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Screening while Controlling an Externality

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Screening while Controlling an Externality

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Abstract
We propose a tractable framework to introduce externalities into a monopolist screening model. Agents differ both in their payoff type and their influence, i.e. how strongly their action affects the aggregate externality. Applications range from non-linear pricing of a network good, to taxation or subsidization of industries that produce externalities (e.g. pollution and human capital formation). When both dimensions are unobserved (full screening) the optimal allocation satisfies *lexicographic monotonicity*: within a payoff-type, the monopolist optimally tilts the allocation towards influential agents to increase the externality, while standard IC drives monotonicity across payoff-types. We characterize the solution through a two-step ironing procedure that addresses the nonmonotonicity in virtual values arising from the countervailing impact of payoff-types and influence. The allocation is inefficient if and only if the payoff-type is unobservable. Only influence is observable, equilibrium utility can vary across the latter as it is used as a signal of the payoff-type. We provide sufficient conditions for (expected) rents from influence to emerge.

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1 Introduction

In many settings ranging from production with a polluting factor to the consumption of a network good, individual activities affect the payoff of others and their willingness to act. A planner or firm designing policy in such an environment is often affected by adverse selection as well: A network good monopolist may price discriminate to exploit unobserved heterogeneity in consumers' valuations, while providing large amounts cheaply to influential consumers in order to increase the willingness to pay of the population. Likewise, a regulator that controls pollution through production quotas or taxes may need to discriminate among heterogeneously productive firms. Given the attention received by the study of externalities and screening separately and the abundance of applications that are affected by both forces, it is perhaps surprising that their interaction has only received limited attention in the literature on screening. A possible explanation is that the most natural framework for conducting this analysis is a screening model with at least two dimensions of heterogeneity: agents differ both in their taste for the activity and their impact on others. Multidimensional screening problems, however, present notorious difficulties and often require a case by case analysis of the particulars of the setting.[1]

In this paper, we propose a tractable framework for analysing the interplay of screening and externalities. The principal faces a population of agents whose payoffs are interdependent through a global externality. Agents can be characterized by a two-dimensional type, one parametrizing the returns from the activity as in a standard screening model, the other parametrizing the impact of his activity on the externality. There is no aggregate uncertainty, the principal knows the distribution of types and costs or benefits from the externality in aggergate, but is ignorant of each individual agent's type. The crucial assumption is that even though both dimensions of the agents type enter the principal's objective, only one dimension affects their utility. Apart from single-crossing, we are permissive on the functional form of the payoff and externality function. We provide bounds ensuring the existence of a solution and derive its properties in the general case, though we we make specific functional form assumption that relate to the applications presented to illustrate our results in specific settings. A crucial intermediate step is to show that despite the apparent multidimensionality, there is a fixed total order of types determining the binding sorting constraints. We study how the observability of each of the dimensions affects the allocation of the activity and rents as well as the aggregate externality.

We now preview our results in more detail. Clearly, if both characteristics are observed by the principal, she can implement the efficient contract and extract all rents.

[1] Rochet and Stole (2003) survey the literature on multidimensional screening, highlighting tractable cases and the source of the general difficulty of such problems: the lack of fixed order on types along which sorting constraints bind. This justifies the usage of multi-dimensional for the discrete setting as well.

This result holds as long as the payoff type is observed:[2] as the influence type does not affect an agent's utility, sorting constraint imply that utility is flat along this dimension; the first best (through full surplus extraction) satisfies this feasibility condition, and hence is optimal.

We then turn to the screening problem where both dimensions are unobserved (*full screening* contract), which constitutes the core contribution of this paper. We exploit the special structure of the resulting two-dimensional screening problem with externalities to arrive at a tractable solution. As in a standard screening problem, incentive compatibility requires that the allocation is increasing in the payoff type. Within a payoff type slice, the monopolist tilts the allocation and provides higher consumption to more influential consumers in order to create a larger externality (we ensure the higher surplus/higher rents tension always favors the former and profits are increasing in the aggregate activity). We show (Theorem 1) that the full screening allocation is increasing along the lexicographic order where the payoff type is the dominant dimension. We can hence transform the problem into a one-dimensional problem along this order. Even under the usual regularity conditions, the virtual value will typically be non-monotonic in the lexicographic order for two reasons, both arising around the switching types (types with the highest level of influence, that are consequently adjacent to a type with higher payoff type in the lexicographic order): First, only the consumption of these types directly causes information rents and hence only their virtual value is downward distorted. Second, the subsequent type in the lexicographic order has the lowest influence; as the virtual value is increasing in influence, this downward jump is a source of non-monotonicity. We generalize standard techniques to take into account that the network effect creates interdependence among individuals' virtual values and provide a two-step ironing procedure (Theorem 2) to obtain allocations and aggregate activity that solve the full screening problem. In contrast to efficiency at the boundary results (e.g. Rochet and Choné, 1998), we show that bunching can occur even for agents with the highest payoff-type and that every bunching region contains agents of the highest-influence type.

One feature shared by the efficient and the full screening contract is that individuals' influence is not rewarded: In the former case, full surplus extraction leaves everyone without any rent, while in the latter case incentive compatibility prevents any rent to emerge along a dimension (influence) that does not directly affect individuals' utility. This implication seems at odds with evidence of large rents enjoyed by influencers inside a network. To see whether such rents can emerge we study the problem with observable influence (but unobserved payoff type) in our linear-quadratic application to the consumption of network goods. In this case, a condition on primitives ensures that

[2] Indeed, full extraction at the first-best quantities is feasible even when consumers can misreport but only *over*-report their payoff-types, since it is the downward constraints that are violated in the first-best.

the optimal contract exhibits rents for influential consumers:[3] Even when influencers have no market power, they can gain from their position. For such gains to emerge it is however necessary that influence is verifiable, and even in that case it emerges indirectly as a reward for higher payoff types.

The paper proceeds as follows. We conclude this introductory section by discussing the relevant literature. Section 2 presents the general model and three applications which will also be used as running examples to illustrate our results. As a benchmark, we characterize the decentralized equilibrium of the game if the technology is available to every agent and the efficient allocation in Section 3, and show that the efficient allocation is implementable as long as the payoff-type is observed. We then analyze the full screening problem in Section 4. Section 5 analyses the problem when influence is observed but payoff-types are private information, Section 6 concludes. We gather all proofs in Appendix A and derivations for the examples in Appendix B.

1.1 Literature

Our model relates to the classic literature on contracting with network effects (Segal, 1999). Jadbabaie and Kakhbod (2016) compare bilateral and multilateral contracting in this setting when there are finitely many consumers and consequently, there is aggregate uncertainty about the realized distribution of types. This literature focuses on the externality of contracting in a setting with finitely many agents, in particular on the effect on the outside option of the agents, while we focus on a continuum of consumers with public contracting in a setting where the outside option of the agents is independent of the contract accepted by others. Sundararajan (2004) and Csorba (2008) study screening with externalities in consumption when consumers have private information about their valuation of the good. We study screening on the payoff-type and influence to the externality.

The application to the sale of a network good relates to the classic literature on externalities in consumption following the seminal Farrell and Saloner (1985) and Katz and Shapiro (1985). A recent literature focuses on the use of network information by a monopolist, both in the case of an explicit finite network (e.g. Bloch and Quérou, 2013; Candogan et al., 2012) and when consumers only know their level of susceptibility and influence (Fainmesser and Galeotti, 2016a,b). We adopt the demand and interaction specification developed in the latter in our example, but focus on the screening problem and arrive at different implications for pricing.

There is a growing literature on monopolist screening for these characteristics. Zhang and Chen (2017) consider an explicit stochastic network formation model, where the out-degree of agents is fixed and consider screening along the in-degree. They consider two specifications, susceptibility is either a consumers' in- or out-degree.

[3] In particular, the condition puts an upper bound on the affiliation between the payoff- and influence-type.

Depending on this choice, their model can generate both quantity discounts and premia. Gramstad (2016) consider screening in a undirected network when network effects only depend on the number of neighbors that adopted the good, not their intensity of consumption. We analyze both dimensions of private information – payoff type and influence – at the same time and study their interaction in screening. In a contemporaneous paper, Shi and Xing (2018) study screening with the same demand specification as our application to network goods and assume a continuum type space. Consequently, the optimal allocation is constant in influence and the solution is one-dimensional. They focus on the implications for the value of network information, while we focus on how the allocation and rents depend on influence (and its observability) with discrete types and general single-crossing utility and externality specification. Galeotti et al. (2020) use a principal component approach to characterize optimal interventions (like taxes and consumption subsidies) when local externalities flow on a known finite network. We focus on screening of a global externality when agents types are private information. Weber (2006) characterizes the implementable allocations in a general multidimensional model with externalities and provides a set of necessary conditions for the optimal control problem characterizing the optimal screening contract. We focus on a setting where despite its underlying multidimensionality, the problem can be reduced to a single-dimensional screening problem and use this tractablity to provide a tight charaterization of the solution and its properties.

2 Model and Applications

We construct a parsimonious model of screening with externalities and two dimensions of (potentially unobserved) heterogeneity: an influence type and a payoff type. Agents choose actions which have aggregate effects. The influence type determines the impact of individual action in the creation of this externality, while the payoff type parametrizes the surplus from the individual action and the aggregate effect.

2.1 Setup and Primitives

There is a unit mass of agents characterized by a type $\theta \in \Theta$ distributed according to a full support distribution F. Each agent takes an action $x \in \mathbb{R}_+$ whose payoff is subject to network (or aggregate) effects: The attractiveness of the action is dependent on an aggregate variable, $\bar{x}$. For a given $\bar{x}$, an agent of type θ derives utility

$$u(\theta, x, \overline{x}) - t \tag{1}$$

from a action x and transfer $t \in \mathbb{R}$. The aggregate effect $\bar{x}$ is a weighted average of individual actions

$$\bar{x} = \int v(x(\theta), \theta) \, \mathrm{d}F(\theta) \tag{2}$$

We assume that the individual payoff characteristics and externality production can each be summarized by a one-dimensional type. On other words, we can write

$$u(\theta, x, \overline{x}) = u(x, k(\theta), \overline{x}) \tag{3}$$

$$v(x, \theta) = v(x, l(\theta)) \tag{4}$$

for a pair of functions $(l, k) : \Theta \to [k_0, K] \times [l_0, L] \subset \mathbb{R}^2$. We assume concavity in the agents action ($u_{xx} < 0$) to ensure the existence of an interior optimum and the following properties

$$u_k \geq 0, \quad u_{kx} \geq 0, \quad u_{xxk} \geq 0 \tag{5}$$

$$u_{\bar{x}} \geq 0, \quad u_{\bar{x}\bar{x}} \leq 0, \quad u_{x\bar{x}} \geq 0, u_{\bar{x}\bar{x}k} \geq 0 \tag{6}$$

The payoff-type k behaves as in a standard screening model with single-crossing and the usual condition ensuring convexity of the information rent.[4] The global externalty $\bar{x}$ describes a (weakly) positive externality with diminishing returns that also increases the marginal payoff from the activity x. For the influence function v, we maintain

$$v_{xx} < 0, \quad v_l > 0, \quad v_{xl} > 0 \tag{7}$$

This setup allows us to encompass both negative and positive externalities from the agents' actions in a common framework. If $v_x > 0$, the activity produces a positive shift in individual payoffs (per $u_{\bar{x}} \geq 0$) and we assume that this technology is concave. If $v_x < 0$, we have a negative externality of the activity with convex costs ($v_{xx} < 0$). In either case, agents with high influence type l are the "good types", either because they produce a larger positive or smaller negative externality as $v_{xl} > 0$.

Even though our type-space is two-dimensional in principle, only one of the coordinates enters the agents utility. As we will show, the principal can implement different contracts along this payoff-irrelevant dimension, indeed, this will be optimal in many cases. To satisfy incentive compatibility in the full screening problem, however, the allocation is required to be increasing the in agents payoff type k. Therefore, we have

Lemma. *Suppose $k(\Theta)$ is uncountable and the induced distribution $F \circ k^{-1}$ is atomless. Then, for almost all k, $x(k, l)$ is constant in l.*

To see why, consider $x(k) = \min_l x(k, l)$. This has to be an increasing function and has an upwards jump wherever $x(k, l)$ is not constant in l. As an increasing function can have at most countably many points of discontinuity, we have the result.

As our interest is in exploring the impact of heterogeneous influence on screening, we rule out this case. Let $\mathcal{K} \times \mathcal{L}$ denote the induced types space. By abuse of notation,

[4]Since the function $k(\theta)$ is a derived objects, this assumption simply implies that there exists such a linear order on Θ, which implicitly defines k.

we denote the induced distribution by F. We assume that $\mathcal{K}\times\mathcal{L}$ is finite and, as a normalization, let $\mathcal{K}\times\mathcal{L} = \{k_0, k_0+1, \ldots, \mathrm{K}\}\times\{l_0, l_0+1, \ldots, \mathrm{L}\}$.[5] Hence, we write $f_{k,l}$ for the probability mass and $x_{k,l}$ for the allocation of type k, l, etc. Apart from full support, we make no restriction on $f_{k,l}$. Importantly, we don't impose any correlation structure, though we will explore the consequences of correlation between payoff and influence type.

