taking the limit for $\nu \to +\infty$ and using the continuity of Dt_j and the "free disposal" property (see Points 1 and 4 of Assumption 1), the sequence $(\alpha^\nu_j)_{\nu \in \mathbb{N}}$ converges to $\alpha^*_j := \frac{1}{D_{y^C_j} t_j(y^*_j, y^*_{-j}(\tau^*), x^*(\tau^*))} > 0$. By $\Gamma^{j.1}(\xi^\nu, \tau^\nu) = 0$, for every commodity $c \neq C$ and for all $\nu \in \mathbb{N}$ we have $p^{\nu\, c} = \alpha^\nu_j D_{y^c_j} t_j(y^\nu_j, y^\nu_{-j}(\tau^\nu), x^\nu(\tau^\nu))$. Taking the limit and using Points 1 and 4 of Assumption 1, for all $c \neq C$ we get $p^{*\,c} = \alpha_j^* D_{y_j^c} t_j(y_j^*, y_{-j}^*(\tau^*), x^*(\tau^*)) > 0$. Thus, $p^{*\,\setminus} \in \mathbb{R}_{++}^{C-1}$.

Step 2.4. *Up to a subsequence, $(\lambda^\nu, \mu^\nu)_{\nu\in\mathbb{N}}$ converges to some $(\lambda^*, \mu^*) \in \mathbb{R}_{++}^H \times \mathbb{R}_+^H$.* We have two possible cases, in Case a), $\tau^* = 0$, and in Case b), $\tau^* \in (0, 1]$.

Case a). $\tau^* = 0$. Using $\Gamma^{h.3}(\xi^\nu, \tau^\nu) = 0$, we first claim that there exists $\nu^* \in \mathbb{N}$ such that for every $\nu \geq \nu^*$, $\mu_h^\nu = 0$. Since $\tau^* = 0$, the sequence $(\tau^\nu x_h^\nu + (1-\tau^\nu)\widehat{x}_h(x_{-h}^\nu, y^\nu, e_h), x_{-h}^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to $(\widehat{x}_h(x_{-h}^*, y^*, e_h), x_{-h}^*, y^*)$. By Proposition 8, $\chi_h(\widehat{x}_h(x_{-h}^*, y^*, e_h), x_{-h}^*, y^*) > 0$. The continuity of the functions $\widehat{x}_h$ and χ_h (see Proposition 8 and Point 1 of Assumption 5) imply that there is $\nu^* \in \mathbb{N}$ such that for every $\nu \geq \nu^*$, $\chi_h(\tau^\nu x_h^\nu + (1-\tau^\nu)\widehat{x}_h(x_{-h}^\nu, y^\nu, e_h), x_{-h}^\nu, y^\nu) > 0$, which proves the claim. Thus, the sequence $(\mu_h^\nu)_{\nu\in\mathbb{N}}$ converges to $\mu_h^* = 0$.

By $\Gamma^{h.1}(\xi, \tau) = 0$, considering commodity C for every $\nu \geq \nu^*$, we get $\lambda_h^\nu = D_{x_h^C} u_h(x_h^\nu, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu))$. Taking the limit and using the continuity of Du_h (Point 1 of Assumption 4), we get $\lambda_h^* = D_{x_h^C} u_h(x_h^*, \overline{x}_{-h}, \overline{y})$ which is strictly positive by Point 2 of Assumption 4.

Case b). $\tau^* \in (0, 1]$. We first claim that up to a subsequence, $(\lambda^\nu, \mu^\nu)_{\nu\in\mathbb{N}} \subseteq \mathbb{R}_{++}^H \times \mathbb{R}_+^H$ converges to some $(\lambda^*, \mu^*) \in \mathbb{R}_+^H \times \mathbb{R}_+^H$. Second, we show that $\lambda^* \gg 0$. In order to prove the claim, it is enough to show that $(\lambda_h^\nu, \mu_h^\nu)_{\nu\in\mathbb{N}}$ is bounded for every $h \in \mathcal{H}$. Otherwise, suppose that there is a subsequence that without loss of generality we continue to denote with $(\lambda_h^\nu, \mu_h^\nu)_{\nu\in\mathbb{N}}$ such that $\|(\lambda_h^\nu, \mu_h^\nu)\|$ diverges to $+\infty$. Consider the sequence $\left(\frac{(\lambda_h^\nu, \mu_h^\nu)}{\|(\lambda_h^\nu, \mu_h^\nu)\|}\right)_{\nu\in\mathbb{N}}$ in the sphere, which is a compact set. [10] Up to a subsequence, $\left(\frac{(\lambda_h^\nu, \mu_h^\nu)}{\|(\lambda_h^\nu, \mu_h^\nu)\|}\right)_{\nu\in\mathbb{N}}$ converges to some $(\lambda_h, \mu_h) \neq (0, 0)$. [11] Obviously, $\lambda_h \geq 0$ and $\mu_h \geq 0$, since $\lambda_h^\nu > 0$ and $\mu_h^\nu \geq 0$ for all $\nu \in \mathbb{N}$.

