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Tristan Gagnon-Bartsch[*]**, Marco Pagnozzi**[**] **and Antonio Rosato**[***]

Abstract

We explore how taste projection – the tendency to overestimate how similar others' tastes are to one's own – affects bidding in auctions. Taste-projecting bidders underestimate the dispersion in valuations and exaggerate the intensity of competition. Consequently, they overbid in firstprice auctions – irrespective of whether values are independent, correlated, or (a)symmetrically distributed – but not in second-price auctions. Hence, first-price auctions raise more revenue. Moreover, the optimal reserve price in first-price auctions is lower than the rational benchmark, and decreasing in the extent of projection and the number of bidders. With an uncertain common-value component, projecting bidders draw distorted inferences about others' information. This misinference is stronger in second-price and English auctions, reducing their allocative efficiency compared to first-price auctions.

Keywords: Auctions; Projection Bias; False-Consensus Effect; Overbidding.

JEL Classification: D03; D44; D82; D83.

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1 Introduction

Evidence from psychology and economics suggests that people mispredict others' preferences in a systematic way: we tend to believe that others' tastes are more similar to our own than they actually are (Ross et al., 1977; Marks and Miller, 1987; Krueger and Clement, 1994; Engelmann and Strobel, 2012; Orhum and Urminsky, 2013). For instance, those with a strong taste for particular varieties of art, sports, or wine tend to overestimate how many share these tastes. Such misperceptions also pervade socially important domains, such as preferences for income redistribution (Cruces et al., 2013), political candidates (Delavande and Manski, 2012), and paternalistic interventions (Ambuehl et al., 2019). While a large literature provides empirical evidence documenting this bias—known as the "false-consensus effect" or "taste projection"—there is little research studying its economic implications.

In this paper, we develop a model of taste projection and explore how it affects bidding strategies, efficiency, and revenue across a variety of auction formats and environments. In many auction formats, a bidder's strategy crucially hinges on her beliefs about the distribution of her opponents' valuations. Taste projection implies that these beliefs are excessively influenced by one's own private value, and thus bidders with different preferences perceive this distribution differently. In particular, bidders with high private values overestimate the likelihood that others have high valuations, whereas bidders with low private values think the opposite.[1] These misperceptions naturally distort bidding behavior whenever idiosyncratic tastes influence participants' valuations for the good on sale; e.g., in auctions for artwork, wine, books, automobiles, real estate, memorabilia, and most consumer goods sold in internet auctions.

To give a concrete example, consider Bruce and Sheila bidding for an apartment in downtown Melbourne. Suppose their values are drawn independently from a distribution with an average value equal to 500k (Australian) dollars. Bruce grew up in the country, and is not fond of living in the congested city (although he lives there for work); he has a below-average valuation of $300k. Sheila, on the other hand, grew up in Melbourne, just blocks away from the apartment up for sale; she has an above-average valuation of $700k. If Bruce and Sheila were fully rational, they would realize that their own valuation should not affect their estimate of an anonymous rival's valuation—both would correctly expect it to match the average value (i.e., $500k). With taste projection, however, Bruce projects his dislike for city living onto Sheila, estimating her valuation to be similar to his, and hence lower than $500k. Likewise, Sheila projects her fond memories of her childhood neighborhood onto Bruce, mistaking his valuation to be similar to hers, and hence

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[1]Such taste-dependent beliefs also arise in rational models with affiliated (or correlated) values. The crucial difference is that, in those models, bidders share a common prior about the distribution of values and correctly understand how others' posterior beliefs about this distribution depend on their values. In our model of projection, instead, a bidder fails to realize the (excessive) extent to which her own private value shapes her beliefs about the distribution of others' valuations and similarly fails to appreciate the heterogeneity in perceptions across players with different values.

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higher than $500k.

Now imagine these two face each other in a first-price sealed-bid auction, where it is optimal to bid less than one's value. It is easy to see that taste projection leads Sheila to shade her bid too little compared to what a rational bidder would do. What is maybe less obvious at first glance, however, is that taste projection induces Bruce to make the same mistake. Since Bruce thinks Sheila's value is similar to his own, he not only underestimates her unconditional expected value, but also *overestimates* her value conditional on it being lower than his own. Thus, when deciding how much to shade his bid, he exaggerates the likelihood that Sheila's value is just below his and therefore bids too aggressively. More generally, projection leads bidders to underestimate the heterogeneity in values and exaggerate the competitiveness of others' bids, which, in turn, induces them to bid above the rational benchmark. Hence, our model provides an explanation for the widely-documented phenomenon of overbidding in first-price auctions with private values (Harrison, 1989; Cox et al., 1992; Kagel and Levin, 1993; Goeree et al., 2002).

In contrast, suppose that Bruce and Sheila face each other in a second-price sealed-bid auction for that same apartment. Their mistaken beliefs about each others' valuations now have no bearing on their bidding, because it is a weakly dominant strategy to bid one's value, regardless of the competitors' behavior. The key difference is that misperceptions of others' values distort behavior only when bidding incentives are linked to the perceived intensity of competition, and this link is absent in the second-price auction. Therefore, in private-value environments, the first-price format yields a higher expected revenue than the second-price one, thereby breaking the famous revenue-equivalence principle.

Taste projection also introduces an inferential error when the good for sale has an unknown common-value component. In such settings, bids reveal private information about the common value, and a bidder's strategy should therefore account for this information. A taste projector, however, systematically develops biased estimates of others' information and hence of the common value. To illustrate, suppose that Bruce and Sheila participate in an English auction for the apartment. This time, however, bidders also care about the cost of a pending maintenance project—which is the same for all bidders—and each of them makes an independent estimate of this cost. Moreover, suppose that Bruce and Sheila observe that another bidder, Priscilla, drops out of the auction at a relatively low price. Projection leads Bruce and Sheila to draw different, and biased, inferences regarding Priscilla's estimate of the maintenance cost. Bruce—who is not fond of the city—over-attributes Priscilla's low willingness to pay to a low private taste for the apartment. Sheila, on the other hand, presumes that Priscilla shares her preference for the neighborhood and thus over-attributes Priscilla's low willingness to pay to pessimistic information about the maintenance cost. Consequently, upon observing exactly the same event, Sheila develops a more pessimistic view than Bruce about the common value. This may lead Sheila to lose the auction, even

though she has the highest valuation for the apartment. In this way, the inferential errors induced by taste projection generally harm allocative efficiency.

To formalize these insights, in Section 2 we develop a tractable model of taste projection (building on Gagnon-Bartsch, 2016) that is applicable to Bayesian games, and we incorporate it within an otherwise standard auction environment with private values. In our baseline model, we assume private values are independently drawn from a common distribution F. Under projection, a bidder with private value t_i wrongly thinks that the private values of his opponents are drawn from a distribution $\widehat{F}(\cdot|t_i)$ that is overly concentrated around his own value, t_i. Specifically, Bidder i with private value t_i perceives the private value of Bidder j as $\hat{t}_j = \alpha t_i + (1 - \alpha) t_j$—that is, a convex combination of his own value and Bidder j's true value. This approach is an interpersonal analogue of Loewenstein, O'Donoghue, and Rabin's (2003) (henceforth LOR) model of *intra*personal projection bias in which an individual exaggerates the similarity between his future and current tastes. The parameter α measures the extent of this bias: $\alpha = 0$ reflects the rational benchmark, while $\alpha = 1$ captures full projection whereby a bidder believes others share his exact taste.[2]

Our solution concept assumes players are naive about their own bias and that of others, but are otherwise rational. Hence, each Player i believes he faces a Bayesian game in which all players agree that private values are distributed according to $\widehat{F}(\cdot|t_i)$. Solving the model is then relatively straightforward: Player i plays a Bayesian-Nash-Equilibrium (BNE) strategy of the auction where $\widehat{F}(\cdot|t_i)$ is commonly known.

In Section 3, we analyze bidding with independent private values. In first-price and Dutch auctions—where a bidder must estimate his strongest competitor's valuation conditional on himself having the highest—projection leads all bidders to exaggerate the intensity of competition and overbid. We call this distortion the *competition effect*. In contrast, projection has no effect on bidding in second-price and English auctions. We thus show in Section 4 that the competition effect breaks revenue equivalence: first-price (or Dutch) auctions fetch a higher expected revenue than second-price (or English) auctions.

In addition to choosing the auction format, a seller may increase revenue by posting a (public) reserve price. Section 5 analyzes the reserve price chosen by a sophisticated seller who is aware of bidders' projection bias. With an optimal reserve price, a first-price auction continues to raise more revenue than a second-price one due to the competition effect. Moreover, the optimal reserve price differs across formats: it matches the rational benchmark in a second-price auction, while it is strictly lower in a first-price auction. Intuitively, the benefit of a reserve price in a first-price auction is to induce high-value bidders to bid more aggressively at the cost of excluding (some) low-value bidders; yet, since the competition effect induces all bidders to bid more aggressively, the seller is less willing to exclude low-value bidders. We show that the optimal reserve price is

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[2]Section 2 also reviews the evidence motivating our model of taste projection.

consequently decreasing in both the extent of projection and the number of bidders. Since this diminished price raises the probability that the good is sold, projection also increases efficiency relative to the rational benchmark. These results contrast with classical theory, which predicts an optimal reserve price that is substantially higher than the seller's own value and independent of the number of bidders (Myerson, 1981; Riley and Samuelson, 1981). Taste projection can therefore help explain the broad empirical observation that, in practice, reserve prices are often lower than predicted by these canonical models (Ashenfelter, 1989; Ashenfelter and Graddy, 2003; Bajari and Hortaçsu, 2003; Hasker and Sickles, 2010).[3]

In Section 6, we consider settings where bidders observe private signals about an unknown common-value component of the good for sale. We show that taste projection causes high-value bidders to form more pessimistic inferences about the common value and hence bid less aggressively. This distortion—termed the *misinference effect*—harms allocative efficiency. Intuitively, for a bidder with a high private value, taste projection exacerbates the bad news conveyed by winning: when such a bidder wins, he infers that other bidders with similar private values were unwilling to bid as much as him and thus must have especially low estimates of the common value. In other words, projection causes high-value bidders to exaggerate the potential winner's curse associated with winning the auction. In second-price and English auctions, relative to rational bidding, this misinference effect makes it more likely that a player with an optimistic common-value signal outbids the player with the highest private value, thereby increasing the probability of an inefficient outcome. In first-price auctions, however, the misinference effect is countered by the tendency for bidders with high private values to bid more aggressively due to the competition effect. We thus show that, under taste projection, the efficiency of first-price auctions always exceeds that of second-price and English auctions (and it may even exceed the rational benchmark).

We also consider two additional extensions of our baseline framework in Section 7. First, Section 7.1 analyzes asymmetric auctions with independent private values. As in Maskin and Riley (2000), a strong bidder has a valuation drawn from a distribution that dominates that of a weak bidder. In a first-price auction, a weak bidder bids more aggressively than a strong bidder conditional on both of them having the same value, which may result in an inefficient allocation. Projection, however, reduces the perceived asymmetry between bidders: a weak (resp. strong) bidder underestimates how "strong" (resp. "weak") his opponent really is and hence has a *lower* (resp. *greater*) incentive to bid aggressively. In addition, both types of players bid too aggressively due to the competition effect. As a result, projection increases both efficiency and revenue. Second, Section 7.2

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[3]Relatedly, Crawford et al. (2009) derive the optimal reserve price for symmetric first-price and second-price auctions when bidders have independent private values and engage in "level-k" thinking. Like in our model, "level-k" thinking only influences the optimal reserve price in the first-price auction, and the seller may be able to raise more revenue than with rational bidders. Yet, differently from our model, the optimal reserve price under "level-k" can be either lower or higher than in the rational benchmark depending on the distribution of values. Moreover, they focus on two-bidder auctions, and hence do not analyze how the optimal reserve price varies with the number of bidders.

contrasts bidding under projection with rational bidding when valuations are affiliated. The notable difference is that rational bidders realize the extent to which those with different tastes hold different beliefs about the distribution of values, but projectors underappreciate this heterogeneity in beliefs. The competition effect again causes bidders to bid too aggressively in first-price auctions. Projection can therefore overturn the revenue ranking implied by Milgrom and Weber's (1982) celebrated "linkage principle": the first-price format can actually yield a higher expected revenue than the second-price one.

Overall, our analysis reveals that taste projection promotes revenue and efficiency in first-price auctions, but it hurts them in second-price and English auctions. Indeed, this pattern emerges in all of the various environments we consider, irrespective of whether the bidders' values are purely private, (a)symmetrically distributed or correlated, and of whether the auction has a reserve price. Hence, when selling an object to taste-projecting bidders, both sellers and social planners might favor first-price auctions over other common auction formats.

This paper contributes to the literature incorporating various psychological biases into the analysis of auctions and strategic reasoning more broadly; see Eyster (2019) for a thorough review. For instance, Eyster and Rabin's (2005) "cursed equilibrium" provides a compelling explanation for overbidding in common-value auctions wherein people neglect the informational content of others' behavior and consequently fail to appreciate that bids reveal private information in equilibrium. Taste projection complements this explanation given that overbidding is also widely observed in private-value auctions where cursedness has no traction.[4] Our solution concept is also closely related to Madarász's (2012, 2016) "information projection equilibrium" in which players exaggerate the extent to which their private information is known by others. However, our paper differs from Madarász (2012, 2016) both because we focus on projection of preferences rather than information and in our application—he considers social investment, performance evaluation, strategic communication, and bilateral trade with adverse selection.[5] Relatedly, in a paper developed independently yet simultaneously with ours, Breitmoser (2019) shows that a particular combination of information and taste projection explains overbidding in experimental auctions. In contrast to our paper, Breitmoser (2019) primarily focuses on reanalyzing data from prior laboratory experiments and hence abstracts from a general theoretical treatment of auctions (e.g., different auction formats, revenue, efficiency, and optimal reserve prices). We further compare our framework with his after presenting our model.

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[4]Furthermore, when the good for sale has a common-value component, our model generates a form of misinference from others' behavior that is distinct from cursedness. Unlike in Eyster and Rabin (2005), taste projecting players are fully aware that others' bids depend on their private information but, because they under-appreciate the heterogeneity in beliefs about the distribution of private values, they mispredict the mapping from others' signals to their bids.

[5]In an auction context, information projection (roughly) implies that Bidder i with private value t_i wrongly assumes that his opponent (i) likely knows t_i and (ii) best responds to this information. Thus, in contrast to taste projection, information projection does not necessarily cause Bidder i to think that his opponent's valuation is like his.

We also contribute to a small but growing theoretical literature studying the implications of taste projection and the false-consensus effect.[6] For instance, Goeree and Grosser (2007) explore the consequences of a false-consensus effect in two-party voting settings and show that voters' miscalculated probabilities of being pivotal can lead to inefficient election outcomes. More recently, Gagnon-Bartsch (2016), Bohren and Hauser (2018), and Frick, Iijima, and Ishii (2020) examine how misspecified beliefs about others' preferences interfere with social learning. Finally, Frick, Iijima, and Ishii (2019) show how the false-consensus effect may arise when agents neglect the assortative nature of matching when interacting with one another.

2 Model

Consider a good up for auction with $N \geq 2$ risk-neutral bidders. Each Bidder $i \in \{1, \ldots, N\}$ has a private value for the good equal to $t_i \in \mathcal{T} \equiv [\underline{t}, \overline{t}] \subseteq \mathbb{R}$. Values are independently and identically distributed across bidders according to CDF $F : \mathcal{T} \to [0, 1]$. We assume that F admits a smooth, positive density $f \equiv F'$ and a monotone hazard rate.

We focus on two canonical auction formats: first-price and second-price sealed-bid auctions. Each Bidder i simultaneously submits a bid $b_i \in \mathbb{R}$, and the highest bidder is allocated the good. The winner pays his bid in the first-price format and the second-highest bid in the second-price format. Letting p denote the auction price, Bidder i's payoff is thus $t_i - p$ if he wins and 0 otherwise.

While our baseline setup has symmetric bidders and independent private values, in Sections 6 and 7 we also consider asymmetric bidders, affiliated private values, and a good with both private and common-value elements. Furthermore, with purely private values, our analysis and results for the first-price (resp. second-price) sealed-bid auction immediately apply to its strategically equivalent open format, the Dutch (resp. English) auction. We discuss the differences that arise between second-price and English auctions with common-value elements in Section 6.

In our formulation of taste projection, we assume each bidder's perceived distribution of private values is excessively concentrated around his own taste. Before describing this model, we provide a brief overview of the motivating evidence.

2.1 Motivating Evidence

Several strands of research suggest that people systematically mispredict others' preferences. A large literature in psychology studies "interpersonal projection bias" and the "false-consensus ef-

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[6]The term "false-consensus effect" is often used broadly to describe situations where individuals overestimate the prevalence of their personal traits, preferences, beliefs, or actions. In this paper, we focus specifically on projection of preferences. For other models that capture the "false-consensus effect" via projection of either actions or beliefs, see Williams (2013), Rubinstein and Salant (2016), Wang (2020), Jimenez-Gomez (2019) and Levin and Zhang (2019).

fect": the tendency for people to perceive their own tastes and behaviors as more common than they really are. The seminal study by Ross et al. (1977)—along with many similar studies that followed—find a positive correlation between subjects' own stated preferences and their estimates of others' preferences across many domains (e.g., art, sports, wine, consumer products, politics, risk).[7] While this correlation may be rational when there is uncertainty about others' preferences (Dawes 1989, 1990; Prelec, 2004), later studies suggest that these perceptions reflect a systematic bias, whereby subjects rely on their own preference too heavily when making predictions about others. For instance, Krueger and Clement (1994) show that a false-consensus bias remains even when subjects have information about others' preferences—subjects weight their own preference significantly more than those of anonymous others when making population predictions.[8] In incentivized experiments, Engelmann and Strobel (2012) and Ambuehl et al. (2019) similarly find that a false-consensus bias remains if subjects must exert minimal effort to view information on others' choices. Preference misperceptions therefore appear robust even in settings with ample opportunity to observe others (where rational explanations due to limited information are tenuous).[9]

Relatedly, economists have documented several instances of *intra*personal projection bias, where people exaggerate the degree to which their future tastes will resemble their current tastes (Chang et al. 2018; Busse et al. 2015; Simonsohn 2010; Conlin et al. 2007; and LOR 2003). To the extent that others' tastes are as difficult to predict as one's own, we would expect the logic underlying intrapersonal projection bias—that we "mentally trade places" with our future selves, and in doing so, project our current preferences—to similarly apply when empathizing with others. Indeed, Van Boven and Loewenstein (2003) show that transient preference states known to warp subjects' predictions of their own future preferences also distort their predictions of *others'* preferences. For instance, subjects tasked with predicting whether others would prefer food or water made predictions that were strongly biased in the direction of their own exercise-induced thirst. More economically relevant, Van Boven et al. (2000, 2003) show that sellers who experience an endowment effect project their high valuation of a good onto the valuations of potential buyers, causing sellers to set inefficiently high prices. Our model captures a similar intuition, yet we focus on buyers projecting their own valuations onto competing buyers.

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[7]Marks and Miller (1987) document the false-consensus effect in 45 studies published in the decade following Ross et al. (1977). Mullen et al. (1985) find robust evidence of the effect in a large meta-study, and show that it is insensitive to differences in the actual degree of consensus or the generality of the predicted population. Evidence on the false-consensus effect also spans a broad range of domains, including political preferences (e.g., Brown, 1982); preferences over income redistribution (e.g., Cruces et al., 2013); and risk preferences (e.g., Faro and Rottenstreich, 2006).

[8]When predicting the percentage of others who endorse a preference, subjects in Krueger and Clement (1994) weight their own preference nearly twice as much as that of an anonymous other.

[9]Relatedly, Delavande and Manski (2012) show that survey respondents in the American Life Panel demonstrate a false-consensus bias with respect to preferences over political candidates in both the 2008 U.S. presidential election and 2010 U.S. congressional election. Furthermore, they find that respondents continue to exaggerate the similarity between their own and others' preferences even after the release of poll results, further indicating that taste-dependent (mis)perceptions can persist despite abundant information regarding others' tastes.

2.2 Projection of Private Values

Building on Gagnon-Bartsch (2016), our model of taste projection assumes a player's own private value t has undue influence on his perceived distribution of others' tastes: the player misperceives this distribution to be $\widehat{F}(\cdot|t)$, which—relative to the true distribution—overweights the likelihood of values near t. We will also assume players are naive about this bias: each neglects that he (and others) misperceive the distribution.

To model these distorted perceptions, we follow an approach analogous to LOR's (2003) model of intrapersonal projection bias: Player i with a realized value t_i believes the value of any other Player j is

$$\hat{t}_j(t_i) \equiv \alpha t_i + (1-\alpha)t_j, \tag{1}$$

for some $\alpha \in [0,1]$. Player i perceives j's valuation as closer to his own than it really is. Namely, similar to LOR, Player i treats j's value as if it were a convex combination of j's true value and i's own value. The parameter α captures the "degree of projection": $\alpha = 0$ is the rational benchmark, while $\alpha = 1$ represents the extreme case where a player believes others share his exact taste. For simplicity, we assume the degree of projection is equal across players.

Perceptions of the Value Distribution. Since others' valuations are not observed, we must consider how Player i's misperception in Equation (1) translates into beliefs about the distribution of values. Replacing t_j in Equation (1) with its associated random variable, Player i's perception of others' valuations is then described by the random variable

$$\widehat{T}(t_i) \equiv \alpha t_i + (1-\alpha)T, \tag{2}$$

where $T \sim F$ is the true random variable describing each individual's valuation.[10] In other words, each Player i perceives a distribution of tastes that, relative to reality, is overly concentrated around his own taste, t_i.

Under the specification above, a projecting player misperceives the support of private values, $\mathcal{T}$, whenever $\mathcal{T}$ is bounded. A bidder with valuation t_i thinks the lowest possible value is $\underline{t}(t_i) \equiv \alpha t_i + (1-\alpha)\underline{t}$ and the highest is $\bar{t}(t_i) \equiv \alpha t_i + (1-\alpha)\bar{t}$, and hence his perceived support is $\widehat{\mathcal{T}}(t_i) \equiv [\underline{t}(t_i), \bar{t}(t_i)] \subset \mathcal{T}$. It is worth emphasizing, however, that our results do not hinge on this: all of our qualitative results would hold with a perceived distribution that is approximately the same as in Equation (2), yet slightly modified to have support $\mathcal{T}$.[11]

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[10]This definition naturally extends to cases where players are not symmetric—and thus values are not identically distributed—and to cases where values are correlated. We consider both of these extensions in Section 7.