Principal

The agents action is produced at zero marginal cost to the planner. The planner offers a menu of contracts $\{(x_{k,l}, t_{k,l})\}_{k,l\in\mathcal{K}\times\mathcal{L}}$ to maximize expected transfers plus possibly a direct payoff from the aggregate action, $\kappa(\bar{x})$, subject to sorting and participation constraints.[6] We assume that $\kappa' \geq 0$ and $\kappa'' \leq 0$. This aggregate term captures the impact of the externality that is not mediated through the payoffs of the agents, for example the impact of pollution on society at large. Finally, we assume as a non-triviality condition that the externality has an impact on the principal's problem, either directly or indirectly through some type's utility,

$$\forall \overline{x}, \quad \max\left\{\max_{k\in\mathcal{K}}\{u_x(0,k,\overline{x})\}, \kappa'(\overline{x})\right\} > 0. \tag{NT}$$

We will consider several monopolist problems, each corresponding to a different assumption on which consumer characteristics are observable. Throughout these problems, the objective, the aggregate network effect and the participation constraints will remain the same. Depending on the misreports that are feasible (i.e. on what characteristics are verifiable), the problem will have different sorting constraints. We identify the sorting constraint with the associated (pair of) types. That is, denote the set of feasible deviations by $\mathrm{A} \subset (\mathcal{K}\times\mathcal{L})^2$, where $(k,l),(k',l') \in \mathrm{A}$ means that type k, l can imitate type k', l' and consequently a feasible allocation must satisfy the sorting constraint

$$u\left(x_{k,l}, k, \overline{x}\right) - t_{k,l} \geq u\left(x_{k',l'}, k, \overline{x}\right) - t_{k',l'}. \tag{IC$_{k,l\to k',l'}$}$$

The problem corresponding to a set of feasible deviations A is

$$\pi(\mathrm{A}) := \max_{\bar{x},\left\{(x_{k,l},t_{k,l})\right\}_{k,l\in\mathcal{K}\times\mathcal{L}}} \sum f_{k,l} t_{k,l} + \kappa(\bar{x}) \tag{8}$$

$$\text{s.t.} \qquad \bar{x} = \sum f_{k,l} v\left(x_{k,l}, l\right) \tag{ANE}$$

$$\forall k,l: \quad u\left(x_{k,l}, k, \overline{x}\right) - t_{k,l} \geq 0 \tag{P$_{k,l}$}$$

[5]Note that this is for simplicity, we could allow for continuum $\mathcal{L}$ and - assuming a distribution with atoms – even continuum $\mathcal{K}$ at the cost of more complex notation.

[6]Note that every set of contracts induces a game among the consumers at the consumption stage, as aggregate consumption is endogenous. Without loss of generality, we restrict attention to menus inducing a pure strategy equilibrium. This is implied by the concavity of the planer problem, which we impose throughout.

$$\forall\left((k,l),(k',l')\right)\in \mathrm{A}: \quad u\left(x_{k,l},k,\overline{x}\right)-t_{k,l}\geq u\left(x_{k',l'},k,\overline{x}\right)-t_{k',l'}\geq \qquad (\mathrm{IC}_{k,l\to k',l'})$$

To save on notation, we suppress the non-negativity constraints $x_{k,l}\geq 0$. Table 1 specifies the set of feasible deviations associated to each observability assumption. Throughout the paper, we will let ζ denote the Lagrange multiplier associated to the

	payoff type observable	payoff type not observable
influence observable	$\varnothing$	$\bigcup_{l\in\mathcal{L}}\left(\mathcal{K}\times l\right)^2$
influence not observable	$\bigcup_{k\in\mathcal{K}}\left(k\times\mathcal{L}\right)^2$	$\left(\mathcal{K}\times\mathcal{L}\right)^2$

Table 1: Four different observability assumptions as sets of feasible deviations.

ANE constraint, i.e. marginal increase in the principal's objective associated to an exogenous increase in the externality.

2.2 Applications

We now present three economic applications that fit our general framework. We will return to simple 2-by-2 examples of these models to illustrate our results and their implications throughout the paper.

2.2.1 Sale of a Network Good

Consider a good with externalities in consumption, like the internet services discussed in the introduction. We follow the network formation model formulated and applied in Galeotti and Goyal (2009) and Fainmesser and Galeotti (2016a,b). There is a continuum of consumers connected by a directed network. When there is a link from i to j we say that consumer i is influenced by agent j. A consumer's marginal utility of consumption increases as others who influence her increase their consumption. Formally, let I_i be the set of consumers who influence i; the utility of consumer i is given by

$$u_i\left(\left(x_j\right)_{j\in[0,1]},t_i\right)=x_i+\gamma x_i\sum_{j\in \mathrm{I}_i}x_j-\frac{1}{2}x_i^2-t_i \tag{9}$$

where γ is the intensity of network effects.

The influence parameter l coincides with the agent's in-degree, while the payoff parameter k is his out-degree. When making consumption choices, consumers don't know the network structure, but only their in- and out-degree. They take expectations over their realized utility conditional on this information alone.[7] So, the utility can be expressed as

[7] Formally, we model the network formation as follows: There is a unit interval of consumers, ordered by in-degree l. Denote the in-degree of consumers at $i\in[0,1]$ by $l(i)$, an increasing step function with finite range. After consumption decisions are made, a consumer with out-degree k draws k consumers independently from the unit interval with density $\frac{l(i)}{\mathbb{E}[l]}$ and links to them. In expectation, a consumer is drawn and linked to by l other consumers.

$$u_i(x_i, p_i, \overline{x}) = x_i + \gamma k_i \bar{x} x_i - \frac{1}{2}x_i^2 - p_i \tag{10}$$

where

$$\mathbb{E}\left[x_j | j \in \mathrm{I}_i\right] = \sum_{k,l} f_{k,l} \frac{l}{\mathbb{E}[l]} x_{k,l} = \bar{x}. \tag{11}$$

When forming expectations, individuals take account of the fact that they are more likely to link to influential individuals which consequently need to be over-counted relative to their frequency in determining the expected consumption of a neighbor.[8] Clearly, equations (10) and (11) fit into our general framework with

$$u(k, x_i, \overline{x}) = (1 + \gamma \overline{x} k)x - \frac{1}{2}x^2, \qquad v(x, l) = \frac{l}{\mathbb{E}(\boldsymbol{l})}x, \qquad \kappa \equiv 0 \tag{12}$$

Since the payoff type k also parametrizes the returns from the aggregate action, it can be interpreted as an agent's susceptibility to the network effect.

The reduced from can also be interpreted as an aggregate network effect: Agents directly care about the weighted population average of x, e.g. because of a desire to conform. Agents differ both in their desire to conform k and their intensity of creating network effects for others l (visibility or social status).

2.2.2 Nonlinear taxation of externality producing goods

Firms produce goods using a polluting process. They differ both in their productivity and in their pollution intensity. A regulator desiring to control the aggregate level of externality while raising tax revenue designs nonlinear production taxes.

A firm of type k produces quantity x employing a perfect complement decreasing returns technology $x = \left(\min\left\{\frac{1}{\theta_1}z_1, \frac{1}{\theta_2}z_2\right\}\right)^{\frac{1}{2}}$, where z_1 is a clean factor and z_2 is a pollutant factor. Let w_1, w_2 denote the factor prices, we normalize the price of output to one. Hence, profits are given by $x - (w_1\theta_1 + w_2\theta_2)x^2$, and the externality is $-\theta_2 x^2$. The planner faces a disutility of pollution $\kappa(\bar{x}) = \kappa\bar{x}$ for $\kappa > 0$. This fits our framework with $k(\theta) = \frac{2}{w_1\theta_1 + w_2\theta_2} - 1$, $u(x, k, \bar{x}) = x - \frac{1}{2(k+1)}x^2$, $l(\theta) = \max\{\theta_2\} - \theta_2$, and $v(x, l) = -(\max\{\theta_2\} - l)x^2$.[9] Notice that "fundamental" parameters of the production function determine payoff and influence type. In particular, the pollutant factor requirement θ_2 determines not only the impact on the externality but also a firm's payoff type. Even if fundamental parameters $\boldsymbol{\theta}$ are independently distributed, factor prices determine a correlation structure between payoff and influence types in the rewriting that fits our general framework. Since we have no restriction on the (joint) distribution over the two dimensional type space, such induced correlation can be analyzed as a comparative statics over the distribution primitive. The DRS perfect

[8] For discussion of further effects of this "friends paradox", see Jackson (2017).

[9] Note that this joint definition of k, l rules out a rectangular type space with full support unless $w_2 = 0$. Factor prices determine the correlation between types. What is crucial for our analysis is that we keep single dimensionality which requires a perfect complements production function.

complement production function is instead needed to fit the general framework: DRS to have concavity and no substitutability to keep single dimensionality of the payoff type.[10]

2.2.3 Human capital

Consider a steady state model of a labor market in with firms and workers are matched randomly for only one period. Work at time t produces human capital that is carried over into the next period (but then is forgotten: only last period employment determines human capital).

Let $\overline{h}$ the average human capital in an economy. By random matching, the effective labor units of a firm with productivity k employing a measure h of workers are $k\overline{h}h$. The firm operates a Cobb-Douglas technology, generating profits

$$u(h,k,\bar{h}) = \left(k\overline{h}h\right)^{\alpha} - wh \tag{13}$$

for a given wage w.[11] Human capital formation depends on the type of employment, we parameterize human capital by worker by l, $v(h,l) = hl$. It is easy to see that this specification fits our general setting as long as production has decreasing returns, $\alpha < 1$. The ministry of economic development chooses a nonlinear employment subsidy to maximize human capital subject to a cost of funds λ, i.e. we write $\kappa(\bar{h}) = \frac{\bar{h}}{\lambda}$.[12]

3 Benchmark Allocations

We first characterize the decentralized solution. We then turn to the efficient allocation. Clearly, this allocation is implemented by a principal who can observe the payoff and influence types, i.e. it solves $\pi(\varnothing)$. Finally, we show that this solution is implemented even if the principal can observe only the payoff type. The unobservability of influence does not create any rents and distortions in this case.

3.1 The Decentralized Solution

As a benchmark, consider the case where every agent has access to the production technology and chooses $x^{\mathrm{D}}_{k,l}$ to maximize $u(x,k,\bar{x})$.[13] Observe that $x^{\mathrm{D}}_{k,l} = x^{\mathrm{D}}_{k}$, as influence does not enter utility. A decentralized allocation solves

$$u_x(x^{\mathrm{D}}_k,k,\bar{x}) = 0, \quad \bar{x}^{\mathrm{D}} = \sum f_{k,l} v(x^{\mathrm{D}}_k,l) \tag{14}$$

[10] Under a more permissive functional form, we would obtain $\pi(\theta,w,x) = \sum w_i z_i(\theta,w,x)$ which cannot in general be written as $\pi(k(\theta),x,w)$ for a single dimensional type $k(\theta)$. Indeed this fails if $w_1 \cdot w_2 \neq 0$ whenever any substitutability across factors is permitted.

[11] The wage rate is fixed, as there is a reserve army of the unemployed working in the traditional sector.

[12] To ensure that condition (NT) is met even at $\overline{h} = 0$, we must ensure κ' is bounded away from zero.

[13] Equivalently, consider $n > 1$ firms competing in price-schedules.

For any given $\bar{x}$, the privately optimal allocation is unique by concavity. An equilibrium $\left(\boldsymbol{x}^{\mathrm{D}}, \bar{x}^{\mathrm{D}}\right)$ exists but may not be unique. All equilibria are Pareto-ranked in $\bar{x}$, as higher aggregate activity increases private utility.

Example (**Human Capital**)**.** In the setting of Section 2.2.3, the decentralized conditional labor demand is given by

$$h_k^{\mathrm{D}} = \left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} \left(k\overline{h}\right)^{\frac{\alpha}{1-\alpha}} \tag{15}$$

notice that the non-triviality condition (NT) does not rule out a degenerate decentralized equilibrium; a non-trivial equilibrium exists if $\alpha < \frac{1}{2}$ and is given by

$$\overline{h} = \left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} \mathbb{E}\left[lk^{\frac{\alpha}{1-\alpha}}\right]^{\frac{1-\alpha}{1-2\alpha}} \tag{16}$$

Example (**Network Good**)**.** In the setting of Section 2.2.1, there is a unique decentralized equilibrium given by[14]

$$x_k^{\mathrm{D}} = 1 + \gamma k \bar{x}^{\mathrm{D}}, \qquad \bar{x}^{\mathrm{D}} = \frac{1}{1 - \gamma \frac{\mathbb{E}[kl]}{\mathbb{E}[l]}} \tag{17}$$

The externality in the decentralized case is merely a byproduct of privately chosen consumption. Fixing the marginal distribution over k and l, it is increasing in the covariance of payoff- and influence-type. The first-best (which we characterize in the subsequent section) coincides with the decentralized solution if $\gamma = 0$. Aggregate consumption and total surplus in the first best exceed their decentralized values when $\gamma > 0$.

3.2 The First Best

The efficient allocation solves the principal's problem without incentive compatibility constraints, $\pi(\varnothing)$. In order to guarantee the existence of a solution it is not sufficient to assume concavity of the agents utility function. Instead, concavity of the planner's problems arises jointly from the agents utility and the aggregate externality. The following Lemma leverages the fact that we can focus the attention only on two dimensions of the allocation, the subspace along which externalities are produced and consumed.

[14]When $\kappa \equiv 0$, condition (NT) ensures that even when the aggeregate externality is 0 some agents are willing to act, effectively excluding a degenerate decentralized equilibrium.

Lemma 1. *Let* $\Phi = [\sqrt{f}][\sqrt{f}]^{\mathrm{T}} \in \mathbb{R}^{|\mathcal{K}\times\mathcal{L}|^2}$ *with typical element* $\sqrt{f_{k,l} f_{k',l'}}$*. The first-best planner problem is globally concave if and only if the value of the maximization problem*[15]

$$\max y^{\mathrm{T}} \left(\mathrm{dg}\left([u_{xx}] + (\mathbb{E}u_{\bar{x}} + \kappa')[v_{xx}]\right) + (\mathbb{E}u_{\bar{x}\bar{x}} + \kappa'')[v_x]\Phi[v_x]^{\mathrm{T}} + 2\,\mathrm{Sym}\left([u_{\bar{x}x}]\Phi[v_x]^{\mathrm{T}}\right)\right) y \tag{18}$$

$$s.t.\ y \in \mathrm{span}\left(\left[\sqrt{f}\right] \odot [u_{\bar{x}x}], \left[\sqrt{f}\right] \odot [v_x]\right), \quad \|y\| = 1$$

is negative for all $\boldsymbol{x} \geq 0$*. Then, the first-best contract induces a pure strategy equilibrium among agents.*

Note that the all the expressions in (18) depend implicitly on the full allocation x. The condition provides a tight bound for the general nonparametrized case and simplifies to a familiar upper bound on the degree of complementarities in the application to network good

$$\gamma < \frac{\mathbb{E}[l]}{\left(\sqrt{\mathbb{E}[k^2]\mathbb{E}[l^2]} + \mathbb{E}[kl]\right)} \tag{19}$$

From now on, we assume that the condition of Lemma 1 is met.

