CSEF
Centre for Studies in Economics and Finance

WORKING PAPER NO. 593

Best response algorithms in ratio-bounded games: convergence of affine relaxations to Nash equilibria

Francesco Caruso, Maria Carmela Ceparano and Jacqueline Morgan

December 2020

University of Naples Federico II   University of Salerno   Bocconi University, Milan

CSEF - Centre for Studies in Economics and Finance
DEPARTMENT OF ECONOMICS – UNIVERSITY OF NAPLES
80126 NAPLES - ITALY
Tel. and fax +39 081 675372 – e-mail: csef@unina.it
ISSN: 2240-9696

Working Paper no. 593

Best response algorithms in ratio-bounded games: convergence of affine relaxations to Nash equilibria

Francesco Caruso[*], Maria Carmela Ceparano[†] and Jacqueline Morgan [‡]

Abstract
In two-player non-cooperative games whose strategy sets are Hilbert spaces, in order to approach Nash equilibria we are interested in the affine relaxations of the best response algorithm (where a player's strategy is exactly a best response to the strategy of the other player that comes from the previous step, sometimes called as "fictitious play"). For this purpose we define a class of games, called *ratio-bounded games*, that relies on explicit assumptions on the data and that contains large classes of games already known in literature, both in finite and in infinite dimensional setting: extended quadratic games including potential and antipotential games, non-quadratic games with a bilinear interaction, and linear state differential games. We provide a classification of the ratio-bounded games in four subclasses such that, for each of them, the following issues are examined: the existence and uniqueness of Nash equilibria, the convergence of affine relaxations of the best response algorithm and the estimation of related errors. In particular, the results on convergence of convex relaxations of the best response algorithm include those obtained for zero-sum games in Morgan [*Int. J. Comput. Math.*, 4 (1974), pp. 143-175], and the results on convergence of affine non-convex relaxations include those obtained for non-zero-sum games in Caruso, Ceparano, Morgan [*SIAM J. Optim.*, 30 (2020), pp. 1638-1663].

Keywords: Two-player non-cooperative game; Nash equilibrium; existence and uniqueness; fixed point; contraction mapping; non-expansive mapping; super monotone operator; best response algorithm; convex relaxation; affine non-convex relaxation; convergence and error bound.

[*] Università di Napoli Federico II. E-mail: francesco.caruso@unina.it.

[†] Università di Napoli Federico II. E-mail: mariacarmela.ceparano@unina.it.

[‡] Università di Napoli Federico II and CSEF. E-mail: morgan@unina.it.

Table of contents

1 Introduction

The issue of finding Nash equilibria of non-cooperative games has been (and still is) a deeply investigated topic in Game Theory literature since their definition in [32]. Starting from there, many papers has been devoted to the algorithms for computing Nash equilibria and one of the most explored methods involves the so-called *Best Response Algorithm*. In its best-known version, that goes back to the *alternating fictitious play process* introduced in [8] for finite strategy games (see also [34, 26, 27] for first results), the best response algorithm generates a sequence of strategy profiles as follows: at a given step, the strategy of player i is obtained by selecting a best response to the strategy profile of players other than i coming from the previous step.

For instance, when there are two players and each player has a unique best response to a strategy of the other player, the step n of the best response algorithm can be synthesized in the following way

$$(Step\ n) \quad \begin{cases} v_n = b_2(u_{n-1}) \\ u_n = b_1(v_n), \end{cases}$$

where $b_2(u_{n-1})$ is the best response of player 2 to the strategy u_{n-1} of player 1 and $b_1(v_n)$ is the best response of player 1 to the strategy v_n of player 2.

In this paper, we will consider algorithms where, at a given step, the strategy of at least one player is obtained by taking into account both his best response to the other players' strategies (coming from the previous step) and his own previous step strategy. In particular, we will focus on the so-called affine relaxations for two-player games where only for player 1 the relaxation is allowed: the generic step n of the *affine relaxation of the best response algorithm* we consider is illustrated below

$$(Step\ n) \quad \begin{cases} v_n = b_2(u_{n-1}) \\ u_n = \delta u_{n-1} + (1-\delta) b_1(v_n), \end{cases}$$

where $\delta \in \mathbb{R}$. Depending on the value of δ, three types of relaxations can be recognized

(a) when $\delta = 0$: the *classical best response algorithm*,

(b) when $\delta \in]0, 1[$: a *convex relaxation of the best response algorithm*,

(c) when $\delta \in] - \infty, 0[$ or $\delta \in]1, +\infty[$: an *affine non-convex relaxation of the best response algorithm*.

Focusing on games where the strategy sets of the players are unconstrained spaces, various authors, in different situations, employed these algorithms and showed their convergence to a Nash equilibrium under suitable conditions on the data of the game.

(a) As well-known, the convergence of the classical best response algorithm is obtained under assumptions of contraction on $b_1 \circ b_2$ or $b_2 \circ b_1$. Sufficient conditions on the data for such assumptions have been given in [11] for two-player zero-sum games and in [23] for two-player non-zero-sum games.

	Literature	Game class	Strategy sets	Payoff functions
Classical best response algorithm (no relaxation)	Cherruault, Loridan [11]	two-player zero-sum	finite dimensional spaces	strictly convex and coercive in its argument, differentiable
	Li, Başar [23]	two-player	Hilbert spaces	strongly convex in its argument and differentiable
Convex relaxations	Morgan [29]	two-player zero-sum	Hilbert spaces	strictly convex and coercive in its argument, differentiable
Affine non-convex relaxations	Caruso, Ceparano, Morgan [10]	two-player non-zero-sum	Hilbert spaces	strongly convex in its argument and differentiable
Affine relaxations	Başar [1]	two-player	$\mathbb{R}$ or $\mathbb{R}^2$	strongly convex in its argument and quadratic

Table 1: Affine relaxations of the best response algorithm for games on Hilbert spaces

(b) Convex relaxations of the best response algorithm have been introduced in [29] for two-player zero-sum games in real Hilbert spaces: the convergence is proved for suitable values of $\delta \in]0, 1[$ and error bounds are obtained.

(c) Affine non-convex relaxations have been presented in [10] for two-player non-zero-sum games in real Hilbert spaces: the convergence is proved for suitable values of $\delta \in]1, +\infty[$ and error bounds are computed.

Note that, when the best response functions are linear and the strategy sets are $\mathbb{R}$ or $\mathbb{R}^2$, convergence of affine (convex and non-convex) relaxations has been investigated in [1] for two-players games. The results just illustrated, together with additional information about the assumptions on payoff functions used in each paper, are summarized in Table 1.

When the players have constrained strategy sets, we mention that best response dynamics has been investigated in [20, 18, 22] for two-player zero-sum games, in [18, 3] for weighted potential games, and in [17, 4, 2] for N-player games (see [21, 35] for further discussion); whereas convex relaxations of the best response dynamics have been examined in [16] for N-player games.

In each of the results mentioned in Table 1 about an unconstrained setting, a determinate situation has been considered and a particular affine relaxation of the best response algorithm has been used for that situation. In this paper we deal with two-player games whose strategy sets are Hilbert spaces and our aim is to propose a class of games whereby:

(i) the above mentioned situations are enclosed,

(ii) new situations can be incorporated,

(iii) in each of the different cases identified by the class, one investigates

- existence and uniqueness of Nash equilibria,
- which types of affine relaxations of the best response algorithm converge,
- what is the "best" algorithm (in the sense of speed of convergence) when there are more convergent algorithms.

Having in mind these motivations, we will define a non-restrictive class of games, called *ratio-bounded games*, relying on explicit assumptions on the data and depending on three parameters.

The values of such parameters with respect to each other and to the number 1 will allow to identify different situations that, according to the similarity of the results shown afterward, will provide a partition in four subclasses of the class of the ratio-bounded games. For each of the four subclasses, the existence and uniqueness of Nash equilibria, the convergence of affine relaxations of the best response algorithm and the estimation of related errors will be examined.

The paper is structured as follows. In Section 2, we introduce the class of ratio-bounded games and we state associated properties, how they have been used in the existing related literature and some key preliminary results. In Section 3, we show that the class of ratio-bounded games contains large classes of games already known in literature, both in finite and in infinite dimensional setting: extended quadratic games, including potential and antipotential games (Section 3.1), non-quadratic games with a bilinear interaction (Section 3.2) and linear state differential games (Section 3.3). The core results of the paper are illustrated in Section 4: firstly, we carry out the partition of the class of ratio-bounded games in four subclasses and we formally present the algorithm we consider, called *Affine-Relaxed Best Response Algorithm*; then, for each of the four cases identified by the partition, we investigate the following issues: existence and uniqueness of the Nash equilibria, convergence of the Affine-Relaxed Best Response Algorithm and estimation of the errors. In the first three cases we prove that all the issues are positively answered (Sections 4.1 to 4.3); instead in the fourth one we show by counterexamples that it is not possible to obtain a positive result as in the previous cases (Section 4.4).

2 Blanket assumptions

In the whole paper we consider a two-person noncooperative game $\Gamma = \{2, X_1, X_2, f_1, f_2\}$ where, for $i \in I \coloneqq \{1, 2\}$, the strategy set X_i of player i is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_{X_i}$ and the associated norm $\|\cdot\|_{X_i}$, and the payoff function f_i of player i is a real-valued function defined on the set of strategy profiles $X_1 \times X_2$ (equipped with the inner product defined by $\langle(x_1', x_2'), (x_1'', x_2'')\rangle_{X_1\times X_2} \coloneqq \langle x_1', x_1''\rangle_{X_1} + \langle x_2', x_2''\rangle_{X_2}$ and the associated norm $\|\cdot\|_{X_1\times X_2}$).