[11]For instance, the perceived distribution could be such that Player i believes that any opponent's value is drawn from $\widehat{T}(t_i)$ with probability $1-\varepsilon$, and from $U(\underline{t}, \bar{t})$ with probability ε. For ε sufficiently small, this distribution has the same support as the true one, yet leads to the same qualitative conclusions delivered by our simpler approach.
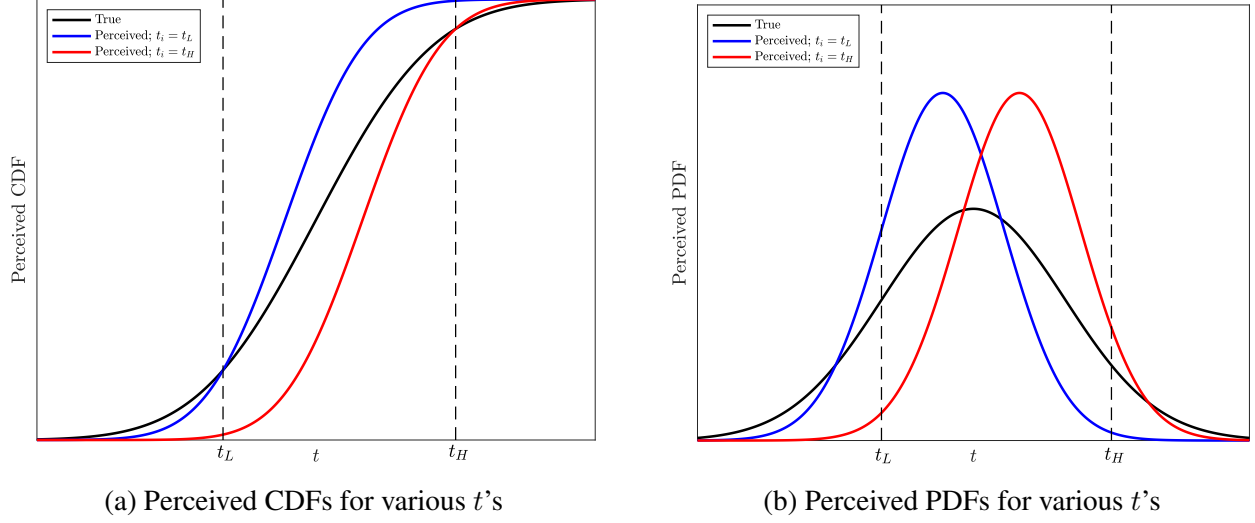
(a) Perceived CDFs for various t's

(b) Perceived PDFs for various t's

Figure 1: *Perceived CDFs and PDFs of a bidder with a below-average valuation, t_L, and an above-average valuation, t_H.*

Our formulation of projection pins down the perceived distributions held by projecting players, $\{\widehat{F}(\cdot|t)\}_{t\in\mathcal{T}}$, in terms of the true distribution, F, and the projection parameter, α. Each player perceives a distribution with the same shape as F, but with the probability mass compressed around his own value. In particular, Player i with private value t_i believes the CDF of valuations is

$$\widehat{F}(t|t_i) = \Pr\left(\widehat{T}(t_i) \leq t\right) = \begin{cases} 0 & \text{if } t < \underline{t}(t_i) \\ F\left(\frac{t-\alpha t_i}{1-\alpha}\right) & \text{if } t \in [\underline{t}(t_i), \overline{t}(t_i)] \\ 1 & \text{if } t > \overline{t}(t_i). \end{cases} \tag{3}$$

Notice that these perceived distributions inherit our assumptions on F: each $\widehat{F}(\cdot|t_i)$ admits a smooth, positive density and a monotone hazard rate.[12] Additionally, let $\widehat{\mathbb{E}}[\cdot|t_i]$ denote expectations with respect to Player i's perceived distribution, $\widehat{F}(\cdot|t_i)$, and let $\mathbb{E}[\cdot]$ denote expectations with respect to the truth, F. For instance, Player i's perceived average valuation is $\widehat{\mathbb{E}}[T|t_i] = \int_{\underline{t}(t_i)}^{\overline{t}(t_i)} t\hat{f}(t|t_i)dt$.

The family of perceived distributions described above exhibits several intuitive properties.

Observation 1. *Perceived distributions $\{\widehat{F}(\cdot|t)\}_{t\in\mathcal{T}}$ have the following properties:*

1. *A projecting bidder with private value t thinks the mean value is $\widehat{\mathbb{E}}[T|t] = \alpha t + (1-\alpha)\mathbb{E}[T]$.*

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[12] We similarly denote Player i's perceived density of valuations by $\hat{f}(\cdot|t_i)$, which is obtained by differentiating (3):

$$\hat{f}(t|t_i) = \left(\frac{1}{1-\alpha}\right) f\left(\frac{t-\alpha t_i}{1-\alpha}\right) \text{ for } t \in \widehat{\mathcal{T}}(t_i).$$
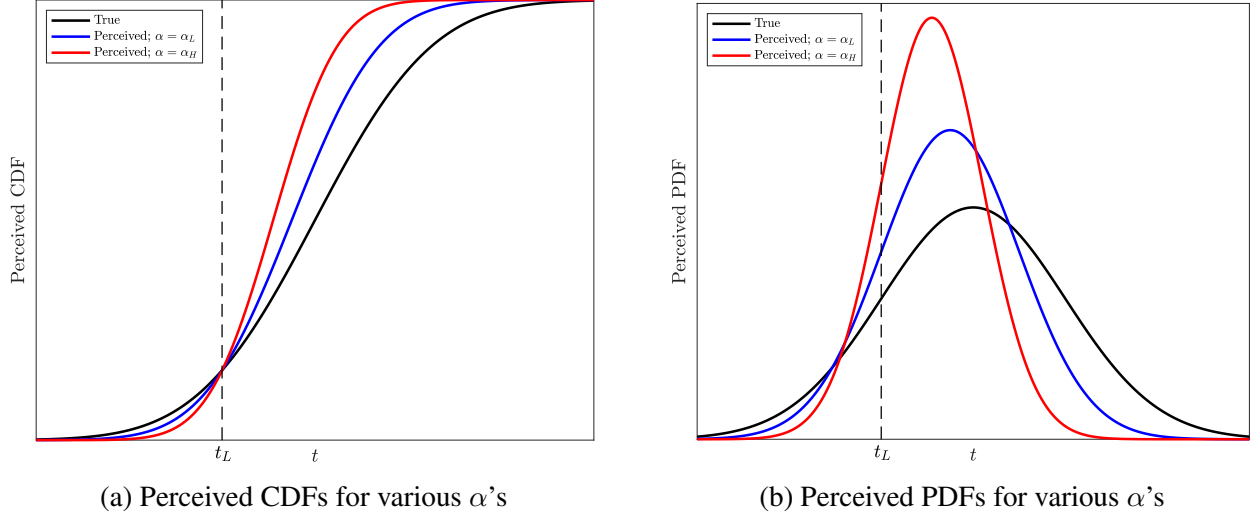
(a) Perceived CDFs for various α's

(b) Perceived PDFs for various α's

Figure 2: *Perceived CDFs and PDFs of a bidder with a below-average valuation, t_L, for two different degrees of projection, α_L and $\alpha_H > \alpha_L$.*

2. *A projecting bidder believes the variance in private values is $(1-\alpha)^2 \mathrm{Var}[T]$.*

3. *The perceived distribution of a projecting bidder with private value t first-order stochastically dominates (FOSD) that of a bidder with private value $t' < t$.*

4. *The perceived distribution of a projecting bidder with private value t is a counterclockwise rotation of the true distribution. That is:* (a) $\widehat{F}(x|t) < F(x)$ if $x < t$; (b) $\widehat{F}(x|t) > F(x)$ if $x > t$; and (c) $\widehat{F}(t|t) = F(t)$. *Furthermore, for all $x \neq t$, $|\widehat{F}(x|t) - F(x)|$ is strictly increasing in α.*

A bidder's perceived distribution of values below his own often plays an important role in his bidding strategy. Notice that Property 4, above, implies that each bidder overestimates this conditional distribution (in the sense of FOSD).

To give an example of a perceived distribution, suppose that in reality $T \sim U(0, \bar{t})$. Our model implies that Player i still thinks T is uniform, but over a support compressed around t_i; namely, $T \sim U(\alpha t_i, \alpha t_i + (1-\alpha)\bar{t})$. For a visual example, Figure 1 considers normally-distributed values and shows the perceived CDFs and PDFs of two bidders—one with a below-average valuation and the other with an above-average valuation. Notice that the perceived CDF of the high-value bidder first-order stochastically dominates that of the low-value bidder, and both perceived distributions are less dispersed than the true one. Figure 2 displays how a bidder's perceived distribution varies with α.

Higher-Order Beliefs. We assume that a projector is *naive* about his bias: he neglects that he and others mispredict the distribution of preferences and therefore fails to appreciate that others

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form discrepant perceptions of this distribution. A bidder with private value t thus believes that (i) all others think that private values are distributed according to $\widehat{F}(\cdot|t)$, and that (ii) this mutual perception is common knowledge. In essence, people imagine they are playing a game with common knowledge of the environment when in fact perceptions are heterogeneous across players. Our naivete assumption is motivated by the idea that people who are ignorant about their own projection bias are likely not carefully attending to others' projection bias.[13]

Naivete is a key feature differentiating our model of taste projection from rational models in which a player's own private value influences his beliefs about others' values (e.g., models with correlated values or uncertainty about the distribution). A rational player in such settings correctly understands how other agents' beliefs about the value distribution depend on their own value. In Section 7.2, we demonstrate how our projection framework naturally extends to these settings where values are not independent, leading players to form posterior beliefs about others' valuations that are too tightly concentrated around their own. Our naivete assumption naturally extends to these settings as well: a projecting player thinks all others share his misperception of the map between any given player's own value and her posterior beliefs about others' values.[14]

Solution Concept.　We close the model with a modified version of Bayesian Nash Equilibrium (BNE). Aside from misperceptions about F (and about others' misperceptions of F), we assume bidders are otherwise rational and believe their opponents are rational. Each player maximizes his expected payoff according to his distorted beliefs and the presumption that others share his misspecified model. Therefore, each Player i plays a BNE strategy of the "perceived auction" in which $\widehat{F}(\cdot|t_i)$ is indeed the commonly-known taste distribution. We call the resulting profile of strategies a *Naive Bayesian Equilibrium*.

To formalize this concept, let Γ denote the true auction game under consideration. To analyze "near by" games with different distributions of values, let $\Gamma(\widehat{F})$ denote that same game when the distribution is instead $\widehat{F}$; all other elements of $\Gamma(\widehat{F})$ are identical to Γ. Player i thinks he is playing the game $\Gamma(\widehat{F}(\cdot|t_i))$ and presumes that behavior will constitute a BNE of $\Gamma(\widehat{F}(\cdot|t_i))$. For each $j \in \{1, \dots, N\}$, let $\tilde{\beta}_j^i : \widehat{\mathcal{T}}(t_i) \to \mathbb{R}$ denote a pure strategy for Player j within the perceived game $\Gamma(\widehat{F}(\cdot|t_i))$, and let $\tilde{\beta}^i = (\tilde{\beta}_1^i, \dots, \tilde{\beta}_N^i)$ denote a profile of such strategies; let $B\left(\Gamma(\widehat{F}(\cdot|t_i))\right)$ denote the set of such profiles that are a BNE of $\Gamma(\widehat{F}(\cdot|t_i))$.[15] Our solution concept is as follows:

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[13]Although studies on the false-consensus effect rarely elicit second-order beliefs, the few that do, e.g. Egan et al. (2014), find that people greatly overestimate how many share their second-order beliefs, which suggests naivete.

[14]Our naivete assumption also departs from much of the literature on non-common priors, which assumes that individuals have rational expectations about the distribution of heterogeneous beliefs across players; see Harrison and Kreps (1978) or Morris (1996).

[15]As $\widehat{F}(\cdot|t)$ inherits our assumptions on F, existence of such a BNE in the perceived game $\Gamma(\widehat{F}(\cdot|t))$ follows from the existence of a BNE in the original game Γ.

Definition 1. *A strategy profile* $\widehat{\beta} = (\widehat{\beta}_1, \ldots, \widehat{\beta}_N)$ *is a* Naive Bayesian Equilibrium *(NBE) of* Γ *if, for all* $i \in \{1, \ldots, N\}$, *there exists a strategy profile* $\tilde{\beta}^i = (\tilde{\beta}_1^i, \ldots, \tilde{\beta}_N^i) \in B\left(\Gamma\left(\widehat{F}(\cdot|t_i)\right)\right)$ *such that* $\tilde{\beta}_i^i = \widehat{\beta}_i$.

Hence, each Player i introspects about others' behavior within his perceived game, and this process leads him to a BNE $\tilde{\beta}^i$ of that game. He then follows the strategy $\tilde{\beta}_i^i$ dictated by this conjectured equilibrium. A NBE is the resulting strategy profile when each player engages in this reasoning. Finally, note that players do not best respond to the true distribution of actions—they best respond to their *predicted* distributions, and these predictions are systematically biased since each player mispredicts both the distribution of types and the strategies that other types follow.

NBE only requires that each player follows *some* Bayesian Nash Equilibrium strategy of his perceived game. To discipline the analysis, we focus on equilibria with symmetric monotone strategies.[16] Since the symmetric monotone pure-strategy equilibrium is generically unique in each auction environment we consider, this assumption precisely pins down a unique NBE.

Deriving this NBE follows a simple process. First, we solve for the unique symmetric monotone BNE bidding function under the assumption that private values are actually distributed according to $\widehat{F}(\cdot|t)$, which we denote by $\widehat{\beta}(\cdot|t) : \widehat{\mathcal{T}}(t) \to \mathbb{R}$. Player i with private value t_i predicts that all bidders will bid according to $\widehat{\beta}(\cdot|t_i)$, and thus he himself bids $b_i = \widehat{\beta}(t_i|t_i)$. The true map from a player's valuation to his bid in a NBE is then $\widehat{\beta} : \mathcal{T} \to \mathbb{R}$ defined by $\widehat{\beta}(t) \equiv \widehat{\beta}(t|t)$. It is worth emphasizing the meaning of the notation above as it will play a central role in the paper: $\widehat{\beta}(t|t_i)$ is Player i's *perceived bidding function*—what he thinks a player with valuation t will bid; $\widehat{\beta}(t) = \widehat{\beta}(t|t)$ is what a player with valuation t actually bids.

We refer to $\widehat{\beta}(t)$ as the *NBE bidding function*. The remainder of the paper analyzes $\widehat{\beta}(t)$ across various auction formats and environments. To understand the effects of projection, we often compare $\widehat{\beta}(t)$ to the rational symmetric monotone bidding function given the true distribution of values, F. We denote this rational benchmark by $\beta^* : \mathcal{T} \to \mathbb{R}$.

Our solution concept is related to others where players have wrong beliefs about the link between their competitors' types and behavior (e.g., Eyster and Rabin, 2005; Jehiel and Koessler, 2008; Esponda, 2008; Madarász, 2016). In particular, Breitmoser (2019) considers "type projection equilibrium" (TPE) in auctions. However, our model differs from TPE in two significant ways. First, in our model, a player's perceived distribution of others' values smoothly exaggerates the true probability density at values near his own. In a TPE, a projecting player believes his opponents share his exact type with positive probability. While both models predict overbidding in first-price auctions, they do so in very different ways. Specifically, TPE precludes pure-strategy equilibria; hence, in contrast to our approach, overbidding in a TPE stems entirely from mixing. Second, TPE assumes that players fully anticipate and best respond to each others' projection. We

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[16]In Section 7.1, where we consider asymmetric auctions, we can impose an analogous assumption.

think that our naivete assumption—whereby players ignore projection when strategizing—is more realistic insofar as it does not simultaneously assume players are biased yet acutely aware of this bias. We suspect that a player who is sophisticated enough to best respond to others' projection bias *and* determine how others respond to his own bias (as in TPE) would likely recognize and correct his mistake.

3   Bidding Strategies

In this section, we analyze how taste projection affects bidding. First consider a second-price auction (SPA). With rational bidders, the "standard" symmetric BNE bidding strategy is $\beta_{II}^*(t_i) = t_i$: a player bids his valuation of the good. No matter the extent of the bias, this is still an equilibrium under projection.[17] Hence, projection has no effect on bidding:

Proposition 1. *In a second-price auction with independent private values, the strategy profile in which all players bid their private value is a NBE no matter the extent of projection. That is, $\widehat{\beta}_{II}(t_i) = t_i$ and $\widehat{\beta}_{II}$ is thus independent of α.*

Note that this result is entirely independent of any assumptions on the parametric form of projection. It follows entirely from the fact that a bidder's strategy does not depend on his beliefs about others' private values.

Next, suppose bidders engage in a first-price sealed-bid auction (FPA).[18] The symmetric (rational) BNE calls for Bidder i to bid his estimate of the second-highest valuation conditional on him having the highest valuation: $\beta_I^*(t_i) = \mathbb{E}[\max_{j \neq i} T_j | \max_{j \neq i} T_j < t_i]$. Unlike in the SPA, this strategy clearly depends on a bidder's beliefs about others' valuations, and hence projection will distort it.

To clearly demonstrate how our solution concept operates, we derive the NBE strategy step by step.[19] Let $T_{i,1} \equiv \max_{j \neq i} T_j$ denote the highest valuation among bidders other than i. Let $\widehat{F}_1(\cdot|t_i)$ and $\widehat{f}_1(\cdot|t_i)$ denote Player i's perceived CDF and PDF of $T_{i,1}$, respectively.[20] Player i's perceived (symmetric) bidding function, $\widehat{\beta}_I(\cdot|t_i)$, is defined as follows: Player i believes a bidder with valuation t solves

$$\max_b \widehat{F}_1\left(\widehat{\beta}_I^{-1}(b|t_i)|t_i\right) \times (t - b),$$

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[17]The English auction has an equivalent equilibrium: each bidder remains in the auction until the price reaches his private value. Therefore, our results for the second-price format immediately extend to the English one.

[18]Our results for the FPA equally apply to the Dutch auction, since the two formats are strategically equivalent.

[19]For the remainder of the paper, we will omit this step-by-step derivation.

[20]Given our assumption of independent private values, $\widehat{F}_1(t|t_i) = \widehat{F}(t|t_i)^{N-1}$.

yielding the first-order condition

$$\frac{\hat{f}_1\left(\widehat{\beta}_I^{-1}\left(b|t_i\right)|t_i\right)}{\widehat{\beta}_I'\left(\widehat{\beta}_I^{-1}\left(b|t_i\right)|t_i\right)}\left(t-b\right)-\widehat{F}_1\left(\widehat{\beta}_I^{-1}\left(b|t_i\right)|t_i\right)=0.^{21} \tag{4}$$

The symmetric BNE bidding strategy within Player i's model is such that $b = \widehat{\beta}_I\left(t|t_i\right)$ for all $t \in \widehat{\mathcal{T}}(t_i)$, and therefore Equation (4) yields the following differential equation:

$$\frac{d}{dt}\left(\widehat{F}_1\left(t|t_i\right)\widehat{\beta}_I\left(t|t_i\right)\right)=t\hat{f}_1\left(t|t_i\right).$$

Using the initial condition $\widehat{\beta}_I\left(\underline{t}(t_i)|t_i\right)=\underline{t}(t_i)$ to solve this differential equation, Player i thinks an opponent with value t bids[22]

$$\widehat{\beta}_I\left(t|t_i\right)=\frac{\int_{\underline{t}(t_i)}^{t} y\hat{f}_1\left(y|t_i\right)dy}{\widehat{F}_1\left(t|t_i\right)}.$$

The associated NBE bidding function is thus given by

$$\widehat{\beta}_I(t_i) = \widehat{\beta}_I(t_i|t_i) = \widehat{\mathbb{E}}[T_{i,1}|T_{i,1} \le t_1; t_i] = \frac{\int_{\underline{t}(t)}^{t_i} y\hat{f}_1\left(y|t_i\right)dy}{\widehat{F}_1\left(t_i|t_i\right)}. \tag{5}$$

It follows from Equation (5) that the NBE bidding function can be re-written in terms of the rational bidding strategy:

$$\widehat{\beta}_I\left(t_i\right)=\left(1-\alpha\right)\beta_I^*(t_i)+\alpha t_i. \tag{6}$$

Hence, the bidding strategy with taste projection is a convex combination of the rational bid and the bidder's own valuation.[23] It is easy to see then that the bidding strategy under projection is more aggressive than the rational one, and, for $\alpha = 1$, it reduces to bidding one's own value. Intuitively, if a bidder thinks that all of his competitors have the same private value as him, there is no room for shading. The next proposition summarizes these results.

Proposition 2. *Consider a first-price auction with independent private values.*

1. *The NBE bidding function is $\widehat{\beta}_I\left(t\right)=\frac{\int_{\underline{t}(t)}^{t} y\hat{f}_1(y|t)dy}{\widehat{F}_1(t|t)}$.*

2. *$\widehat{\beta}_I\left(t\right)\ge\beta_I^*(t)$, and this inequality is strict for all $t>\underline{t}$. Moreover, $\widehat{\beta}_I\left(t\right)$ is increasing in α.*

The intuition for Part 2 of Proposition 2 is as follows. Bidder i estimates the value of his strongest opponent (conditional on himself having the highest valuation) in order to slightly outbid

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[21]Note that $\widehat{\beta}^{-1}(\cdot|t_i)$ denotes the inverse of $\widehat{\beta}(\cdot|t_i)$; that is, $t=\widehat{\beta}^{-1}(\widehat{\beta}(t|t_i)|t_i)$ for all $t\in\widehat{\mathcal{T}}(t_i)$.

[22]As in the rational model, a bidder believes the lowest type within his model bids his value.