Proposition 1. *The efficient allocation* $(x^\star_{k,l})_{k,l \in \mathcal{K}\times\mathcal{L}}$ *solves*

$$0 = u_x\left(k, x^\star_{k,l}, \bar{x}^\star\right) + v_x\left(x^\star_{k,l}, l\right)\zeta^\star \tag{20}$$

$$\zeta^\star = \sum f_{k,l} u_{\bar{x}}\left(k, x^\star_{k,l}, \bar{x}^\star\right) + \kappa'(\bar{x}^\star) \tag{21}$$

$$\bar{x}^\star = \sum f_{k,l} v(x^\star_{k,l}, l) \tag{22}$$

The effecient allocation maximizes individual utility with the adjustment term $v_x\left(x^\star_{k,l}, l\right)\zeta^\star$ taking the spillover into account. This adjustment is proportional to the shadow value of $\bar{x}$, which corresponds to the surplus generated by the externality.

A monopolist observing both the payoff- and influence-type implements the efficient allocation and extracts all surplus. Agents receive the same level of utility (zero) in the optimal contract, in particular, influence is neither rewarded nor punished.

Example (Network Good cont'd). For the sale of a network good, we can solve (20) in closed form

$$x^\star_{k,l} = 1 + \gamma\bar{x}^\star k + \zeta^\star \frac{l}{\mathbb{E}[l]}. \tag{23}$$

In order to produce the efficient level of $\bar{x}$, the planner induces all types to overconsume relative to their privately optimal level $1 + \gamma\bar{x}k$. This is especially pronounced at high levels of influence. Such "influencers" are not compensated with higher utility, but they

[15] dg(a) denotes the diagonal matrix with entries provided by the vector a, Sym(A) = $\frac{A+A^{\mathrm{T}}}{2}$ and $\odot$ denotes element-wise multiplication of vectors.

are held indifferent through lower unit prices,

$$\frac{t^{\star}_{k,l}}{x^{\star}_{k,l}} = \frac{1}{2}\left(1 + \gamma\overline{x}^{\star}k - \frac{l}{\mathbb{E}[l]}\zeta^{\star}\right). \tag{24}$$

3.3 Observable Payoff-Type

We now turn to the case where to principal observes the payoff-type of all agents while their influence is private information. The revenue maximization problem needs to satisfy the sorting constraints $\bigcup_{k\in\mathcal{K}}(k\times\mathcal{L})^2$.[16] Note that l does not directly enter the utility function. Consequently, sorting is equivalent to the requirement that the utility of type (k,l) in their respective contract is independent of the level of influence l; Should this condition fail, every consumer with a given k would mimic the type k,l' whose contract delivers the highest level of utility.

Lemma 2. *A menu of contracts satisfies the* $\bigcup_{k\in\mathcal{K}}(k\times\mathcal{L})^2$ *sorting constraints if and only if for each* k,l,l'

$$u(x_{k,l},k,\bar{x}) - t_{k,l} = u(x_{k,l'},k,\bar{x}) - t_{k,l'} \tag{H}$$

Influence doesn't interact with the contract terms, so it can not introduce distortions in the form of information rents. Because the principal observes the only dimension in which she can actively screen, eliciting influence does not create any rents by itself. Even though the problem has a full dimension of incomplete information, it collapses for given k. Henceforth, let $\mathrm{U}_k := u(x_{k,l},k,\bar{x}) - t_{k,l}$ denote the utility of agents with payoff-type k as a (usually implicit) function of the contract.

The first best contract with full rent-extraction satisfies condition (H) since all types receive zero utility, so by Lemma 2 above it is feasible and hence optimal.

Proposition 2. *The efficient allocation with full extraction solves the problem with known payoff-type.*

No Underreporting of Payoff-Type

Suppose agents cannot underreport their payoff type. This may occur because of a technological constraint or because the seller has correct information about a lower bound of their payoff type. Such a constraint is reasonable in social media if the payoff type is tightly linked to the time spent on the social network which is identifiable by the provider and cannot be easily hidden or split across multiple accounts.

Agents can still exaggerate their payoff type or misreport their influence which must therefore be ruled out by sorting constraints. That is, the monopolist faces the problem $\pi(\bigcup_{k\in\mathcal{K}}(k\times\mathcal{L})\times(k^{+}\times\mathcal{L}))$ where $k^{+} := \{k'\in\mathcal{K} : k'\geq k\}$. As in a standard model without an externality, only the downward sorting constraints will bind in the second best

[16] Recall we identify sorting constraints with pairs of types.

problem of Section 4. Prohibiting this deviation makes the first best implementable and allows the principal to extract all surplus just as with observable k.

Remark 1. The first-best allocation also solves $\pi(\bigcup_{k\in\mathcal{K}}(k\times\mathcal{L})\times(k^+\times\mathcal{L}))$.

4 Full Screening

We now turn to the case in which both payoff- and influence-type are not observed. The principal then solves the 2-dimensional screening problem with consumption externality $\pi(\mathcal{K}\times\mathcal{L})^2$. We call this the full screening problem.

We first characterize the implementable allocations. Following the usual argument combining upward and downward incentive compatibility between two types, any implementable allocation has to satisfy monotonicity along the payoff type. Importantly, this is required for any combination of influence types. Conversely, for any such allocation, we can find transfers that satisfy the incentive compatibility and participation constraints.

Proposition 3. *There exists a vector of transfers implementing* $\boldsymbol{x}$ *if and only if it satisfies k-monotonicity; that is, for every* k, k', l, l',

$$(x_{k,l} - x_{k',l'})(k - k') \geq 0 \tag{25}$$

In contrast to a well-behaved screening problem without externalities, the first-best may fail to be implementable in our setting. In the first, the allocation of agents with high influence but low payoff-type is inflated in order to create the externality. The allocation may violate k-monotonicity as a result. To illustrate, consider

Example (**Pollution** 2×2)**.** Consider the setting of Section 2.2.2 with $\mathcal{K} = \{0,1\}$ and $\mathcal{L} = \{0,1\}$: Apart from differences in productivity, there is one polluting sector $l = 0$ and one green sector $l = 1$. Let $\max\{\theta_2\} = 1$, i.e. the green sector does not pollute at all. As $\bar{x}$ does not enter firm profits and the marginal social cost of pollution is constant, we have $\zeta^\star = \kappa$ and the first best is given by

$$(x_{0,0}, x_{0,1}, x_{1,0}, x_{1,1})^\star = \left(\frac{1}{1+\kappa}, 1, \frac{1}{\frac{1}{2}+\kappa}, 2\right) \tag{26}$$

The first best is implementable if and only if $\kappa \leq \frac{1}{2}$. Intuitively, when damage from pollution is limited, dirty high productivity firms are allowed to produce more than clean low productivity firms, which is required by incentive compatibility in the full screening problem.

Remark 2. The decentralized solution, by contrast, is always implementable as it is flat in l and increasing in k.

Extremal Sorting

Towards characterizing the full screening allocation, we start by simplifying the set of constraints. Since the problem contains all sorting constraints along the influence dimension, Lemma 2 implies that utility has to be constant along the l dimension: Condition H remains necessary in the full screening problem. Then, a slice $k \times \mathcal{L}$ of the type space can be treated as a single type for the purpose of outward deviations. In addition, we can rank the attractiveness of contracts in each $k \times \mathcal{L}$ slice by the level of the allocation: Higher payoff types will prefer the highest allocation contract, while lower payoff types will prefer the lowest allocation contract (higher transfer). Furthermore, we can restrict attention to local misrepresentation of the payoff-type. Consequently, for types in $k \times \mathcal{L}$, the relevant downward deviation is towards the contract giving the highest consumption in the $k-1 \times \mathcal{L}$ slice, whereas the relevant upward deviation is towards the contract giving the lowest consumption in the $k+1 \times \mathcal{L}$ slice. If it is not profitable to deviate to the contract with the largest (smallest) level of consumption in the slice, it isn't profitable to deviation into the slice at all.

Definition 1. A menu of contracts $\{(x_{kl}, p_{k,l})\}_{kl \in \mathcal{K} \times \mathcal{L}}$ satisfies **extremal sorting (ES)** if, for each k,

$$U_k \geq u(\min_l x_{k+1,l}, \bar{x}, k) - t_{k+1, \arg\min_l x_{k-1,l}} \tag{ES-A$_k$}$$

$$U_k \geq u(\max_l x_{k-1,l}, \bar{x}, k) - t_{k-1, \arg\max_l x_{k-1,l}} \tag{ES-B$_k$}$$

Finally, by the sorting constraints, it is sufficient to consider the participation constraint of the lowest payoff type as all other participation constraints will be implied.

$$U_{k_0} \geq 0 \tag{P}$$

Formalizing this discussion, we have

Proposition 4. *An allocation satisfies k-monotonicity, (H), (P), and extremal sorting if and only if it satisfies all participation and incentive constraints* $(\mathcal{K} \times \mathcal{L})^2$.

Lexicographic Monotonicity

The next step is to identify the extremal types (i.e. those that have highest and lowest allocation in an optimal contract) within a slice $k \times \mathcal{L}$. Let $\succ_L$ on $\mathcal{K} \times \mathcal{L}$ where $\mathcal{K}$ is the dominant dimension, i.e.

$$(k,l) \succ_L (k',l') \iff k > k' \text{ or } k = k', l > l' \tag{27}$$

and denote by $M := \{\boldsymbol{x} \geq 0 : \boldsymbol{x} \text{ is weakly increasing in } \succ_L\}$ denote the set of lex-monotonic allocations.

Theorem 1 (**Lexicographic Monotonicity**). *If $\boldsymbol{x}$ solves $\pi(\mathcal{K} \times \mathcal{L})^2$, then $\boldsymbol{x} \in M$.*

The proof of the theorem proceeds in three steps. First, we show that the upward sorting constraints are implied in the optimal allocation. We then write the problem in utility space (anticipating Proposition 5 below) and establish that the principal always benefits from a marginal (windfall) increase in $\bar{x}$, all other things equal. In other words, the associated increase in surplus dominates the increase in information rents and we have $\zeta > 0$.[17] Consequently, we establish with a variational argument that within a $k \times \mathcal{L}$ slice, the principal always desires to allocate higher levels of consumption to more influential types – a change that increases $\bar{x}$ while even increasing the surplus generated within the slice. Since Proposition 4 implies that only the largest $x_{k,l}$ for a slice $k \times \mathcal{L}$ is relevant for deviation, adding the sorting constraints does not alter this property. Combining this fact with k-monotonicity we have that the optimal allocation has to be lexicographic monotonic.

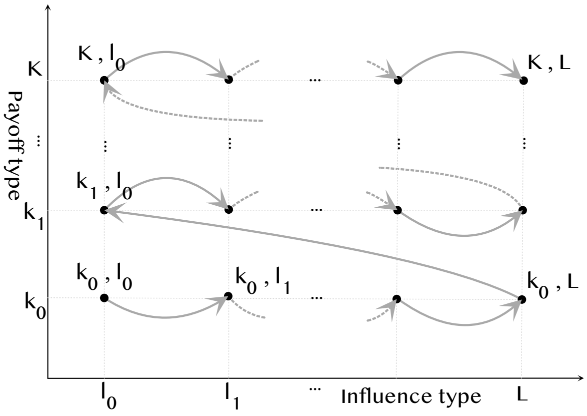

Figure 1: The Lexicographic Order $>_L$.

The Relaxed Problem

Using the results derived in the previous section, we can rewrite the principal's problem as a monotonicity constrained optimization in terms of virtual values. This problem is one-dimensional along the lexicographic order.

Proposition 5. *The problem* $\pi(\mathcal{K} \times \mathcal{L})^2$ *is equivalent to*

$$\max_{\boldsymbol{x} \in M} \sum f_{k,l} \left[u\left(k, x_{k,l}, \bar{x}\right) - \chi_{l=L} \left\{ \frac{1 - F_k}{f_{kl}} \int_k^{k+1} u_k\left(j, x_{k,l}, \bar{x}\right) \mathrm{d}j \right\} \right] + \kappa(\bar{x}) \tag{UP}$$

$$s.t. \bar{x} = \sum f_{k,l} v\left(x_{k,l}, l\right) \tag{ζ}$$

In the appendix, we provide general conditions that ensure that the principal's problem is strictly concave. As for the first best (Lemma 1), they involve maximizing a two-dimensional quadratic form. Under this condition, the full screening contract induces a pure strategy equilibrium between agents. In the sale of a network good

[17] A positive marginal value of aggregate consumption for the monopolist is a natural though not immediate result. In contrast to the symmetric information benchmark, under asymmetric information aggregate consumption $\bar{x}$ impacts revenues in two opposing ways: On the one hand, increasing $\bar{x}$ increases total surplus; on the other hand, it increases the information rents paid to consumers. We show that the first force dominates. Hence consumption is increasing in l: this increases aggregate consumption $\bar{x}$, counteracting some of the downward distortion due to screening.

from Section 2.2.1, we get a bound on the degree of complementarities

$$\gamma < \frac{\mathbb{E}[l]}{\sqrt{\left[\sum_k \frac{(1-F_k)^2}{f_{k,L}} + \mathbb{E}[k]\right]\mathbb{E}[l^2]} + [\mathbb{E}[kl] - L(\mathbb{E}[k] - k_0)]} \tag{28}$$

From now on, we assume that the concavity condition is satisfied. Therefore, we have a unique solution to our program which we denote by $\boldsymbol{x}^{FS}$.