As usual in Game Theory, for any $i \in I$ we denote by $-i$ the player who is not i, that is $\{-i\} = I \setminus \{i\}$; hence $(x_1, x_2) \in X_1 \times X_2$ could be denoted with (x_i, x_{-i}). The best response correspondence of player $i \in I$ is the set-valued map $B_i \colon X_{-i} \rightrightarrows X_i$ defined by

$$B_i(x_{-i}) \coloneqq \operatorname*{Arg\,max}_{x_i \in X_i} f_i(x_i, x_{-i}) \coloneqq \{x_i' \in X_i : f(x_i', x_{-i}) \geq f(x_i, x_{-i}), \text{ for any } x_i \in X_i\}.$$

We recall that $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ is a Nash equilibrium of Γ if and only if $\bar{x}_i \in B_i(\bar{x}_{-i})$ for any $i \in I$ (see [32]). When B_i is single valued, the best response function of player i is the function $b_i \colon X_{-i} \to X_i$ such that $\{b_i(x_{-i})\} = B_i(x_{-i})$ for any $x_{-i} \in X_{-i}$.

Given two real normed vector spaces S and T equipped with the norms $\|\cdot\|_S$ and $\|\cdot\|_T$ respectively, we denote by $\mathcal{L}(S, T)$ the normed vector space of all continuous linear operators from S to T, with the usual norm $\|\Lambda\|_{\mathcal{L}(S,T)} \coloneqq \sup\{\|\Lambda(s)\|_T \colon s \in S \text{ and } \|s\|_S = 1\}$, and by $\mathcal{GL}(S,T) \subseteq \mathcal{L}(S,T)$ the set of all bijective continuous linear operators from S to T with continuous (and linear) inverse.

We deal with games satisfying the following assumptions:

($\mathcal{A}_1$) the function $f_i(\cdot, x_{-i})$ is strongly concave on X_i for any $x_{-i} \in X_{-i}$, for any $i \in I$;

($\mathcal{A}_2$) f_i is twice continuously Fréchet differentiable on $X_1 \times X_2$ and $D^2_{x_i} f_i(x_1, x_2) \in \mathcal{GL}(X_i, X_i^*)$ for any $(x_1, x_2) \in X_1 \times X_2$, for any $i \in I$;

($\mathcal{A}_3$) λ_i is a real number for any $i \in I$, where

$$\lambda_i := \sup_{(x_1,x_2)\in X_1\times X_2} \|[D^2_{x_i} f_i(x_1, x_2)]^{-1} \circ D_{x_{-i}}(D_{x_i} f_i)(x_1, x_2)\|_{\mathcal{L}(X_{-i},X_i)}. \tag{1}$$

Remark 2.1 For the sake of completeness, recall that the following properties of the best response correspondences hold when the assumptions ($\mathcal{A}_1$)–($\mathcal{A}_3$) are satisfied (see, e.g., [10, Remark 2.1 and Lemma 2.5(i)(ii)]):

(i) the best response correspondences are single valued;

(ii) the function b_i is continuously differentiable on X_{-i} and Lipschitz continuous with Lipschitz constant no greater than λ_i, for any $i \in I$.

Let $\vartheta\colon X_1 \to X_1$ be the composition of b_1 and b_2, that is

$$\vartheta(x_1) := (b_1 \circ b_2)(x_1) = b_1(b_2(x_1)) \quad \text{for any } x_1 \in X_1. \tag{2}$$

Therefore, $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ is a Nash equilibrium of Γ if and only if $\bar{x}_1$ is a fixed point of ϑ and $\bar{x}_2 = b_2(\bar{x}_1)$.

Finally, let $H\colon X_1 \times X_1 \times X_2 \to \mathcal{L}(X_1, X_1)$ be the operator defined by

$$\begin{aligned} H(x_1', x_1'', x_2) :=& [D^2_{x_1} f_1(x_1', x_2)]^{-1} \circ D_{x_2}(D_{x_1} f_1)(x_1', x_2) \\ & \circ [D^2_{x_2} f_2(x_1'', x_2)]^{-1} \circ D_{x_1}(D_{x_2} f_2)(x_1'', x_2). \end{aligned} \tag{3}$$

The properties of ϑ and its connections with the operator H are summarized below.

Remark 2.2 Assume ($\mathcal{A}_2$) and ($\mathcal{A}_3$). Then, we have (see, e.g., [10, Lemma 2.5(iii) and Remark 2.8])

(i) the function ϑ is continuously differentiable on X_1 and Lipschitz continuous with Lipschitz constant no greater than $\lambda := \lambda_1 \cdot \lambda_2$;

(ii) the derivative $D\vartheta\colon X_1 \to \mathcal{L}(X_1, X_1)$ of ϑ is defined on X_1 by

$$\begin{aligned} D\vartheta(x_1) =& [D^2_{x_1} f_1(\vartheta(x_1), b_2(x_1))]^{-1} \circ D_{x_2}(D_{x_1} f_1)(\vartheta(x_1), b_2(x_1)) \\ & \circ [D^2_{x_2} f_2(x_1, b_2(x_1))]^{-1} \circ D_{x_1}(D_{x_2} f_2)(x_1, b_2(x_1)); \end{aligned}$$

(iii) $H(\vartheta(x_1), x_1, b_2(x_1)) = D\vartheta(x_1)$ for any $x_1 \in X_1$.

The Lipschitz property in (i) and the connection illustrated in (iii) between the derivative of ϑ and the operator H (involving explicitly the players' payoff functions) will be key tools for the convergence results on algorithms for finding Nash equilibria that we present in the next sections. Such fundamental properties have been already used

- firstly in a zero-sum games framework: in [11] when ϑ is a contraction mapping (classical best response dynamics), and in [29] when ϑ is not necessarily a contraction (convex relaxation of best response dynamics). See also [30, 31, 12] for the case of constrained strategy sets;

- afterwards, in a non-zero-sum games framework: in [23] when ϑ is a contraction (classical best response dynamics), in [1] when the strategy sets are $\mathbb{R}$ or $\mathbb{R}^2$ and the best response functions are linear (affine relaxation), in [9] when Γ is a weighted potential game (non-convex relaxation), and in [10] when ϑ is a super monotone and Lipschitz mapping (non-convex relaxation).

We point out that the following properties on H have been crucial in [29] and in [10]:

- in [29] (zero-sum games), the condition

$$\frac{\langle H(x_1', x_1'', x_2)\varphi, \varphi\rangle_{X_1}}{\|\varphi\|_{X_1}^2} \leq \beta < 0 \quad \text{for any } x_1', x_1'' \in X_1,\ x_2 \in X_2,\ \varphi \in X_1 \text{ with } \|\varphi\|_{X_1} \neq 0,$$

  implying that $-H(x_1', x_1'', x_2)$ is strongly monotone, has been required for the improvement of the convergence of a "convex relaxation" of the best response algorithm;

- in [10] (non-zero-sum games), the condition

$$1 < \alpha \leq \frac{\langle H(x_1', x_1'', x_2)\varphi, \varphi\rangle_{X_1}}{\|\varphi\|_{X_1}^2} \quad \text{for any } x_1', x_1'' \in X_1,\ x_2 \in X_2,\ \varphi \in X_1 \text{ with } \|\varphi\|_{X_1} \neq 0,$$

  implying that $H(x_1', x_1'', x_2)$ is super monotone, has been required for the convergence of a "non-convex relaxation" of the best response algorithm.

Moreover, in both the cases above illustrated, the inequality

$$\frac{\langle H(x_1', x_1'', x_2)\varphi, \varphi\rangle_{X_1}}{\|\varphi\|_{X_1}^2} \leq \lambda \quad \text{for any } x_1', x_1'' \in X_1,\ x_2 \in X_2,\ \varphi \in X_1 \text{ with } \|\varphi\|_{X_1} \neq 0$$

is guaranteed.

Starting from this, the aim of the paper is to identify a new class of games (that includes those considered in the above mentioned papers) allowing to determine other games for which affine relaxations of the best response algorithms converge to Nash equilibria. This class is introduced in the following definition.

Definition 2.1 Let $\alpha, \beta \in \mathbb{R}$. A game Γ is (α, β)-*ratio-bounded* if Γ satisfies $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ and the operator H has the (α, β)-*ratio bounded property*, that is,

$(\mathcal{A}_4)$ for any $x_1', x_1'' \in X_1$ and $x_2 \in X_2$

$$\alpha \le \frac{\langle H(x_1', x_1'', x_2)\varphi, \varphi\rangle_{X_1}}{\|\varphi\|_{X_1}^2} \le \beta \quad \text{for any } \varphi \in X_1 \text{ with } \|\varphi\|_{X_1} \neq 0.$$

We denote by $\mathfrak{R}_{\alpha,\beta}$ the class of all (α,β)-ratio-bounded games. Moreover, $\mathfrak{R} := \cup_{\alpha,\beta\in\mathbb{R}}\mathfrak{R}_{\alpha,\beta}$ will be called the *class of ratio-bounded games*.

The next results illustrate some implications of assumption $(\mathcal{A}_4)$ which concern the monotonicity properties of ϑ and the relations among α, β and λ.