[23]This convex-combination result—which we derive in the proof of Proposition 2—is specific to auctions with independent private values and is not a general implication of our model of taste projection.

her. Hence, it is intuitive that taste projection leads a bidder with a relatively high value to bid more aggressively than the rational benchmark. What is perhaps less obvious is that taste projection leads all bidders to bid more aggressively than the rational benchmark, irrespective of their value. Given that low-value bidders project their values onto others, it is tempting to think they will consequently underbid—after all, projection leads them to underestimate the average value of their opponents. Yet, a bidder's strategy in the FPA depends on his estimate of his strongest opponent's value conditional on himself winning. Because a taste-projecting bidder underappreciates the dispersion in valuations, he both underestimates the values of bidders with higher values than him, and overestimates those of bidders with lower values. This, in turn, implies that all bidders overestimate the intensity of competition and hence bid too aggressively.

Although all players overbid, those with higher private values do so to a greater extent. As a result, taste projection has no effect on efficiency in this setting: the good is still allocated to the bidder with the highest value.

Our overbidding result is consistent with a vast experimental literature documenting a widespread tendency for subjects to bid higher than the risk-neutral Bayesian-Nash-Equilibrium benchmark in first-price auctions with independent private values; see Kagel (1995) and Kagel and Levin (2016).[24] One strand of literature attempts to explain overbidding by appealing to bidders' preferences; e.g., risk aversion (and related concepts such as loss or regret aversion) or "joy of winning" (see Cox et al., 1992; Harrison, 1989; Kagel and Levin, 1993; Filiz-Ozbay and Ozbay, 2007; Lange and Ratan, 2010). A second strand of literature attributes overbidding to cognitive errors. For instance, Goeree et al. (2002) explain overbidding via Quantal-Response equilibrium (QRE), while Crawford and Iriberri (2007) predict overbidding in first-price auctions under Level-k thinking if the distribution of values is not uniform.[25] Our explanation for overbidding shares some similarities with the latter group, but it differs in one important aspect. Unlike Level-k and QRE models, we do not assume that bidders fail to reach the correct equilibrium bid due to a limited depth of reasoning or inability to best respond: taste-projecting players would indeed bid optimally if they held correct beliefs about the distribution of valuations. In other words, strategic errors are not assumed a priori but arise from players best responding to their misspecified models of others' preferences.

Beyond overbidding, taste projection also influences how bidding behavior responds to the number of bidders, N. With rational bidders, an increase in N leads participants to bid more aggressively since "shading" now exposes bidders to a greater chance of losing the auction. With projection, however, this incentive to increase one's bid is diminished: the greater is α, the less

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[24]We should point out, however, that our model does not aptly apply to laboratory experiments on auctions that use induced monetary values—people likely have a reasonable sense of how much others value a given amount of money, which leaves little scope for projection. We therefore suspect that taste projection has larger effects in domains with "homegrown" values and is better suited for testing in the field.

[25]In contrast, projection predicts overbidding even with uniformly distributed private values.

responsive bids are to an increase in the number of participants. Roughly put, since a projecting bidder already exaggerates the intensity of competition, there is less scope for him to increase his bid in response to an increase in N. This is immediate from Equation (6): $\frac{\partial^2}{\partial\alpha\partial N}\widehat{\beta}_I(t) = -\frac{\partial}{\partial N}\beta_I^*(t) < 0$.

Will bidders eventually correct their bias if they participate in multiple auctions? Perhaps surprisingly, in many natural environments the answer seems negative. Although our framework involves bidders mispredicting others' strategies, the NBEs described above exhibit a particular consistency property: each bidder correctly predicts the likelihood that he wins an auction. Thus, a taste-projecting agent who repeatedly participates in first-price auctions may never observe data inconsistent with his misspecified model. For instance, when participants only observe the price when they win, the outcomes that a taste-projecting bidder observes precisely accord with his model's predictions.[26] Furthermore, in second-price auctions, a taste-projecting winner typically observes a price that is lower than expected, but this data should not alter his behavior since the NBE is based on a dominant strategy. Together, these points suggest that the behavior predicted by an NBE may persist even in repeated auctions.

The presence of taste-projecting bidders, however, cannot be exploited by rational bidders: projecting bidders are irrelevant in the SPA, and they actually hurt rational bidders in the FPA by driving up the auction price. Hence, rational bidders are worse off when competing against projecting bidders, even when they are fully aware of their presence.

4 Revenue Implications

We now consider revenue under projection, beginning with a comparison to the rational benchmark.

Proposition 3. *Consider an auction environment with independent private values.*

1. *In a first-price (or Dutch) auction, expected revenue is higher with projection than with rational bidders and is strictly increasing in α.*

2. *In a second-price (or English) auction, expected revenue with projecting bidders is the same as with rational bidders.*

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[26]Matters are different if the winning bid is made public. In this case, losing bidders may observe that winning bids are—on average—higher than they expected and hence come to realize that their predictions are biased. Yet, overbidding might persist even in this case because, to remain competitive against these higher-than-expected bids, losing bidders may bid even more aggressively than predicted by Proposition 2. Moreover, bidders in our model do not expect to learn anything from carefully attending to the distribution of winning bids and may therefore "rationally" ignore this data. Indeed, bidders' misspecified models in this setting are "attentionally stable" in the sense of Gagnon-Bartsch et al.'s (2020) framework whereby seemingly rational inattention can allow misspecified models to persist.

Proposition 3 follows immediately from Propositions 1 and 2: projection does not affect the symmetric equilibrium of the SPA but leads to more aggressive bidding than the rational benchmark in the FPA. This implies that the two formats are not revenue equivalent.

Corollary 1. *Revenue equivalence does not hold with projection: if $\alpha > 0$, then a first-price (or Dutch) auction revenue dominates a second-price (or English) auction.*

Hence, a revenue-maximizing auctioneer who suspects that bidders might suffer from taste projection should always prefer the FPA over the SPA. This does not imply, however, that taste-projecting bidders have the opposite preference. As the following result shows, such bidders expect to attain the same utility in both auction formats, even though their actual payoff is lower in the FPA than in the SPA.[27]

Proposition 4. *Under taste projection, bidders perceive a second-price auction and a first-price auction to be payoff equivalent.*

Intuitively, taste-projecting bidders use their misspecified beliefs to evaluate their expected utility. While these incorrect beliefs do not affect bidding strategies in the SPA, they do affect bidders' expected payments. In particular, a taste-projecting bidder in the SPA overestimates the bid of his strongest opponent and hence overestimates the price he will pay upon winning. Consequently, projecting bidders are equally willing to participate in the format preferred by the auctioneer—the FPA—and the one that they should actually prefer—the SPA.

These results provide a direct testable prediction of our model. Fixing the number of bidders and the good for sale, Corollary 1 implies that the FPA should raise more revenue than the SPA. Proposition 4 additionally removes any concern for "selection" effects, whereby different formats systematically attract different numbers and types of bidders because the seller and bidders openly disagree over the preferred format. Taste-projecting bidders are indifferent among all standard auctions and hence equally willing to participate in any of them. This feature of our model makes it easier, in principle, to test revenue non-equivalence in the field relative to other models.[28]

Finally, in contrast to first-price auctions, projection in other formats can induce underbidding and hence *reduce* revenue compared to the rational benchmark. Consider, for instance, a third-price auction where the highest bidder wins and pays the second-highest losing bid. In a symmetric equilibrium of this auction, risk-neutral rational bidders bid more than their value. Intuitively, since the

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[27]The result in Proposition 4 extends to all "standard" auctions; that is, auctions where (i) the bidder with the lowest value always gets a payoff of zero, and (ii) the winner is the bidder with the highest value.

[28]For instance, risk aversion also breaks revenue equivalence, but it gives rise to selection effects: a risk-neutral seller prefers the FPA to the SPA when bidders are risk averse; yet, buyers prefer the SPA to the FPA if they exhibit decreasing absolute risk aversion (Matthews, 1987). Similarly, with affiliated private values, the SPA raises more revenue than the FPA and, exactly for this reason, bidders prefer the FPA to the SPA. Besides the seminal contribution of Lucking-Reiley (1999), we are not aware of studies testing revenue equivalence in the field.

winner expects to pay a price significantly lower than his bid, bidders have an incentive to "gamble" by bidding more than their value in an attempt to win. However, a taste-projecting bidder—while still bidding more than his value—bids less aggressively than a rational one. Indeed, since he underestimates the dispersion in his competitors' bids, a taste-projecting bidder overestimates the likelihood that, conditional on winning, the third-highest bid is above his value. Hence, he is less willing to gamble.[29]

5 Optimal Reserve Price

In this section, we characterize the optimal reserve price for a seller who faces taste-projecting bidders. We consider a sophisticated seller who is aware of the bidders' bias and sets the revenue-maximizing reserve price accordingly. We show that projection lowers the optimal reserve price in a first-price auction and, contrary to the canonical result under rational bidding, this reserve price is strictly decreasing the number of bidders.

Let $\widehat{\beta}_I(\cdot; r)$ and $\widehat{\beta}_{II}(\cdot; r)$ denote the NBE bidding functions with a public reserve price r in a first- and second-price format, respectively. In the SPA, the symmetric BNE strategy—where each player bids his value—is again the NBE bidding function: $\widehat{\beta}_{II}(t_i; r) = t_i$ (for all $t_i \geq r$).[30] Now consider the FPA. For $t_i \geq r$, the rational symmetric bidding function is equal to Bidder i's expectation of his strongest competitor's value given that t_i is the highest value, taking into account that the "strongest competitor" may be the reserve price (see Myerson, 1981):

$$\beta_I^*(t_i; r) \equiv \frac{\int_r^{t_i} y f_1(y) dy + r F_1(r)}{F_1(t_i)}. \tag{7}$$

Following the logic of Proposition 2, a projecting bidder adopts this same formula, but uses his misspecified distribution of values. Hence, the NBE bidding function is

$$\widehat{\beta}_I(t_i; r) \equiv \frac{\int_r^{t_i} y \widehat{f}_1(y|t_i) dy + r \widehat{F}_1(r|t_i)}{\widehat{F}_1(t_i|t_i)}. \tag{8}$$

Moreover, $\widehat{\beta}_I(t; r) > \beta_I^*(t; r)$ for all $t > r$, and $\widehat{\beta}_I(t; r)$ is strictly increasing in α for $t > r$.[31]

Therefore, for a fixed reserve price, the SPA under projection delivers the same expected revenue as the rational benchmark, while the FPA generates a higher expected revenue. But what

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[29]More generally, for any $k \in \{1, \ldots, N\}$, the symmetric-equilibrium strategy of a taste-projecting bidder in a kth-price auction is $\widehat{\beta}_k(t_i) = (1 - \alpha) \beta_k^*(t_i) + \alpha t_i$. Hence, projection leads to overbidding if and only if $\beta_k^*(t_i) \leq t_i$.

[30]We focus on the unique bidding functions for $t \geq r$. Bidders with $t < r$ are irrelevant and thus ignored.

[31]This follows form an argument similar to the one underlying Proposition 2. Also, note that a projecting player with a sufficiently high valuation might believe that r is below the lower bound of his perceived support of valuations. When this happens, such a player deems r irrelevant and the bidding function in (8) reduces to the one in (5).

reserve price maximizes this revenue?

We assume that the seller knows α and F, and thus can accurately predict how bidders behave for any given reserve price. While we have so far been silent regarding the seller's valuation for the good, the following analysis assumes it is $v_s \in [0, \bar{t})$ and is independent of all t_i. In the rational benchmark, the optimal reserve price, denoted by r^*, solves

$$r^* = \frac{1 - F(r^*)}{f(r^*)} + v_s. \tag{9}$$

This price is identical in first- and second-price formats, it is independent of the number of bidders, and it exceeds the seller's value. Moreover, as shown by Myerson (1981) and Riley and Samuelson (1981), any standard auction with a reserve price equal to r^* is a revenue-optimal mechanism. As the optimal reserve price in the SPA under projection is also equal to r^*, this format yields the same expected revenue as the optimal rational benchmark.

In the FPA under projection, the optimal reserve price maximizes the seller's expected revenue,

$$\Pi_I(r, v_s) \equiv N \int_r^{\bar{t}} \widehat{\beta}_I(t; r) f(t) F(t)^{N-1} \, dt + F(r)^N v_s. \tag{10}$$

We denote this optimal reserve price by $\hat{r}(\alpha, N)$ and consider comparative statics on both the extent of projection and the number of bidders. With rational bidders, $\hat{r}(0, N) = r^*$ for all N.

Proposition 5. *Consider a first-price auction with independent private values. For all $\alpha > 0$, the revenue-maximizing reserve price $\hat{r}(\alpha, N)$ is strictly lower than r^*. Furthermore, $\hat{r}(\alpha, N)$ is decreasing in both α and N, with $\hat{r}(1, N) = v_s$.*

Intuitively, as bids are increasing in the extent of projection, the larger is α, the smaller is the effect of the reserve price on the strategy of high-value bidders. These bidders underestimate the prevalence of competitors who are not willing to pay the reserve price. Hence, projection weakens the competition-enhancing effect of the reserve price on the bids of buyers who are not excluded (i.e., those with $t > r$). In contrast, the probability of excluding some bidders depends on the *true* distribution of values and is thus unaffected by α. Stronger projection, therefore, reduces the benefits of a reserve price, but it does not affect its cost. The seller thus lowers the reserve price if α increases.

With rational bidders, the optimal reserve price is independent of the number of bidders. A higher reserve price pushes bidders who are not excluded to bid more aggressively, but comes at the cost of increasing the likelihood of not selling at all. As N grows, both of these effects become smaller but, remarkably, they perfectly offset each other. This is not the case with taste-projecting bidders: as N grows, the bidding function of bidders who are not excluded increases *by less* than in the rational benchmark (as in Section 3). In contrast, a change in N has exactly the same effect
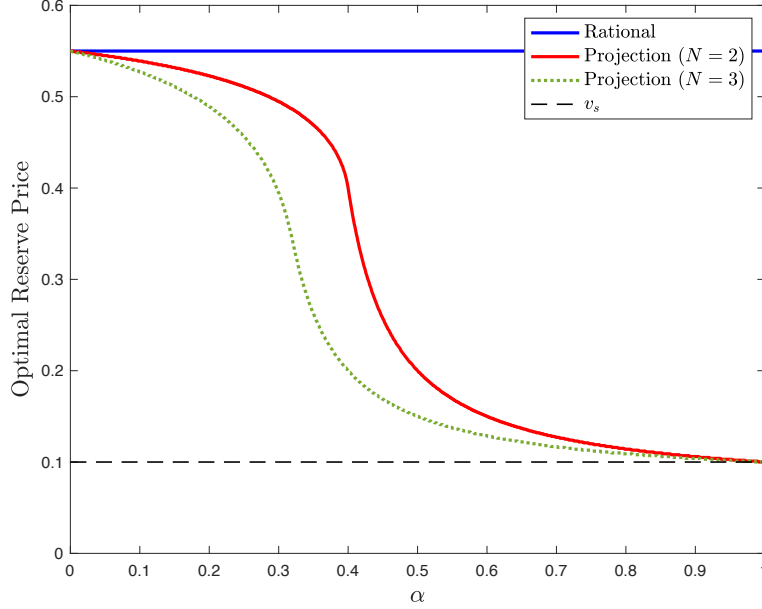
19

Figure 3: *The optimal reserve price in a first-price auction as a function of α for two different values of N (assuming T is uniform on $[0, 1]$).*

on the likelihood of not selling regardless of projection. Thus, with taste projection, the cost of increasing the reserve price is more sensitive to changes in N than the benefit. The optimal reserve price is therefore decreasing in N. Figure 3 shows how $\hat{r}$ changes with both N and α.

A corollary of Proposition 5 is that, at the optimal reserve price, a FPA with projecting bidders is more efficient than both (i) a SPA under projection, and (ii) the optimal auction under rational bidding. Indeed, the probability of not selling the good—which happens when all bids fall below r—is lower in a FPA under projection because the seller posts a lower reserve price.

Proposition 5 also implies that, with an optimal reserve price, the FPA with projecting bidders generates a higher expected revenue than both the SPA with projecting bidders and the optimal auction with rational bidders. With projection, revenue is higher in the FPA than the SPA at the rational-benchmark price, r^*, due to overbidding. Hence, in the FPA, setting the optimal reserve price $\hat{r}(\alpha, N) < r^*$ generates even greater revenue. To summarize:

Corollary 2. *For all $\alpha > 0$, a first-price auction with the optimal reserve price is more efficient and yields a higher expected revenue than a second-price auction with the optimal reserve price. It is also more efficient and yields a higher expected revenue than the optimal auction under rational bidding.*

Figure 4 shows the expected revenue of the FPA with the optimal reserve price as a function of α and compares it to the case without a reserve price. The figure reveals that, with sufficiently strong projection, the expected revenue *without* a reserve price is higher than the expected revenue under rational bidding with the optimal reserve price. This is in fact a general result:

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Proposition 6. *Consider a first-price auction with independent private values. There exists $\tilde{\alpha} \in (0, 1)$ such that, if $\alpha > \tilde{\alpha}$, then the auctioneer's expected revenue under projection with no reserve price is greater than the expected revenue under rational bidding with the optimal reserve price.*
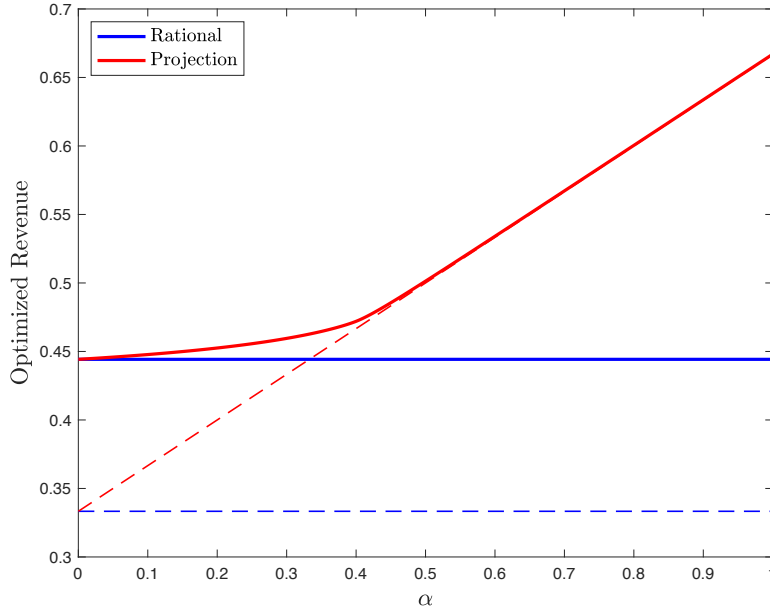

Figure 4: *Expected revenue under the optimal reserve price (solid plots) and expected revenue without a reserve price (dotted plots) assuming T is uniform on $[0, 1]$.*

Since Proposition 5 predicts reserve prices below the rational benchmark, taste projection may help explain why many empirical studies find that reserve prices are typically lower than what the rational theory would predict (Ashenfelter, 1989; Ashenfelter and Graddy, 2003; Bajari and Hortaçsu, 2003). Furthermore, in light of Proposition 6, the auctioneer may find it optimal to forgo a reserve price altogether if there is concern that posting a low reserve price might "de-bias" taste-projecting bidders. Since the optimal reserve price $\hat{r}(\alpha, N)$ deviates from what projecting bidders expect, posting such a price might trigger bidders to realize their mistake.[32] As such, taste projection may also shed light on the frequent use of "absolute auctions" —those with no reserve price at all—despite the seller having a positive reservation value (Hasker and Sickles, 2010).[33]

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[32]If bidders are uncertain about the seller's valuation, however, then they can rationalize a range of reserve prices as optimal. Hence, in this case, they are not necessarily surprised by the seller's chosen reserve price.

[33]Other theoretical explanations for low reserve prices include departures from the independent-private-value (IPV) setting (Levin and Smith, 1996; Hu et al., 2019) and bidders' selection neglect if sellers are privately informed about quality (Jehiel and Lamy, 2015). In contrast, our model predicts low reserve prices even in the most basic IPV setting.

6 Private and Common Values

In this section, we enrich our baseline framework by considering auctions where the good has an unknown common-value element and bidders receive private signals about this common value. In contrast to pure private-value auctions, the good is no longer necessarily acquired by the bidder with the highest taste, and the allocation may therefore be inefficient. This section shows how projection alters the allocative efficiency of different auction formats. With a common-value element, bids convey information about the object's value, yet projection distorts players' inferences from bids: those with high private values over-attribute a competitor's bid to her taste, while those with low private values over-attribute a competitor's bid to her common-value signal.

Our setup follows Goeree and Offerman (2003) who characterize the rational equilibria of a tractable model of auctions with private and common values.[34] In addition to his private taste t_i, each Bidder i observes the realization of a private signal S_i about the common value. Signals are i.i.d. across players with support $\mathcal{S} \equiv [\underline{s}, \overline{s}] \subseteq \mathbb{R}$ and CDF $G : \mathcal{S} \to [0, 1]$. We assume that Bidder i's valuation (net of price) is equal to

$$v_i = t_i + \gamma \sum_{i=1}^{N} s_i,$$

where $\sum_{i=1}^{N} s_i$ denotes a common-value component which, for simplicity, is equal to the sum of all bidders' signals.[35] The parameter $\gamma > 0$ captures the relative weight that bidders' preferences put on the common-value component and is commonly known among them. As in our introductory example, imagine a real-estate auction where bidders have idiosyncratic private values over attributes of the property (e.g., architectural design, neighborhood, etc.), but also commonly value some of its features which are independent of tastes (e.g., the amount of money necessary to maintain or renovate it; the possibility of a large firm moving its headquarters nearby and raising the neighborhood's market value; or the risk of a disaster that may harm the property). There is natural uncertainty surrounding such common-value elements, and bidders obtain independent estimates of them through idiosyncratic sources of information, such as inspection or appraisal.

As in the previous sections, private tastes are i.i.d. with CDF F but, because of projection, a bidder with taste t believes that private tastes are distributed according to $\widehat{F}(\cdot|t)$ defined in (3). To isolate the effects of taste projection, we assume bidders have correct perceptions of G; hence, they project tastes but not information.[36]

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[34]For other models of auctions with private and common values see, e.g., Pesendorfer and Swinkels (2000), Compte and Jehiel (2002) and Pagnozzi (2007).

[35]This formulation of the common-value component follows the "Wallet Game" introduced in Klemperer (1998) and Bulow and Klemperer (2002). Goeree and Offerman (2003) similarly assume that the common value is equal to the average of bidders' signals, which corresponds to $\gamma = 1/N$ in our setting.