To achieve a closer characterization of the optimal allocation we need to deal with the monotonicity constraints. As opposed to the textbook screening model, a simple condition on primitives is not sufficient to rule out violations of monotonicity. Instead, by the nature of our problem, there are two sources of monotonicity violations in this candidate allocation. First, sorting constraints only affect types with $l = L$ directly: screening distortions on the whole $k \times \mathcal{L}$ slice accumulate on the k, L type. The resulting downward distortion will typically be propagated along the l-dimension by lexicographic monotonicity. This would happen even if single-crossing and a monotone hazard rate are satisfied. The second source of violations of monotonic virtual values is the jump between type k, L and $k+1, l_0$: One the one hand, the latter has a higher payoff type, on the other hand, he is less influential, which depresses his virtual value. The strength of those distortions depends on the endogenous objects $\zeta, \bar{x}$ and no general conditions consistent with our analysis can ensure the lex-monotonicity of the candidate allocation.

Example (**Pollution** 2×2)**.** Let us illustrate these issues in the pollution setting where we can solve for the allocation in closed form. The pointwise maximizer of the objective (UP) is

$$(\check{x}_{0,0}, \check{x}_{0,1}, \check{x}_{1,0}, \check{x}_{1,1}) = \left(\frac{1}{1+\kappa}, 1 - \frac{1 - f_{0,0} - f_{0,1}}{1 - f_{0,0} + f_{0,1}}, \frac{1}{\frac{1}{2}+\kappa}, 2\right) \tag{29}$$

The pointwise maximum violates lexicographic monotonicity between $(0,0)$ and $(0,1)$ if

$$f_{0,1} < \frac{1 - f_{0,0}}{2\kappa} \tag{30}$$

i.e. whenever the downward distortion due to sorting constraints is larger than the reduction in output due to pollution for type $(0,0)$. This condition is always met if $\kappa \approx 0$ or if the downward distortion is large as $f_{0,1} \approx 0$. In this case, the two lowest types will be bunched.

There is a non-monotonicity induced by the downward jump in influence between $(0,1)$ and $(1,0)$ if

$$f_{0,1} > \frac{1 - f_{0,0}}{1 + 2\kappa} \tag{31}$$

i.e. whenever the downward distortion due to sorting is smaller than the reduction in output due to pollution for type $(1,0)$. Combining both inequalities, we see that the pointwise maximizer $\check{x}$ solves the full screening problem if and only if

$$\kappa \in \left[\frac{1-f_{0,0}}{2f_{0,1}} - \frac{1}{2}, \frac{1-f_{0,0}}{2f_{0,1}}\right]. \tag{32}$$

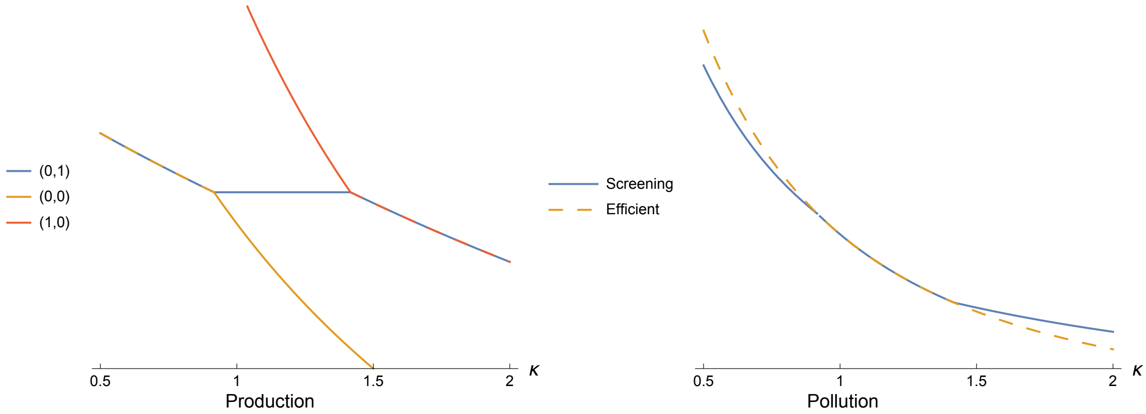

Figure 2: Production and aggregate pollution as a function of social cost κ.

The resulting ironing procedure has important consequences for the impact of screening on pollution, as illustrated in the second panel of Figure 2. In the unconstrained case, the downward distortion only affects the "green" type $(0,1)$. Therefore, the pollution mitigation and rent extraction motive are (locally) independent and $\bar{x}^{\star}$ and $\bar{x}^{\mathrm{FS}}$ coincide. When the monotonicity constraint between $(0,0)$ and $(0,1)$ is binding, the "dirty" type is distorted downwards and the rent extraction motive leads to pollution lower than in the first best. When instead the monotonicity constraint between $(0,1)$ and $(1,0)$ is binding, the need for screening high from low productivity types depresses the output of the inefficient green type in favor of the efficient dirty type, leading to pollution in excess of the first-best level.

In general, the ironing conditions depend not only on primitives, but the endogenous objects $\bar{x}$ and ζ.[18] In the sale of a network good, for example, the solution requires ironing unless for all k

$$\zeta\frac{\mathrm{L}-l_0}{\mathbb{E}[l]} < \gamma\bar{x}\left(1+\frac{1-\mathrm{F}(k)}{f_{k,\mathrm{L}}}\right) < \gamma\bar{x}+\zeta\frac{1}{\mathbb{E}[l]}. \tag{33}$$

4.1 The Ironing Procedure

The ironing procedure therefore proceeds in two steps. We first solve for the candidate allocation for fixed aggregate variables $(\bar{x}, \zeta)$ and then solve for the aggregate quantities.

[18]In the pollution example, ζ is pinned down exogenously since we have a constant (marginal) social benefit of $\overline{x}$ and $\bar{x}$ does not affect agents' returns.

We write the objective function as the weighted sum of virtual values $J(x_{k,l}, k, l, \overline{x}, \zeta)$.

$$\sum_{k,l} f_{k,l} \underbrace{\left[u\left(k, x_{k,l}, \overline{x}\right) - \chi_{l=L} \left\{ \frac{1 - F_k}{f_{kl}} \left[u\left(k+1, x_{k,l}, \overline{x}\right) - u\left(k, x_{k,L}, \overline{x}\right) \right] \right\} + \zeta v\left(x_{k,l}, l\right) \right]}_{:=J(x_{k,l}, k, l, \overline{x}, \zeta)} \tag{34}$$

which defines a set of proposed allocations $\check{x}_{k,l}$ maximizing the (rescaled) virtual value J. By concavity of the virtual value, $\check{x}_{k,l}\left(\overline{x}, \zeta\right)$ solves $J_x\left(\check{x}_{k,l}, k, l, \overline{x}, \zeta\right) = 0$. Clearly, if $\check{x} \in M$, then it is the solution, $\check{x}\left(\overline{x}, \zeta\right) = x^{FS}\left(\overline{x}, \zeta\right)$. As discussed above, this generally won't be the case.

Allocation Conditional on Aggregate Variables

We adapt standard techniques from Toikka (2011) to the problem rendered one dimensional in the lexicographic order by virtue of Theorem 1.

Let $k(q), l(q) : [0,1] \mapsto \mathcal{K} \times \mathcal{L}$ trace out the distribution f on $\mathcal{K} \times \mathcal{L}$ along the lexicographic order. In other words, if $q \in \left[\sum_{i,j <_L k,l} f_{i,j}, \sum_{i,j \lesssim_L k,l} f_{i,j}\right)$, we have $k(q) = k$ and $l(q) = q$. Denote an inverse by $q(k,l) = \sum_{i,j <_L k,l} f_{i,j}$. The cumulative virtual value is given by

$$H(x, q) = \int_0^q J_x(x, k(r), l(r), \bar{x}, \zeta) \mathrm{d}r \tag{35}$$

It follows from Toikka (2011) that ironing the original problem is equivalent to convexifying H. For every x, let $G(x, \cdot) := \operatorname{Conv} H(x, \cdot) := \max\{g(x, \cdot) \leq H(x, \cdot) | g \text{ is convex}\}$ which is continuously differentiable almost everywhere on $[0,1]$.

$$\bar{J}(x, k, l, \overline{x}, \zeta) = J(0, k, l, \overline{x}, \zeta) + \int_0^x G_q(y, q(k,l)) \mathrm{d}y \tag{36}$$

The conditionally optimal allocation then solves

$$x^{FS}(\bar{x}, \zeta) := \arg\max_{\boldsymbol{x}} \sum \bar{J}(x, k, l, \overline{x}, \zeta) \tag{37}$$

Remark 3. As our type space is finite, the convexification is easy to compute. A simple algorithm proceeds downwards in the lexicographic order and "greedily" irons out violations of convexity as it encounters them. It finishes in at most $|\mathcal{K} \times \mathcal{L}| + 1$ steps.

If $H(x_{k,l}, q) < G(x_{k,l}, q)$, the lex-monotonicity constraints are active at the corresponding type and there is bunching. Indeed, since H is piecewise linear, with kinks where $k(q), l(q)$ jumps between types, we obtain an intuitive characterization of $x^{FS}(\bar{x}, \zeta)$.

Lemma 3. *The ironing procedure induces a partition of types $\mathcal{B}$, with typical element* B, *ordered by $>_L$. The optimal allocation for a given $(\bar{x}, \zeta)$ is constant within cells and strictly increasing across cells. For $(k,l) \in B$, $x_{k,l} = x_B$, solving*

$$u_x\left(k_B, x_B, \overline{x}\right) + \zeta \mathbb{E}\left[v_x\left(x_B, l\right) | B\right] - \frac{\sum_{k,l >_L B} f_{k,l}}{\sum_{k,l \in B} f_{k,l}} \left[u_x\left(k^B + 1, x_B, \overline{x}\right) - u_x\left(k^B, x_B, \overline{x}\right) \right] = 0 \tag{38}$$

where $k_{\mathrm{B}} = \min_{(k,l)\in \mathrm{B}} k$ *and* $k^{\mathrm{B}} = \max_{(k,l)\in \mathrm{B}} k$.

In the network good application, types coincide with marginal utility and the marginal externality and hence condition (38) simplifies to

$$x_{\mathrm{B}} = \max\left\{1+\gamma\bar{x}\left(\mathbb{E}[k|\mathrm{B}] - \frac{\sum_{k,l>_{\mathrm{L}}\mathrm{B}} f_{k,l}}{\sum_{(k,l)\in \mathrm{B}} f_{k,l}}\right) + \zeta\frac{\mathbb{E}[l|\mathrm{B}]}{\mathbb{E}[l]}, 0\right\} \tag{39}$$

Agents are allocated consumption according to the expected payoff-type and expected externality in their partition cell.

The bunching regions have the following properties

Proposition 6.

1. *There is no bunching at the top of the lexicographic order:* $\{(\mathrm{K},\mathrm{L})\} \in \mathcal{B}$.
2. *Every nontrivial cell of* $\mathcal{B}$ *contains a switching type in* $>_{\mathrm{L}}$*:* $|\mathrm{B}| > 1 \implies \exists\,(k,\mathrm{L}) \in \mathrm{B}$.
3. *There is active influence tilting within a payoff slice only if the highest influence type consumes more than his decentralized allocation:* $x^{\mathrm{FS}}_{k,l} > x^{\mathrm{FS}}_{k,l-1} \implies u_x\left(k, x^{\mathrm{FS}}_{k,l}, \overline{x}\right) < 0$.

The first property provides a weak analogue to the "no distortion at the top" results common across screening models: For a given externality $\bar{x}$ and value of the externality ζ, the highest type is not affected by the ironing procedure. His consumption, however, is distorted relative to the decentralized and first-best consumption, *even given* $\bar{x}$, as the value of the externality for the monopolist generally differs from $\zeta^{\star}$.[19] Agents with the highest payoff-type but lower influence, by contrast, can be affected by ironing as shown in the above example.The second property says every bunching region includes an agent with the highest influence. It is around these types that the nonmonotonic virtual values can arise: Either their action is heavily downward distorted (to reduce information rents of higher types) and they are bunched with less influential agents of the same (or lower) payoff-type, or their action is distorted upward (to promote the externality) and they are bunched with less influential agents of higher payoff-type. Bunching regions "strictly" within a payoff slice are never optimal as virtual values are locally increasing. As for the third property, notice that $u_x\left(k, x^{\mathrm{FS}}_{k,l}, \overline{x}\right) < 0 \iff x^{\mathrm{FS}}_{k,l} > x^{\mathrm{D}}_{k}(\overline{x})$ where the latter is the decentralized solution associated to aggregate activity from full screening. Whenever the principal discriminates based on influence alone, the more influential agents consume more than their privately optimal level. In other words, such tilting is only optimal if the provision of the externality overpowers the usual downward-distortion motive.

[19] In the pollution example, we have $u_{\bar{x}} \equiv 0$ and linear κ, and therefore always $\zeta = \kappa$. Hence, if $\bar{x}^{\mathrm{FB}} = \bar{x}^{\mathrm{FS}}$ – which is the case for an open set of parameters, see (32) – the higest type produces at the (unconditionally) efficient level.