Lemma 2.1. *Let $\Gamma \in \mathfrak{R}_{\alpha,\beta}$. Then the following inequalities hold*

$$\alpha\|x_1' - x_1''\|_{X_1}^2 \le \langle \vartheta(x_1') - \vartheta(x_1''), x_1' - x_1''\rangle_{X_1} \le \min\{\beta,\lambda\}\|x_1' - x_1''\|_{X_1}^2 \quad \textit{for any } x_1', x_1'' \in X_1.$$

Proof. Let $x_1', x_1'' \in X_1$ with $x_1' \neq x_2''$. For any $x_1 \in X_1$, the Cauchy-Schwarz inequality and the assumption $(\mathcal{A}_3)$ imply that

$$\frac{\langle H(\vartheta(x_1), x_1, b_2(x_1))(x_1' - x_1''), x_1' - x_1''\rangle_{X_1}}{\|x_1' - x_1''\|_{X_1}^2} \le \|H(\vartheta(x_1), x_1, b_2(x_1))\|_{\mathcal{L}(X_1,X_1)} \le \lambda_1\lambda_2 = \lambda.$$

So, from assumption $(\mathcal{A}_4)$ it follows that

$$\alpha \le \frac{\langle H(\vartheta(x_1), x_1, b_2(x_1))(x_1' - x_1''), x_1' - x_1''\rangle_{X_1}}{\|x_1' - x_1''\|_{X_1}^2} \le \min\{\beta,\lambda\}. \tag{4}$$

Moreover, by applying the Mean Value Theorem to the real-valued function h defined by $h(s) := \langle \vartheta(sx_1' + (1-s)x_1''), x_1' - x_1''\rangle_{X_1}$ for any $s \in [0,1]$, there exists $\bar{s} \in]0,1[$ such that $\langle \vartheta(x_1') - \vartheta(x_1''), x_1' - x_1''\rangle_{X_1} = \langle D\vartheta(\bar{s}x_1' + (1-\bar{s})x_1'')(x_1' - x_1''), x_1' - x_1''\rangle_{X_1}$. So, in light of Remark 2.2(iii), we get

$$\langle \vartheta(x_1') - \vartheta(x_1''), x_1' - x_1''\rangle_{X_1} = \langle H(\vartheta(\tilde{x}_1), \tilde{x}_1, b_2(\tilde{x}_1))(x_1' - x_1''), x_1' - x_1''\rangle_{X_1}, \tag{5}$$

where $\tilde{x}_1 = \bar{s}x_1' + (1-\bar{s})x_1''$.
Therefore, from (4) and (5) the result is proved. □

Lemma 2.2. *Let $\Gamma \in \mathfrak{R}_{\alpha,\beta}$. Then the following inequalities hold:*

$$\alpha \le \lambda \quad \textit{and} \quad -\beta \le \lambda.$$

Proof. The first inequality immediately comes from Lemma 2.1. By exploiting the bilinearity property of the inner product, the Cauchy-Schwarz inequality and the assumptions $(\mathcal{A}_3)$ and $(\mathcal{A}_4)$, we have

$$\begin{aligned} -\beta \le \frac{\langle H(\vartheta(x_1), x_1, b_2(x_1))(-\varphi), \varphi\rangle_{X_1}}{\|\varphi\|_{X_1}^2} &\le \frac{\|(H(\vartheta(x_1), x_1, b_2(x_1))(-\varphi)\|_{X_1}}{\|\varphi\|_{X_1}} \\ &\le \|H(\vartheta(x_1), x_1, b_2(x_1))\|_{\mathcal{L}(X_1,X_1)} \le \lambda, \end{aligned}$$

for any $x_1 \in X_1$ and $\varphi \in X_1$ with $\|\varphi\|_{X_1} \neq 0$; so even the second inequality is proved.
Note that the inequality $-\beta \le \lambda$ is informative only if $\beta < 0$. □

In the next section, we illustrate examples of games showing that the class of ratio-bounded games is non-restrictive and contains widely used games in literature.

3 About the class of ratio-bounded games

We start in a finite dimensional setting and show that $\mathfrak{R}$ contains "large" classes of quadratic games (including potential games and antipotential games) and non-quadratic games with a bilinear interaction. Then, moving towards an infinite dimensional setting, we present a class of differential games that belong to $\mathfrak{R}$.

3.1 Extended quadratic games

Let $\Gamma = \{2, X_1, X_2, f_1, f_2\}$ be the game where $X_1 = X_2 = \mathbb{R}^n$ is equipped with the usual inner product $(\cdot,\cdot)_{\mathbb{R}^n}$ and the associated Euclidean norm $\|\cdot\|_{\mathbb{R}^n}$, and the payoff functions are defined by

$$f_i(x_1, x_2) = -a_i\|x_i\|^2_{\mathbb{R}^n} + \ell_i(x_i) + s_i + p_i(x_{-i}) + d_i\langle x_1, x_2\rangle_{\mathbb{R}^n} \quad \text{for any } i \in I, \tag{6}$$

where $a_i > 0$, $\ell_i : \mathbb{R}^n \to \mathbb{R}$ is a linear function, $s_i \in \mathbb{R}$, $p_i : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, and $d_i \in \mathbb{R}$.

Obviously, the Nash equilibria of Γ are independent on the functions p_1 and p_2 and, when $p_1 \equiv p_2 \equiv 0$, we recover the so-called class of quadratic games.

Let $i \in I$. The function f_i is twice continuously differentiable, $f_i(\cdot, x_{-i})$ is strongly concave for any $x_{-i} \in \mathbb{R}^n$, and $D^2_{x_i} f_i(x_1, x_2) \equiv -2a_i\mathcal{I}_n$ is invertible for any $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$ (where $\mathcal{I}_n$ denotes the identity matrix of size n). Moreover, denoted with $\|\cdot\|_\infty$ the infinity matrix norm defined on $\mathbb{R}^{n\times n}$ by $\|U\|_\infty := \max_{1\le j\le n}\sum_{k=1}^n |u_{jk}|$, we get

$$\lambda_i = \|[-2a_i\mathcal{I}_n]^{-1}[d_i\mathcal{I}_n]\|_\infty = \frac{|d_i|}{2a_i}.$$

Hence, Γ satisfies $(\mathcal{A}_1)$–$(\mathcal{A}_3)$.

Furthermore, for any $x_1', x_1'', x_2 \in \mathbb{R}^n$ and any $\varphi \in \mathbb{R}^n$ we have

$$(H(x_1', x_1'', x_2)\varphi, \varphi)_{\mathbb{R}^n} = ([-2a_1\mathcal{I}_n]^{-1}[d_1\mathcal{I}_n][-2a_2\mathcal{I}_n]^{-1}[d_2\mathcal{I}_n]\varphi, \varphi)_{\mathbb{R}^n} = \frac{d_1 d_2}{4a_1a_2}\|\varphi\|^2_{\mathbb{R}^n},$$

so the operator H has the (α, β)-ratio bounded property with $\alpha = \beta = \frac{d_1d_2}{4a_1a_2} \in \mathbb{R}$ and Γ belongs to $\mathfrak{R}$.

Depending on d_1 and d_2, the class of games illustrated above contains special types of games already employed in literature.

Potential games: $d_1 > 0$ and $d_2 > 0$. In light of [9, Proposition 2], the game Γ is a weighted potential game (see [28] for the definition). A weighted potential function $P\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$P(x_1, x_2) = -\sum_{i\in I}\left[\frac{a_i}{d_i}\|x_i\|^2_{\mathbb{R}^n} - \frac{\ell_i(x_i)}{d_i}\right] + \langle x_1, x_2\rangle_{\mathbb{R}^n},$$

and the associated weights are d_1 and d_2. For a further discussion on potential games, see for example [14, 33, 6].

Antipotential games: $d_1 = -d_2$**.** The payoff functions in Γ can be rewritten as

$$f_1(x_1, x_2) = Q(x_1, x_2) + h_1(x_2) \quad \text{and} \quad f_2(x_1, x_2) = -Q(x_1, x_2) - h_2(x_1)$$

where $Q(x_1, x_2) = -a_1\|x_1\|^2_{\mathbb{R}^n} + \ell_1(x_1) + d_1\langle x_1, x_2\rangle_{\mathbb{R}^n} + a_2\|x_2\|^2_{\mathbb{R}^n} - \ell_2(x_2)$ and
$h_i(x_{-i}) = (-1)^{i+1}[-a_{-i}\|x_{-i}\|^2_{\mathbb{R}^n} + \ell_{-i}(x_{-i}) + s_i + p_i(x_{-i})]$ for any $i \in I$ and any $x_1, x_2 \in \mathbb{R}^n$.
$h_1(x_2) = -a_2\|x_2\|^2_{\mathbb{R}^n} + \ell_2(x_2) + s_1 + p_1(x_2)$ and $h_2(x_1) = a_1\|x_1\|^2_{\mathbb{R}^n} - \ell_1(x_1) - s_2 - p_2(x_1)$.
So the game above illustrated belongs to the class of *antipotential games*, introduced in [24], where lower semicontinuity properties of approximate Nash equilibria have been investigated in the framework of one-leader two-follower Stackelberg games. Recall that Γ is an antipotential game if the game $\Omega = \{2, X_1, X_2, f_1, -f_2\}$ is an (exact) potential game.
Note that, if

$$p_i(x_{-i}) = a_{-i}\|x_{-i}\|^2_{\mathbb{R}^n} - \ell_{-i}(x_{-i}) - s_i \quad \text{for any } i \in I,$$

then Γ is a zero-sum game. More generally, the class of antipotential games includes the constant-sum games.