[36]This stands in contrast to Breitmoser (2019), who assumes that a player projects both his taste and his signal. In

Although bidders now have a two-dimensional type, we can describe Bidder i's interim expected valuation using a unidimensional summary statistic $\theta_i \equiv t_i + \gamma s_i$, which we define as his "aggregate type". Specifically, Bidder i's interim expected valuation is

$$\mathbb{E}[V_i|\theta_i] = \theta_i + \gamma\,(N-1)\,\mathbb{E}[S], \tag{11}$$

where $\mathbb{E}[S]$ is the unconditional mean of an individual signal. Let $\Theta \equiv [\underline{t} + \gamma\underline{s}, \overline{t} + \gamma\overline{s}]$ denote the true support of the aggregate types whereas $\widehat{\Theta}(t_i) \equiv [\underline{t}(t_i) + \gamma\underline{s}, \overline{t}(t_i) + \gamma\overline{s}]$ denotes Bidder i's perception of this support. For each $\theta \in \Theta$, let $\mathcal{T}(\theta) \subseteq \mathcal{T}$ denote the set of private values that treat θ as a plausible outcome—i.e., $t \in \mathcal{T}(\theta)$ if and only if $\theta \in \widehat{\Theta}(t)$.

For technical purposes, we assume that both $f \equiv F'$ and $g \equiv G'$ are log-concave. This implies that, in the rational benchmark, there exists a unique symmetric equilibrium where each player's strategy depends only on his aggregate type θ. More specifically, our log-concavity assumption guarantees that a higher aggregate type reveals good news about a bidder's common-value signal: $\mathbb{E}[S_i|\theta_i = \theta]$ and $\mathbb{E}[S_i|\theta_i \leq \theta]$ are increasing in θ.[37] We additionally assume that $\mu(\theta) \equiv \mathbb{E}[S_i|\theta_i = \theta] - \mathbb{E}[S_i|\theta_i \leq \theta]$ is increasing in θ.[38] Finally, the definition of a bidder's perceived distribution (Equation 3) implies that the relationships described above extend to projecting bidders' expectations.

The next section describes equilibrium bidding strategies for both rational and projecting bidders. With a common-value element in bidders' valuations, the English auction is no longer strategically equivalent to the SPA; hence, we analyze the English auction separately in Section 6.3.

6.1 Bidding Strategies

First consider the rational benchmark. Let $\theta_{i,1} \equiv \max_{j \neq i} \theta_j$. In a second-price auction, the symmetric BNE calls for Bidder i to bid his expected value of the object conditional on tying with his strongest opponent, since in this event he is perfectly indifferent between winning and losing:

$$\beta_{II}^*(\theta_i) = \mathbb{E}\left[V_i|\theta_{i,1} = \theta_i\right] = \theta_i + \gamma\mathbb{E}[S_j|\theta_j = \theta_i] + \gamma\,(N-2)\,\mathbb{E}[S_j|\theta_j \leq \theta_i].$$

fact, we show that taste projection alone leads a bidder (in equilibrium) to think others have signals excessively similar to his own.

[37]Without independence of tastes and signals, we cannot conveniently "rank" bidders according to a one-dimensional summary statistic. Furthermore, as argued by Goeree and Offerman (2003), log-concavity allows us to use standard techniques for auctions with unidimensional types to derive equilibrium bids. Notice that many common densities are log-concave, including uniform, normal, and exponential.

[38]This regularity condition ensures that as θ increases, a bidder's expected signal increases by more if his aggregate type is equal to, rather than lower than, θ. This condition holds if the conditional density of S_i given $\theta_i = \theta$ is log-concave, and has been assumed in similar models of auctions for goods with both private and common-value elements (see, e.g., Hernando-Veciana, 2009). For instance, it holds when tastes and signals are normally distributed or identically uniformly distributed.

In a first-price auction, the symmetric BNE calls for Bidder i to bid his expectation of his strongest opponent's valuation conditional on winning. This can be expressed as Bidder i's expected valuation conditional on having the highest aggregate type, minus the expected difference between his aggregate type and that of his strongest opponent:[39]

$$\beta_I^*(\theta_i) = \mathbb{E}\left[V_i | \theta_{i,1} \leq \theta_i\right] - \mathbb{E}\left[\theta_i - \theta_{i,1} | \theta_{i,1} \leq \theta_i\right] = \mathbb{E}\left[\theta_{i,1} | \theta_{i,1} \leq \theta_i\right] + \gamma(N-1)\mathbb{E}\left[S_j | \theta_j \leq \theta_i\right].$$

With taste projection, Bidder i's strategy depends on both θ_i and t_i—even if two bidders have identical aggregate types, they bid differently whenever their tastes differ. The NBE strategies for the SPA and FPA take the same form as in the rational case, yet the expectation terms are now with respect to the bidder's misspecified model:[40]

$$\widehat{\beta}_{II}(\theta_i | t_i) = \theta_i + \gamma\widehat{\mathbb{E}}[S_j | \theta_{i,1} = \theta_i; t_i] + \gamma(N-2)\widehat{\mathbb{E}}[S_j | \theta_j \leq \theta_i; t_i], \tag{12}$$

and

$$\widehat{\beta}_I(\theta_i | t_i) = \widehat{\mathbb{E}}\left[\theta_{i,1} | \theta_{i,1} \leq \theta_i; t_i\right] + \gamma(N-1)\widehat{\mathbb{E}}\left[S_j | \theta_j \leq \theta_i; t_i\right]. \tag{13}$$

Importantly, projection systematically distorts Bidder i's equilibrium inference about his opponent's common-value signal and aggregate type. In particular, the higher is Bidder i's own private taste t_i: (i) the more pessimistic is his equilibrium inference about his opponent's signal s_j and hence about the common value of the good, and (ii) the higher is his estimate of his strongest opponent's aggregate type.

Lemma 1. *Fixing $\theta \in \Theta$, the following properties hold over the domain $\mathcal{T}(\theta)$:*

1. *Misinference Effect: $\widehat{\mathbb{E}}[S_j | \theta_j = \theta; t_i]$ and $\widehat{\mathbb{E}}[S_j | \theta_j \leq \theta; t_i]$ are decreasing in t_i.*

2. *Competition Effect: $\widehat{\mathbb{E}}[\theta_{i,1} | \theta_{i,1} \leq \theta; t_i]$ is increasing in t_i.*

To see the intuition for the misinference effect in the FPA and SPA, notice that Bidder i conditions his bid on the event of winning. His expectation of an opponent's common-value signal is then decreasing in his estimate of that opponent's taste. Since projection causes a bidder with a high taste to overestimate his opponents' tastes, it also leads him to underestimate their common-value signals. Simply put, a bidder who desires the good—and thinks others do too—underestimates the signal that a typical competitor would need in order to bid a given amount; a bidder who is lukewarm about the good instead overestimates this signal.

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[39]The second term represents the optimal amount by which Bidder i should shade his bid.

[40]Although a player's bidding behavior in a NBE depends jointly on θ and t, the projecting player thinks that bidders will follow a strategy that depends solely on θ. Hence, each individual's *perceived* bidding function is unidimensional and can thus be simply derived as in the rational case.

Figure 5 displays how the misinference effect distorts bidding under projection by showing the perceived SPA bidding functions of two players with different private tastes. Conditional on both players having the same aggregate type θ, the player with the *lower* private taste bids more. Furthermore, notice that both perceived bidding functions are excessively sensitive to θ relative to the rational bidding function: fixing a player's taste, he overbids if he has a sufficiently high common-value signal, and underbids otherwise. This is most transparent when bidders fully project their tastes: when Bidder i conditions on being tied with his opponent (i.e., $\theta_j = \theta_i$) *and* on having the same taste as his opponent (i.e., $t_j = t_i$), then Bidder i must additionally think he has the same signal as his opponent (i.e., $s_j = s_i$). More generally, when a bidder projects his taste, he naively bids as if his opponent has a similar signal as well. Consequently, a bidder's estimate of the common value, and thus his bid, are unduly influenced by his own signal.
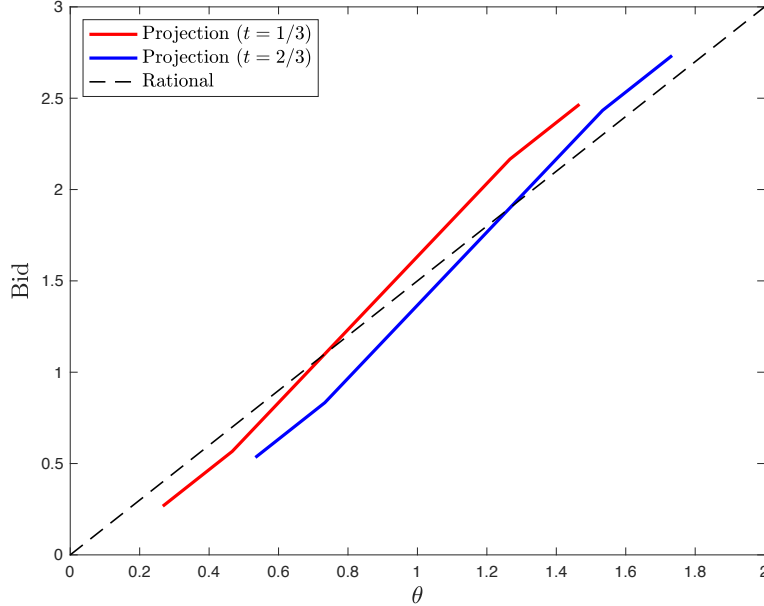

Figure 5: *Perceived bidding strategies in the SPA. The two solid plots show the bidding strategies followed by two players with differing private values (assuming both T and S are uniform on $[0, 1]$). The dotted plot shows the rational symmetric bidding strategy.*

In addition, in first-price auctions, projection also induces a competition effect that is analogous to the one that emerged with purely private values. This effect goes in the opposite direction of the misinference effect: although in equilibrium a bidder with a higher taste makes more pessimistic inferences regarding the common value, he also overestimates the aggregate type of his opponents.

6.2 Efficiency Comparison

Even with rational bidders, auctions with private and common values may lead to inefficient allocations: since the rational bidding function is strictly increasing in a player's aggregate type, the

winner is the player with the highest aggregate type, not the highest private value. Thus, the efficient bidder might lose to an opponent with a sufficiently strong common-value signal. With projection, the additional dependence of bidding functions on t above and beyond θ alters the efficiency property of auctions relative to the rational benchmark. For our analysis, we define efficiency as the probability that the bidder with the highest private value wins.[41]

In second-price auctions, efficiency is harmed by the fact that taste projectors' bids overweight their common-value signals: relative to the rational benchmark, the good will too often go to the player with the highest signal rather than the highest private taste. Intuitively, the misinference effect tends to bias a high private-taste bidder's estimate of the common value downward and bias a low private-taste bidder's estimate upward. In other words, taste projection makes more efficient bidders less optimistic about the common value and hence induces them to bid less aggressively.

In first-price auctions, the efficiency-enhancing competition effect partially offsets the negative effect of misinference and thus increases the chance that the player with the highest private value wins the auction. Therefore, under projection, a first-price auction is more efficient than a second-price one. Furthermore, the competition effect may be strong enough to entirely overturn the misinference effect. In such cases, a first-price auction is more efficient under projection than under rational bidding. Intuitively, this happens when γ, the weight assigned to the common value in bidder's valuations, is sufficiently small. The next proposition summarizes these results.

Proposition 7. *Consider a setting with private and common values.*

1. *In a second-price auction, efficiency with projection is lower than with rational bidders.*

2. *In a first-price auction, there exists $\bar{\gamma} > 0$ such that, if $\gamma < \bar{\gamma}$, then efficiency with projection is higher than with rational bidders.*

3. *For all $\alpha > 0$, efficiency is higher in a first-price auction than in a second-price auction.*

It is important to notice that, as we show in the proof of Proposition 7, projection never improves efficiency compared to the rational benchmark in a second-price auction. Indeed, if the winner in the rational benchmark is inefficient, projection will actually increase the gap in bids between the winner and the other bidders, thereby preserving the inefficient allocation. Yet, because of the misinference effect, whenever the winner with projection is different from the winner in the rational benchmark, the winner with projection is a less efficient bidder.

6.3 English Auction

In an English auction, the price is raised continuously by the auctioneer, and a bidder who wishes to be active at the current price depresses a button. After a bidder quits, he cannot become active

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[41]With rational bidders, all standard auction formats are equally inefficient; see Goeree and Offerman (2003).

again, and all remaining bidders observe the price at which he quit. The winner is the last remaining active bidder. With more than two bidders, English and second-price auctions are not strategically equivalent, because the public drop-out prices contain information about the common value that is relevant for the remaining bidders. In this section, we restrict attention to full-support signals so that a projecting bidder's inference from these prices is always well-defined.[42]

In the symmetric BNE, each bidder follows a profile of strategies $\beta_E^* = \left(\beta_0^*, \beta_1^*, \ldots, \beta_{N-2}^*\right)$, where each element β_D^* specifies how high a bidder is willing to bid after D others have dropped out. Each β_D^* depends on a bidder's aggregate type and the history of the dropout prices, $p_1 < p_2 < \cdots < p_D$. Letting $\theta_{i,n}$ denote the nth-highest aggregate type among all bidders other than i, these strategies are defined recursively as follows:

$$
\begin{aligned}
\beta_0^*(\theta_i) &= \mathbb{E}\left[V_i | \theta_{i,1} = \ldots = \theta_{i,N-1} = \theta_i\right] = \theta_i + \gamma(N-1)\mathbb{E}[S_j|\theta_j = \theta_i], \\
\beta_D^*(\theta_i; p_1, \ldots, p_D) &= \mathbb{E}\left[V_i | \theta_{i,1} = \cdots = \theta_{i,N-D-1} = \theta_i, \theta_{i,N-D} = \tilde{\theta}_D, \ldots, \theta_{i,N-1} = \tilde{\theta}_1\right] \\
&= \theta_i + \gamma\left(N-1-D\right)\mathbb{E}[S_j|\theta_j = \theta_i] + \gamma\sum_{d=1}^{D}\mathbb{E}[S_d|\theta_d = \tilde{\theta}_d],
\end{aligned}
$$

where $\tilde{\theta}_d$ denotes a bidder's inference about the aggregate type of an opponent who drops out at price p_d, and, slightly abusing notation, S_d denotes that opponent's common-value signal.[43]

With taste projection, Bidder i initially plans to stay active up to a price equal to his (biased) expected valuation conditional on being tied with all of his opponents; i.e.,

$$
\widehat{\beta}_0\left(\theta_i|t_i\right) = \theta_i + \gamma(N-1)\widehat{\mathbb{E}}[S_j|\theta_j = \theta_i; t_i].
$$

Then, as the auction unfolds, the bidder revises his bid downwards based on the observed dropout prices. Thus, for $D \in \{1, \ldots, N-2\}$ the NBE bidding function is

$$
\widehat{\beta}_D(\theta_i; p_1, \ldots, p_D|t_i) = \theta_i + \gamma\left(N-1-D\right)\widehat{\mathbb{E}}[S_j|\theta_j = \theta_i; t_i] + \gamma\sum_{d=1}^{D}\widehat{\mathbb{E}}[S_d|\theta_d = \tilde{\theta}_d^i; t_i], \quad (14)
$$

where $\tilde{\theta}_d^i$ denotes Bidder i's inference about the aggregate type of an opponent who drops out at price p_d. This inference is warped by Bidder i's own private taste. Hence, differently from the rational benchmark where all active bidders draw the same inference about the common value when observing a competitor drop out, projection causes bidders with different tastes to draw divergent inferences.

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[42]Specifically, we assume that for every possible dropout price p and private value $t \in \mathcal{T}$, there exists a possible signal realization that is consistent with p given the misspecified model held by type t.

[43]These inferences are defined recursively; initially, $\tilde{\theta}_1$ solves $p_1 = \beta_0^*(\tilde{\theta}_1)$, and then for each $d > 1$, $\tilde{\theta}_d$ solves $p_d = \beta_d^*(\tilde{\theta}_d; p_1, \ldots, p_{d-1})$.

Importantly, projection causes Bidder i to misinfer both a competitor's aggregate type and his common-value signal after every dropout. These inferences, however, are distorted in opposite directions. The first part of Lemma 1 shows that if two bidders draw the same inference about the aggregate type of a competitor, then the bidder with the lower taste forms a higher expectation about the competitor's common-value signal. Yet, the second part of Lemma 1 shows that the lower is a bidder's taste, the lower is his estimate of an opponent's aggregate type. Put differently: Bidder i's inferred aggregate type of a competitor who drops out, $\tilde{\theta}_d^i$, is increasing in his own taste t_i, but fixing this inference, Bidder i's expectation of the competitor's signal, S_d, is decreasing in t_i. Therefore, it is not obvious how to rank bidders' inferences about a competitor's common-value signal. The following result, however, shows that the latter effect always outweighs the former one:

Lemma 2. Dynamic Misinference Effect: *For each $d \in \{1, \ldots, N-1\}$, $\widehat{\mathbb{E}}[S_d | \theta_d = \tilde{\theta}_d^i; t_i]$ is decreasing in t_i.*

Hence, analogous to the misinference effect that emerges in sealed-bid auctions, a bidder with a higher private taste forms more pessimistic inferences about the common-value signals of those who drop out before him. As the following proposition shows, this harms efficiency. In line with our focus on full-support signals, the result considers normally-distributed signals and tastes.[44]

Proposition 8. *Consider a setting with private and common values, and suppose that signals and tastes are normally distributed. For any $\alpha > 0$, efficiency in an English auction is (i) lower than with rational bidders, and (ii) lower than in a second-price or first-price auction.*

The misinference effect reduces the efficiency of the English auction, and does so more than in the static SPA. The intuition is as follows. In the SPA, bidders condition their bids on tying with the strongest opponent; in the English auction, they condition on tying with all active opponents. Hence, the misinference effect is more severe at the start of the English auction than in the SPA, and thus a bidder with a low taste is more likely to begin the English auction "in the lead" than he is to win the SPA. Lemma 2 further implies that, as the auction unfolds, a bidder with a low taste progressively draws more optimistic inferences than the efficient bidder about the common-value signals of the opponents who drop out. Hence, an inefficient bidder who begins the English auction "in the lead" will eventually win. In light of Proposition 7, the English auction is therefore the worst among all common auction formats in terms of efficiency, while the FPA is the best.

In terms of expected revenue, with rational bidders revenue equivalence holds because their strategies depend solely on their aggregate types, which are independent across players. As noted above, however, a taste-projecting player's bidding behavior cannot be described in terms of a

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[44]We make this assumption for tractability and to simplify the exposition, but the results of Proposition 8 hold for other log-concave distributions.

unidimensional statistic; this introduces familiar challenges for attempting to rank auction formats in the presence of multi-dimensional types (see, e.g., Che and Gale, 2006; Fang and Morris, 2006; Heumann, 2019). Nonetheless, we conjecture that the first-price auction yields a higher expected revenue with projection because of the competition effect, which is absent in second-price and English auctions. Consistent with our findings in Section 5, this would make the FPA both the more efficient and the more profitable format.

7 Extensions

In this section, we examine some implications of taste projection when we relax our assumptions that bidders' valuations are (i) symmetrically distributed, and (ii) independent.

7.1 Asymmetric Auctions

Our first extension considers asymmetric auctions with private values. Following Maskin and Riley's (2000) seminal contribution, we restrict attention to auctions with only two bidders and suppose that a "strong" bidder has a valuation drawn from a distribution that first-order stochastically dominates that of a "weak" bidder. Using two leading examples from their paper, we show that, as in Section 3, taste projection induces bidders to bid too aggressively in first-price auctions and hence increases the seller's revenue. Moreover, as in Section 5, projection increases efficiency in the FPA (in our second example).[45]

Our model of taste projection in Section 2 easily extends to asymmetric bidders. Let T_W and T_S denote the true random variables representing the weak and the strong bidder's values, respectively. Analogous to Equation (2), a taste-projecting Bidder i's perception of the weak type's valuation is described by the random variable $\widehat{T}_W(t_i) \equiv \alpha t_i + (1-\alpha)T_W$; similarly, his perception of the strong type's valuation is described by $\widehat{T}_S(t_i) \equiv \alpha t_i + (1 - \alpha)T_S$.[46] Our naivete assumption implies that each bidder thinks that his opponent shares these perceptions and that this is common knowledge. Thus, a weak (resp. strong) bidder correctly realizes that his opponent is a strong (resp. weak) bidder, but he wrongly thinks that they agree on the distributions from which values are drawn.

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[45]Models of asymmetric auctions are usually not very tractable and fully general treatments are still missing from the literature. Kirkegaard (2014) extends the analysis of Maskin and Riley (2000) to auction environments that lie between their first two examples, while Kirkegaard (2012) provides a mechanism-design approach for ranking asymmetric auctions; both papers restrict attention to the case of two bidders. Kirkegaard (2009) studies asymmetric auctions with more than two bidders but assumes the distributions of bidders' values have identical supports.

[46]In the symmetric model of Section 2, each player's perceived support is always a subset of the true one. In contrast, with asymmetric distributions, a player's perceived support might contain some types that are not present in the true supports. In a sense, these are two sides of the same coin. In a symmetric environment, misperceptions of the support arise because bidders believe that values far from theirs are impossible; in an asymmetric environment, these misperceptions may also arise because bidders believe that values close to theirs are possible when in fact they are not.

While players are aware of an asymmetry, they both think the strong and weak distributions are closer together than they really are.

As values are private, it is still weakly dominant for bidders to bid their values in second-price and English auctions. Therefore, we focus on the first-price auction.

Example 1: The strong bidder's distribution is "shifted" to the right. Suppose the weak bidder's value is uniform on $[\underline{t}, \underline{t} + k]$ with $\underline{t} \geq 0$ and $k > 0$ whereas the strong bidder's value is distributed (in any way) on an interval with lower bound $\underline{t} + 2k$ (the upper bound is irrelevant). Hence, the strong bidder always has a higher value than the weak bidder. Under taste projection, a bidder—either weak or strong—with value t believes that the weak bidder's value is uniform on $[(1 - \alpha)\underline{t} + \alpha t, (1 - \alpha)(\underline{t} + k) + \alpha t]$ and that the strong bidder's value is at least $(1 - \alpha)(\underline{t} + 2k) + \alpha t$.