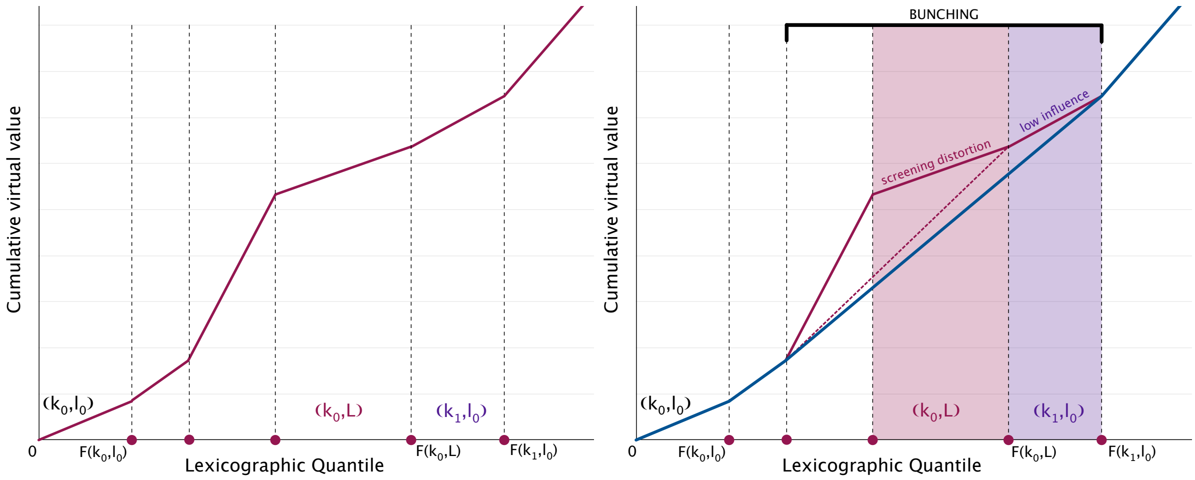

Figure 3: A typical ironing region.

Endogenizing Aggregate Variables

The final step in the derivation of the optimal contract is to endogenize $\bar{x}$. We solve for the aggregate variables in a fixed point problem. We do not establish that the fixed point mapping is a contraction. However, since a solution in this two-step procedure corresponds to the solution of the relaxed problem, there exists a unique fixed point. Therefore, the original problem and the two-step procedure lead to the same solution.

Theorem 2. *The allocation* $\boldsymbol{x}^{\mathrm{FS}}$ *solving* $\pi(\mathcal{K}\times\mathcal{L})^2$ *satisfies*

$$\boldsymbol{x}^{\mathrm{FS}} = x^{\mathrm{FS}}(\bar{x}^{\mathrm{FS}}, \zeta^{\mathrm{FS}}) \tag{40}$$

where $(\bar{x}^{\mathrm{FS}}, \zeta^{\mathrm{FS}})$ *is the unique fixed point of the self-map* $\Gamma : \mathbb{R}^2 \mapsto \mathbb{R}^2$ *given by*

$$\Gamma\begin{pmatrix} \bar{x} \\ \zeta \end{pmatrix} = \begin{pmatrix} \sum f_{k,l}\, v\left(x^{\mathrm{FS}}_{k,l}(\bar{x},\zeta), l\right) \\ \sum f_{k,l}\left(u_{\bar{x}}(x^{\mathrm{FS}}_{k,l}(\bar{x},\zeta), k, \bar{x}) - \chi_{l=\mathrm{L}}\frac{1-\mathrm{F}_k}{f_{kl}}\int_k^{k+1} u_{k\bar{x}}\left(j, x^{\mathrm{FS}}_{k,l}(\bar{x},\zeta), \bar{x}\right)\mathrm{d}j\right) + \kappa'(\bar{x}) \end{pmatrix} \tag{41}$$

Establishing that the two step procedure yields the unique solution of the general problem $\pi(\mathcal{K}\times\mathcal{L})^2$ proves that all properties of the optimal allocation conditional on aggregate variables $\bar{x}, \zeta$ also characterize the solution to $\pi(\mathcal{K}\times\mathcal{L})^2$. In particular, Proposition 6 characterizes unconditionally optimal bunching regions. We illustrate those properties in a 2×2 network good application, detailed derivations are given in B.2.

Example (Network Good 2×2**).** Consider the setting of Section 2.2.1 with $\mathcal{K} = \{0,1\}$ and $\mathcal{L} = \{0,1\}$: agents with payoff type 0 are not susceptible at all to the network good, $u_{\bar{x}}(0,\cdot) \equiv 0$; agents with influence type 0 do not create any consumption externality, $v(0,\cdot) \equiv 0$.[20] Given this parametrization three cases emerge as a full screening solu-

[20]The first best is implementable if and only if $f_{0,1} > f_{1,0}$, i.e. there are influential but not susceptible agents then susceptible but not influential ones.

tion,[21] depending on the residual primitives (complementarities γ and the distribution of types f).

1. Low susceptibility agents are bunched and excluded, $x_{1,1} > x_{1,0} > x_{0,1} = x_{0,0} = 0$.
2. Low susceptibility agents are bunched at a positive level of consumption, $x_{1,1} > x_{1,0} > x_{0,1} = x_{0,0} > 0$
3. The allocation satisfies strict monotonicity along the lexicographic order, $x_{1,1} > x_{1,0} > x_{0,1} > x_{0,0} = x^\star_{0,0} = 1$.

Figure 4 displays these possible regions. For every distribution, at $\gamma = 0$ every type consumes 1 (the first best allocation) as there are effectively no externalities. For low γ, there is always bunching of the non-susceptible agents; local to $\gamma = 0$, this level is decreasing in the degree of complementarity. As γ approaches his upper bound γ^{SB}, given in (28), two things can happen (depending on the distribution of types): either the bunching level drops to 0 and the non-susceptible agents are excluded, or it bends back to 1 and the allocation is strictly monotonic. The latter case occurs if $\gamma > 1$, and is therefore relevant only if the bound (28) exceeds 1. Notice all properties of Proposition 6 hold: Type $(1,1)$ is never bunched. The second property holds vacuously in a 2×2 example. To check the third property, recall from Example 3.1 that $x^{\text{D}}_k(\overline{x}) = 1 + k\gamma\overline{x}$. Agents that are not susceptible are separated in the full screening solution if and only if $x_{1,0} > 1 = x^{\text{D}}_0$, that is for large γ in the right panel of Figure 4. Agents that are susceptible are always separated since

$$x^{\text{FS}}_{1,1} > 1 + \gamma\overline{x} + \frac{\zeta}{\mathbb{E}[l]} > 1 + \gamma\overline{x} = x^{\text{FS}}_{1,0} = x^{\text{D}}_1(\overline{x}) \tag{42}$$

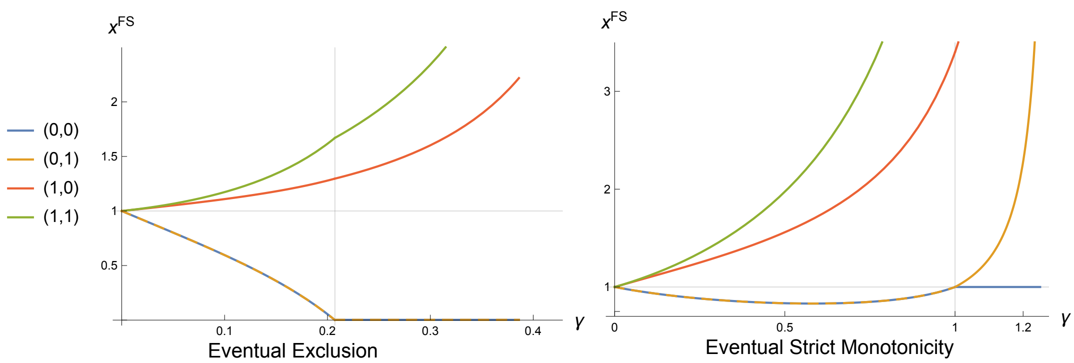

Figure 4: Consumption in the 2×2 example as a function of complementarities. Type distributions $f = (.1,.1,.2,.6)$ (left, switching from region 2 to 1) and $f = (.3,.3,.3,.1)$ (right, from region 2 to 3).

As for aggregate variables and welfare, recall that aggregate consumption is inefficiently low in the decentralized outcome. The ranking of total surplus between the

[21] The $\{0,1\}^2$ type space is restrictive: for example, it prevents inter-payoff bunching the would otherwise emerge.

decentralized and second best allocation is ambiguous. On the one hand, the screening motive of the principal induces a downward distortion, on the other hand, the principal internalizes the aggregate externality. We show by means of example (Figure 5) that the decentralized solution dominates the screening solution in terms of total surplus and consumer surplus for low γ, but screening performs better for sufficiently high γ, even in terms of consumer surplus.[22]

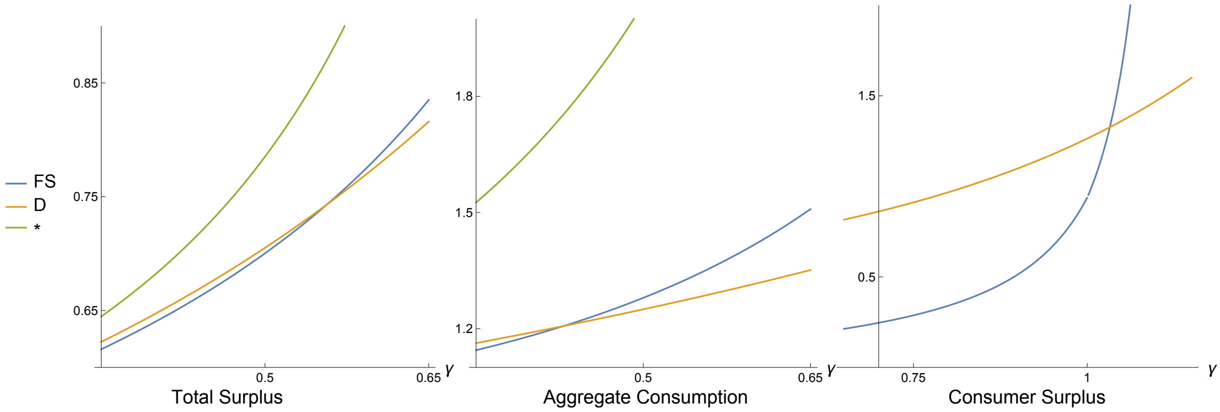

Figure 5: The 2x2 consumption example with $f = (.45,.3,.05,.2)$.

5 Observable Influence

Neither the efficient nor the full screening contract produce influence rents: either agents are fully extracted or the horizontal sorting condition, (H), implies that utility has to be constant in a payoff-type slice. Influence rents, however, seem to characterize some markets, especially network goods where celebrities often get lucrative deals to promote products. In a model where such influential agents have market power those rents can emerge as a result of bargaining. We investigate now whether a model like ours where such market power is excluded (as there is a continuum of agents in each type), can still generate rents from influence. To this end we investigate the final observability assumption, i.e. the influence type l is observed but the payoff type is private information. Motivated by the application, and to simplify the exposition we will focus on the pricing of a network good with linear quadratic utility (Section (2.2.1)).

Now, the monopolist can condition consumption on the observable l but has to ensure types k sort into their contract. Therefore – for a given $\overline{x}$ – the planner is solving a sequence of $\mathcal{L}$ one-dimensional screening problems. Per the standard arguments, we can rewrite each of these problems as the maximization of virtual value subject to a monotonicity constraint. The components for different l are however coupled through aggregate consumption $\bar{x}$.

[22]Clearly, the latter result depends crucially on the distribution of types. No matter the externality, if the type of the agent is almost known there will be almost full extraction. In the example, type $(0,1)$ is relatively abundant, linking information rents to the creation of the externality.

Proposition 7. *The maximization problem* $\pi\left(\bigcup_{l\in\mathcal{L}} (\mathcal{K}\times l)^2\right)$ *is equivalent to*

$$\max_{\boldsymbol{x},\bar{x}} \sum_{k,l} f_{k,l}\left\{\left(1+\gamma\bar{x}\left(k-\frac{\mathrm{F}(\mathrm{K}|l)-\mathrm{F}(k|l)}{f(k|l)}\right)\right)x_{kl}-\frac{1}{2}x_{kl}^2\right\} \tag{43}$$

subject to the aggregate effect, non-negativity and monotonicity conditional on l.

In this case, violations of monotonicity are solely the mechanical consequence of a nonmonotonic inverse hazard rate of the conditional type distribution. We hence restrict attention to the regular case in which the monotonicity constraints are slack.

Assumption 1. *For every l, the virtual value* $k-\frac{\mathrm{F}(\mathrm{K}|l)-\mathrm{F}(k|l)}{f(k|l)}$ *is increasing in k.*

Analogous to the full information case, the first-order conditions of this problem have two components. The first part is the familiar screening formula, the second adjusts consumption upward for influential individuals in order to provide a stronger network effect.

$$x_{k,l}^{\mathrm{OI}} = \max\Big\{0, \underbrace{1+\gamma\bar{x}^{\mathrm{OI}}\left(k-\frac{\mathrm{F}(\mathrm{K}|l)-\mathrm{F}(k|l)}{f(k|l)}\right)}_{\text{optimal screening for fixed }\bar{x}} + \underbrace{\frac{l}{\mathbb{E}[l]}\zeta^{\mathrm{OI}}}_{\text{provide public good }\bar{x}}\Big\} \tag{44}$$

Agents receive information rents for their level of susceptibility k. The magnitude of these rents depends on the level of consumption of agents with the same l but lower k. Therefore, the rent of type k,l is dependent on his (observable) level of influence. We say that there are *rents from influence* if, for every fixed k, the information rent is increasing in l. There are *expected rents from influence*, if the expected rent is increasing in l.

The information rent of type k,l can be written as $\gamma\bar{x}\sum_{j<k} x_{j,l}^{\mathrm{OI}}$. Influence affects optimal consumption and hence information rents through two channels. First, more influential individuals consume more and high levels of consumption cause high rents. Second, influence has an effect on the downward distortion of consumption by the monopolist. If susceptibility and influence are affiliated, high influence makes it more likely that the agent also has high susceptibility and the monopolist distorts consumption downwards more. The outcome depends on the balance of these two forces whose relative strength is determined by a moment Ξ of the type-distriution measuring both the scale of externalities and the (unsigned) association between k,l, i.e. how informative the observable influence is about the payoff type of the agent. Formally, the rents of type k,l are proportional to

$$\xi(k,l) := \underbrace{k\frac{l}{\mathbb{E}[l]}\Xi}_{\text{provision of }\bar{x}} - \underbrace{\textstyle\sum_{j=0}^{k-1}\frac{1-\mathrm{F}(j|l)}{f(j|l)}}_{\text{screening distortion}}, \tag{45}$$

$$\text{where } \Xi := \gamma\left(\mathbb{E}[k] + \mathbb{E}[\left(\frac{1-F(k|l)}{f(k|l)}\right)^2]\right) > 0. \tag{46}$$

Proposition 8. *Suppose the nonnegativity constraints are slack.*

1. *There are rents from influence if and only if, for all k, $\xi(k,l)$ is increasing in l.*
2. *There are expected rents from influence if and only if $\mathbb{E}[\xi(k,l)|l]$ is increasing in l.*

If k and l are independent, the screening distortion is independent of l and we have both rents from influence and expected rents from influence. In general, one can happen without the other, as we will illustrate in our running example.