3.2 Non-quadratic games with a bilinear interaction

Let $\Gamma = \{2, X_1, X_2, f_1, f_2\}$ be the game where $X_1 = X_2 = \mathbb{R}^n$ and the payoff functions are defined by

$$f_i(x_1, x_2) = q_i(x_i) + p_i(x_{-i}) + d_i\langle x_1, x_2\rangle_{\mathbb{R}^n} \quad \text{for any } i \in I$$

where $q_i : \mathbb{R}^n \to \mathbb{R}$, $p_i : \mathbb{R}^n \to \mathbb{R}$ and $d_i \in \mathbb{R}$. For the sake of simplicity of notation and calculus, we consider $n = 1$.
Assume that $q_i : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and

$$M_i := -\inf_{x_i \in \mathbb{R}} D^2 q_i(x_i) \in \mathbb{R} \quad \text{and} \quad m_i := -\sup_{x_i \in \mathbb{R}} D^2 q_i(x_i) \in]0, +\infty[\quad \text{for any } i \in I. \tag{7}$$

Given the above, $f_i(\cdot, x_{-i})$ is strongly concave, as $D^2_{x_i} f_i(x_1, x_2) \le -m_i < 0$ for any $(x_1, x_2) \in \mathbb{R}^2$, and

$$\lambda_i = \sup_{(x_1, x_2) \in \mathbb{R}^2} \left| \frac{D_{x_{-i}}(D_{x_i} f_i)(x_1, x_2)}{D^2_{x_i} f_i(x_1, x_2)} \right| = \frac{|d_i|}{m_i},$$

for any $i \in I$. Hence Γ satisfies $(\mathcal{A}_1)$–$(\mathcal{A}_3)$.
Furthermore, the (α, β)-ratio bounded property $(\mathcal{A}_4)$ holds when

$$\alpha \le \min\left\{\frac{d_1 d_2}{M_1 M_2}, \frac{d_1 d_2}{m_1 m_2}\right\} \quad \text{and} \quad \beta \ge \max\left\{\frac{d_1 d_2}{M_1 M_2}, \frac{d_1 d_2}{m_1 m_2}\right\};$$

so Γ belongs to $\mathfrak{R}$.
The assumptions in (7) are satisfied if, for example

$$q_i(x_i) = \frac{1}{1 + x_i^2} - 4x_i^2 + x_i \quad \text{for any } i \in I.$$

Examples of games involving a non-bilinear interaction will be considered in the next section.

3.3 Linear state differential games

Here we consider the following class of two-player differential games (see, e.g., [13, 19] for definitions and applications):

- time varies in $[0, T]\!]$, where $[0, T]\!] = [0, T]$ if $T < +\infty$ and $[0, T]\!] = [0, +\infty[$ if $T = +\infty$;
- the state equation is given by

$$\dot{x}(t) = c_1(t) + c_2(t) - mx(t), \tag{8}$$

  where $x \colon [0, T]\!] \to \mathbb{R}^n$ with $x(0) = x_0 \in \mathbb{R}^n$, $m \in \mathbb{R}$ and the control variable of player $i \in I$, $c_i \colon [0, T]\!] \to \mathbb{R}^n$, belongs to $L^2([0, T]\!])$;
- the instantaneous payoff of player $i \in I$ at time t is

$$\pi_i(x(t), c_1(t), c_2(t)) \coloneqq \langle r_i(t), x(t)\rangle_{\mathbb{R}^n} - a_i\|c_i(t)\|^2_{\mathbb{R}^n} + d_i\langle c_1(t), c_2(t)\rangle_{\mathbb{R}^n}, \tag{9}$$

  where $r_i \colon [0, T]\!] \to \mathbb{R}^n$, $a_i > 0$ and $d_i \in \mathbb{R}$;
- the objective function of player $i \in I$ is

$$J_i(x, c_1, c_2) = \int_0^T e^{-\rho_i t}\pi_i(x(t), c_1(t), c_2(t))\, dt + e^{-\rho_i T}\langle w_i, x(T)\rangle_{\mathbb{R}^n},$$

  where $\rho_i \geq 0$ is the individual discount rate and $w_i \in \mathbb{R}^n$ is the individual residual (when $T = +\infty$, it is assumed that $w_i = 0$).

The game described above belongs to the class of linear state differential games (for a further discussion in an economic framework, see [13, sections 7.2, 9.5 and 11.3]). In order to show that such a game is ratio-bounded, first we substitute the solution to the first-order differential equations (8), that is $x(t) = x_0 e^{-mt} + e^{-mt}\int_0^T [c_1(s) + c_2(s)]e^{ms}\, ds$, in the payoff defined in (9). Therefore, we can rewrite the players' objective functions as functions of the control variables only and, denoted such functions by f_i for any $i \in I$, we obtain

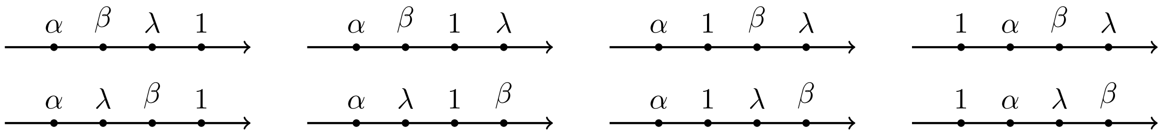
$$\begin{aligned} f_i(c_1, c_2) = &\int_0^T e^{-\rho_i t}\Big\{e^{-mt}\langle r_i(t), x_0\rangle_{\mathbb{R}^n} - a_i\|c_i(t)\|^2_{\mathbb{R}^n} + d_i\langle c_1(t), c_2(t)\rangle_{\mathbb{R}^n} \\ &+ e^{-mt}\Big\langle r_i(t), \int_0^t [c_1(s) + c_2(s)]e^{ms}\, ds\Big\rangle_{\mathbb{R}^n}\Big\}\, dt \end{aligned}$$

for any $(c_1, c_2) \in L^2([0, T]\!]) \times L^2([0, T]\!])$. By arguing similarly to [10, Example 2.13] (see also [9, Subsection 4.1] where weighted potential games are examined), it follows that the game $\Gamma = \{2, L^2([0, T]\!]), L^2([0, T]\!]), f_1, f_2\}$ belongs to $\mathfrak{R}$ as

$$\lambda_i = \frac{|d_i|}{2a_i} \text{ for any } i \in I \quad \text{and} \quad \alpha = \beta = \frac{d_1 d_2}{4a_1 a_2}.$$

4 Affine relaxations and algorithms

Let $\Gamma \in \mathfrak{R}$. To such a game we can associate three important constants: α and β, bounds of the ratio in Definition 2.1, and λ, as defined in Remark 2.2. Therefore, since $\alpha \le \beta$ and $\alpha \le \lambda$ (by Lemma 2.2), we can identify eight situations depending on the values of such constants with respect to each other and to the number 1:



For the sake of exposition, we will collect the situations that lead to the same results. So we will consider the different cases in the following order:

C_1) $\lambda < 1$ (situation 1 or 5 or 6),

C_2) $\alpha > 1$ (situation 4 or 8),

C_3) $\beta < 1 \le \lambda$ (situation 2),

C_4) $\alpha \le 1 \le \min\{\beta, \lambda\}$ (situation 3 or 7).

For each of these four cases we will analyze the following issues: existence and uniqueness of the Nash equilibrium, convergence of algorithms based on affine relaxations of the best response algorithm and related error bounds. We will answer differently in each of the four cases.
The algorithms we are interested in are based on the combination of the identity map of X_1 and of the composition of the best response functions of Γ. More precisely, denoted by $t_\delta \colon X_1 \to X_1$ the operator defined by

$$t_\delta(x_1) := \delta x_1 + (1 - \delta)\vartheta(x_1), \tag{10}$$

where $\delta \in \mathbb{R}$, we state below the *Affine-Relaxed Best Response Algorithm.*

Affine-Relaxed Best Response Algorithm $(\mathcal{A}^\delta)$

Let $v_0 \in X_2$ and $u_0 = b_1(v_0)$. For any $n \in \mathbb{N}$

$$(\textit{Step } n) \qquad \begin{cases} v_n = b_2(u_{n-1}) \\ u_n = \delta u_{n-1} + (1 - \delta)\, b_1(v_n) = t_\delta(u_{n-1}). \end{cases}$$

Note that, for the sake of readability, we labelled the points generated by the algorithm as u_n and v_n in order to avoid the use of two indexes.
The algorithm $(\mathcal{A}^\delta)$ is an affine relaxation of the widely used best response algorithm and, depending on the value of δ, it will be named

- when $\delta = 0$: the classical best response algorithm,
- when $\delta \in]0, 1[$: an affine convex relaxation of the best response algorithm,

- when $\delta \in]-\infty, 0[$ or $\delta \in]1, +\infty[$: an affine non-convex relaxation of the best response algorithm.

4.1 Contractive case C_1: $\lambda < 1$

Let Γ be a game belonging to $\mathfrak{R}_{\alpha,\beta}$ and C_1 be satisfied, that is

$$\alpha \leq \beta \leq \lambda < 1 \quad \text{or} \quad \alpha \leq \lambda \leq \beta < 1 \quad \text{or} \quad \alpha \leq \lambda < 1 \leq \beta.$$

In light of Remark 2.2(i) and since $\lambda \in [0,1[$, the function ϑ is a contraction and Γ has a unique Nash equilibrium. In this case, the classical best response algorithm is well-known converging to the Nash equilibrium; moreover its affine, convex as well as non-convex, relaxations can provide improvements in the speed of convergence. The introduction of the classical best response algorithm goes back to Brown in [8], where it was named as *fictitious play* (see, e.g. [15, Chapter 2] for a further discussion). The convergence analysis of such an algorithm, in the framework defined by assumptions $(\mathcal{A}_1)$-$(\mathcal{A}_3)$, has been shown first in [11, Theorem 3.1, Corollaries 3.1 and 3.2] for two-player zero-sum games and then in [23, Theorem 1] for two-player non-zero-sum games. Affine convex relaxations of the best response algorithm have been shown to converge to the Nash equilibrium for zero-sum games in [29] (even for $\lambda \geq 1$). When $X_1 = X_2 = \mathbb{R}$ and the best response functions are linear, affine non-convex relaxations of the best response algorithms have been proved to converge in [1, Section 4] (even for $\lambda \geq 1$).
The following more general result for two-player games, which includes the previous ones when $\lambda < 1$, investigates the convergence of the Affine-Relaxed Best Response Algorithm $(\mathcal{A}^\delta)$, $\delta \in \mathbb{R}$.