In Appendix B, we show that it is an equilibrium for the strong bidder to bid (his perception of) the highest possible value of the weak bidder, irrespective of his own valuation—that is, $\widehat{\beta}_S(t) = (1 - \alpha)(\underline{t} + k) + \alpha t$—and for the weak bidder to bid his valuation—that is, $\widehat{\beta}_W(t) = t$.[47] Since the strong bidder always wins the auction and his bid is strictly increasing in α, the competition effect that we identified in the symmetric model emerges here as well: the strong bidder bids too aggressively since he believes the distribution of the weak bidder is "stochastically higher" than it actually is.

It follows that projection raises revenue in the FPA relative to the rational benchmark, and the FPA yields a higher revenue than the SPA. As in the rational benchmark, the FPA and SPA are equally efficient under projection.

Example 2: The strong bidder's distribution is "stretched". Suppose that the weak bidder's valuation is distributed uniformly on $\left[0, \frac{1}{1+z}\right]$ while the strong bidder's distribution is distributed uniformly on $\left[0, \frac{1}{1-z}\right]$, where $z \geq 0$ ($z = 0$ represents the symmetric case). With taste projection, a bidder with value t thinks that the weak bidder's valuation is uniform on $\left[\alpha t, \frac{(1-\alpha)}{1+z} + \alpha t\right]$ and that the strong bidder's valuation is uniform on $\left[\alpha t, \frac{(1-\alpha)}{1-z} + \alpha t\right]$. In Appendix B, following the analysis of Maskin and Riley (2000), we show that the equilibrium bidding functions are

$$\widehat{\beta}_S(t) = \alpha t + \frac{(1 - \alpha)}{4zt}\left(\sqrt{1 + 4zt^2} - 1\right) = \alpha t + (1 - \alpha)\beta_S^*(t)$$

and

$$\widehat{\beta}_W(t) = \alpha t + \frac{(1 - \alpha)}{4zt}\left(1 - \sqrt{1 - 4zt^2}\right) = \alpha t + (1 - \alpha)\beta_W^*(t),$$

where $\beta_S^*(t)$ and $\beta_W^*(t)$ are the equilibrium bidding functions in the rational benchmark.

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[47]These strategies are analogous to those derived by Maskin and Riley (2000) for rational bidders.

Since $\widehat{\beta}_S(t) > \beta_S^*(t)$ and $\widehat{\beta}_W(t) > \beta_W^*(t)$, projection leads both the strong and weak bidder to bid too aggressively. There are two reasons for this. The first one is the "usual" competition effect. The second one is a reduction in perceived asymmetry: the strong (resp. weak) bidder perceives the weak (resp. strong) bidder to be less weak (resp. strong) than he actually is. This reduction in perceived asymmetry reinforces the competition effect for the strong bidder, whereas it goes in the opposite direction for the weak one; yet, the competition effect still dominates, leading both types to overbid. Hence, projection increases the expected revenue of the FPA.

Projection also increases efficiency in the FPA. Maskin and Riley (2000) showed that $\beta_W^*(t) - \beta_S^*(t) > 0$: if the two bidders have the same value, then the weak one bids more aggressively than the strong one. Intuitively, a strong bidder shades his bid more because he faces a stochastically lower distribution of bids. This implies that, with positive probability, the allocation is inefficient: a weak bidder might outbid a strong one with a higher valuation. With projection, a weak bidder still bids more aggressively than a strong bidder with the same value; however, it is easy to verify that $\widehat{\beta}_W(t) - \widehat{\beta}_S(t) = (1 - \alpha)\left[\beta_W^*(t) - \beta_S^*(t)\right]$. That is, projection narrows the gap between the bidding strategies of the strong and weak bidders. Intuitively, since projection reduces the perceived asymmetry between bidders, the strong bidder raises his bid above the rational benchmark by a greater amount than the weak bidder. This implies that the allocation is less likely to be inefficient.

7.2 Affiliated Private Values

Our final extension considers auctions with private values that are symmetrically but not independently distributed. We show that our result on overbidding in first-price auctions (Proposition 2) extends to the case of affiliated private values (APV). Taste projection can thus overturn the standard revenue ranking with rational bidders: if projection is sufficiently strong, then a first-price auction generates a higher expected revenue than a second-price one. We also show that bidding under projection and independent private values (IPV) is behaviorally distinct from rational bidding with affiliated private values.

Overbidding in First-Price Auctions. With affiliation, a rational bidder's beliefs about others' values depend on his own value. Our model of projection with independent values readily extends to correlated values. In particular, suppose $(T_1, \ldots, T_N)$ are symmetrically distributed according to a joint distribution that exhibits affiliation (i.e., the joint density is log-supermodular). Let $F(\cdot|t)$ denote the Bayesian posterior distribution a rational player forms about an opponent's value, conditional on himself having valuation t, and let $T(t)$ denote a random variable with distribution $F(\cdot|t)$. Analogous to Equation (2), a projecting Player i thinks an opponent's value is described by the random variable $\alpha t_i + (1 - \alpha)T(t_i)$. That is, Player i's perception of an opponent's value is a convex

combination of his own value and the random variable he ought to believe in if he were rational, which now depends on t_i.

In this environment, second-order beliefs are common knowledge among rational bidders—it is common knowledge that a player with value t believes an opponent's value is distributed according to $F(\cdot|t)$. Extending our naivete assumption to this setting implies that a projecting Player i thinks it is common knowledge that a player with value t believes an opponent's value is described by the random variable $\alpha t_i + (1 - \alpha)T(t)$, which has CDF $\widehat{F}(x|t; t_i) \equiv F\left(\frac{x-\alpha t_i}{1-\alpha}|t\right)$. Notice that $\widehat{F}(\cdot|\cdot; t_i)$ pins down Player i's first- and higher-order beliefs: Player i thinks an opponent's value is distributed according to $\widehat{F}(x|t_i; t_i)$ and, additionally, he thinks it is common knowledge that an opponent with a value equal to t believes values are distributed according to $\widehat{F}(x|t; t_i)$. In essence, Player i's perceived posterior distributions are all overly concentrated around his own value.

Bidder i's NBE bidding strategy in the FPA follows the standard formula from Milgrom and Weber (1982), but uses i's perceived distribution in place of the true one. Focusing on an auction with two bidders ($N = 2$) for simplicity,

$$\widehat{\beta}_{APV}(t_i) = \int_{\underline{t}(t_i)}^{t_i} y d\widehat{L}(y|t_i), \tag{15}$$

where

$$\widehat{L}(y|t_i) = \exp\left(-\int_y^{t_i} \frac{\widehat{f}(z|z; t_i)}{\widehat{F}(z|z; t_i)} dz\right) \tag{16}$$

for $y \in [\underline{t}(t_i), t_i]$ and 0 elsewhere. When $\widehat{F}(z|z; t_i) = F(z|z)$, (15) reduces to the rational bid.

Proposition 9. *Consider a first-price auction with affiliated private values.*

1. *The NBE bidding function is $\widehat{\beta}_{APV}(t)$ described by Equation (15) and, for every $t > \underline{t}$, $\widehat{\beta}_{APV}(t)$ is strictly increasing in α.*

2. *There exists $\bar{\alpha} \in (0, 1)$ such that, if $\alpha > \bar{\alpha}$, then expected revenue is higher in a first-price auction than in a second-price auction.*

As with independent private values, a taste-projecting bidder's conditional distribution of his opponent's value is too concentrated around his own. He thus overestimates his opponent's value conditional on it being lower than his own and consequently overbids.

Therefore, taste projection can reverse the well-known revenue ranking of auction formats with affiliated private values. Milgrom and Weber's (1982) "linkage principle" shows that, with rational bidders, a second-price auction generates a higher expected revenue than a first-price one. Taste projection, however, increases the expected revenue of the FPA relative to the rational benchmark, but has no effect on revenue in the SPA (since it is weakly dominant for bidders to bid their values).

Thus, if projection is strong enough, revenue is higher in the first-price auction than the second-price one. To see this, notice that as α approaches 1, each player's bid in the FPA approaches his value. Bids are then (approximately) the same in the FPA and SPA. However, the price in the FPA is equal to the highest bid rather than the second-highest one, so the FPA generates a higher revenue.

Projection Differs from Imagined Affiliation. It is tempting to think that bidding in a first-price auction with IPV under taste projection is essentially equivalent to rational bidding with APV. After all, a projecting bidder treats others' valuations as if they are similar to his own. Do taste-projecting bidders therefore bid as if their valuations were affiliated (when in fact they are not), in the same way that rational agents would bid in a setting where valuations were truly affiliated? While this intuition is compelling, the answer is no. The key difference concerns higher-order beliefs. In a rational setting with affiliated values, a bidder is fully aware that those with different private values hold different beliefs about the distribution of values, and he correctly predicts these beliefs for each possible type of his opponent. With projection, instead, a bidder forms incorrect beliefs about others' perceptions of the distribution. In the IPV case, a projecting bidder thinks that others share his perception of the distribution; in the APV case, a projecting bidder thinks others' perceptions are closer to his own than they actually are.

These differences between rational and taste-projecting bidding generate testable predictions in the FPA. In particular, relative to the rational IPV benchmark, a taste-projecting bidder in an IPV setting bids more aggressively for all possible values of t. By contrast, a rational bidder with value t bids more aggressively in an APV setting (relative to the rational IPV benchmark) only when his valuation t is sufficiently high; when t is low, he bids *less* aggressively. (See Appendix C for an illustrative example.) That is, relative to the rational IPV benchmark, biased bidding with IPV leads to overbidding for all types, while rational bidding with APV leads to overbidding for high types and underbidding for low types. In this sense, there is a difference between the behavior of bidders who believe that values are affiliated and agree on the joint distribution and the behavior of taste-projecting bidders.

8   Conclusion

In this paper, we developed a model of taste projection—the tendency to overestimate how similar others' tastes are to one's own—and studied how this bias affects bidding, revenue, and efficiency in several auction formats. In private-value settings, projection leads to overbidding in first-price auctions but has no effect in second-price (and English) auctions. This implies that projection breaks the revenue equivalence principle with independent private values: a first-price auction fetches a higher expected revenue than a second-price one. In fact, a first-price auction can yield a higher

revenue even when private values are affiliated, thereby overturning the revenue ranking implied by the linkage principle. We have also identified several instances where projection affects efficiency. In first-price auctions with purely private values, projection improves efficiency because it lowers the optimal reserve price and, with asymmetric bidders, it induces stronger types to bid more aggressively. With a common-value element, however, there is a misinference effect which can reduce efficiency, particularly in second-price and English auctions. Overall, our analysis suggests that, with taste-projecting bidders, first price auctions raise more revenue and yield more efficient allocations than other common auction formats.

Although our model makes some specific assumptions about the nature of projection, many of our results hold more generally. First, in order to streamline the analysis, we assume that a projector's misperceptions of others' tastes are a convex combination of his own and others' tastes. Yet, our overbidding results do not require this structure; in fact, assuming that perceived distributions are stochastically ordered across types often suffices. Second, our revenue implications continue to hold if α—the degree of projection—varies across players. Third, we assumed that a player projects his entire private value, t_i. While this is surely an oversimplification—bidders likely have accurate perceptions about some aspects of their tastes—our revenue implications continue to hold if bidders project only some of the attributes comprising their private values.[48]

Our model sheds light on some stylized facts in the auction literature and delivers new testable predictions for future empirical work. For instance, taste projection provides an alternative explanation for the well-documented observation of overbidding in first-price auctions with private values.[49] Our model can also rationalize why reserve prices observed in the field are often lower than what the classical model predicts. In particular, with projection, the optimal reserve price in first-price auctions is lower than the rational benchmark and decreasing in the number of bidders.

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[48]Generalizing our model to allow for either heterogeneity in α or projection of only some private-value attributes does influence our efficiency results. For instance, a bidder with the highest private value may be outbid if, relative to other bidders, his α is sufficiently low or if his private value on the projected dimension is sufficiently low.

[49]Overbidding can also be induced by risk aversion; however, risk aversion alone is insufficient to explain the experimental evidence (see Kagel, 1995).

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Appendix

A Proofs

Proof of Proposition 1. Let $\widehat{\beta}_{II}(\cdot|t_i)$ denote Bidder i's perceived bidding strategy. From our naivete assumption, Bidder i thinks it is common knowledge that $T \sim \widehat{F}(\cdot|t_i)$. If this were indeed common knowledge, it is well known that $\widehat{\beta}_{II}(t|t_i) = t$ constitutes a BNE. Hence, the collection of perceived strategies such that $\widehat{\beta}_{II}(t|t_i) = t$ for all $t_i \in \mathcal{T}$ constitutes a NBE. ∎

Proof of Proposition 2. The proof of Part 1 is in the text. For Part 2, notice that

$$\widehat{\beta}_I(t_i) = \frac{\int_{\underline{t}(t_i)}^{t_i} y \widehat{f}_1(y|t_i)\, dy}{\widehat{F}_1(t_i|t_i)} = \frac{\frac{1}{1-\alpha}\int_{\underline{t}(t_i)}^{t_i} y f_1\left(\frac{y-\alpha t_i}{1-\alpha}\right)\, dy}{F(t_i)}, \tag{A.1}$$

where the second equality follows from the fact that $\widehat{F}_1(t_i|t_i) = F_1(t_i)$ and the relationship between the true and perceived densities. Next, consider the following change of variables: $x = \frac{y-\alpha t_i}{1-\alpha}$. Then, the lower- and upper-bounds of integration in (A.1) become $\frac{\underline{t}(t_i)-\alpha t_i}{1-\alpha} = \frac{\alpha t_i + (1-\alpha)\underline{t} - \alpha t_i}{1-\alpha} = \underline{t}$ and $\frac{t_i - \alpha t_i}{1-\alpha} = t_i$, respectively; hence, (A.1) becomes

$$
\begin{aligned}
\widehat{\beta}_I(t_i) &= \frac{\int_{\underline{t}}^{t_i} [(1-\alpha)x + \alpha t_i] f_1(x)\, dx}{F(t_i)} \\
&= (1-\alpha)\frac{\int_{\underline{t}}^{t_i} x f_1(x)\, dx}{F(t_i)} + \alpha t_i \frac{\int_{\underline{t}}^{t_i} f_1(x)\, dx}{F(t_i)} \\
&= (1-\alpha)\beta_I^*(t_i) + \alpha t_i.
\end{aligned}
$$

Both claims then easily follow since $\beta_I^*(t_i) \le t_i$, with a strict inequality for all $t_i > \underline{t}$. ∎

Proof of Proposition 3. Part 1 follows immediately from Part 2 of Proposition 2. Part 2 follows from Proposition 1. ∎

Proof of Proposition 4. In any standard symmetric auction, a bidder with type t_i expects to win the auction with the same probability, $\widehat{F}_1(t_i|t_i)$. Hence, it suffices to show that, conditional on winning, the bidder's expected payment is the same in all standard formats. In the FPA, this expected payment, denoted by $\widehat{P}_I(t_i)$, is his own bid. Thus, by Proposition 2 (and Equation 5), we have:

$$\widehat{P}_I(t_i) = \widehat{\beta}_I(t_i) = \widehat{\mathbb{E}}[T_{i,1}|T_{i,1} \le t_i; t_i].$$

In the SPA, a bidder's expected payment conditional on winning, denoted by $\widehat{P}_{II}(t_i)$, is the bid of his strongest opponent. Hence,

$$\widehat{P}_{II}(t_i) = \widehat{\mathbb{E}}\left[\widehat{\beta}_{II}(T_{i,1}|t_i)\,|T_{j,1} \le t_i; t_i\right] = \widehat{\mathbb{E}}[T_{i,1}|T_{i,1} \le t_i; t_i] = \widehat{P}_I(t_i),$$

where the second equality follows from the fact that $\widehat{\beta}_{II}(t|t_i) = t$ by Proposition 1. ∎

Proof of Proposition 5. First, consider the effect of a change in α on the optimal reserve price. Let $\Pi(r, \alpha)$ denote the seller's expected revenue as a function of r and α; Equation (10) implies

$$\Pi(r, \alpha) \equiv N \int_r^{\bar{t}} \widehat{\beta}_I(t; r) f(t) F(t)^{N-1} \, dt + F(r)^N v_s. \tag{A.2}$$

Using Topkis's Theorem, it suffices to show that $\frac{\partial^2}{\partial \alpha \partial r} \Pi(r, \alpha) < 0$. Note that

$$
\begin{aligned}
\frac{\partial}{\partial r} \Pi(r, \alpha) &= -N \widehat{\beta}_I(r; r) f(r) F(r)^{N-1} + N \int_r^{\bar{t}} \frac{\partial}{\partial r} \widehat{\beta}_I(t; r) f(t) F(t)^{N-1} dt + N F(r)^{N-1} f(r) v_s \\
&= -N f(r) F(r)^{N-1} (r - v_s) + N \int_r^{\bar{t}} \frac{\partial}{\partial r} \widehat{\beta}_I(t; r) f(t) F(t)^{N-1} dt,
\end{aligned} \tag{A.3}
$$

where the second line follows from the boundary condition $\widehat{\beta}_I(r; r) = r$. Thus

$$\frac{\partial^2}{\partial \alpha \partial r} \Pi(r, \alpha) = N \int_r^{\bar{t}} \frac{\partial^2}{\partial \alpha \partial r} \widehat{\beta}_I(t; r) f(t) F(t)^{N-1} dt.$$

Hence, if $\frac{\partial^2}{\partial \alpha \partial r} \widehat{\beta}_I(t; r) \leq 0$ for all $t > r$ (and strictly so for a measurable subset of $t > r$), then $\frac{\partial^2}{\partial \alpha \partial r} \Pi(r, \alpha) < 0$. Applying Equation (8) and the fact that $\widehat{F}(t_i | t_i) = F(t_i)$, we have

$$
\begin{aligned}
\frac{\partial}{\partial r} \widehat{\beta}_I(t_i; r) &= \Bigg(-(N-1) r \widehat{f}(r|t_i) \widehat{F}(r|t_i)^{N-2} + N \int_r^{t_i} \frac{\partial}{\partial r} x \widehat{f}(x|t_i) \widehat{F}(x|t_i)^{N-1} dx \\
&\quad + \widehat{F}(r|t_i)^{N-1} + (N-1) r \widehat{f}(r|t_i) \widehat{F}(r|t_i)^{N-2} \Bigg) \Bigg/ F(t_i)^{N-1} \\
&= \left(\frac{\widehat{F}(r|t_i)}{F(t_i)} \right)^{N-1} = \left(\frac{F\left(\frac{r - \alpha t_i}{1 - \alpha}\right)}{F(t_i)} \right)^{N-1}.
\end{aligned} \tag{A.4}
$$

We now show that for all $r \in (v_s, \bar{t})$, $\frac{\partial^2}{\partial \alpha \partial r} \widehat{\beta}_I(t_i; r)$ is negative for a measurable subset of types and zero otherwise. (The case in which $r = v_s$ is irrelevant since r will never decrease below v_s.) Note that the set $\{t_i \mid t_i > r \text{ and } r \geq \alpha t_i + (1 - \alpha) \underline{t}\}$ has positive measure and $\widehat{F}(r|t_i) > 0$ for all t_i in this set. This implies that

$$\frac{\partial^2}{\partial \alpha \partial r} \widehat{\beta}_I(t_i; r) = (N-1) \frac{F\left(\frac{r - \alpha t_i}{1 - \alpha}\right)^{N-2}}{F(t_i)^{N-1}} f\left(\frac{r - \alpha t_i}{1 - \alpha}\right) \frac{r - t_i}{(1 - \alpha)^2} < 0. \tag{A.5}$$

The final inequality holds because $t_i > r$. For any $t_i > 0$ such that $r < \alpha t_i + (1 - \alpha) \underline{t}$, $\widehat{F}(r|t_i) = 0$ and thus $\frac{\partial^2}{\partial \alpha \partial r} \widehat{\beta}_I(t_i; r) = 0$. Hence, we conclude that $\hat{r}(\alpha, N)$ is decreasing in α. Furthermore, since Condition (A.5) holds at $\alpha = 0$, it follows that $\hat{r}(\alpha, N) < r^*$ for all $\alpha > 0$ and $N \geq 2$.

We now show that $\lim_{\alpha \to 1} \hat{r}(\alpha, N) = v_s$. Since $\lim_{\alpha \to 1} \widehat{\beta}(t_i; r) = t_i$, we have $\lim_{\alpha \to 1} \Pi(r, \alpha) = N \int_r^\infty t f(t) F(t)^{N-1} \, dt + F(r)^N v_s$, which is maximized at $\hat{r}(\alpha, N) = v_s$ (immediate from the first-order condition).