Example (**Network Good** 2×2 **cont'd**)**.** We further parameterize the 2×2 network good setting by the covariance of k and l, letting $f_{0,0} = f_{1,1} = .25+\rho$, $f_{0,1} = f_{1,0} = .25-\rho$.[23] We highlight some features of the solution (detailed in B.2) in Fig. 6.

In the first two panels we compare the full screening contract with the observable influence contract. Aggregate consumption is smaller with observable influence when there is moderate positive correlation. This results from the large downward distortion of $x_{0,1}$ chosen in order to depress the information rents of the relatively common type $(1,1)$, a motive that is attenuated when l is not observed. Clearly, the profit of the seller is weakly higher when influence is observable, with equality only when $x^{OI}_{0,0} = x^{OI}_{0,1}$, i.e. when the observable-l contract is incentive compatible in the full screening problem (the tangency point in the top right panel).

In the bottom panels we plot rents and expected rents from influence. For $\rho < 0$, there are always (pointwise) rents from influence. Both the relative abundance of low payoff types and the motive to provide the consumption externality push towards a relatively high $x^{OI}_{0,1}$ which results in these rents. There is a cutoff $\bar{\rho} > 0$ above which the high-payoff low-influence type obtains a higher rent. The question of expected influence rents is more subtle, as there is the additional composition effect: as ρ increases, the influential agents also become more abundant relative to the non influential one (in the slice of high payoff agents that receive some rents). As long as γ is not too large, this composition effect dominates for moderately negative correlation. Even though type $(1,1)$ obtains a higher rent, the relative abundance of $(0,1)$ types means that on average high influence consumers have a lower rent. This highlights the interaction of the conditional rent above (which is positive if correlation is negative) and the shift in relative mass from low to high payoff types (which favors rents for high influence types if there is positive correlation, at least initially). The relative strength of these effects is mediated by γ as it scales up the magnitude of rents: For large γ expected rents are in line with pointwise rents.

[23] Note that Assumption 1 is satisfied trivially in a two payoff-type example and that

$$Cov(k,l) = f_{11} - \left(f_{0,1}+f_{1,1}\right)\left(f_{1,0}+f_{1,1}\right) = .25+\rho-(.5)^2 = \rho$$

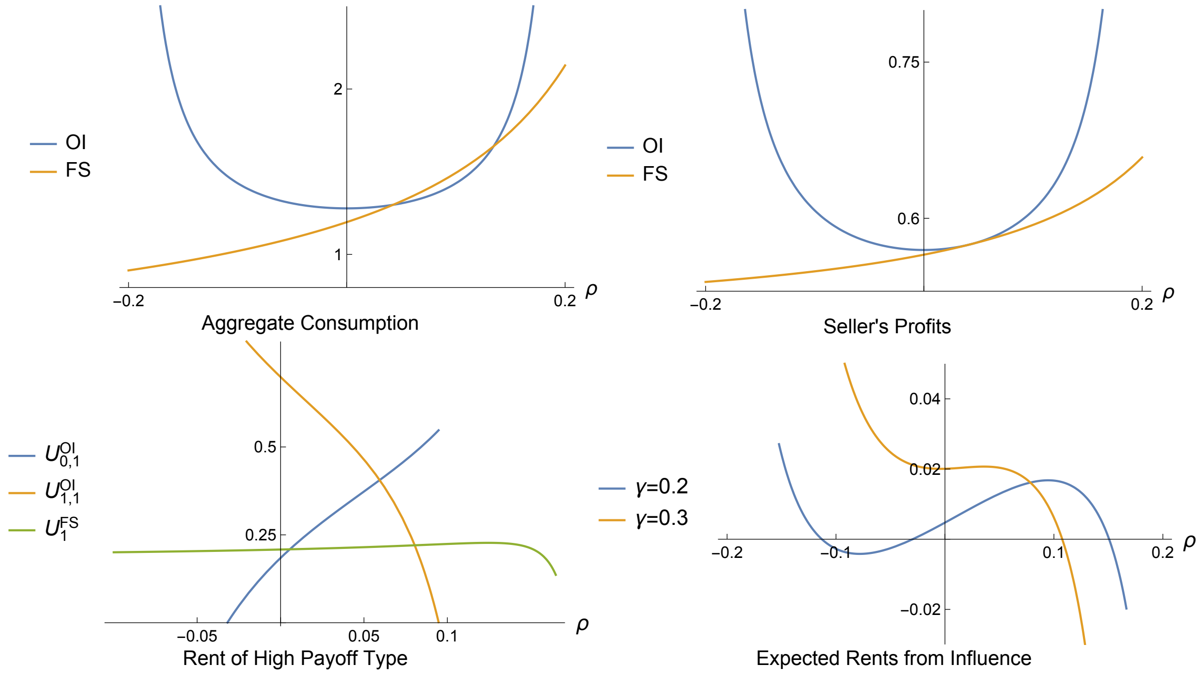

Figure 6: Observable influence in the 2x2 consumption example.

Comparing the results in this section to the previous one, we see that influence affects an agent's utility *only if it is observable and only indirectly, through its impact on information rents.*

6 Conclusion

We analyze a screening problem with externalities. Agents have private information about their payoff-type and their influence on the externality. A monopolist principal provides a menu of actions and trades off revenues with the direct payoff from the externality. Several problems fall into this framework, for example a monopolist firm using nonlinear pricing when there are consumption externalities or a government designing a tax when there are externalities between firms, positive through external economies of scale or negative through pollution. Even though the problem is two-dimensional at the surface and contracts are linked "globally" through the externality, we show that it is nevertheless tractable. The principal screens along the payoff-type while tilting the allocation along the influence-type to correct for the externality. Eliciting influence is for free: As long as the payoff-type is observable, the principal can implement the first-best. If both characteristics are unobservable, we show that the problem can be transformed into a one-dimensional problem along the lexicographic order, with the payoff-type as the dominant dimension. There are rents for high payoff-types, but no rents for influence. If influence is observable, the problem is equivalent to a family of one-dimensional screening problems coupled through the externality. Influence affects utility only if it is observed and even then only indirectly, through its effect on information rents. Highly influential consumers obtain higher rents if payoff- and influence-type are not too affiliated.

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A Proof Appendix

Proof of Lemma 1: Consider the Hessian of total surplus.

$$\mathrm{H}^{\mathrm{FB}} = \mathrm{dg}[f \odot (u_{xx} + \mathbb{E}[u_{\bar{x}} + \kappa']v_{xx})] + \mathbb{E}[u_{\bar{x}\bar{x}} + \kappa''][f \odot v_x][f \odot v_x]^{\mathrm{T}} + 2\,\mathrm{Sym}\left([f \odot u_{\bar{x}x}][f \odot v_x]^{\mathrm{T}}\right)$$

Let $\mathrm{S} = \mathrm{dg}[\sqrt{f}]$. Then $\mathrm{H}^{\mathrm{FB}} = \mathrm{S}\widehat{\mathrm{H}}^{\mathrm{FB}}\mathrm{S}$, where

$$\widehat{\mathrm{H}}^{\mathrm{FB}} = \mathrm{dg}[(u_{xx} + \mathbb{E}[u_{\bar{x}} + \kappa']v_{xx})] + \mathbb{E}[u_{\bar{x}\bar{x}} + \kappa'']\left[\sqrt{f} \odot v_x\right]\left[\sqrt{f} \odot v_x\right]^{\mathrm{T}} + 2\,\mathrm{Sym}\left(\left[\sqrt{f} \odot u_{\bar{x}x}\right]\left[\sqrt{f} \odot v_x\right]^{\mathrm{T}}\right)$$

Since S is positive definite, the Hessian is negative definite whenever the inner sum is n.d.. The first two summands are n.d. since $u_{xx} < 0$, $u_{\bar{x}} \geq 0$, $\kappa' \geq 0$ $v_{xx} \leq 0$, $u_{\bar{x}\bar{x}} \leq 0$, and $\kappa'' \leq 0$. Hence, the only threat to concavity comes from the two final terms. Note that the two matrices annihilate the component of any vector outside of $\mathrm{span}\left(\left[\sqrt{f} \odot u_{\bar{x}x}\right], \left[\sqrt{f} \odot v_x\right]\right)$. To establish concavity it is hence sufficient to show that the quadratic form defined by H^{FB} is negative for unit vectors of the form $x = \alpha\left[\sqrt{f} \odot u_{\bar{x}x}\right] + \beta\left[\sqrt{f} \odot v_x\right]$.

First, note that

$$\|x\| = \alpha^2\mathbb{E}[u_{\bar{x}x}^2] + \beta^2\mathbb{E}[v_x^2] + 2\alpha\beta\mathbb{E}[u_{\bar{x}x}v_x]$$

The quadratic form evaluates to

$$\begin{aligned} Q(\alpha,\beta) :=& y(\alpha,\beta)^{\mathrm{T}}\widehat{\mathrm{H}}^{\mathrm{FB}}y(\alpha,\beta) \\ =& \alpha^2\mathbb{E}[u_{\bar{x}x}^2(u_{xx} + \mathbb{E}[u_{\bar{x}} + \kappa']v_{xx})] + 2\alpha\beta\mathbb{E}[u_{\bar{x}x}v_x(u_{xx} + \mathbb{E}[u_{\bar{x}} + \kappa']v_{xx})] \\ &+ \beta^2\mathbb{E}[v_x^2(u_{xx} + \mathbb{E}[u_{\bar{x}} + \kappa']v_{xx})] + \mathbb{E}[u_{\bar{x}\bar{x}} + \kappa'']\left\{\alpha^2\mathbb{E}[u_{x\bar{x}}v_x]^2 + 2\alpha\beta\mathbb{E}[u_{x\bar{x}}v_x]\mathbb{E}\left[v_x^2\right]\right. \end{aligned}$$

$$+\beta^2\mathbb{E}\left[v_x^2\right]^2\Big\} + 2\Big\{\alpha^2\mathbb{E}[u_{\bar{x}x}^2]\mathbb{E}[u_{\bar{x}x}v_x] + \alpha\beta\Big[\mathbb{E}[u_{\bar{x}x}^2]\mathbb{E}[v_x^2] + (\mathbb{E}[u_{\bar{x}x}v_x])^2\Big] + \beta^2\mathbb{E}[u_{\bar{x}x}v_x]\mathbb{E}[v_x^2]\Big\}$$

and we have concavity if the value of

$$\max_{\alpha,\beta} Q(\alpha,\beta)$$
$$\text{s.t.}\,\alpha^2\mathbb{E}[u_{\bar{x}x}^2] + \beta^2\mathbb{E}[v_x^2] + 2\alpha\beta\mathbb{E}[u_{\bar{x}x}v_x] = 1 \tag{47}$$

is negative. The solution to this problem is conceptually simple as we are maximizing a quadratic form over an elliptic constraint, but analytically cumbersome. We hence restrict attention to our examples where we can derive meaningful bounds on the parameters.

In the consumption example, we have $(u_{xx} + \mathbb{E}[u_{\bar{x}} + \kappa']v_{xx}) = -1$ and $\mathbb{E}[u_{\bar{x}\bar{x}} + \kappa''] = 0$. Hence we get

$$\begin{aligned} Q(\alpha,\beta) =& -\left(\alpha^2\mathbb{E}[u_{\bar{x}x}^2] + \beta^2\mathbb{E}[v_x^2] + 2\alpha\beta\mathbb{E}[u_{\bar{x}x}v_x]\right) \\ &+ 2\Big\{\alpha^2\mathbb{E}[u_{\bar{x}x}^2]\mathbb{E}[u_{\bar{x}x}v_x] + \alpha\beta\Big[\mathbb{E}[u_{\bar{x}x}^2]\mathbb{E}[v_x^2] + (\mathbb{E}[u_{\bar{x}x}v_x])^2\Big] + \beta^2\mathbb{E}[u_{\bar{x}x}v_x]\mathbb{E}[v_x^2]\Big\} \\ =& -1 + 2\Big\{\mathbb{E}[u_{\bar{x}x}v_x] + \alpha\beta\Big[\mathbb{E}[u_{\bar{x}x}^2]\mathbb{E}[v_x^2] - (\mathbb{E}[u_{\bar{x}x}v_x])^2\Big]\Big\} \end{aligned}$$

by plugging in the constraint. Note that the coefficient of $\alpha\beta$ is nonnegative by Cauchy-Schwartz. Hence, it is sufficient to find $\max\alpha\beta$ subject to the constraint. It follows from straightforward computation that

$$\max_{s.t.47}\alpha\beta = \frac{1}{2\left(\sqrt{\mathbb{E}[u_{\bar{x}x}^2]\mathbb{E}[v_x^2]} + \mathbb{E}[u_{\bar{x}x}v_x]\right)}$$

Plugging back and using that $\mathbb{E}[u_{\bar{x}x}^2] = \gamma^2\mathbb{E}[k^2]$, $\mathbb{E}[v_x^2] = \frac{\mathbb{E}[l^2]}{\mathbb{E}[l]^2}$, $\mathbb{E}[u_{\bar{x}x}v_x] = \gamma\frac{\mathbb{E}[kl]}{\mathbb{E}[l]}$, after straightforward manipulation we get

$$\gamma < \frac{\mathbb{E}[l]}{\left(\sqrt{\mathbb{E}[k^2]\mathbb{E}[l^2]} + \mathbb{E}[kl]\right)}.$$

In the pollution case, we have $u_{\bar{x}} = 0$ and hence

$$Q(\alpha,\beta) = \beta^2\left(\mathbb{E}[v_x^2(u_{xx} + \kappa' v_{xx})] + \kappa''\mathbb{E}\left[v_x^2\right]^2\right) = \frac{1}{\mathbb{E}[v_x^2]}\mathbb{E}[v_x^2(u_{xx} + \kappa v_{xx})] < 0$$

which is always satisfied. □

Proof of Proposition 1: As the problem is concave, differentiation of the objective – treating $\bar{x}$ as a constraint – yields the desired conditions. □

For the derivation of the unit price in the network good application, note that solving $u(x^\star_{k,l}, k, \bar{x}^\star) = 0$ yields

$$t^\star_{k,l} = \frac{1}{2}\left(1 + \gamma\bar{x}^\star k\right)^2 - \frac{1}{2}\left(\frac{l}{\mathbb{E}[l]}\zeta^\star\right)^2 = \frac{1}{2}x^\star_{k,l}\left(1 + \gamma\bar{x}^\star k - \frac{l}{\mathbb{E}[l]}\zeta^\star\right).$$

Proof of Lemma 2: Fix an arbitrary k and suppose the set of contracts $\{x_{k,l}, p_{k,l}\}_{l\in\mathcal{L}}$ delivers the same utility $u((k,l), x, \bar{x})$ for all $l \in \mathcal{L}$. Clearly, there is no incentive to misrepresent the influence-type.