Theorem 4.1. *For any $\Gamma \in \mathfrak{R}_{\alpha,\beta}$ with $\lambda < 1$ we have:*

(i) the game Γ has a unique Nash equilibrium $(\bar{u}, \bar{v})$, where $\bar{u}$ is the unique fixed point of ϑ and $\bar{v} = b_2(\bar{u})$;

(ii) for any $\delta \in \left]\frac{\lambda^2-1}{\lambda^2-2\alpha+1}, 1\right[$ the sequence $(u_n, v_n)_n$ generated by algorithm $(\mathcal{A}^\delta)$ is strongly convergent to $(\bar{u}, \bar{v})$ in $X_1 \times X_2$ and

$$\lim_{n \to +\infty} f_i(u_n, v_n) = f_i(\bar{u}, \bar{v}) \quad \textit{for any } i \in I;$$

(iii) for any $\delta \in \left]\frac{\lambda^2-1}{\lambda^2-2\alpha+1}, 1\right[$ the error estimations hold:

$$\|u_n - \bar{u}\|_{X_1} \leq \frac{\kappa(\delta)^n}{1-\kappa(\delta)} \|u_1 - u_0\|_{X_1} \textit{ and } \|v_{n+1} - \bar{v}\|_{X_2} \leq \frac{\lambda\,\kappa(\delta)^n}{1-\kappa(\delta)} \|v_1 - v_0\|_{X_2} \textit{ for any } n \in \mathbb{N},$$

where $\kappa \colon \left]\frac{\lambda^2-1}{\lambda^2-2\alpha+1}, 1\right[\to \mathbb{R}$ is defined by:

$$\kappa(\delta) = \begin{cases} [(\lambda^2 - 2\alpha + 1)\delta^2 - 2(\lambda^2 - \alpha)\delta + \lambda^2]^{1/2}, & \textit{if } \frac{\lambda^2-1}{\lambda^2-2\alpha+1} < \delta < 0 \\ [(\lambda^2 - 2\min\{\beta,\lambda\} + 1)\delta^2 - 2(\lambda^2 - \min\{\beta,\lambda\})\delta + \lambda^2]^{1/2}, & \textit{if } 0 \leq \delta < 1. \end{cases} \tag{11}$$

Proof. Point (i) comes from the Contraction Mapping Theorem and the definition of ϑ in (2). To show (ii) and (iii), we preliminarly note that the function κ in (11) is well-defined since: on the one hand $\lambda^2 - 2\alpha + 1 > 0$ and for any $\delta \in \left]\frac{\lambda^2-1}{\lambda^2-2\alpha+1}, 0\right[$ we have $(\lambda^2 - 2\alpha + 1)\delta^2 - 2(\lambda^2 - \alpha)\delta + \lambda^2 \geq 0$; on the other hand $\lambda^2 - 2\min\{\beta, \lambda\} + 1 > 0$, $-\beta \leq \lambda$ (see Lemma 2.2) and for any $\delta \in [0, 1[$ we have $(\lambda^2 - 2\min\{\beta, \lambda\} + 1)\delta^2 - 2(\lambda^2 - \min\{\beta, \lambda\})\delta + \lambda^2 \geq 0$.
Let $\delta \in \mathbb{R} \setminus \{1\}$ and let $x_1', x_1'' \in X_1$ with $x_1' \neq x_1''$. By (10) and Remark 2.2(i),

$$\begin{aligned} \|t_\delta(x_1') - t_\delta(x_1'')\|_{X_1}^2 \leq & [\delta^2 + (1-\delta)^2\lambda^2]\|x_1' - x_1''\|_{X_1}^2 \\ & + 2\delta(1-\delta)\langle \vartheta(x_1') - \vartheta(x_1''), x_1' - x_1'' \rangle_{X_1}. \end{aligned} \tag{12}$$

In light of Lemma 2.1, from (12) we get

$$\|t_\delta(x_1') - t_\delta(x_1'')\|_{X_1}^2 \leq \begin{cases} [\delta^2 + (1-\delta)^2\lambda^2 + 2\delta(1-\delta)\alpha]\|x_1' - x_1''\|_{X_1}^2, & \text{if } \delta < 0 \text{ or } \delta > 1 \\ [\delta^2 + (1-\delta)^2\lambda^2 + 2\delta(1-\delta)\min\{\beta, \lambda\}]\|x_1' - x_1''\|_{X_1}^2, & \text{if } 0 \leq \delta < 1. \end{cases}$$

Since $\delta^2 + (1-\delta)^2\lambda^2 + 2\delta(1-\delta)\alpha = (\lambda^2 - 2\alpha + 1)\delta^2 - 2(\lambda^2 - \alpha)\delta + \lambda^2 < 1$ if $\delta \in \left]\frac{\lambda^2-1}{\lambda^2-2\alpha+1}, 0\right[$ and $\delta^2 + (1-\delta)^2\lambda^2 + 2\delta(1-\delta)\min\{\beta, \lambda\} = (\lambda^2 - 2\min\{\beta, \lambda\} + 1)\delta^2 - 2(\lambda^2 - \min\{\beta, \lambda\})\delta + \lambda^2 < 1$ if $\delta \in [0, 1[$, it follows that

$$\|t_\delta(x_1') - t_\delta(x_1'')\|_{X_1} \leq \kappa(\delta)\|x_1' - x_1''\|_{X_1} \quad \text{and} \quad \kappa(\delta) \in [0, 1[\quad \text{for any } \delta \in \left]\frac{\lambda^2 - 1}{\lambda^2 - 2\alpha + 1}, 1\right[.$$

Therefore, t_δ is a contraction mapping for any $\delta \in \left]\frac{\lambda^2-1}{\lambda^2-2\alpha+1}, 1\right[$ and, since t_δ has the same fixed points of ϑ, the sequence $(u_n)_n$ in $(\mathcal{A}^\delta)$ strongly converges to $\bar{u}$. Furthermore, by Remark 2.1(ii) we have

$$\|v_n - \bar{v}\|_{X_2} = \|b_2(u_{n-1}) - b_2(\bar{u})\|_{X_2} \leq \lambda_2\|u_{n-1} - \bar{u}\|_{X_1}, \quad \text{for any } n \in \mathbb{N}. \tag{13}$$

As $(u_n)_n$ strongly converges to $\bar{u}$, the sequence $(v_n)_n$ is strongly convergent to $\bar{v}$. So, the sequence $(u_n, v_n)_n$ strongly converges to $(\bar{u}, \bar{v})$. The second part of (ii) follows from the continuity of f_i for any $i \in I$. Finally, since t_δ is a contraction and $\kappa(\delta)$ is the estimated contraction constant, the error estimation on $(u_n)_n$ in (iii) is a straightforward consequence of the Contraction Mapping Theorem (see, for example, [5, Theorem 2.1(iii)]). Given the above, inequality (13) proves the error estimation on $(v_n)_n$ in (iii). □

Remark 4.1 When $\delta \in]0, 1[$ the sequence $(u_n)_n$ in $(\mathcal{A}^\delta)$ can be seen as generated via the *Mann iteration procedure* (see [25] and, e.g. [5, Chapter 4]). If in addition $\beta \in]0, 1[$, by Lemma 2.1 the function ϑ is a *generalized pseudo-contraction* (as introduced in [36]) and the convergence of $(u_n)_n$ can be shown by using [36, Theorem 2.1].

We highlight that Theorem 4.1(ii) ensures the convergence to the Nash equilibrium by employing three kinds of affine relaxations of the best response algorithm: the classical best response algorithm $(\mathcal{A}^0)$, its convex relaxation $(\mathcal{A}^\delta)$ for any $\delta \in]0, 1[$ and its non-convex relaxation $(\mathcal{A}^\delta)$ for any $\delta \in \left]\frac{\lambda^2-1}{\lambda^2-2\alpha+1}, 0\right[$.

As regards the speed of convergence of ($\mathcal{A}^\delta$), improvements have been obtained: in [29, Theorem 1.3] for affine convex relaxations of the classical best response algorithm in two-player zero-sum games whereby $\beta < 0$; and in [1, Section 4.1.3] for affine non-convex relaxations when $X_1 = X_2 = \mathbb{R}$ and the best response functions are linear.
The next result, that includes the above mentioned ones, shows for which values of δ the highest speed of convergence is obtained and which kind of affine relaxation of the classical best response algorithm is the fastest one, depending on α, β and λ. We call "best algorithm" the algorithm ensuring the highest speed of convergence.

Theorem 4.2. *Let $\Gamma \in \mathfrak{R}_{\alpha,\beta}$ with $\lambda < 1$. Then*

- *when $\alpha > 0$ and $\lambda < \sqrt{\alpha}$ the best algorithm is the affine non-convex relaxation of the classical best response algorithm corresponding to $\delta^* = \frac{\lambda^2-\alpha}{\lambda^2-2\alpha+1}$,*
- *when $\beta < 0$ or ($\beta > 0$ and $\sqrt{\beta} < \lambda$) the best algorithm is the affine convex relaxation of the classical best response algorithm corresponding to $\delta^* = \frac{\lambda^2-\beta}{\lambda^2-2\beta+1}$,*
- *in the remaining cases the best algorithm is the classical best response algorithm corresponding to $\delta^* = 0$.*

Proof. By Theorem 4.1(iii), the decrease of $\kappa(\delta)$ implies the increase in the speed of convergence of the algorithm ($\mathcal{A}^\delta$). Hence, since the function κ has a unique minimizer over $\left]\frac{\lambda^2-1}{\lambda^2-2\alpha+1}, 1\right[$, that is

$$\delta^* = \begin{cases} \frac{\lambda^2-\alpha}{\lambda^2-2\alpha+1}, & \text{if } \alpha > 0 \text{ and } \lambda < \sqrt{\alpha} \\ \frac{\lambda^2-\beta}{\lambda^2-2\beta+1}, & \text{if } \beta < 0 \text{ or } \left(\beta > 0 \text{ and } \sqrt{\beta} < \lambda\right) \\ 0, & \text{otherwise,} \end{cases}$$

the thesis follows by noting that $\lambda < \sqrt{\alpha}$ implies $\delta^* < 0$, $\beta < 0$ implies $\delta^* \in]0, 1[$, and $\sqrt{\beta} < \lambda$ implies $\delta^* \in]0, 1[$. □

4.2 Super monotonicity case C_2: $\alpha > 1$

Let Γ be a game belonging to $\mathfrak{R}_{\alpha,\beta}$ and C_2 be satisfied, that is

$$1 < \alpha \leq \beta \leq \lambda \quad \text{or} \quad 1 < \alpha \leq \lambda \leq \beta.$$

When $\alpha > 1$, uniqueness of Nash equilibria and convergence of algorithms have been investigated in [10]. Note that, by Lemma 2.1, ϑ is super monotone with constant α, that is

$$(\vartheta(x_1') - \vartheta(x_1''), x_1' - x_1'')_{X_1} \geq \alpha \|x_1' - x_1''\|^2_{X_1} \quad \text{for any } x_1', x_1'' \in X_1$$

(see [10, Definition 2.6]), and so ϑ is not a contraction (see [10, Proposition 2.7]). In this case, in light of [10, theorems 2.10 and 3.2], the game Γ has a unique Nash equilibrium and it can be approached via affine non-convex relaxations of the best response algorithm, as summarized in the following result.