Second, consider the effect of a change in N on the optimal reserve price. Without loss of

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generality, normalize $v_s = 0$. Substituting (A.4) in (A.3) yields:

$$\frac{\partial}{\partial r}\Pi(r,\alpha) = -Nrf(r)F(r)^{N-1} + N\int_r^{\bar{t}} \widehat{F}(r|t)^{N-1}f(t)dt. \tag{A.6}$$

We use the Implicit Function Theorem (IFT) to examine how the solution to $\frac{\partial}{\partial r}\Pi(r,\alpha) = 0$ varies with N. Define the function $L(r,N;\alpha) \equiv \frac{\partial}{\partial r}\Pi(r,\alpha)$ and let $\hat{r}$ solve $L(r,N;\alpha) = 0$. Using the IFT,

$$\frac{\partial \hat{r}}{\partial N} = -\left(\frac{\partial}{\partial r}L(r,N;\alpha)\right)^{-1}\left(\frac{\partial}{\partial N}L(r,N;\alpha)\right)\Big|_{r=\hat{r}}. \tag{A.7}$$

Assuming the SOC holds, then

$$\frac{\partial}{\partial r}L(\hat{r},N;\alpha)\Big|_{r=\hat{r}} = \frac{\partial^2}{\partial r^2}\Pi(\hat{r},\alpha)\Big|_{r=\hat{r}} < 0.$$

(Otherwise, we are at the corner solution $\hat{r} = 0$, in which case our claim that $\hat{r}$ is weakly decreasing in N holds). Hence, by (A.7), showing that $\frac{\partial}{\partial N}L(\hat{r},N;\alpha) < 0$ will imply $\frac{\partial \hat{r}}{\partial N} < 0$ and thus complete the proof. From (A.6),

$$\frac{\partial}{\partial N}L(r,N;\alpha) = -rf(r)F(r)^{N-1} + \int_r^{\bar{t}} \widehat{F}(r|t)^{N-1}f(t)dt$$
$$+ N\frac{\partial}{\partial N}\left(-rf(r)F(r)^{N-1} + \int_r^{\bar{t}} \widehat{F}(r|t)^{N-1}f(t)dt\right)$$
$$= -rf(r)F(r)^{N-1}[1 + N\log F(r)] + \int_r^{\bar{t}}[1 + N\log \widehat{F}(r|t)]\widehat{F}(r|t)^{N-1}f(t)dt. \tag{A.8}$$

At the optimal reserve price, the following must hold:

$$\int_r^{\bar{t}} \widehat{F}(r|t)^{N-1}f(t)\,dt = rf(r)F(r)^{N-1}. \tag{A.9}$$

Therefore, (A.8) is negative if and only if

$$N\int_r^{\bar{t}} \log \widehat{F}(r|t)\widehat{F}(r|t)^{N-1}f(t)\,dt < Nrf(r)F(r)^{N-1}\log F(r). \tag{A.10}$$

Recall that $\widehat{F}(r|t) = F\left(\frac{r-\alpha t}{1-\alpha}\right)$, so $\widehat{F}(r|r) = F(r)$ and $\widehat{F}(r|t) < F(r)$ for all $t > r$. Multiplying each side of (A.9) by $N\log F(r)$ thus yields

$$Nrf(r)F(r)^{N-1}\log F(r) = N\int_r^{\infty} \log F(r)\widehat{F}(r|t)^{N-1}f(t)\,dt$$
$$> N\int_r^{\infty} \log \widehat{F}(r|t)\widehat{F}(r|t)^{N-1}f(t)\,dt,$$

which implies that the desired condition (A.10) indeed holds. ∎

Proof of Corollary 2. For a given reserve price r, the probability that the good is not sold is $F(r)^N$.

If $\alpha > 0$, then $\hat{r}(\alpha, N) < r^*$ by Proposition 5. Thus $F(\hat{r}(\alpha, N))^N < F(r^*)^N$.

Let $\Pi^*(r^*, v_s)$ be the expected revenue under rational bidding given the optimal rational reserve price r^*, which is identical in the FPA and SPA (see Myerson, 1981). Since bidding behavior is identical to rational bidding in the SPA with projection, the optimal reserve price is also r^* and the expected revenue is equal to $\Pi^*(r^*, v_s)$. By contrast, since $\widehat{\beta}_I(t; r^*) > \beta_I^*(t; r^*)$ for all $t > r^*$, expected revenue in the FPA with projection, $\Pi_I(r^*, v_s)$, exceeds $\Pi^*(r^*, v_s)$. Furthermore, since the optimal reserve price under projection in the FPA is $\hat{r}(\alpha, N) \neq r^*$, it must be that $\Pi_I(\hat{r}, v_s) > \Pi_I(r^*, v_s) > \Pi^*(r^*, v_s)$. ∎

Proof of Proposition 6. Without loss of generality, assume $v_s = \underline{t} = 0$. Consider $\alpha > 0$. The auctioneer's expected revenue facing rational bidders with a reserve price r^* is

$$\Pi^*(r^*) \equiv N \int_{r^*}^{\bar{t}} \beta_I^*(t; r^*) f(t) F(t)^{N-1} \, dt, \tag{A.11}$$

where $\beta^*(t; r^*)$ is given by Equation (7). Under projection of degree α, the expected revenue without a reserve price is

$$\Pi_I(\alpha) \equiv N \int_0^{\bar{t}} \widehat{\beta}_I(t) f(t) F(t)^{N-1} \, dt, \tag{A.12}$$

where $\widehat{\beta}_I(t)$ is given by Equation (5). Since $r^* > 0$, $\Pi_I(\alpha) > \Pi^*(r^*)$ if $\widehat{\beta}(t) > \beta^*(t; r^*)$ for all $t > r^*$. Using Equation (6) to write $\widehat{\beta}_I(t)$ in terms of $\beta^*(t)$,

$$\widehat{\beta}(t) > \beta^*(t; r^*) \quad \Leftrightarrow \quad \alpha t + (1 - \alpha) \frac{\int_0^t y f_1(y) dy}{F_1(t)} > \frac{\int_{r^*}^t y f_1(y) dy + r^* F_1(r^*)}{F_1(t)}$$

$$\Leftrightarrow \quad \alpha F_1(t) t - \int_0^t y f_1(y) dy > F_1(r^*) r^* - \int_0^{r^*} y f_1(y) dy. \tag{A.13}$$

Let $Q(x) \equiv F_1(x) x - \int_0^x y f_1(y) dy$; note that Q is strictly increasing since $Q'(x) = F_1(x) > 0$ for $x > 0$. Moreover, $Q(r^*)/Q(t) < 1$ for all $t > r^*$. Let $\tilde{t}$ denote the greatest value in $[r^*, \bar{t}]$ such that

$$\int_0^{r^*} \widehat{\beta}(t) f_1(t) dt \geq \int_{r^*}^{\tilde{t}} \beta^*(t; r^*) f_1(t) dt. \tag{A.14}$$

This value is well-defined and strictly greater than r^* since the left-hand side of (A.14) is strictly positive and the right-hand side is (i) strictly increasing and continuous in $\tilde{t}$, and (ii) equal to zero at $\tilde{t} = r^*$. Let $\tilde{\alpha} \equiv Q(r^*)/Q(\tilde{t}) < 1$. If $\alpha > \tilde{\alpha}$, it follows from Condition (A.13) that all players with private value $t \geq \tilde{t}$ bid more aggressively with projection and no reserve price than with rational bidding and reserve price r^*. Hence, $\alpha > \tilde{\alpha}$ implies that $\Pi_I(\alpha) > \Pi^*(r^*)$. ∎

Proof of Lemma 1. *Part 1.* Fix $\theta \in \Theta$. For any $t_i \in \mathcal{T}(\theta)$, let $\widehat{\mathcal{S}}(\theta|t_i)$ denote the set of realizations of S_j consistent with $\theta_j = \theta$. Note that if $\bar{t}(t_i) + \gamma \underline{s} < \underline{t}(t_i) + \gamma \bar{s}$, then

$$\widehat{\mathcal{S}}(\theta|t_i) = \begin{cases} [\underline{s}, (\theta - \underline{t}(t_i))/\gamma] & \text{if } \theta < \bar{t}(t_i) + \gamma \underline{s} \\ [(\theta - \bar{t}(t_i))/\gamma, (\theta - \underline{t}(t_i))/\gamma] & \text{if } \theta \in [\bar{t}(t_i) + \gamma \underline{s}, \underline{t}(t_i) + \gamma \bar{s}] \\ [(\theta - \bar{t}(t_i))/\gamma, \bar{s}] & \text{if } \theta > \underline{t}(t_i) + \gamma \bar{s}. \end{cases} \tag{A.15}$$

If instead $\bar{t}(t_i) + \gamma \underline{s} > \underline{t}(t_i) + \gamma \bar{s}$, then $\widehat{\mathcal{S}}(\theta|t_i)$ is identical to (A.15) except the middle region of θ has reversed bounds. Let $\hat{g}(s|\theta; t_i)$ denote Player i's perceived PDF of S_j conditional on $\theta_j = \theta$:

$$\hat{g}(s|\theta; t_i) = \frac{\hat{f}(\theta - \gamma s|t_i)g(s)}{\int_{\widehat{\mathcal{S}}(\theta|t_i)} \hat{f}(\theta - \gamma \tilde{s}|t_i)g(\tilde{s})d\tilde{s}}. \tag{A.16}$$

Thus

$$\widehat{\mathbb{E}}[S_j|\theta_j = \theta; t_i] = \int_{\widehat{\mathcal{S}}(\theta|t_i)} s\hat{g}(s|\theta; t_i)ds. \tag{A.17}$$

Let $M_S(\theta) \equiv \widehat{\mathbb{E}}[S_j|\theta_j = \theta; \underline{t}]$ denote the expectation above according to a player with the lowest private value. We now show that the expectation according to any other player can be written in terms of M_S; namely, $\widehat{\mathbb{E}}[S_j|\theta_j = \theta; t_i] = M_S(\theta - \delta(t_i))$ where $\delta(t_i) \equiv \alpha(t_i - \underline{t})$. Using the relationship between the perceived and true PDF, notice that

$$M_S(\theta - \delta(t_i)) = \int_{\widehat{\mathcal{S}}(\theta - \delta(t_i)|\underline{t})} s\hat{g}(s|\theta - \delta(t_i); \underline{t})ds = \frac{\int_{\widehat{\mathcal{S}}(\theta - \delta(t_i)|\underline{t})} sf\left(\frac{\theta - \gamma s - \delta(t_i) - \alpha \underline{t}}{1 - \alpha}\right)g(s)ds}{\int_{\widehat{\mathcal{S}}(\theta - \delta(t_i)|\underline{t})} f\left(\frac{\theta - \gamma s - \delta(t_i) - \alpha \underline{t}}{1 - \alpha}\right)g(s)ds}$$

$$= \frac{\int_{\widehat{\mathcal{S}}(\theta - \delta(t_i)|\underline{t})} s\hat{f}(\theta - \gamma s|t_i)g(s)ds}{\int_{\widehat{\mathcal{S}}(\theta - \delta(t_i)|\underline{t})} \hat{f}(\theta - \gamma s|t_i)g(s)ds}. \tag{A.18}$$

Evaluating $\widehat{\mathcal{S}}(\theta|t_i)$ (Equation A.15) at $\theta - \delta(t_i)$ and $\underline{t}$ yields $\widehat{\mathcal{S}}(\theta - \delta(t_i)|\underline{t}) = \widehat{\mathcal{S}}(\theta|t_i)$. It thus follows from Equation (A.18) that

$$M_S(\theta - \delta(t_i)) = \frac{\int_{\widehat{\mathcal{S}}(\theta|t_i)} s\hat{f}(\theta - \gamma s|t_i)g(s)ds}{\int_{\widehat{\mathcal{S}}(\theta|t_i)} \hat{f}(\theta - \gamma s|t_i)g(s)ds} = \int_{\widehat{\mathcal{S}}(\theta|t_i)} s\hat{g}(s|\theta; t_i)ds = \widehat{\mathbb{E}}[S_j|\theta_j = \theta; t_i]. \tag{A.19}$$

Since log-concavity of f and g implies that M_S is increasing, it follows that $\widehat{\mathbb{E}}[S_j|\theta_j = \theta; t_i] = M_S(\theta - \alpha(t_i - \underline{t}))$ is decreasing in t_i.

Next, let $\hat{g}(s|\theta_j \leq \theta; t_i)$ denote Player i's perceived PDF of S_j conditional on $\theta_j \leq \theta$:

$$\hat{g}(s|\theta_j \leq \theta; t_i) = \frac{\widehat{F}(\theta - \gamma s|t_i)g(s)}{\widehat{H}(\theta|t_i)}, \tag{A.20}$$

where $\widehat{H}(\theta|t_i)$ is Player i's perceived CDF of θ. Hence, $\widehat{H}(\theta|t_i) = \int_{\underline{t}(t_i)+\gamma \underline{s}}^{\theta} \hat{h}(\tilde{\theta}|t_i)d\tilde{\theta}$, where $\hat{h}(\tilde{\theta}|t_i) = \int_{\widehat{\mathcal{S}}(\tilde{\theta}|t_i)} \hat{f}(\tilde{\theta} - \gamma s|t_i)g(s)ds$ is Player i's perceived PDF of θ. Notice that

$$\widehat{\mathbb{E}}[S_j|\theta_j \leq \theta; t_i] = \int_{\widehat{\mathcal{S}}(\theta|t_i)} s\hat{g}(s|\theta_j \leq \theta; t_i)ds. \tag{A.21}$$

Let $\widetilde{M}_S(\theta) \equiv \widehat{\mathbb{E}}[S_j|\theta_j \leq \theta; \underline{t}]$ denote the expectation above according to a player with the lowest

private value. We will show that $\widehat{\mathbb{E}}[S_j | \theta_j \leq \theta; t_i] = \widetilde{M}_S(\theta - \delta(t_i))$. Notice that

$$\widetilde{M}_S(\theta - \delta(t_i)) = \int_{\widehat{\mathcal{S}}(\theta - \delta(t_i)|\underline{t})} s\hat{g}(s|\theta_j \leq \theta - \delta(t_i); \underline{t})ds = \frac{\int_{\widehat{\mathcal{S}}(\theta - \delta(t_i)|\underline{t})} sF\left(\frac{\theta - \gamma s - \delta(t_i) - \alpha \underline{t}}{1 - \alpha}\right) g(s)ds}{\widehat{H}(\theta - \delta(t_i)|\underline{t})}$$

$$= \frac{\int_{\widehat{\mathcal{S}}(\theta|t_i)} s\widehat{F}(\theta - \gamma s|t_i) g(s)ds}{\widehat{H}(\theta - \delta(t_i)|\underline{t})}, \tag{A.22}$$

where the final inequality follows from the definition of $\widehat{F}(\cdot|t_i)$ and the fact that $\widehat{\mathcal{S}}(\theta - \delta(t_i)|\underline{t}) = \widehat{\mathcal{S}}(\theta|t_i)$ (see above). Furthermore,

$$\widehat{H}(\theta - \delta(t_i)|\underline{t}) = \int_{\underline{t} + \gamma\underline{s}}^{\theta - \delta(t_i)} \hat{h}(\tilde{\theta}|\underline{t})d\tilde{\theta} = \int_{\alpha t_i + (1 - \alpha)\underline{t} + \gamma\underline{s}}^{\theta} \hat{h}(\tilde{\theta} - \delta(t_i)|\underline{t})d\tilde{\theta}, \tag{A.23}$$

and

$$\hat{h}(\tilde{\theta} - \delta(t_i)|\underline{t}) = \int_{\widehat{\mathcal{S}}(\tilde{\theta} - \delta(t_i)|\underline{t})} \hat{f}(\tilde{\theta} - \gamma s - \delta(t_i)|\underline{t})g(s)ds$$

$$= \int_{\widehat{\mathcal{S}}(\tilde{\theta}|t_i)} \hat{f}(\tilde{\theta} - \gamma s)|t_i)g(s)ds = \hat{h}(\tilde{\theta}|t_i). \tag{A.24}$$

Thus Equation (A.23) along with the fact that $\underline{t}(t_i) = \alpha t_i + (1 - \alpha)\underline{t}$ implies that

$$\widehat{H}(\theta - \delta(t_i)|\underline{t}) = \int_{\underline{t}(t_i) + \gamma\underline{s}}^{\theta} \hat{h}(\tilde{\theta}|t_i)d\tilde{\theta} = \widehat{H}(\theta|t_i), \tag{A.25}$$

and Equation(A.22) then implies that

$$\widetilde{M}_S(\theta - \delta(t_i)) = \frac{\int_{\widehat{\mathcal{S}}(\theta|t_i)} s\widehat{F}(\theta - \gamma s|t_i) g(s)ds}{\widehat{H}(\theta|t_i)} = \widehat{\mathbb{E}}[S_j | \theta_j \leq \theta; t_i]. \tag{A.26}$$

Since log-concavity of f and g implies that $\widetilde{M}_S$ is increasing, $\widehat{\mathbb{E}}[S_j | \theta_j \leq \theta; t_i]$ is therefore decreasing in t_i.

Part 2. Notice that Player i believes $\theta_{i,1} \sim \widehat{H}(\theta|t_i)^{N-1}$, and hence

$$\widehat{\mathbb{E}}[\theta_{i,1} | \theta_{i,1} \leq \theta; t_i] = (N - 1) \int_{\underline{t}(t_i) + \gamma\underline{s}}^{\theta} \tilde{\theta} \frac{\hat{h}(\tilde{\theta}|t_i)\widehat{H}(\tilde{\theta}|t_i)^{N-2}}{\widehat{H}(\theta|t_i)^{N-1}} d\tilde{\theta}. \tag{A.27}$$

Let $M_\theta(\theta) \equiv \widehat{\mathbb{E}}[\theta_{i,1} | \theta_{i,1} \leq \theta; \underline{t}]$ and note that Equation (A.27) along with (A.24) and (A.25) yields

$$M_\theta(\theta - \delta(t_i)) = (N - 1) \int_{\underline{t} + \gamma\underline{s}}^{\theta - \delta(t_i)} \tilde{\theta} \frac{\hat{h}(\tilde{\theta}|\underline{t})\widehat{H}(\tilde{\theta}|\underline{t})^{N-2}}{\widehat{H}(\theta - \delta(t_i)|\underline{t})^{N-1}} d\tilde{\theta}$$

$$= (N - 1) \int_{\alpha t_i + (1 - \alpha)\underline{t} + \gamma\underline{s}}^{\theta} (\tilde{\theta} - \delta(t_i)) \frac{\hat{h}(\tilde{\theta} - \delta(t_i)|\underline{t})\widehat{H}(\tilde{\theta} - \delta(t_i)|\underline{t})^{N-2}}{\widehat{H}(\theta - \delta(t_i)|\underline{t})^{N-1}} d\tilde{\theta}$$

$$= (N - 1) \int_{\underline{t}(t_i) + \gamma\underline{s}}^{\theta} (\tilde{\theta} - \delta(t_i)) \frac{\hat{h}(\tilde{\theta}|t_i)\widehat{H}(\tilde{\theta}|t_i)^{N-2}}{\widehat{H}(\theta|t_i)^{N-1}} d\tilde{\theta} = \widehat{\mathbb{E}}[\theta_{i,1} | \theta_{i,1} \leq \theta; t_i] - \delta(t_i),$$

and thus

$$\widehat{\mathbb{E}}[\theta_{i,1}|\theta_{i,1} \leq \theta; t_i] = M_\theta(\theta - \delta(t_i)) + \delta(t_i). \tag{A.28}$$

While Equation (A.28) will be useful in later proofs, it is not enough to establish that $\widehat{\mathbb{E}}[\theta_{i,1}|\theta_{i,1} \leq \theta; t_i]$ is increasing in t_i. From Equation (A.27), this result follows if $\widehat{H}(\theta|t_i)^{N-1}$ conditionally stochastically dominates $\widehat{H}(\theta|t'_i)^{N-1}$ for all $t'_i < t_i$; that is, for each $\theta \in \widehat{\Theta}(t_i) \cap \widehat{\Theta}(t'_i)$, $\widehat{H}(\tilde{\theta}|t_i)/\widehat{H}(\theta|t_i) \leq \widehat{H}(\tilde{\theta}|t'_i)/\widehat{H}(\theta|t'_i)$ for all $\tilde{\theta} \leq \theta$ and strictly so for some $\tilde{\theta}$. It is well known that conditional stochastic dominance holds if and only if $\hat{h}(\theta|t_i)/\widehat{H}(\theta|t_i) \geq \hat{h}(\theta|t'_i)/\widehat{H}(\theta|t'_i)$ for all $\theta \in \widehat{\Theta}(t_i) \cap \widehat{\Theta}(t'_i)$ and strictly so for some θ. From equations (A.24) and (A.25), the previous condition is equivalent to

$$\frac{\hat{h}(\tilde{\theta} - \delta(t_i)|\underline{t})}{\widehat{H}(\theta - \delta(t_i)|\underline{t})} \geq \frac{\hat{h}(\tilde{\theta} - \delta(t'_i)|\underline{t})}{\widehat{H}(\theta - \delta(t'_i)|\underline{t})}, \tag{A.29}$$

for all $\theta \in \widehat{\Theta}(t_i) \cap \widehat{\Theta}(t'_i)$. Since $\delta(t_i) > \delta(t'_i)$, Condition (A.29) holds for all such θ if $\hat{h}(x|\underline{t})/\widehat{H}(x|\underline{t})$ is decreasing in x. This is indeed the case since $\hat{h}(x)$ is log-concave given that it is the density of the convolution of two independent random variables that each have log-concave densities. ∎

Proof of Proposition 7. Let $x = (t_1, s_1, t_2, s_2, \ldots, t_N, s_N) \in \mathcal{X} = (\mathcal{T} \times \mathcal{S})^N$ denote the vector of all players' private values and signals. Without loss of generality, normalize $\underline{t} = 0$ and let $t_1 > \max_{i \neq 1} t_i$—i.e., Player 1 is the efficient winner—and let $\mathcal{X}_1 \equiv \{x \in \mathcal{X}|t_1 > \max_{i \neq 1} t_i\}$.

Part 1. For all $\alpha \in [0,1]$, we partition $\mathcal{X}_1$ into two non-empty subsets: $\mathcal{W}(\alpha) \equiv \{x \in \mathcal{X}_1|\widehat{\beta}_{II}(\theta_1|t_1) > \max_{i \neq 1} \widehat{\beta}_{II}(\theta_i|t_i)\}$ and $\mathcal{L}(\alpha) \equiv \{x \in \mathcal{X}_1|\widehat{\beta}_{II}(\theta_1|t_1) < \max_{i \neq 1} \widehat{\beta}_{II}(\theta_i|t_i)\}$. $\mathcal{W}(\alpha)$ contains all realizations where the SPA is efficient (because Player 1 wins), and $\mathcal{L}(\alpha)$ contains all those where it is not.

We first show that, in the SPA, projection preserves inefficient outcomes under rational bidding; that is, $\mathcal{L}(0) \subseteq \mathcal{L}(\alpha)$ whenever $\alpha > 0$. Let $x \in \mathcal{L}(0)$, which implies that there exists $j \neq 1$ such that $\theta_1 < \theta_j$. Fixing $\alpha > 0$, by Equation (12) Player i bids $\widehat{\beta}_{II}(\theta_i|t_i) = \theta_i + \gamma M_S(\theta_i - \delta(t_i)) + \gamma(N-2)\widetilde{M}_S(\theta_i - \delta(t_i))$, where the $M_S(\cdot)$ and $\widetilde{M}_S(\cdot)$ are defined in Equations (A.19) and (A.26), respectively. Therefore, (since $\delta(t_i) = \alpha t_i$ when $\underline{t} = 0$) $\widehat{\beta}_{II}(\theta_1|t_1) < \widehat{\beta}_{II}(\theta_j|t_j)$ if and only if

$$\theta_1 - \theta_j < \gamma[M_S(\theta_j - \alpha t_j) - M_S(\theta_1 - \alpha t_1)] + \gamma(N-2)[\widetilde{M}_S(\theta_j - \alpha t_j) - \widetilde{M}_S(\theta_1 - \alpha t_1)]. \tag{A.30}$$

This condition holds because the left-hand side is negative, and the right-hand side is positive since M_S and $\widetilde{M}_S$ are increasing, $\theta_1 < \theta_j$, and $t_1 > t_j$. Thus, $x \in \mathcal{L}(\alpha)$ as desired.