That this is necessary is immediate from $\text{IC}_{k,l\to k,l'}$ and $\text{IC}_{k,l'\to k,l}$:

$$u(x_{k,l},k,\bar{x})-t_{k,l}\geq u(x_{k,l'},k,\bar{x})-t_{k,l'}\geq u(x_{k,l},k,\bar{x})-t_{k,l} \qquad \square$$

Proof of Lemma 2: Recall that the relevant set of constraints for this problem are given by $\bigcup_{k\in\mathcal{K}}(k\times\mathcal{L})$. Consider the first-best allocation. The participation constraints are satisfied and the $u_{k,l}$ is independent of l. Hence, by Lemma (2), the sorting constraints of this problem are satisfied. Clearly, this is the maximal profit the principal can achieve and hence the first-best allocation is the optimal menu of contracts. $\square$

Proof of Remark 1: Suppose an agent with type k,l deviates to k',l' with $k'>k$. Since we have full extraction ($\text{U}_k=0$), the utility under this deviation is

$$\text{U}_{k'}+u(x^{\text{FB}}_{k',l'},k,\bar{x})-u(x^{\text{FB}}_{k',l'},k',\bar{x})=-\int_k^{k'}u_k<0=\text{U}_k$$

Hence, it is not profitable. Similarly, there is also no incentive to misrepresent only influence. $\square$

Proofs for the Full Screening Problem

Proof of Proposition 3: Consider the constraints $\text{IC}_{k,l\to k',l'}$ and $\text{IC}_{k',l'\to k,l}$:

$$\begin{aligned} u(x_{k,l},\bar{x},k)-p_{k,l}&\geq u(x_{k',l'},\bar{x},k)-p_{k',l'}\\ u(x_{k',l'},\bar{x},k')-p_{k',l'}&\geq u(x_{k,l},\bar{x},k')-p_{k,l}\end{aligned}$$

Taking differences we arrive at

$$u(x_{k,l},\bar{x},k)-u(x_{k',l'},\bar{x},k)\geq u(x_{k,l},\bar{x},k')-u(x_{k',l'},\bar{x},k')$$

which implies $k'<k \iff x_{k',l'}<x_{k,l}$ since u has increasing differences in x,k. $\square$

Notation. Fix a menu of contracts $\{(x_{kl},p_{k,l})\}_{kl\in\mathcal{K}\times\mathcal{L}}$ and, for each k, pick

$$l_k\in\arg\min_{\widetilde{l}}x_{k,\widetilde{l}},\qquad l^k\in\arg\max_{\widetilde{l}}x_{k,\widetilde{l}}.$$

Proof of Proposition 4: Consider the sorting constraint from type k,l to type k',l', where $k>k'$. It is implied since

$$\begin{aligned} u(k,x_{k,l},\overline{x})-p_{k,l}=\text{U}_k&\geq u(k,x_{k-1,l^{k-1}},\overline{x})-t_{k-1,l^{k-1}}\\ &=\text{U}_{k-1}+u(k,x_{k-1,l^{k-1}},\overline{x})-u(k-1,x_{k-1,l^{k-1}},\overline{x})\\ &\geq\cdots\geq\text{U}_{k'}+\sum_{j=k'+1}^{k}\Big(u(j,x_{j-1,l^{j-1}},\overline{x})-u(j-1,x_{j-1,l^{j-1}},\overline{x})\Big)\\ &\geq\text{U}_{k'}+\sum_{j=k'+1}^{k}\left(u(j,x_{k',l'},\overline{x})-u(j-1,x_{k',l},\overline{x})\right)\\ &\geq u(k,x_{k',l'},\overline{x})-t_{k',l'}\end{aligned}$$

where the first inequality is the extremal downward sorting constraint (ES-B$_k$) and the equalities follow from condition H. We apply this argument iteratively and estimate the sum of differences using that $\boldsymbol{x}$ in k-monotonic and u has increasing differences. An analogous argument leveraging (ES-A$_k$) establishes the upward IC. Hence, all IC constraints are implied. The sufficiency of P for all participation constraints follows from the above argument, noting that the LHS of the penultimate line for $k' = k_0$ is nonnegative by P and $u_k \geq 0$. □

Lemma 4. *Consider an allocation satisfying the conditions of Proposition (4). If the downward ES-constraints (ES-B$_k$) are binding, the upward ES-constraints (ES-A$_k$) are inactive. Furthermore, in any second best contract, the downward ES-constraints (ES-B$_k$) and participation for k_0, P, are binding.*

Proof of Lemma: Consider k, l and k', l' with $k < k'$. Then

$$
\begin{aligned}
u(k, x_{k',l'}, \overline{x}) - p_{k',l'} &= \mathrm{U}_{k'} + u(k, x_{k',l'}, \overline{x}) - u(k', x_{k',l'}, \overline{x}) \\
&= \mathrm{U}_k + \sum_{j=k+1}^{k'} \Big(u(j, x_{j-1,l^{j-1}}, \overline{x}) - u(j-1, x_{j-1,l^{j-1}}, \overline{x}) \Big) - \big(u(k', x_{k',l'}, \overline{x}) - u(k, x_{k',l'}, \overline{x}) \big) \\
&\geq \mathrm{U}_k + \sum_{j=k+1}^{k'} \big(u(j, x_{k',l'}, \overline{x}) - u(j-1, x_{k',l'}, \overline{x}) \big) - \big(u(k', x_{k',l'}, \overline{x}) - u(k, x_{k',l'}, \overline{x}) \big) = \mathrm{U}_k
\end{aligned}
$$

where the second line follows by expressing $\mathrm{U}_{k'}$ via the binding downward IC and the inequality follows by (i) increasing differences and (ii) k-monotonicity as $x_{k',l'} \geq x_{j-1,l^{j-1}}$ for all $j \leq k'$. Hence, we have the downward IC.

Furthermore, suppose that a downward ES-constraint is strictly slack. We can increase the transfer from all affected types without implicating any other constraints, which increases the principal's objective. The same holds if $\mathrm{U}_{k_0} > 0$. □

Lemma 5. *If $\boldsymbol{x}, \boldsymbol{t}, \bar{x}, \zeta$ solves the Lagrangian associated to $\pi(\mathcal{K} \times \mathcal{L})^2$, then $\zeta > 0$.*

Proof of Lemma. By the previous lemma, the downward ES constraints and P are binding and hence we have that

$$
\begin{aligned}
\mathrm{U}_k &= u(\max_l x_{k-1,l}, k, \bar{x}) - t_{k-1, \arg\max x_{k-1,l}} \\
&= \mathrm{U}_{k-1} + u(\max_l x_{k-1,l}, k, \bar{x}) - u(\max_l x_{k-1,l}, k-1, \bar{x}) \\
&= \sum_{j=k_0}^{k-1} u(\max_l x_{j,l}, j+1, \bar{x}) - u(\max_l x_{j,l}, j, \bar{x})
\end{aligned}
$$

Then, we can rewrite the principal's objective as

$$
\sum f_{k,l} \left(u(x_{k,l}, k, \bar{x}) - \mathrm{U}_k \right) = \sum f_{k,l} \left(u(x_{k,l}, k, \bar{x}) - \sum_{j=k_0}^{k-1} u(\max_l x_{j,l}, j+1, \bar{x}) - u(\max_l x_{j,l}, j, \bar{x}) \right)
$$

Consider the Lagrangian with this objective, k-monotonicity, and the ζ constraint. Then, in a candidate optimum, we have

$$0 = \frac{\partial \mathcal{L}}{\partial \bar{x}} = \sum f_{k,l}\left(u_{\bar{x}}(x_{k,l},k,\bar{x}) - \sum_{j=k_0}^{k-1} u_{\bar{x}}(\max_l x_{j,l}, j+1, \bar{x}) - u_{\bar{x}}(\max_l x_{j,l}, j, \bar{x})\right) - \zeta + \kappa'(\bar{x})$$

$$\zeta = \sum f_{k,l}\left(u_{\bar{x}}(x_{k,l},k,\bar{x}) - \sum_{j=k_0}^{k-1} u_{\bar{x}}(\max_l x_{j,l}, j+1, \bar{x}) - u_{\bar{x}}(\max_l x_{j,l}, j, \bar{x})\right) + \kappa'(\bar{x})$$

$$= \sum f_{k,l}\left(\underbrace{u_{\bar{x}}(x_{k,l},k,\bar{x}) - u_{\bar{x}}(\max_l x_{k-1,l}, k, \bar{x})}_{\geq 0 \text{ k-mono and } u_{x\bar{x}}\geq 0} + \sum_{j=k_0}^{k-1} \underbrace{u_{\bar{x}}(\max_l x_{j,l}, j, \bar{x}) - u_{\bar{x}}(\max_l x_{j-1,l}, j, \bar{x})}_{\geq 0 \text{ k-mono and } u_{x\bar{x}}\geq 0}\right) + \kappa'(\bar{x})$$

$$> 0$$

where strictness follows from nontriviality of the allocation and the condition (NT) that either $u_{\bar{x}} > 0$, $u_{x\bar{x}} > 0$ for a positive measure of types, or $\kappa' > 0$. □

Proof of Theorem 1: Suppose $\boldsymbol{x} \notin \mathrm{M}$. Then, there exists a $k, l' > l$ such that $x_{k,l'} < x_{k,l}$. Consider

$$x^{\epsilon}_{k,l} = x_{k,l} - \epsilon, \quad t^{\epsilon}_{k,l} = t_{k,l} - \epsilon u_x(x_{k,l}, k, \bar{x})$$
$$x^{\epsilon}_{k,l'} = x_{k,l'} + \epsilon \frac{f_{k,l}}{f_{k,l'}}, \quad t^{\epsilon}_{k,l'} = t_{k,l} + \epsilon \frac{f_{k,l}}{f_{k,l'}} u_x(x_{k,l'}, k, \bar{x})$$

To the first order, this change keeps the utility of agents k,l and k,l' unchanged (for fixed $\bar{x}$). Furthermore, this does not tighten any constraints since the range of $x_{k,\cdot}$ contracts while utilities are held constant for type k. Furthermore, consider the expected transfers. we have

$$f_{k,l}t^{\epsilon}_{k,l} + f_{k,l'}t^{\epsilon}_{k,l'} = f_{k,l}t_{k,l} + f_{k,l'}t_{k,l'} + \epsilon f_{k,l}\left(u_x(x_{k,l'},k,\bar{x}) - u_x(x_{k,l},k,\bar{x})\right) > f_{k,l}t_{k,l} + f_{k,l'}t_{k,l'}$$

by concavity of u. For $\bar{x}$, we get

$$\bar{x}^{\epsilon} = \bar{x} + \epsilon\left(\frac{f_{k,l}}{f_{k,l'}} f_{k,l'} v_x(x_{k,l'}, l') - f_{k,l} v_x(x_{k,l}, l)\right)$$
$$= \bar{x} + \epsilon f_{k,l}\left(v_x(x_{k,l'}, l') - v_x(x_{k,l}, l)\right) > \bar{x}$$

and hence the principal, by judicious adjustment of transfers, can obtain an additional payoff $\zeta \epsilon f_{k,l}\left(v_x(x_{k,l'}, l') - v_x(x_{k,l}, l)\right)$, establishing that the original allocation was not optimal. □

Proof of Proposition 5: By Lemma 4 and Theorem 1we can write

$$\mathrm{U}_k = \sum_{j=k_0+1}^{k}\left(u(j, x_{j-1,\mathrm{L}}, \bar{x}) - u(j-1, x_{j-1,\mathrm{L}}, \bar{x})\right)$$

and hence the objective of the principal reads

$$\sum f_{k,l}\left(u(x_{k,l},k,\bar{x}) - \sum_{j=k_0+1}^{k}\left(u(j, x_{j-1,\mathrm{L}}, \bar{x}) - u(j-1, x_{j-1,\mathrm{L}}, \bar{x})\right)\right) + \kappa(\bar{x}) = \tag{48}$$

$$\sum f_{k,l}\left[u(x_{k,l},k,\bar{x})-\chi_{l=\mathrm{L}}\frac{1-\mathrm{F}_k}{f_{k,l}}\left(u(x_{k,\mathrm{L}},k+1,\bar{x})-u(x_{k,\mathrm{L}},k,\bar{x})\right)\right]+\kappa(\bar{x})$$

where $\mathrm{F}_k=\sum_{j>k}\sum_l f_{k,l}$. Note that this objective subsumes all participation and sorting constraints, subject to monotonicity and $\bar{x}$, which establishes the proposition. □