Theorem 4.3. *For any $\Gamma \in \mathfrak{R}_{\alpha,\beta}$ with $\alpha > 1$ we have:*

(i) the game Γ has a unique Nash equilibrium $(\bar{u}, \bar{v})$;

(ii) for any $\delta \in \left]1, \frac{\lambda^2-1}{\lambda^2-2\alpha+1}\right[$ the sequence $(u_n, v_n)_n$ generated by algorithm $(\mathcal{A}^\delta)$ strongly converges to $(\bar{u}, \bar{v})$ in $X_1 \times X_2$ and

$$\lim_{n\to+\infty} f_i(u_n, v_n) = f_i(\bar{u}, \bar{v}) \quad \textit{for any } i \in I;$$

(iii) for any $\delta \in \left]1, \frac{\lambda^2-1}{\lambda^2-2\alpha+1}\right[$ the following error estimations hold:

$$\|u_n - \bar{u}\|_{X_1} \le \frac{\iota(\delta)^n}{1-\iota(\delta)} \|u_1 - u_0\|_{X_1} \;\; \textit{and} \;\; \|v_{n+1} - \bar{v}\|_{X_2} \le \frac{\iota(\delta)^n \lambda}{1-\iota(\delta)} \|v_1 - v_0\|_{X_2} \;\; \textit{for any } n \in \mathbb{N},$$

where $\iota\colon \left]1, \frac{\lambda^2-1}{\lambda^2-2\alpha+1}\right[\to \mathbb{R}$ is defined by $\iota(\delta) = [(\lambda^2 - 2\alpha + 1)\delta^2 - 2(\lambda^2 - \alpha)\delta + \lambda^2]^{1/2}$;

(iv) the value of δ corresponding to the highest speed of convergence of $(\mathcal{A}^\delta)$ is the unique minimizer of ι, that is $\delta^* = \frac{\lambda^2-\alpha}{\lambda^2-2\alpha+1}$.

We point out that, in this case, the classical best response algorithm and its affine convex relaxations may not converge to the Nash equilibrium, as illustrated in the following example.

Example 4.1 Let Γ be the game where $X_1 = X_2 = \mathbb{R}$ and

$$f_1(x_1, x_2) = -x_1^2 + 4x_1x_2, \quad f_2(x_1, x_2) = -x_2^2 + 6x_1x_2.$$

This game belongs to the class illustrated in Section 3.1 (in particular Γ is a weighted potential game) with $a_i = 1$, $\ell_i \equiv p_i \equiv 0$, $s_i = 0$ for any $i \in I$, $d_1 = 4$ and $d_2 = 6$. Since $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda = 6$ and

$$\frac{D_{x_2}(D_{x_1} f_1)(x_1', x_2) D_{x_1}(D_{x_2} f_2)(x_1'', x_2)}{D_{x_1}^2 f_1(x_1', x_2) D_{x_2}^2 f_2(x_1'', x_2)} = 6 \quad \text{for any } x_1', x_1'', x_2 \in \mathbb{R},$$

the game Γ belongs to $\mathfrak{R}_{\alpha,\beta}$ with $\alpha > 1$ and there is a unique Nash equilibrium, namely $(0, 0)$. Given an initial point $v_0 \in X_2$ and got $u_0 = 2v_0$, the algorithm $(\mathcal{A}^\delta)$ generates the sequence $(u_n, v_n)_n$ defined by $v_n = 3u_{n-1}$ and $u_n = \delta u_{n-1} + 2(1-\delta)v_n = (6 - 5\delta)u_{n-1}$ for any $n \in \mathbb{N}$. Therefore, unless one chooses $v_0 = 0$, the sequence $(u_n, v_n)_n$ diverges for any $\delta \in [0, 1[$, that is both the classical best response algorithm and its affine convex relaxations do not converge to the Nash equilibrium of Γ. Instead, according to Theorem 4.3, the affine non-convex relaxations $(\mathcal{A}^\delta)$ for any $\delta \in]1, 7/5[$ converge to $(0, 0)$ and the highest speed of convergence is achieved when $\delta = 6/5$.

Further investigations regarding to the uniqueness of Nash equilibria, to its numerical approximation and to related error bounds can be found in [10].

4.3 Presque-contractive case C_3: $\beta < 1 \leq \lambda$

Let Γ be a game belonging to $\mathfrak{R}_{\alpha,\beta}$ and C_3 be satisfied, that is

$$\alpha \leq \beta < 1 \leq \lambda.$$

In this case, only zero-sum games have been investigated: in [29] the convergence of affine convex relaxations of the classical best response algorithm is proved and an improvement in the speed of convergence is obtained when $\beta < 0$ ([29, theorems 1.2 and 1.3]). The following result concerns affine convex relaxations and more general two-player games.

Theorem 4.4. *For any $\Gamma \in \mathfrak{R}_{\alpha,\beta}$ with $\alpha \leq \beta < 1 \leq \lambda$ we have:*

(i) the game Γ has a unique Nash equilibrium $(\bar{u}, \bar{v})$;

(ii) for any $\delta \in \left]\frac{\lambda^2-1}{\lambda^2-2\beta+1}, 1\right[$ the sequence $(u_n, v_n)_n$ generated by algorithm $(\mathcal{A}^\delta)$ strongly converges to $(\bar{u}, \bar{v})$ in $X_1 \times X_2$ and

$$\lim_{n\to+\infty} f_i(u_n, v_n) = f_i(\bar{u}, \bar{v}) \quad \textit{for any } i \in I;$$

(iii) for any $\delta \in \left]\frac{\lambda^2-1}{\lambda^2-2\beta+1}, 1\right[$ the following error estimations hold:

$$\|u_n - \bar{u}\|_{X_1} \leq \frac{\sigma(\delta)^n}{1-\sigma(\delta)}\|u_1 - u_0\|_{X_1} \textit{ and } \|v_{n+1} - \bar{v}\|_{X_2} \leq \frac{\sigma(\delta)^n \lambda}{1-\sigma(\delta)}\|v_1 - v_0\|_{X_2} \textit{ for any } n \in \mathbb{N},$$

where $\sigma\colon \left]\frac{\lambda^2-1}{\lambda^2-2\beta+1}, 1\right[\to \mathbb{R}$ is defined by $\sigma(\delta) = [(\lambda^2 - 2\beta + 1)\delta^2 - 2(\lambda^2 - \beta)\delta + \lambda^2]^{1/2}$;

(iv) the value of δ corresponding to the highest speed of convergence of $(\mathcal{A}^\delta)$ is the unique minimizer of σ, that is $\delta^* = \frac{\lambda^2-\beta}{\lambda^2-2\beta+1}$.

Proof. We preliminarly note that $\frac{\lambda^2-1}{\lambda^2-2\beta+1} \in]0,1[$ and the function σ is well-defined.
Let $\delta \in]0,1[$ and let $x_1', x_1'' \in X_1$ with $x_1' \neq x_1''$. By (10), Remark 2.2(i) and Lemma 2.1,

$$\begin{aligned}\|t_\delta(x_1') - t_\delta(x_1'')\|_{X_1}^2 \leq & [\delta^2 + (1-\delta)^2\lambda^2]\|x_1' - x_1''\|_{X_1}^2 + 2\delta(1-\delta)\langle\vartheta(x_1') - \vartheta(x_1''), x_1' - x_1''\rangle_{X_1} \\ \leq & [\delta^2 + (1-\delta)^2\lambda^2 + 2\delta(1-\delta)\beta]\|x_1' - x_1''\|_{X_1}^2.\end{aligned}$$

Since $\delta^2 + (1-\delta)^2\lambda^2 + 2\delta(1-\delta)\beta = (\lambda^2 - 2\beta + 1)\delta^2 - 2(\lambda^2 - \beta)\delta + \lambda^2 \in]0,1[$ if and only if $\delta \in \left]\frac{\lambda^2-1}{\lambda^2-2\beta+1}, 1\right[$, we have

$$\|t_\delta(x_1') - t_\delta(x_1'')\|_{X_1} \leq \sigma(\delta)\|x_1' - x_1''\|_{X_1} \quad \text{and} \quad \sigma(\delta) \in [0,1[\quad \text{for any } \delta \in \left]\frac{\lambda^2-1}{\lambda^2-2\beta+1}, 1\right[.$$