We now show that an inefficient outcome in the SPA is more likely with projection because $\mathcal{W}(0) \cap \mathcal{L}(\alpha)$ has positive measure. Fix $\bar{x} = (\bar{t}_1, \bar{s}_1, \ldots, \bar{t}_N, \bar{s}_N)$ such that: (i) $\bar{x} \in \mathcal{X}_1$; (ii) for some j, $\bar{\theta}_1 \equiv \bar{t}_1 + \gamma \bar{s}_1 = \bar{t}_j + \gamma \bar{s}_j \equiv \bar{\theta}_j$; and (iii) $\bar{\theta}_k < \bar{\theta}_1$ for all $k \neq 1, j$. Let $\bar{x}(\varepsilon)$ be a vector of types identical to $\bar{x}$ except Player j's signal is $s_j = \bar{s}_j - \varepsilon/\gamma$ for some $\varepsilon \geq 0$. At $\bar{x}(\varepsilon)$, Player j's aggregate type is $\bar{\theta}_1 - \varepsilon$, and thus $\bar{x}(\varepsilon) \in \mathcal{W}(0)$. Furthermore, $\bar{x}(\varepsilon) \in \mathcal{L}(\alpha)$ if Player j outbids Player 1 at $\bar{x}(\varepsilon)$. From (A.30), this happens if and only if

$$\varepsilon < \gamma \left[M_S\left(\bar{\theta}_1 - \varepsilon - \alpha \bar{t}_j\right) - M_S\left(\bar{\theta}_1 - \alpha \bar{t}_1\right) \right]$$
$$+ \gamma(N-2)\left[\widetilde{M}_S\left(\bar{\theta}_1 - \varepsilon - \alpha \bar{t}_j\right) - \widetilde{M}_S\left(\bar{\theta}_1 - \alpha \bar{t}_1\right)\right]. \tag{A.31}$$

When $\varepsilon = 0$, this inequality holds since $\bar{t}_j < \bar{t}_1$. Furthermore, since the right-hand side of (A.31)

is continuously decreasing in ε, it is immediate that there is an open set $\mathcal{E}$ of $\varepsilon > 0$ sufficiently small such that Condition (A.31) holds at $\bar{x}(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$. Hence, $\bar{x}(\varepsilon) \in \mathcal{W}(0) \cap \mathcal{L}(\alpha)$ for $\varepsilon \in \mathcal{E}$. Furthermore, for $\varepsilon \in \mathcal{E}$, all perturbations of $\bar{x}(\varepsilon)$ that change the signals and tastes of Players $k \neq 1, j$, yet preserve the assumption that Player 1 has the highest taste and aggregate type, are also in $\mathcal{W}(0) \cap \mathcal{L}(\alpha)$, and hence $\mathcal{W}(0) \cap \mathcal{L}(\alpha)$ has positive measure.

Part 2. In Part 1, the proof that the SPA is efficient less often under projection than under rational bidding follows entirely from the fact that $\widehat{\beta}_{II}(\theta_i|t_i)$ is decreasing in t_i holding θ_i fixed. Analogously, if $\widehat{\beta}_I(\theta_i|t_i)$ is increasing in t_i holding θ_i fixed, then a symmetric argument (with appropriate swapping of signs) implies that the FPA is efficient *more often* under projection than under rational bidding. By Equations (13), (A.28), and (A.26), in the FPA Player i bids $\widehat{\beta}_I(\theta_i|t_i) = M_\theta(\theta_i - \alpha t_i) + \alpha t_i + \gamma(N-1)\widetilde{M}_S(\theta_i - \alpha t_i)$ and

$$\frac{\partial}{\partial t_i}\widehat{\beta}_I(\theta_i|t_i) = -\alpha[M'_\theta(\theta_i - \alpha t_i) + \gamma(N-1)\widetilde{M}'_S(\theta_i - \alpha t_i)] + \alpha.$$

Since M_θ and $\widetilde{M}_S$ are increasing, this derivative is positive if and only if

$$\gamma < \frac{1 - M'_\theta(\theta_i - \alpha t_i)}{(N-1)\widetilde{M}'_S(\theta_i - \alpha t_i)}. \tag{A.32}$$

Using Equation (A.28), notice that $M'_\theta(\theta - \alpha t_i) < 1$ for all θ since $\frac{\partial}{\partial t_i}\widehat{\mathbb{E}}[\theta_{i,1}|\theta_{i,1} \leq \theta; t_i] > 0$ by Lemma 1. Thus, the right-hand side of Condition (A.32) is positive. Let

$$\bar{\gamma} \equiv \min_{\theta \in \widehat{\Theta}(\underline{t})} \frac{1 - M'_\theta(\theta)}{(N-1)\widetilde{M}'_S(\theta)} > 0. \tag{A.33}$$

It thus follows that, if $\gamma < \bar{\gamma}$, then $\widehat{\beta}_I(\theta_i, t_i)$ is increasing in t_i holding θ_i fixed, for every $(t_i, s_i) \in \mathcal{T} \times \mathcal{S}$, and hence the FPA is more efficient under projection than under rational bidding.

Part 3. Adopting the notation from the proof of Part 1, we show that if the FPA is inefficient at $x \in \mathcal{X}_1$, then the SPA is also inefficient at x. From Equation (13), Player j outbids Player 1 in the FPA if and only if

$$\alpha(t_1 - t_j) - [M_\theta(\theta_j - \alpha t_j) - M_\theta(\theta_1 - \alpha t_1)] < \gamma(N-1)[\widetilde{M}_S(\theta_j - \alpha t_j) - \widetilde{M}_S(\theta_1 - \alpha t_1)]. \tag{A.34}$$

Hence, since M_θ and $\widetilde{M}_S$ are increasing, a necessary condition for the FPA to be inefficient is that, for some $j \neq 1$,

$$\theta_1 - \alpha t_1 < \theta_j - \alpha t_j. \tag{A.35}$$

Furthermore, if $\theta_j > \theta_1$ for some $j \neq 1$, then the SPA is inefficient (because in this case the SPA is inefficient with rational bidders and, hence, it is also inefficient with projection by Part 1). Therefore, we only need to show that, at any $x \in \mathcal{X}_1$ such that Condition (A.35) holds for some $j \neq 1$ and $\theta_1 > \theta_j$, inefficiency in the FPA implies inefficiency in the SPA.

Since $M'_\theta(\theta) < 1$ for all $\theta \in \widehat{\Theta}(\underline{t})$ (see the proof of Part 2), $M_\theta(\theta_j - \alpha t_j) - M_\theta(\theta_1 - \alpha t_1) < (\theta_j - \alpha t_j) - (\theta_1 - \alpha t_1)$. Applying this bound to the left-hand side of Condition (A.34) implies that

a necessary condition for inefficiency in the FPA is

$$\theta_1 - \theta_j < \gamma(N-1)[\widetilde{M}_S(\theta_j - \alpha t_j) - \widetilde{M}_S(\theta_1 - \alpha t_1)]. \tag{A.36}$$

Moreover, from Equation (12), the SPA is inefficient if and only if, for some $j \neq 1$,

$$\theta_1 - \theta_j < \gamma[M_S(\theta_j - \alpha t_j) - M_S(\theta_1 - \alpha t_1)] + \gamma(N-2)[\widetilde{M}_S(\theta_j - \alpha t_j) - \widetilde{M}_S(\theta_1 - \alpha t_1)]. \tag{A.37}$$

Thus, the SPA is necessarily inefficient at any $x \in \mathcal{X}_1$ where the FPA is inefficient if $M_S(\theta_j - \alpha t_j) - M_S(\theta_1 - \alpha t_1) > \widetilde{M}_S(\theta_j - \alpha t_j) - \widetilde{M}_S(\theta_1 - \alpha t_1)$. This condition holds because (i) we are considering $\theta_j - \alpha t_j > \theta_1 - \alpha t_1$ and (ii) $M_S(y) - \widetilde{M}_S(y)$ is increasing (by the assumption that $\mu(x) \equiv \mathbb{E}[S_j|\theta_j = \theta] - \mathbb{E}[S_j|\theta_j \leq \theta]$ is increasing). Hence, the necessary condition for inefficiency in the FPA, Condition (A.36), implies inefficiency in the SPA. Finally, since Condition (A.36) is generically not sufficient, the FPA strictly outperforms the SPA in terms of efficiency. ∎

Proof of Lemma 2. From Equation (A.19), $\widehat{\mathbb{E}}[S_d|\theta_d = \tilde{\theta}_d^i; t_i] = M_S(\tilde{\theta}_d^i - \delta(t_i))$. Therefore, we will show that, fixing $(p_1, \ldots, p_d)$, $t_j < t_i$ implies that $M_S(\tilde{\theta}_d^j - \delta(t_j)) > M_S(\tilde{\theta}_d^i - \delta(t_i))$. Recall that, for each $d \in \{1, \ldots, N-1\}$, $\tilde{\theta}_d^i$ is defined recursively as follows: initially, $\tilde{\theta}_1^i$ solves

$$p_1 = \widehat{\beta}_0(\tilde{\theta}_1^i|t_i) = \tilde{\theta}_1^i + \gamma(N-1)M_S(\tilde{\theta}_1^i - \delta(t_i)), \tag{A.38}$$

and then for $d > 1$, $\tilde{\theta}_d^i$ solves

$$p_d = \widehat{\beta}_{d-1}(\tilde{\theta}_d^i; p_1, \ldots, p_{d-1}|t_i) = \tilde{\theta}_d^i + \gamma(N-d)M_S(\tilde{\theta}_d^i - \delta(t_i)) + \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}_{d'}^i - \delta(t_i)). \tag{A.39}$$

For any integer $d \geq 1$, define the function $m_d(x) \equiv x + \gamma(N-d)M_S(x)$, which is strictly increasing in x and hence invertible. This implies that (A.39) can be written as

$$p_d = m_d(\tilde{\theta}_d^i - \delta(t_i)) + \delta(t_i) + \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}_{d'}^i - \delta(t_i))$$

$$\Leftrightarrow \quad \tilde{\theta}_d^i = m_d^{-1}\left(p_d - \delta(t_i) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}_{d'}^i - \delta(t_i))\right) + \delta(t_i).$$

This inverse is well-defined given our assumption of full-support signals (see footnote 42). Thus

$$M_S(\tilde{\theta}_d^j - \delta(t_j)) - M_S(\tilde{\theta}_d^i - \delta(t_i)) = M_S\left(m_d^{-1}\left(p_d - \delta(t_j) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}_{d'}^j - \delta(t_j))\right)\right)$$

$$- M_S\left(m_d^{-1}\left(p_d - \delta(t_i) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}_{d'}^i - \delta(t_i))\right)\right). \tag{A.40}$$

Since $M_S \circ m_d^{-1}$ is increasing, the difference above is positive if and only if

$$p_d - \delta(t_j) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}_{d'}^j - \delta(t_j)) > p_d - \delta(t_i) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}_{d'}^i - \delta(t_i))$$

$$\Leftrightarrow \quad \delta(t_i) - \delta(t_j) > \sum_{d'=1}^{d-1} \gamma\left(M_S(\tilde{\theta}_{d'}^j - \delta(t_j)) - M_S(\tilde{\theta}_{d'}^i - \delta(t_i))\right). \tag{A.41}$$

When $d = 1$, Condition (A.41) trivially holds if $t_i > t_j$, because the sum terms vanish and $\delta(t_i) - \delta(t_j) > 0$. Hence, to complete the proof we need to show that Condition (A.41) holds for $d \in$

$\{2, \ldots, N-1\}$ if $t_i > t_j$. To do this, we prove by induction that, for $d \geq 2$,

$$\sum_{d'=1}^{d-1} \gamma \left(M_S(\tilde{\theta}_{d'}^j - \delta(t_j)) - M_S(\tilde{\theta}_{d'}^i - \delta(t_i)) \right) < \frac{d-1}{N-1}(\delta(t_i) - \delta(t_j)), \qquad (*)$$

which implies Condition (A.41).

Base Case: $d = 2$. We show that $\gamma M_S(\tilde{\theta}_1^j - \delta(t_j)) - \gamma M_S(\tilde{\theta}_1^i - \delta(t_i)) < \frac{1}{N-1}(\delta(t_i) - \delta(t_j))$. Define the function $Z_d(x) \equiv \gamma M_S(m_d^{-1}(x))$. Hence,

$$\frac{d}{dx} Z_d(x) = \gamma M_S'(m_d^{-1}(x)) \frac{d}{dx} m_d^{-1}(x) = \frac{\gamma M_S'(m_d^{-1}(x))}{1 + \gamma(N-d)M_S'(m_d^{-1}(x))}$$

$$= \frac{1}{N - d + \left(\gamma M_S'(m_d^{-1}(x))\right)^{-1}}, \qquad (A.42)$$

where we have used $\frac{d}{dx} m_d^{-1}(x) = \left(m_d'(m_d^{-1}(x))\right)^{-1}$ and $m_d'(x) = 1 + \gamma(N-d)M_S'(x)$. Note that $\left(\gamma M_S'(m_d^{-1}(x))\right)^{-1} > 0$ since M_S' is positive. Thus (A.42) implies that $Z_d'(x) < \frac{1}{N-d}$. Therefore, from Equation (A.40), we have

$$\gamma M_S(\tilde{\theta}_1^j - \delta(t_j)) - \gamma M_S(\tilde{\theta}_1^i - \delta(t_i)) =$$

$$\gamma M_S\left(m_d^{-1}\left(p_1 - \delta(t_j)\right)\right) - \gamma M_S\left(m_d^{-1}\left(p_1 - \delta(t_i)\right)\right) = Z_1\left(p_1 - \delta(t_j)\right) - Z_1\left(p_1 - \delta(t_i)\right)$$

$$< \frac{1}{N-1}\left((p_1 - \delta(t_j)) - (p_1 - \delta(t_i))\right) = \frac{1}{N-1}(\delta(t_i) - \delta(t_j)).$$

Induction Step: We show that if Condition $(*)$ holds for $d > 2$, then it holds for $d + 1$. Note that

$$\sum_{d'=1}^{d} \gamma \left(M_S(\tilde{\theta}_{d'}^j - \delta(t_j)) - M_S(\tilde{\theta}_{d'}^i - \delta(t_i)) \right) = \sum_{d'=1}^{d-1} \gamma \left(M_S(\tilde{\theta}_{d'}^j - \delta(t_j)) - M_S(\tilde{\theta}_{d'}^i - \delta(t_i)) \right)$$

$$+ \gamma \left(M_S(\tilde{\theta}_d^j - \delta(t_j)) - M_S(\tilde{\theta}_d^i - \delta(t_i)) \right). \qquad (A.43)$$

Following the same approach as in the base case and using Equation (A.40), we can write the second term on the RHS of Equation (A.43) as:

$$\gamma \left(M_S(\tilde{\theta}_d^j - \delta(t_j)) - M_S(\tilde{\theta}_d^i - \delta(t_i)) \right) = \gamma M_S\left(m_d^{-1}\left(p_d - \delta(t_j) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}_{d'}^j - \delta(t_j)) \right) \right)$$

$$- \gamma M_S\left(m_d^{-1}\left(p_d - \delta(t_i) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}_{d'}^i - \delta(t_i)) \right) \right)$$

$$= Z_d\left(p_d - \delta(t_j) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}_{d'}^j - \delta(t_j)) \right) - Z_d\left(p_d - \delta(t_i) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}_{d'}^i - \delta(t_i)) \right)$$

$$< \frac{1}{N-d}\left(\delta(t_i) - \delta(t_j) - \sum_{d'=1}^{d-1} \gamma \left(M_S(\tilde{\theta}_{d'}^j - \delta(t_j)) - M_S(\tilde{\theta}_{d'}^i - \delta(t_i)) \right) \right).$$

Applying this bound to Equation (A.43) reveals that

$$\sum_{d'=1}^{d} \gamma \left(M_S(\tilde{\theta}_{d'}^j - \delta(t_j)) - M_S(\tilde{\theta}_{d'}^i - \delta(t_i)) \right) < \sum_{d'=1}^{d-1} \gamma \left(M_S(\tilde{\theta}_{d'}^j - \delta(t_j)) - M_S(\tilde{\theta}_{d'}^i - \delta(t_i)) \right)$$

$$+ \frac{1}{N-d} \left(\delta(t_i) - \delta(t_j) - \sum_{d'=1}^{d-1} \gamma \left(M_S(\tilde{\theta}_{d'}^j - \delta(t_j)) - M_S(\tilde{\theta}_{d'}^i - \delta(t_i)) \right) \right)$$

$$= \frac{1}{N-d}(\delta(t_i) - \delta(t_j)) + \frac{N-d-1}{N-d} \sum_{d'=1}^{d-1} \gamma \left(M_S(\tilde{\theta}_{d'}^j - \delta(t_j)) - M_S(\tilde{\theta}_{d'}^i - \delta(t_i)) \right)$$

$$< \frac{1}{N-d}(\delta(t_i) - \delta(t_j)) + \frac{N-d-1}{N-d} \frac{d-1}{N-1}(\delta(t_i) - \delta(t_j)) = \frac{d}{N-1}(\delta(t_i) - \delta(t_j)),$$

where the final inequality stems from the induction assumption (i.e., Condition ($*$) holds for d). ∎

Proof of Proposition 8. Fixing $\alpha > 0$, we will show that the English auction is less efficient than the SPA by proving that (i) for any realization of bidders' types where the SPA is inefficient, the English auction is also inefficient, and (ii) there are realizations such that the English auction is inefficient but the SPA is not. Given Parts 1 and 3 of Proposition 7, this will additionally imply that (i) the English auction with projection is less efficient than the English auction with rational bidders, and (ii) for $\alpha > 0$, the English auction is less efficient than the FPA.

Assume $S_i \sim N(\mu, \rho^2)$ and $T_i \sim N(0, \sigma^2)$. It readily follows that, under projection, Player i's expectation regarding γS_j conditional on $\theta_j = \theta$ is

$$\widehat{\mathbb{E}}[\gamma S_j | \theta_j = \theta; t_i] = \lambda (\theta - \alpha t_i) + (1 - \lambda)\gamma\mu, \tag{A.44}$$

where

$$\lambda \equiv \frac{\gamma^2 \rho^2}{\gamma^2 \rho^2 + (1 - \alpha)^2 \sigma^2} \in (0, 1). \tag{A.45}$$

Substituting Equation (A.44) into the bidding strategies in (14) yields:

$$\widehat{\beta}_D(\theta_i; \cdot | t_i) = \theta_i + (N - 1 - D)\left[\lambda (\theta_i - \alpha t_i) + (1 - \lambda)\gamma\mu\right] + \sum_{d=1}^{D} \left[\lambda \left(\tilde{\theta}_d^i - \alpha t_i\right) + (1 - \lambda)\gamma\mu\right]$$

$$= \theta_i \left[1 + \lambda(N - 1 - D)\right] + (N - 1)\left[(1 - \lambda)\gamma\mu - \lambda\alpha t_i\right] + \sum_{d=1}^{D} \lambda\tilde{\theta}_d^i, \tag{A.46}$$

where each $\tilde{\theta}_d^i$ solves $p_d = \widehat{\beta}_{d-1}(\tilde{\theta}_d^i; \cdot | t_i)$ and hence

$$\tilde{\theta}_d^i = \frac{p_d + (N - 1)\left[\lambda\alpha t_i - (1 - \lambda)\gamma\mu\right] - \sum_{d'=1}^{d-1} \lambda\tilde{\theta}_{d'}^i}{1 + \lambda(N - d)}. \tag{A.47}$$

We first show that, in an English auction, the sum of differences between two players' inferences about the aggregate types of competitors who have dropped out is equal to the initial difference in their inferences about this type scaled by the number of players who have dropped out; i.e.,

$$\sum_{d=1}^{D} \left(\tilde{\theta}_d^j - \tilde{\theta}_d^i\right) = D\left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right), \tag{A.48}$$

for all N and $D \leq N - 1$. The proof follows from induction on D:

Base Case: $D = 2$. We show that $\sum_{d=1}^{2} \left(\tilde{\theta}_d^j - \tilde{\theta}_d^i \right) = 2 \left(\tilde{\theta}_1^j - \tilde{\theta}_1^i \right)$. From Equation (A.47),

$$\tilde{\theta}_d^i - \tilde{\theta}_d^j = \frac{(N-1)\lambda\alpha\left(t_i - t_j\right) - \sum_{d'=1}^{d-1} \lambda\left(\tilde{\theta}_{d'}^i - \tilde{\theta}_{d'}^j\right)}{1 + \lambda(N-d)}. \tag{A.49}$$

Hence,

$$\tilde{\theta}_1^j - \tilde{\theta}_1^i = \frac{(N-1)\lambda\alpha\left(t_j - t_i\right)}{1 + \lambda(N-1)}, \tag{A.50}$$

and

$$\begin{aligned}
\sum_{d=1}^{2} \left(\tilde{\theta}_d^j - \tilde{\theta}_d^i \right) &= \frac{(N-1)\lambda\alpha\left(t_j - t_i\right) - \lambda\left(\tilde{\theta}_1^i - \tilde{\theta}_1^j\right)}{1 + \lambda(N-2)} + \left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right) \\
&= \frac{1 + \lambda(N-1)}{1 + \lambda(N-2)} \left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right) + \frac{1 + \lambda(N-3)}{1 + \lambda(N-2)} \left(\tilde{\theta}_1^i - \tilde{\theta}_1^j\right) = 2\left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right).
\end{aligned}$$

Induction Step: Suppose that $\sum_{d=1}^{D} \left(\tilde{\theta}_d^j - \tilde{\theta}_d^i \right) = D \left(\tilde{\theta}_1^j - \tilde{\theta}_1^i \right)$. Using Equations (A.49) and (A.50), we have that

$$\begin{aligned}
\sum_{d=1}^{D+1} \left(\tilde{\theta}_d^j - \tilde{\theta}_d^i \right) &= D\left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right) + \tilde{\theta}_{D+1}^j - \tilde{\theta}_{D+1}^i \\
&= D\left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right) + \frac{(N-1)\lambda\alpha\left(t_j - t_i\right) - \lambda D\left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right)}{1 + \lambda(N-D-1)} \\
&= \frac{D + 1 + (D+1)\lambda(N-D-1)}{1 + \lambda(N-D-1)} \left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right) = (D+1)\left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right),
\end{aligned}$$

which completes the induction step.