Dividing both sides of $\Gamma^{h.1}(\xi^\nu, \tau^\nu) = 0$ by $\|(\lambda_h^\nu, \mu_h^\nu)\|$, and taking the limit, we get

$$\lambda_h p^* = \tau^* \mu_h D_{x_h} \chi_h(\tau^* x_h^* + (1-\tau^*)\widehat{x}_h(x_{-h}^*, y^*, e_h), x_{-h}^*, y^*) \tag{11}$$

Notice that $\mu_h > 0$ and $\lambda_h > 0$. Indeed from Point 4(a) of Assumption 5, we know that $D_{x_h} \chi_h(\tau^* x_h^* + (1-\tau^*)\widehat{x}_h(x_{-h}^*, y^*, e_h), x_{-h}^*, y^*) \neq 0$. Thus, $\mu_h > 0$ because if $\mu_h = 0$, from (11) we get $\lambda_h = 0$ which contradicts the fact that $(\lambda_h, \mu_h) \neq (0, 0)$. Finally, $\mu_h > 0$, $\tau^* > 0$, $p^* \in \mathbb{R}_{++}^C$ and (11) imply $\lambda_h > 0$.

We prove now that

$$\lambda_h p^* \cdot \widehat{x}_h(x_{-h}^*, y^*, e_h) < \lambda_h p^* \cdot x_h^* \tag{12}$$

[10] Since $\|(\lambda_h^\nu, \mu_h^\nu)_{\nu\in\mathbb{N}}\|$ diverges to $+\infty$, without loosing of generality, we suppose that $\|(\lambda_h^\nu, \mu_h^\nu)\| > 0$ for every ν.

[11] Observe that $(\lambda_h, \mu_h) \neq (0, 0)$ since $\|(\lambda_h, \mu_h)\| = 1$.

Since $\lambda_h > 0$, Proposition 8 implies that

$$\lambda_h p^* \cdot \widehat{x}_h(x^*_{-h}, y^*, e_h) < \lambda_h p^* \cdot e_h \tag{13}$$

Multiplying $\Gamma^{h.2}(\xi^\nu, \tau^\nu) = 0$ by λ^ν_h, for every $\nu \in \mathbb{N}$ we get $\lambda^\nu_h p^\nu \cdot e_h + \lambda^\nu_h p^\nu \cdot \sum_{j\in\mathcal{J}} s_{jh} y^\nu = \lambda^\nu_h p^\nu \cdot x^\nu_h$. Thus, dividing both sides by $\|(\lambda^\nu_h, \mu^\nu_h)\|$ and taking the limit, we get

$$\lambda_h p^* \cdot e_h + \lambda_h p^* \cdot \sum_{j\in\mathcal{J}} s_{jh} y^*_j = \lambda_h p^* \cdot x^*_h \tag{14}$$

Therefore, (12) follows from (13) and (14) since $\lambda_h p^* \cdot \sum_{j\in\mathcal{J}} s_{jh} y^*_j \geq 0$. This inequality follows by $\Gamma^{j.1}(\xi^\nu, \tau^\nu) = \Gamma^{j.2}(\xi^\nu, \tau^\nu) = 0$ and the possibility of inactivity (Point 2 of Assumption 1). Indeed, KKT sufficient conditions imply that y^ν_j solves problem (10), and consequently $p^\nu \cdot y^\nu_j \geq 0$ for every $\nu \in \mathbb{N}$. Multiplying both sides by λ^ν_h, dividing by $\|(\lambda^\nu_h, \mu^\nu_h)\|$ and taking the limit, we get $\lambda_h p^* \cdot y^*_j \geq 0$ for every $j \in \mathcal{J}$.

Finally, we show that $\lambda_h p^* \cdot \widehat{x}_h(x^*_{-h}, y^*, e_h) \geq \lambda_h p^* \cdot x^*_h$ which combined with (12) leads to a contradiction. Therefore, our claim is completely proved.

Since $\mu_h > 0$, there exists $n \in \mathbb{N}$ such that $\mu^\nu_h > 0$ for every $\nu \geq n$. From $\Gamma^{h.3}(\xi^\nu, \tau^\nu) = 0$, we get $\chi_h(\tau^\nu x^\nu_h + (1-\tau^\nu)\widehat{x}_h(x^\nu_{-h}, y^\nu, e_h), x^\nu_{-h}, y^\nu) = 0$ for every $\nu \geq n$. Taking the limit, one gets $\chi_h(\tau^* x^*_h + (1-\tau^*)\widehat{x}_h(x^*_{-h}, y^*, e_h), x^*_{-h}, y^*) = 0$. Therefore, (11) and the KKT sufficient conditions imply that x^*_h solves the following problem.

$$\begin{aligned} &\min_{x_h \in \mathbb{R}^C_{++}} \lambda_h p^* \cdot x_h \\ &\text{subject to } \chi_h(\tau^* x_h + (1-\tau^*)\widehat{x}_h(x^*_{-h}, y^*, e_h), x^*_{-h}, y^*) \geq 0 \end{aligned}$$

By Proposition 8, $\widehat{x}_h(x^*_{-h}, y^*, e_h)$ belongs to the constraint of this problem, and so $\lambda_h p^* \cdot \widehat{x}_h(x^*_{-h}, y^*, e_h) \geq \lambda_h p^* \cdot x^*_h$ holds true.