Therefore, t_δ is a contraction mapping for any $\delta \in \left]\frac{\lambda^2-1}{\lambda^2-2\beta+1}, 1\right[$ and, since t_δ has the same fixed points of ϑ, the game Γ has a unique Nash equilibrium $(\bar{u}, \bar{v})$ where $\bar{u}$ is the (unique) fixed point of t_δ (and of ϑ) and $\bar{v} = b_2(\bar{u})$. By the Contraction Mapping Theorem, the sequence $(u_n)_n$ in $(\mathcal{A}^\delta)$ strongly converges to $\bar{u}$. The convergence of the sequence $(v_n)_n$ towards $\bar{v}$, the second part of (ii), and (iii) are obtained by arguing as at the end of the proof of Theorem 4.1. The proof of (iv) is immediate. □

Remark 4.2 In light of Lemma 2.1, the function ϑ satisfies the following inequality:

$$\langle \vartheta(x_1') - \vartheta(x_1''), x_1' - x_1'' \rangle_{X_1} \leq \beta \|x_1' - x_1''\|_{X_1}^2 \quad \text{for any } x_1', x_1'' \in X_1,$$

which implies that ϑ is a *generalized pseudo-contraction* with constant β if $\beta > 0$ and with any positive constant if $\beta \leq 0$ (see [36] for the definition and properties thereof). In particular, when $\beta \leq 0$, since for any $\delta \in \left]\frac{\lambda^2-1}{\lambda^2-2\beta+1}, 1\right[$ the function $r \in]-\infty, 1[\to [(\lambda^2 - 2r + 1)\delta^2 - 2(\lambda^2 - r)\delta + \lambda^2]^{1/2}$ is strictly increasing, the error estimations for the algorithm $(\mathcal{A}^\delta)$ obtained via Theorem 4.4 are better than applying results about generalized pseudo-contraction mappings (see [7, 36] and [5, theorems 3.6 and 3.7]).

Note that the presque-contractive case includes also the situations such that $\beta < 0$ or $\lambda = 1$ (ϑ is a non-expansive mapping), investigated in [29] only for zero-sum games. In order to illustrate examples of non-zero-sum games in $\mathfrak{R}_{\alpha,\beta}$ satisfying C_3 and such that $\beta < 0$ or $\lambda = 1$, we consider games having also a non-bilinear interaction (so, not belonging to any class of games presented in Section 3).
A game satisfying C_3 and such that $\beta < 0$, is described in the following example.

Example 4.2 Let $\Gamma = \{2, \mathbb{R}, \mathbb{R}, f_1, f_2\}$ where

$$f_1(x_1, x_2) = -\frac{3}{4}x_1^2 - \cos x_1 \sin x_2 - 2x_1x_2, \quad f_2(x_1, x_2) = \frac{1}{1 + x_2^2} - 4x_2^2 + x_2 + 4x_1x_2.$$

Since $D_{x_1}^2 f_1(x_1, x_2) \leq -1/2$ and $D_{x_2}^2 f_2(x_1, x_2) \leq -15/2$, the function $f_i(\cdot, x_{-i})$ is strongly concave for any $i \in I$. Moreover, $\lambda_1 = 4$ and $\lambda_2 = 8/15$, so $\lambda = 32/15$ and

$$-\frac{5}{2} \leq \frac{D_{x_2}(D_{x_1}f_1)(x_1', x_2)D_{x_1}(D_{x_2}f_2)(x_1'', x_2)}{D_{x_1}^2 f_1(x_1', x_2)D_{x_2}^2 f_2(x_1'', x_2)} \leq -\frac{1}{5} \quad \text{for any } x_1', x_1'', x_2 \in \mathbb{R}.$$

Therefore Γ is in $\mathfrak{R}_{\alpha,\beta}$ with $\alpha = -5/2$ and $\beta = -1/5 < 0$, thus C_3 is satisfied.

In the next example we illustrate a game satisfying C_3 and such that $\lambda = 1$.

Example 4.3 Let $\Gamma = \{2, \mathbb{R}, \mathbb{R}, f_1, f_2\}$ where

$$f_1(x_1, x_2) = -\frac{3}{4}x_1^2 - \cos x_1 \sin x_2 - \frac{1}{2}x_1x_2, \quad f_2(x_1, x_2) = \frac{1}{1 + x_2^2} - 4x_2^2 + x_2 + \frac{15}{2}x_1x_2.$$

As $D_{x_1}^2 f_1(x_1, x_2) \leq -1/2$ and $D_{x_2}^2 f_2(x_1, x_2) \leq -15/2$, the function $f_i(\cdot, x_{-i})$ is strongly concave for any $i \in I$. Furthermore, $\lambda_1 = \lambda_2 = \lambda = 1$ (that is, ϑ is non-expansive) and

$$-1 \leq \frac{D_{x_2}(D_{x_1}f_1)(x_1', x_2)D_{x_1}(D_{x_2}f_2)(x_1'', x_2)}{D_{x_1}^2 f_1(x_1', x_2)D_{x_2}^2 f_2(x_1'', x_2)} \leq \frac{2}{5} \quad \text{for any } x_1', x_1'', x_2 \in \mathbb{R}.$$

Hence Γ belongs $\mathfrak{R}_{\alpha,\beta}$ with $\alpha = -1$ and $\beta = 2/5$, so C_3 is satisfied.

4.4 Non-positively answered case C_4: $\alpha \leq 1 \leq \min\{\beta, \lambda\}$

Let Γ be a game belonging to $\mathfrak{R}_{\alpha,\beta}$ and C_4 be satisfied, that is

$$\alpha \leq 1 \leq \beta \leq \lambda \quad \text{or} \quad \alpha \leq 1 \leq \lambda \leq \beta.$$

In this case Γ is not guaranteed to have a unique Nash equilibrium, as illustrated in the following three examples showing games that satisfy C_4 and have, respectively, no equilibria, three equilibria and infinitely many equilibria.

Example 4.4 Let $\Gamma = \{2, \mathbb{R}, \mathbb{R}, f_1, f_2\}$ where

$$f_1(x_1, x_2) = -x_1^2 - 2x_1 + 2x_1x_2, \quad f_2(x_1, x_2) = -x_2^2 + 2x_1x_2.$$

The game belongs to the class examined in Section 3.1 (extended quadratic games, in particular Γ is a weighted potential game) with $a_1 = a_2 = 1$, $d_1 = d_2 = 2$, $\ell_1(x_1) = -2x_1$, $\ell_2 \equiv p_1 \equiv p_2 \equiv 0$ and $s_1 = s_2 = 0$. So $\lambda = 1$ and Γ is in $\mathfrak{R}_{\alpha,\beta}$ with $\alpha = \beta = 1$. Therefore Γ satisfies C_4. Since $\vartheta(x_1) = x_1 - 1$ for any $x_1 \in \mathbb{R}$, ϑ has no fixed points so the game has no equilibria.

Example 4.5 Let $\Gamma = \{2, \mathbb{R}, \mathbb{R}, f_1, f_2\}$ where

$$f_1(x_1, x_2) = -\frac{1}{2}x_1^2 - x_1 \cos x_2 + \frac{5}{4}x_1x_2, \quad f_2(x_1, x_2) = -x_2^2 + 2x_1x_2.$$

Since $D^2_{x_1} f_1(x_1, x_2) = -1$ and $D^2_{x_2} f_2(x_1, x_2) = -2$, the function $f_i(\cdot, x_{-i})$ is strongly concave for any $i \in I$. Moreover, $\lambda_1 = 9/4$ and $\lambda_2 = 1$, so $\lambda = 9/4$ and

$$\frac{1}{4} \leq \frac{D_{x_2}(D_{x_1} f_1)(x_1', x_2) D_{x_1}(D_{x_2} f_2)(x_1'', x_2)}{D^2_{x_1} f_1(x_1', x_2) D^2_{x_2} f_2(x_1'', x_2)} \leq \frac{9}{4} \quad \text{for any } x_1', x_1'', x_2 \in \mathbb{R}.$$

Γ is in $\mathfrak{R}_{\alpha,\beta}$ with $\alpha = 1/4$ and $\beta = 9/4$, thus C_4 is satisfied. Since $\vartheta(x_1) = (5/4)x_1 - \cos x_1$ for any $x_1 \in \mathbb{R}$, ϑ has three fixed points so the game has three equilibria.

Example 4.6 Let $\Gamma = \{2, \mathbb{R}, \mathbb{R}, f_1, f_2\}$ where

$$f_1(x_1, x_2) = -x_1^2 + 2x_1x_2, \quad f_2(x_1, x_2) = -x_2^2 + 2x_1x_2.$$

The game belongs to the class examined in Section 3.1 (extended quadratic games, in particular Γ is a weighted potential game) with $a_1 = a_2 = 1$, $d_1 = d_2 = 2$, $\ell_1 \equiv \ell_2 \equiv p_1 \equiv p_2 \equiv 0$ and $s_1 = s_2 = 0$. So $\lambda = 1$ and Γ is in $\mathfrak{R}_{\alpha,\beta}$ with $\alpha = \beta = 1$. Therefore Γ satisfies C_4. Since $\vartheta(x_1) = x_1$ for any $x_1 \in \mathbb{R}$, the set of fixed ponts of ϑ is $\mathbb{R}$ so the game has infinitely many equilibria.

Therefore, the examples above illustrated show that in the case C_4 it is not possible, differently from the previous three cases, to obtain a positive result concerning existence and uniqueness of Nash equilibria together with convergence of affine relaxations of the best response algorithm. In particular, we proved the following result.

Theorem 4.5. *It is not true that for any $\Gamma \in \mathfrak{R}_{\alpha,\beta}$ with $\alpha \leq 1 \leq \min\{\beta, \lambda\}$ the game Γ has a unique Nash equilibrium.*

Moreover, we point out that even when Γ has a unique Nash equilibrium, none of the affine relaxations of the best response algorithm is ensured to converge to the equilibrium. In fact, the next example illustrates a game that satisfies C_4 with a unique equilibrium and such that the operator t_δ defined in (10) is not a contraction mapping, whatever is $\delta \in \mathbb{R}$.