We now prove that, in the English auction, the ranking of bidders' drop-out prices remains fixed as the auction unfolds; i.e., $\widehat{\beta}_0(\theta_j|t_j) > \widehat{\beta}_0(\theta_i|t_i)$ if and only if $\widehat{\beta}_D(\theta_j; p_1, \ldots, p_D|t_j) > \widehat{\beta}_D(\theta_i; p_1, \ldots, p_D|t_i) > 0$, for all $D < N - 1$. This implies that the final winner of the auction is the bidder who plans to bid higher at the beginning of the auction, before any bidder drops out.

From Equation (A.46), notice that $\widehat{\beta}_0(\theta_j|t_j) > \widehat{\beta}_0(\theta_i|t_i)$ if and only if

$$(1 + \lambda(N-1))\left(\theta_j - \theta_i\right) + (N-1)\lambda\alpha\left(t_i - t_j\right) > 0. \tag{A.51}$$

Now consider $0 < D < N - 1$. Using Equations (A.46) and (A.48), $\widehat{\beta}_D(\theta_j; p_1, \ldots, p_D|t_j) > \widehat{\beta}_D(\theta_i; p_1, \ldots, p_D|t_i)$ if and only if

$$(1 + \lambda(N-D-1))(\theta_j - \theta_i) + \lambda(N-1)\alpha\left(t_i - t_j\right) + \lambda\underbrace{\sum_{d=1}^{D} \left(\tilde{\theta}_d^j - \tilde{\theta}_d^i\right)}_{=D\left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right)} > 0$$

$$\Leftrightarrow \quad (1 + \lambda(N-D-1))(\theta_j - \theta_i) + \lambda(N-1)\alpha\left(t_i - t_j\right) + \lambda D\frac{\lambda(N-1)\alpha\left(t_j - t_i\right)}{1 + \lambda(N-1)} > 0$$

$$\Leftrightarrow \quad (1 + \lambda(N-D-1))(\theta_j - \theta_i) - \lambda(N-1)\alpha\left(t_j - t_i\right)\frac{1 + \lambda(N-D-1)}{1 + \lambda(N-1)} > 0,$$

which is identical to condition (A.51).

To complete the proof, we show that if Bidder i is the efficient winner and $\widehat{\beta}_{II}(\theta_j|t_j) > \widehat{\beta}_{II}(\theta_i|t_i)$, then $\widehat{\beta}_0(\theta_j|t_j) > \widehat{\beta}_0(\theta_i|t_i)$; i.e., if the efficient bidder loses a second-price auction, then he does not have the highest drop-out price at the beginning of an English auction and, hence, he also loses the English auction. From Equation (12), $\widehat{\beta}_{II}(\theta_j|t_j) > \widehat{\beta}_{II}(\theta_i|t_i)$ if and only if

$$\theta_i - \theta_j - \gamma[M_S(\theta_j - \delta(t_j)) - M_S(\theta_i - \delta(t_i))] < \gamma(N-2)[\widetilde{M}_S(\theta_j - \delta(t_j)) - \widetilde{M}_S(\theta_i - \delta(t_i))]. \quad \text{(A.52)}$$

Similarly, from Equation (14), $\widehat{\beta}_0(\theta_j|t_j) > \widehat{\beta}_0(\theta_i|t_i)$ if and only if

$$\theta_i - \theta_j - \gamma[M_S(\theta_j - \delta(t_j)) - M_S(\theta_i - \delta(t_i))] < \gamma(N-2)[M_S(\theta_j - \delta(t_j)) - M_S(\theta_i - \delta(t_i))].$$

Hence, the former condition (A.52) implies the latter whenever

$$M_S(\theta_j - \delta(t_j)) - M_S(\theta_i - \delta(t_i)) > \widetilde{M}_S(\theta_j - \delta(t_j)) - \widetilde{M}_S(\theta_i - \delta(t_i)), \quad \text{(A.53)}$$

which holds whenever $\theta_j - \delta(t_j) > \theta_i - \delta(t_i)$ due to our assumption that $\mu(x) \equiv \mathbb{E}[S_j|\theta_j = \theta] - \mathbb{E}[S_j|\theta_j \leq \theta]$ is increasing. Given that inefficiency in the SPA requires $\theta_j - \delta(t_j) > \theta_i - \delta(t_i)$, we have thus established that, with projection, the English auction is always inefficient when the SPA is. Finally, the English auction is strictly less efficient than the SPA since inefficiency in the SPA (Condition A.52) is sufficient for inefficiency in the English auction but not necessary given that (A.53) strictly holds. $\blacksquare$

Proof of Proposition 9. *Part 1*. Consider an arbitrary Bidder i with valuation t_i. We first show that this bidder believes that valuations are affiliated within his misspecified model. This then implies that the unique symmetric monotone BNE strategy within his perceived game is given by the solution from Milgrom and Weber (1982) described in Equations (15) and (16). These perceived strategies also form the NBE. To show affiliation within Bidder i's perceived model, it is sufficient to show that he perceives affiliation between his own valuation and that of his opponent. Without loss of generality, consider Bidder 1 and denote his perceived joint density over (T_1, T_2) by $\hat{f}(x_1, x_2|t_1)$. It suffices to show that this density is log-supermodular—i.e., that $\hat{f}(x_1, x_2|t_1)/\hat{f}(x_1', x_2|t_1)$ is increasing in x_2 whenever $x_1 > x_1'$ and that $\hat{f}(x_1, x_2|t_1)/\hat{f}(x_1, x_2'|t_1)$ is increasing in x_1 whenever $x_2 > x_2'$. Given the relationship between $\hat{f}(\cdot, \cdot|t_1)$ and the true joint density over these variables, f, we have

$$\frac{\hat{f}(x_1, x_2|t_1)}{\hat{f}(x_1', x_2|t_1)} = \frac{f\left(x_1, \frac{x_2 - \alpha t_1}{1-\alpha}\right)}{f\left(x_1', \frac{x_2 - \alpha t_1}{1-\alpha}\right)}.^{50} \quad \text{(A.54)}$$

Since (T_1, T_2) are affiliated, the joint density f exhibits log-supermodularity. Thus, if $x_1 > x_1'$, then expression (A.54) is increasing in x_2 since $(x_2 - \alpha t_1)/(1-\alpha)$ is. Similarly,

$$\frac{\hat{f}(x_1, x_2|t_1)}{\hat{f}(x_1, x_2'|t_1)} = \frac{f\left(x_1, \frac{x_2 - \alpha t_1}{1-\alpha}\right)}{f\left(x_1, \frac{x_2' - \alpha t_1}{1-\alpha}\right)}$$

---

[50] Note that expression (A.54) makes important use of our naivete assumption: Player 1 does not account for the fact that his realized value, t_1, influences his perception of the distribution. Therefore, his perception does not change when considering alternative hypothetical realizations of his own valuation (e.g., x_1 vs x_1'). This is why t_1 appears in the second argument in both the top and bottom of ratio (A.54).

is increasing in x_1 if $x_2 > x'_2$. We can similarly show that Bidder 2 believes that (T_1, T_2) are affiliated. Thus, each bidder bids according to $\widehat{\beta}_{APV}(t)$ defined in Equation (15).

We now show that $\widehat{\beta}_{APV}(t)$ is increasing in α. Notice that $\widehat{\beta}_{APV}(t_i) = \int_{\underline{t}(t_i)}^{t_i} y d\widehat{L}(y|t_i)$, where the function $\widehat{L}(\cdot|t_i)$ (defined in 16) is a well-defined CDF over $[\underline{t}(t_i), t_i]$. Writing $\widehat{L}(\cdot|t_i)$ explicitly as a function of α, it therefore suffices to show that for all $t_i \in \mathcal{T}$, $\widehat{L}(\cdot|t_i, \alpha)$ first-order stochastically dominates $\widehat{L}(\cdot|t_i, \alpha')$ whenever $\alpha > \alpha'$. From the definition of $\widehat{L}(\cdot|t_i, \alpha)$ in (16), $\widehat{L}(\cdot|t_i, \alpha)$ first-order stochastically dominates $\widehat{L}(\cdot|t_i, \alpha')$ if $I(y) \equiv \int_y^{t_i} \frac{\widehat{f}(z|z;t_i)}{\widehat{F}(z|z;t_i)} dz$ is strictly increasing in α for all $y < t_i$. Applying the definition of $\widehat{F}(z|z; t_i)$,

$$I(y) = \int_y^{t_i} \frac{1}{1-\alpha} \frac{f\left(\frac{z-\alpha t_i}{1-\alpha} \Big| z\right)}{F\left(\frac{z-\alpha t_i}{1-\alpha} \Big| z\right)} dz. \tag{A.55}$$

Let $R(t|z) \equiv f(t|z)/F(t|z)$ be the reversed hazard rate of F, and note that affiliation implies $R(t|z)$ is increasing in z for all $t \in \mathcal{T}$ (Lemma 1 in Milgrom and Weber, 1982). Substituting this notation into (A.55) and performing a change of variables with $x = \frac{z-\alpha t_i}{1-\alpha}$ yields

$$I(y) = \int_{\frac{y-\alpha t_i}{1-\alpha}}^{t_i} R\left(x \Big| \alpha t_i + (1-\alpha)x\right) dx. \tag{A.56}$$

Since each point x in the domain of integration above is such that $x \leq t_i$, the value $\alpha t_i + (1-\alpha)x$ is increasing in α, and hence the integrand $R(x|\alpha t_i + (1-\alpha)x)$ is increasing in α (due to affiliation). Furthermore, the lower limit of integration is decreasing in α. Since the integrand is non-negative, it thus follows that $I(y)$ is increasing in α, as desired. This completes the proof of Part 1.

Part 2. For each $t > \underline{t}$, $\lim_{\alpha \to 1} \widehat{\beta}_{APV}(t) = t$. Thus, as $\alpha \to 1$, the expected revenue of the FPA is the expectation of the highest value among the N bidders, while the expected revenue of the SPA is that of the *second* highest value. Thus, as $\alpha \to 1$, the first-price auction yields a strictly higher expected revenue. Since $\widehat{\beta}_{APV}$ is continuous in α, there exists an $\bar{\alpha} \in (0, 1)$ such that preceding inequality holds for all $\alpha > \bar{\alpha}$. ∎

B   Asymmetric Auctions

In this Appendix we derive the bidding strategies reported in Section 7.1.

Example 1. Given his perception of the strong bidder's strategy, a weak bidder with value t expects to always lose the auction and, therefore, it is a best response for him to bid his value; that is, $\widehat{\beta}_W(t) = t$. Hence, we only need to prove that, when the weak bidder bids his value, a strong bidder with value t is willing to bid $\widehat{\beta}_S(t) = (1-\alpha)(\underline{t}+k) + \alpha t$—his perception of the highest possible value of a weak bidder—in order to always win. The strong bidder solves:

$$\max_{b_S} \frac{b_S - [(1-\alpha)\underline{t} + \alpha t]}{k(1-\alpha)} (t - b_S).$$

The FOC yields

$$b_S = \frac{t(1+\alpha) + (1-\alpha)\underline{t}}{2}.$$

This is weakly higher than $(1 - \alpha)(\underline{t} + k) + \alpha t$, for any $t \geq \underline{t} + 2k$.

Example 2. Changing notation for convenience, let the weak and strong bidder's valuations be distributed uniformly on $[\underline{\omega}, \omega_W]$ and $[\underline{\omega}, \omega_S]$, respectively, where $\omega_S > \omega_W$. We first derive the BNE bidding strategies for these generic supports, and then modify them to obtain the NBE strategies. Following Maskin and Riley (2000), the equilibrium bidding functions are the solutions of the following system of differential equations

$$\phi_i'(b) = \frac{\frac{\phi_i(b) - \underline{\omega}}{\omega_i - \underline{\omega}}}{\frac{1}{\omega_i - \underline{\omega}}} \frac{1}{\phi_j(b) - b}, \quad i, j = W, S, \quad i \neq j, \tag{B.1}$$

where ϕ denotes the inverse bidding function. Simplifying and re-arranging yields

$$(\phi_i'(b) - 1)(\phi_j(b) - b) = \phi_i(b) - \phi_j(b) + b - \underline{\omega}, \quad i, j = W, S, \quad i \neq j.$$

Adding these two differential equations and re-arranging yields

$$\frac{d}{db}\{(\phi_j(b) - b)(\phi_i(b) - b)\} = 2(b - \underline{\omega}),$$

and, integrating both sides, we obtain

$$(\phi_j(b) - b)(\phi_i(b) - b) = (b - \underline{\omega})^2. \tag{B.2}$$

(The constant of integration is zero since $\phi_i(\underline{\omega}) = \underline{\omega}$). Now, substituting (B.2) into (B.1) yields

$$\phi_i'(b) = \frac{(\phi_i(b) - \underline{\omega})(\phi_i(b) - b)}{(b - \underline{\omega})^2}, \quad i = W, S. \tag{B.3}$$

In order to solve the differential equation (B.3), we use a change of variables. Let $\psi_i(b)$ be implicitly defined by

$$\phi_i(b) = b + \psi_i(b)(b - \underline{\omega}) \tag{B.4}$$

so that

$$\phi_i'(b) = \psi_i'(b)(b - \underline{\omega}) + \psi_i(b) + 1.$$

It follows that the differential equation (B.3) can be re-written as

$$\psi_i'(b)(b - \underline{\omega}) + \psi_i(b) + 1 = \frac{(b - \underline{\omega})(\psi_i(b) + 1)\psi_i(b)(b - \underline{\omega})}{(b - \underline{\omega})^2}$$

$$\Leftrightarrow \quad \psi_i'(b)(b - \underline{\omega}) = (\psi_i(b) + 1)(\psi_i(b) - 1) \quad \Leftrightarrow \quad \frac{\psi_i'(b)}{\psi_i(b)^2 - 1} = \frac{1}{b - \underline{\omega}},$$

whose solution can be easily verified to be

$$\psi_i(b) = \frac{1 - k_i(b - \underline{\omega})^2}{1 + k_i(b - \underline{\omega})^2},$$

where k_i is a constant of integration.[51]

Substituting for $\psi_i(b)$ into (B.4) yields

$$\phi_i(b) = b + \frac{1 - k_i(b - \underline{\omega})^2}{1 + k_i(b - \underline{\omega})^2}(b - \underline{\omega}) = \frac{2b - \underline{\omega} + \underline{\omega}k_i(b - \underline{\omega})^2}{1 + k_i(b - \underline{\omega})^2}. \tag{B.5}$$

Since $\phi_i(b) = t$, solving for b yields the following equilibrium bidding functions:

$$\beta_i^*(t) = \underline{\omega} + \frac{1}{k_i(t - \underline{\omega})}\left(1 - \sqrt{1 - k_i(t - \underline{\omega})^2}\right), \quad i = W, S. \tag{B.6}$$

To find k_i, let $\bar{b}$ be the bid of the highest-value bidder. Since $\phi_i\left(\bar{b}\right) = \omega_i$, Equation (B.2) yields

$$\left(\omega_j - \bar{b}\right)\left(\omega_i - \bar{b}\right) = \left(\bar{b} - \underline{\omega}\right)^2 \quad \Leftrightarrow \quad \bar{b} = \frac{\omega_i\omega_j - \underline{\omega}^2}{\omega_i + \omega_j - 2\underline{\omega}}.$$

Hence, for $b = \bar{b}$ Equation (B.5) becomes

$$\omega_i = \frac{2\left(\frac{\omega_i\omega_j - \underline{\omega}^2}{\omega_i + \omega_j - 2\underline{\omega}}\right) - \underline{\omega} + \underline{\omega}k_i\left(\frac{\omega_i\omega_j - \underline{\omega}^2}{\omega_i + \omega_j - 2\underline{\omega}} - \underline{\omega}\right)^2}{1 + k_i\left(\frac{\omega_i\omega_j - \underline{\omega}^2}{\omega_i + \omega_j - 2\underline{\omega}} - \underline{\omega}\right)^2}$$

$$\Leftrightarrow \quad k_i\left(\frac{\omega_i\omega_j - \underline{\omega}^2}{\omega_i + \omega_j - 2\underline{\omega}} - \underline{\omega}\right)^2(\omega_i - \underline{\omega}) = \frac{\omega_j(\omega_i - \underline{\omega}) - \omega_i(\omega_i - \underline{\omega})}{\omega_i + \omega_j - 2\underline{\omega}}$$

$$\Leftrightarrow \quad k_i = \frac{(\omega_j - \omega_i)(\omega_i + \omega_j - 2\underline{\omega})}{[\underline{\omega}(\omega_j + \omega_i) - (\omega_j\omega_i + \underline{\omega}^2)]^2}. \tag{B.7}$$

Now that we have the BNE bidding functions, we can obtain the NBE bidding functions by replacing $\underline{\omega}$, ω_i, and ω_j with the appropriate expressions. Substituting $\widehat{\omega} = \alpha t$ (while in reality $\underline{\omega} = 0$) in Equations (B.7) and (B.6) yields

$$\beta_i(t) = \alpha t + \frac{[\alpha t(\omega_j + \omega_i) - (\omega_j\omega_i + \alpha^2 t^2)]^2}{(\omega_j - \omega_i)(\omega_i + \omega_j - 2\alpha t)t(1 - \alpha)}\left[1 - \sqrt{1 + \frac{(\omega_i - \omega_j)(\omega_i + \omega_j - 2\alpha t)t^2(1 - \alpha)^2}{[\alpha t(\omega_j + \omega_i) - (\omega_j\omega_i + \alpha^2 t^2)]^2}}\right].$$

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[51]Indeed, it is easy to verify that

$$\psi_i'(b) = -\frac{4k_i(b - \underline{\omega})}{\left[1 + k_i(b - \underline{\omega})^2\right]^2}$$

and

$$\frac{\psi_i'(b)}{\psi_i(b)^2 - 1} = \frac{-4k_i(b - \underline{\omega})}{\left[1 - k_i(b - \underline{\omega})^2\right]^2 - \left[1 + k_i(b - \underline{\omega})^2\right]^2} = \frac{1}{(b - \underline{\omega})}.$$

Finally, substituting $\widehat{\omega}_i = \alpha t + (1 - \alpha)\,\omega_i$ and $\widehat{\omega}_j = \alpha t + (1 - \alpha)\,\omega_j$, we obtain

$$\widehat{\beta}_i\,(t) = \alpha t + \frac{(1-\alpha)\,(\omega_i\omega_j)^2}{t\left(\omega_j^2 - \omega_i^2\right)}\left[1 - \sqrt{1 - \frac{\left(\omega_j^2 - \omega_i^2\right)t^2}{(\omega_i\omega_j)^2}}\right],$$

and

$$\widehat{\beta}_j\,(t) = \alpha t + \frac{(1-\alpha)\,(\omega_i\omega_j)^2}{t\left(\omega_i^2 - \omega_j^2\right)}\left[1 - \sqrt{1 - \frac{\left(\omega_i^2 - \omega_j^2\right)t^2}{(\omega_i\omega_j)^2}}\right].$$

For $\omega_i = \frac{1}{1-z}$ and $\omega_j = \frac{1}{1+z}$, we obtain the bidding strategies in the text.

C   Example with Affiliated Private Values

The following example illustrates claims from Section 7.2. Namely, bidding under projection with IPV leads all types to overbid (relative to rational IPV benchmark), whereas rational bidding with APV leads high types to overbid and low types to under bid (again, relative to the rational IPV benchmark).

Suppose $N = 2$. Private values for each bidder have a marginal distribution $F(t) = .5t(t+1)$ over $\mathcal{T} = [0,1]$; hence, $f(t) = .5 + t$. First, consider the case where private values are independent across bidders. Under projection, bidder i with type t_i perceives the CDF of valuations as

$$\widehat{F}\,(t|t_i) = F\left(\frac{t - \alpha t_i}{1 - \alpha}\right) = \frac{(t - \alpha t_i)\,(t - \alpha t_i + 1 - \alpha)}{2(1-\alpha)^2}. \tag{C.1}$$

Using Proposition 2, the NBE bidding function is

$$\widehat{\beta}_{IPV}(t_i) = \beta^*_{IPV}(t_i) + \alpha\left[\frac{2t_i^2 + 3t_i}{6(t_i+1)}\right], \tag{C.2}$$

where $\beta^*_{IPV}(t_i) = \frac{t_i(4t_i+3)}{6(t_i+1)}$ is the rational bidding function. It is immediate that $\widehat{\beta}_{IPV}(t) > \beta^*_{IPV}(t)$ for all $t > 0$ whenever $\alpha > 0$.

This "uniform overbidding" relative to the rational IPV benchmark does not emerge when valuations are affiliated and bidders are rational. To see this, now suppose that the joint distribution of valuations (consistent with the marginal distribution above) is $F(t_1, t_2) = \frac{1}{2}t_1 t_2\,(t_1 t_2 + 1)$. Then the posterior CDF of an opponent's valuation is $F(x|t) = \frac{x}{2t+1}\,(2xt + 1)$, and the rational bidding function is

$$\beta^*_{APV}\,(t) = \int_0^t y\,d\left(e^{-\int_y^t \frac{1+4z^2}{z+2z^3}dz}\right) = \frac{(4t^2 - 1)\sqrt{2t^2+1} + 1}{6t\sqrt{2t^2+1}}. \tag{C.3}$$

Importantly, one can show that $\beta^*_{APV}(t)$ crosses $\beta^*_{IPV}(t)$ only once and from below: there exists a $\bar{t} \in (0,1)$ such that $\beta^*_{APV}(t) < \beta^*_{IPV}(t)$ for $t \in (0, \bar{t})$ and $\beta^*_{APV}(t) > \beta^*_{IPV}(t)$ for $t \in (\bar{t}, 1)$.