Therefore, one concludes that the sequence $(\lambda^\nu_h, \mu^\nu_h)_{\nu\in\mathbb{N}}$ is bounded, and consequently it admits a subsequence converging to some $(\lambda^*, \mu^*) \in \mathbb{R}^H_+ \times \mathbb{R}^H_+$.

Now we show that $\lambda^* \gg 0$. From $\Gamma^{h.1}(\xi^\nu, \tau^\nu) = 0$, taking the limit (and using the continuity of functions $D_{x_h} u_h$, $D_{x_h} \chi_h$ and $\widehat{x}_h$) we get

$$\lambda^*_h p^* = D_{x_h} u_h(x^*_h, x^*_{-h}(\tau^*), y^*(\tau^*)) + \tau^* \mu^*_h D_{x_h} \chi_h(\tau^* x^*_h + (1-\tau^*)\widehat{x}_h(x^*_{-h}, y^*, e_h), x^*_{-h}, y^*)$$

Since $\mu^*_h \geq 0$, by Point 2 of Assumption 4 and Point 4 of Assumption 5, there exists a commodity $c(h)$ such that $\lambda^*_h p^{*\,c(h)} = D_{x^{c(h)}_h} u_h(x^*_h, x^*_{-h}, y^*) + \tau^* \mu^*_h D_{x^{c(h)}_h} \chi_h(\tau^* x^*_h + (1-\tau^*)\widehat{x}_h(x^*_{-h}, y^*, e_h), x^*_{-h}, y^*) > 0$. Since $p^{*\,c(h)} > 0$, we get $\lambda^*_h > 0$ which completes the proof of the step. ■

Appendix

We remind the definition of the *degree modulo 2* for continuous functions. See Appendix B in Geanakoplos and Shafer (1990), and Chapter 7 in Villanacci et al. (2002) for further details.

Let M and N be two C^2 manifolds of the same dimension contained in euclidean spaces. Let $\mathcal{A}$ be the set of triples (f, M, y) where

(1) $f : M \to N$ is a continuous function,
(2) $y \in N$ and $f^{-1}(y)$ is compact.

Theorem 15 *There exists a unique function, called degree modulo 2 and denoted by* $\deg_2 : \mathcal{A} \to \{0, 1\}$ *such that*

(1) (Normalisation) $\deg_2(id_M, M, y) = 1$
where $y \in M$ *and* id_M *denotes the identity of* M*.*

(2) (Non–triviality) If $(f, M, y) \in \mathcal{A}$ *and* $\deg_2(f, M, y) = 1$*, then* $f^{-1}(y) \neq \emptyset$*.*

(3) (Excision) If $(f, M, y) \in \mathcal{A}$ *and* U *is an open subset of* M *such that* $f^{-1}(y) \subseteq U$*, then*

$$\deg_2(f, M, y) = \deg_2(f, U, y)$$

(4) (Additivity) If $(f, M, y) \in \mathcal{A}$ *and* U_1 *and* U_2 *are open and disjoint subsets of* M *such that* $f^{-1}(y) \subseteq U_1 \cup U_2$*, then*

$$\deg_2(f, M, y) = \deg_2(f, U_1, y) + deg_2(f, U_2, y)$$

(5) (Local constantness) If $(f, M, y) \in \mathcal{A}$ *and* U *is an open subset of* M *with compact closure such that* $f^{-1}(y) \subseteq U$*, then there is an open neighborhood* V *of* y *in* N *such that for every* $y' \in V$*,*

$$\deg_2(f, U, y') = \deg_2(f, U, y)$$

(6) (Homotopy invariance) Let $L : (z, \tau) \in M \times [0, 1] \to L(z, \tau) \in N$ *be a continuous homotopy. If* $y \in N$ *and* $L^{-1}(y)$ *is compact, then*

$$\deg_2(L_0, U, y) = \deg_2(L_1, U, y)$$

where $L_0 := L(\cdot, 0) : M \to N$ *and* $L_1 := L(\cdot, 1) : M \to N$*.*

If there is no possible confusion on the manifold M, we simply denote $\deg_2(f, y)$ the degree modulo 2 of the triple (f, M, y).

As stated in the following proposition, in the case of C^1 functions and regular values, the degree modulo 2 is computed using the *residue class modulo 2*.

Proposition 16 *If $(g, M, y) \in \mathcal{A}$, g is a C^1 function and y is a regular value of g (i.e., for all $z^* \in g^{-1}(y)$, the differential mapping $Dg(z^*)$ is onto), then $g^{-1}(y)$ is finite (possibly empty) and the degree modulo 2 of g is given by*

$$\deg_2(g, M, y) = [\#g^{-1}(y)]_2 = \begin{cases} 0 & \textit{if } \#g^{-1}(y) \textit{ is even} \\ 1 & \textit{if } \#g^{-1}(y) \textit{ is odd} \end{cases}$$

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