Example 4.7 Let $\Gamma = \{2, \mathbb{R}, \mathbb{R}, f_1, f_2\}$ where

$$f_1(x_1, x_2) = -\frac{1}{2}x_1^2 - x_1 \cos x_2 + \frac{3}{2}x_1 x_2, \quad f_2(x_1, x_2) = -x_2^2 + 2x_1 x_2.$$

Since $D^2_{x_1} f_1(x_1, x_2) = -1$ and $D^2_{x_2} f_2(x_1, x_2) = -2$, the function $f_i(\cdot, x_{-i})$ is strongly concave for any $i \in I$. Moreover, $\lambda_1 = 5/2$ and $\lambda_2 = 1$, so $\lambda = 5/2$ and

$$\frac{1}{2} \leq \frac{D_{x_2}(D_{x_1} f_1)(x_1', x_2) D_{x_1}(D_{x_2} f_2)(x_1'', x_2)}{D^2_{x_1} f_1(x_1', x_2) D^2_{x_2} f_2(x_1'', x_2)} \leq \frac{5}{2} \quad \text{for any } x_1', x_1'', x_2 \in \mathbb{R}.$$

Γ is in $\mathfrak{R}_{\alpha,\beta}$ with $\alpha = 1/2$ and $\beta = 5/2$, thus C_4 is satisfied. Since $\vartheta(x_1) = (3/2)x_1 - \cos x_1$ for any $x_1 \in \mathbb{R}$, ϑ has one fixed points, so the game has a unique equilibrium.
In light of the definition of t_δ in (10), we have

$$t_\delta(x_1) = \delta x_1 + (1 - \delta) \left(\frac{3}{2} x_1 - \cos x_1 \right)$$

By contradiction, suppose there is $\bar{\delta} \in \mathbb{R}$ such that $t_{\bar{\delta}}$ is a contraction mapping. This would imply $Dt_{\bar{\delta}}(x_1) < 1$ for any $x_1 \in \mathbb{R}$, that is equivalent to $(1 - \bar{\delta}) \sin x_1 < -(1 - \bar{\delta})/2$. However the last inequality does not hold for any $x_1 \in \mathbb{R}$, whatever the value of $\bar{\delta}$ is. Hence, there is no value of δ for which t_δ is a contraction.

Remark 4.3 Since the constants α and β in Definition 2.1 are, respectively, a lower and an upper bound of the ratio involved in $(\mathcal{A}_4)$, a more refined estimation of α and β (when achievable) could tell additional information in the analysis of games satisfying C_4. More precisely, if Γ is in $\mathfrak{R}_{\alpha,\beta}$, satisfies C_4 and

- a better estimation of α leads to a value $\alpha' > 1$, then Γ satisfies C_2 and the results of Section 4.2 apply;
- a better estimation of β leads to a value $\beta' < 1$, then Γ satisfies C_1 (if $\lambda \leq \beta$) or C_3 (if $\beta \leq \lambda$) and the results of Section 4.1 or Section 4.3 apply.

Clearly, the best estimations of α and β for a game belonging to $\mathfrak{R}_{\alpha,\beta}$ are given by

$$\alpha = \inf_{\substack{x_1', x_1'' \in X_1, x_2 \in X_2 \\ \varphi \in X_1 \text{ with } \|\varphi\|_{X_1} \neq 0}} \frac{\langle H(x_1', x_1'', x_2)\varphi, \varphi \rangle_{X_1}}{\|\varphi\|^2_{X_1}} \quad \text{and} \quad \beta = \sup_{\substack{x_1', x_1'' \in X_1, x_2 \in X_2 \\ \varphi \in X_1 \text{ with } \|\varphi\|_{X_1} \neq 0}} \frac{\langle H(x_1', x_1'', x_2)\varphi, \varphi \rangle_{X_1}}{\|\varphi\|^2_{X_1}},$$

as obtained in the Examples 4.4 to 4.7. However, in general such values are not always possible to compute explicitly.

References

[1] T. Başar. Relaxation techniques and asynchronous algorithms for on-line computation of non-cooperative equilibria. *J. Econ. Dyn. Control*, 11:531–549, 1987.

[2] D. Balkenborg, J. Hofbauer, and C. Kuzmics. Refined best reply correspondence and dynamics. *Theor. Econ.*, 8:165–192, 2013.

[3] M. Benaïm, J. Hofbauer, and S. Sorin. Perturbations of set-valued dynamical systems, with applications to game theory. *Dyn. Games Appl.*, 2:195–205, 2012.

[4] U. Berger. Fictitious play in $2 \times n$ games. *J. Econ. Theory*, 120:139 – 154, 2005.

[5] V. Berinde. *Iterative Approximation of Fixed Points*. Springer, Berlin Heidelberg, 2007.

[6] R. Brânzei, L. Mallozzi, and S. Tijs. Supermodular games and potential games. *J. Math. Econ.*, 39:39–49, 2003.

[7] F. E. Browder and W. V. Petryshyn. Construction of fixed points of nonlinear mappings in Hilbert space. *J. Math. Anal. Appl.*, 20:197–228, 1967.

[8] G. W. Brown. Iterative solution of games by fictitious play. In T. C. Koopmans, editor, *Activity Analysis of Production and Allocation: Proceedings of a Conference*, pages 374–376. J. Wiley and Sons, New York, 1951.

[9] F. Caruso, M. C. Ceparano, and J. Morgan. Uniqueness of Nash equilibrium in continuous two-player weighted potential games. *J. Math. Anal. Appl.*, 459:1208–1221, 2018.

[10] F. Caruso, M. C. Ceparano, and J. Morgan. An inverse-adjusted best response algorithm for Nash equilibria. *SIAM J. Optim.*, 30:1638–1663, 2020.

[11] Y. Cherruault and P. Loridan. Méthodes pour la recherche de points de selle. *J. Math. Anal. Appl.*, 42:522–535, 1973.

[12] M. R. Crisci and J. Morgan. Convergence results of an approximation method for constrained saddle point problems. *Intern. J. Comput. Math.*, 15:53–64, 1984.

[13] E. J. Dockner, S. Jorgensen, N. Van Long, and G. Sorger. *Differential Games in Economics and Management Science*. Cambridge University Press, Cambridge, 2000.

[14] G. Facchini, F. van Megen, P. Borm, and S. Tijs. Congestion models and weighted bayesian potential games. *Theory Decis.*, 42:193–206, 1997.

[15] D. Fudenberg and D. K. Levine. *The theory of learning in games*. MIT press, 1998.

[16] D. Gabay and H. Moulin. On the uniqueness and stability of Nash equilibrium in noncooperative games. In A. Bensoussan, P. R. Kleindorfer, and C. S. Tapiero, editors, *Applied Stochastic Control in Econometrics and Management Science*, pages 271–293. 1980.

[17] I. Gilboa and A. Matsui. Social stability and equilibrium. *Econometrica*, 59:859–867, 1991.

[18] C. Harris. On the rate of convergence of continuous-time fictitious play. *Games Econ. Behav.*, 22:238 – 259, 1998.

[19] A. Haurie, J.B. Krawczyk, and G. Zaccour. *Games and Dynamic Games.* World Scientific, 2012.

[20] J. Hofbauer. Stability for the best response dynamics. Mimeo, University of Vienna, 1995.

[21] J. Hofbauer and K. Sigmund. *Evolutionary Games and Population Dynamics.* Cambridge University Press, 1998.

[22] J. Hofbauer and S. Sorin. Best response dynamics for continuous zero-sum games. *Discrete Cont. Dyn. - B*, 6:215–224, 2006.

[23] S. Li and T. Başar. Distributed algorithms for the computation of noncooperative equilibria. *Automatica*, 23:523–533, 1987.

[24] L. Mallozzi and J. Morgan. On approximate mixed nash equilibria and average marginal functions for two-stage three-players games. In S. Dempe and V. Kalashnikov, editors, *Optimization with Multivalued Mappings: Theory, Applications, and Algorithms*, pages 97–107. Springer, Boston, MA, 2006.

[25] W. R. Mann. Mean value methods in iteration. *Proc. Am. Math. Soc.*, 4:506–510, 1953.

[26] K Miyazawa. On the convergence of the learning process in a 2× 2 non-zero-sum two-person game. Technical report, Princeton University, Econometric Research Program, Research Memorandum No. 33, 1961.

[27] D. Monderer and L. S. Shapley. Fictitious play property for games with identical interests. *J. Econ. Theory*, 68:258 – 265, 1996.

[28] D. Monderer and L. S. Shapley. Potential games. *Games Econ. Behav.*, 14:124 – 143, 1996.

[29] J. Morgan. Méthode directe de recherche du point de selle d'une fonctionnelle convexe-concave et application aux problèmes variationnels elliptiques avec deux contrôles antagonistes. *Int. J. Comput. Math.*, 4:143–175, 1974.

[30] J. Morgan. Algorithme d'approximation interne de problèmes de point de selle avec contraintes via l'optimisation. *RAIRO Anal. Numer.*, 14:189–202, 1979.

[31] J. Morgan. Sur un procede mixte d'approximation de problemes de point de selle avec contraintes utilisant simultanement les methodes de penalisation interne et de discretisation. *Ric. Mat.*, 28:399–415, 1979.

[32] J. F. Nash. Equilibrium points in n-person games. *Proc. Natl. Acad. Sci.*, 36:48–49, 1950.

[33] A. Neyman. Correlated equilibrium and potential games. *Int. J. Game Theory*, 26:223–227, 1997.

[34] J. Robinson. An iterative method of solving a game. *Ann. Math.*, 54:296–301, 1951.

[35] W. H Sandholm. *Population games and evolutionary dynamics.* MIT press, 2010.

[36] R. U. Verma. A fixed-point theorem involving Lipschitzian generalised pseudo-contractions. *Math. Proc. R. Ir. Acad.*, 97A:83–86, 1997.