Lemma 6. *The relaxed problem UP has a unique solution if the value of the quadratic form defined by the matrix*

$$\begin{aligned}\hat{\mathrm{H}}^{\mathrm{SB}}=&\mathrm{diag}[(u_{xx}-\Delta_{xx}+(\mathbb{E}[u_{\bar{x}}]-\mathbb{E}[\Delta_{\bar{x}}])v_{xx})]\\&+(\mathbb{E}[u_{\bar{x}\bar{x}}]-\mathbb{E}[\Delta_{\bar{x}\bar{x}}])[v_x]\Phi[v_x]^{\mathrm{T}}+2\mathrm{Sym}\left([(u_{\bar{x}x}-\Delta_{\bar{x}x})]\Phi[v_x]^{\mathrm{T}}\right)\end{aligned}$$

along $\mathrm{span}\left(\left[\sqrt{f}\odot(u_{\bar{x}x}-\Delta_{\bar{x}x})\right],\left[\sqrt{f}\odot v_x\right]\right)$ *is strictly bounded above by* 0, *where* $\Delta_{xx}:=\chi_{l=\mathrm{L}}\frac{(1-\mathrm{F}_k)}{f_{l,k}}\int_k^{k+1}u_{kxx}\mathrm{d}s\in\mathbb{R}^{|\mathcal{K}\times\mathcal{L}|}$, $\Delta_{\bar{x}}:=\chi_{l=\mathrm{L}}\left(\frac{1-\mathrm{F}_k}{f_{k,l}}\right)\int_k^{k+1}u_{k\bar{x}}\mathrm{d}s$, $\Delta_{\bar{x}\bar{x}}:=\chi_{l=\mathrm{L}}\left(\frac{1-\mathrm{F}_k}{f_{k,l}}\right)\int_k^{k+1}u_{k\bar{x}\bar{x}}\mathrm{d}s$ *and* $\Delta_{\bar{x}x}:=\chi_{l=\mathrm{L}}\frac{(1-\mathrm{F}_k)}{f_{l,k}}\int_k^{k+1}u_{k\bar{x}x}\mathrm{d}s\in\mathbb{R}^{|\mathcal{K}\times\mathcal{L}|}$.

Proof of Lemma: Using the same approach as for the first best above, the Hessian of the principals objective in where we have substituted for $\bar{x}$ is given by$\mathrm{H}^{\mathrm{SB}}=\mathrm{S}\hat{\mathrm{H}}^{\mathrm{SB}}\mathrm{S}$. Note that we have $u_{kxx}\geq 0$ and $u_{k\bar{x}\bar{x}}\geq 0$. Therefore $\mathbb{E}[\Delta_{\bar{x}\bar{x}}]\geq 0$. Furthermore $\mathbb{E}[u_{\bar{x}}]-\mathbb{E}[\Delta_{\bar{x}}]>0$ is implied by our conditions and lex-monotonicity: We have by (48) and the proof of Lemma (5)

$$\begin{aligned}\mathbb{E}[u_{\bar{x}}]-\mathbb{E}[\Delta_{\bar{x}}]&=\sum f_{k,l}\left[u_{\bar{x}}(x_{k,l},k,\bar{x})-\chi_{l=\mathrm{L}}\frac{1-\mathrm{F}_k}{f_{k,l}}\left(u_{\bar{x}}(x_{k,\mathrm{L}},k+1,\bar{x})-u_{\bar{x}}(x_{k,\mathrm{L}},k,\bar{x})\right)\right]\\&=\sum f_{k,l}\left(u_{\bar{x}}(x_{k,l},k,\bar{x})-\sum_{j=k_0+1}^{k}\left(u_{\bar{x}}(j,x_{j-1,\mathrm{L}},\bar{x})-u_{\bar{x}}(j-1,x_{j-1,\mathrm{L}},\bar{x})\right)\right)=\zeta-\kappa'(\bar{x})\geq 0\end{aligned}$$

Hence, the first two matrices are negative semi-definite. As in the first-best, we can restrict attention to the subspace $\mathrm{span}\left(\left[\sqrt{f}\odot(u_{\bar{x}x}-\Delta_{\bar{x}x})\right],\left[\sqrt{f}\odot v_x\right]\right)$ as the terminal matrix annihilates all others. Let $y(\alpha,\beta)=\alpha\left[\sqrt{f}\odot(u_{\bar{x}x}-\Delta_{\bar{x}x})\right]+\beta\left[\sqrt{f}\odot v_x\right]$. We have concavity if the value of

$$\begin{aligned}&\max_{\alpha,\beta}y(\alpha,\beta)^{\mathrm{T}}\hat{\mathrm{H}}^{\mathrm{SB}}y(\alpha,\beta)\\&\text{s.t.}\alpha^2\mathbb{E}[(u_{\bar{x}x}-\Delta_{\bar{x}x})^2]+\beta^2\mathbb{E}[v_x^2]+2\alpha\beta\mathbb{E}[(u_{\bar{x}x}-\Delta_{\bar{x}x})v_x]=1\end{aligned}$$

is negative (for all x).

In the linear case, we can proceed similar to the first-best

$$\hat{\mathrm{H}}^{\mathrm{SB}}=-\mathrm{I}+2\mathrm{Sym}\left(\left[\sqrt{f}\odot\left(\gamma k-\chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)\right]\left[\sqrt{f}\odot\frac{l}{\mathbb{E}[l]}\right]^{\mathrm{T}}\right)$$

so that the quadratic form evaluates to

$$\begin{aligned}-1+2\mathbb{E}\left[\left(\gamma k-\chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)\frac{l}{\mathbb{E}[l]}\right]\left[\alpha^2\mathbb{E}\left[\left(\gamma k-\chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)^2\right]+\beta^2\mathbb{E}\left[\left(\frac{l}{\mathbb{E}[l]}\right)^2\right]\right]\\+2\alpha\beta\left\{\mathbb{E}\left[\left(\gamma k-\chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)^2\right]\mathbb{E}\left[\left(\frac{l}{\mathbb{E}[l]}\right)^2\right]+\mathbb{E}\left[\left(\gamma k-\chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)\frac{l}{\mathbb{E}[l]}\right]^2\right\}\end{aligned}$$

Using the constraint and simplifying, we arrive at

$$y(\alpha,\beta)^{\mathrm{T}}\widehat{\mathrm{H}}^{\mathrm{SB}}y(\alpha,\beta) = -1 + 2\mathbb{E}\left[\left(\gamma k - \chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)\frac{l}{\mathbb{E}[l]}\right] + $$

$$+ 2\alpha\beta\left\{\underbrace{\mathbb{E}\left[\left(\gamma k - \chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)^2\right]\mathbb{E}\left[\left(\frac{l}{\mathbb{E}[l]}\right)^2\right] - \mathbb{E}\left[\left(\gamma k - \chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)\frac{l}{\mathbb{E}[l]}\right]^2}_{\geq 0 \text{ by Cauchy-Schwarz}}\right\}$$

Maximizing, we get

$$\max_{\alpha,\beta:\|y\|=1} \alpha\beta = \frac{1}{2\sqrt{\mathbb{E}[\left(\gamma k - \chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)^2]\mathbb{E}\left[\left(\frac{l}{\mathbb{E}[l]}\right)^2\right]} + 2\mathbb{E}\left[\left(\gamma k - \chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)\frac{l}{\mathbb{E}[l]}\right]}$$

which implies the bound

$$\sqrt{\mathbb{E}\left[\left(\gamma k - \chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)^2\right]\mathbb{E}\left[\left(\frac{l}{\mathbb{E}[l]}\right)^2\right]} + \mathbb{E}\left[\left(\gamma k - \chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)\frac{l}{\mathbb{E}[l]}\right] < 1$$

Simplifying by using

$$\mathbb{E}\left(k\cdot\chi_{l=\mathrm{L}}\frac{(1-\mathrm{F}_k)}{f_{\mathrm{L},k}}\right) = \sum_k k\cdot(1-\mathrm{F}_k) = \frac{1}{2}\left[\mathbb{E}\left[k^2\right] - \mathbb{E}[k]\right]$$

and

$$\mathbb{E}\left[\left(\gamma k - \chi_{l=\mathrm{L}}\gamma\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)\frac{l}{\mathbb{E}[l]}\right] = \frac{\gamma}{\mathbb{E}[l]}\left[\mathbb{E}[kl] - \mathbb{E}\left[\left(\chi_{l=\mathrm{L}}\frac{(1-\mathrm{F}_k)}{f_{l,k}}\right)l\right]\right]$$

$$= \frac{\gamma}{\mathbb{E}[l]}\left[\mathbb{E}[kl] - \mathrm{L}\sum_k(1-\mathrm{F}_k)\right] = \frac{\gamma}{\mathbb{E}[l]}\left[\mathbb{E}[kl] - \mathrm{L}(\mathbb{E}[k] - k_0)\right]$$

we arrive at equation 28 in the text.

$$\gamma < \frac{\mathbb{E}[l]}{\sqrt{\left[\sum_k \frac{(1-\mathrm{F}_k)^2}{f_{k,\mathrm{L}}} + \mathbb{E}[k]\right]\mathbb{E}\left[(l)^2\right]} + \left[\mathbb{E}[kl] - \mathrm{L}(\mathbb{E}[k] - k_0)\right]}$$

□

Lemma 7. *The virtual value* J *is concave in* x *for all* $\bar{x}, \zeta > 0$.

Proof of Lemma: By direct computation, we have

$$\frac{\partial^2}{\partial x^2}\mathrm{J} = u_{xx} - \chi_{l=\mathrm{L}}\left\{\frac{1-\mathrm{F}_k}{f_{kl}}\left[u_{xx}(k+1,x,\bar{x}) - u_{xx}(k,x,\bar{x})\right]\right\} + \zeta v_{xx} < 0$$

since $u_{xx} < 0$, $v_{xx} \leq 0$ and $u_{xxk} \geq 0$. □

Proof. Since we have a finite type space, $\mathrm{H}(x,q)$ is piece-wise linear in q. Therefore, the convexification induces a partition of q which is a coarsening of the partition induced by the map $q(k,l)$. Therefore, we obtain an induced partition of types, which we denote by $\mathcal{B}$ in both spaces by abuse of notation. For $q \in \mathrm{B}$, we have

$$\mathrm{G}_q(x,q) = \int_{r\in \mathrm{B}} \mathrm{H}_q(x,r)\mathrm{d}r = \int_{r\in \mathrm{B}} \mathrm{J}_x(x,k(r),l(r),\bar{x},\zeta)\mathrm{d}r$$

Since $x_{k,l}$ is constant on B, it solves

$$\begin{aligned} 0 &= \frac{1}{|\mathrm{B}|}\sum_{k,l\in \mathrm{B}} \bar{\mathrm{J}}(x,k,l,\bar{x},\zeta) = \frac{1}{|\mathrm{B}|}\sum_{k,l\in \mathrm{B}} \mathrm{G}_q(x,q(k,l)) = \int_{r\in \mathrm{B}} \mathrm{J}_x(x,k(r),l(r),\bar{x},\zeta)\mathrm{d}r \\ &= \sum_{k,l\in \mathrm{B}} f_{k,l}\left[u_x(k,x_{k,l},\bar{x}) - \chi_{l=\mathrm{L}}\left\{\frac{1-\mathrm{F}_k}{f_{kl}}\left[u_x(k+1,x_{k,l},\bar{x}) - u_x(k,x_{k,\mathrm{L}},\bar{x})\right]\right\} + \zeta v_x(x_{k,l},l)\right] \end{aligned}$$

Finally, we divide by $\sum_{k,l\in \mathrm{B}} f_{k,l}$ and rearrange

$$\begin{aligned} &\frac{1}{\sum_{k,l\in \mathrm{B}} f_{k,l}}\sum_{k,l\in \mathrm{B}} f_{k,l}\left\{u_x(k,x_{k,l},\bar{x}) - \chi_{l=\mathrm{L}}\left\{\frac{1-\mathrm{F}_k}{f_{kl}}\left[u_x(k+1,x_{k,l},\bar{x}) - u_x(k,x_{k,\mathrm{L}},\bar{x})\right]\right\}\right\} = \\ &u_x(k_\mathrm{B},x_\mathrm{B},\bar{x}) + \frac{1}{\sum_{k,l\in \mathrm{B}} f_{k,l}}\left\{\begin{array}{l} \sum_{k,l\in \mathrm{B}} f_{k,l}\left[u_x(k,x_{k,l},\bar{x}) - u_x(k_\mathrm{B},x_\mathrm{B},\bar{x})\right] \\ -\sum_{k,\mathrm{L}\in \mathrm{B}}(1-\mathrm{F}_k)\left[u_x(k+1,x_{k,l},\bar{x}) - u_x(k,x_{k,\mathrm{L}},\bar{x})\right] \end{array}\right\} = \\ &u_x(k_\mathrm{B},x_\mathrm{B},\bar{x}) + \frac{1}{\sum_{k,l\in \mathrm{B}} f_{k,l}}\left\{\begin{array}{l} \sum_{k,l\in \mathrm{B}} f_{k,l}\left[u_x(k,x_{k,l},\bar{x}) - u_x(k_\mathrm{B},x_\mathrm{B},\bar{x})\right] \\ -\left(1-\mathrm{F}_{k_\mathrm{B}}\right)\left[u_x(k_\mathrm{B}+1,x_{k,l},\bar{x}) - u_x(k_\mathrm{B},x_{k,\mathrm{L}},\bar{x})\right] \\ -\sum_{k,\mathrm{L}\in \mathrm{B}\setminus\{k_\mathrm{B},\mathrm{L}\}}(1-\mathrm{F}_k)\left[u_x(k+1,x_{k,l},\bar{x}) - u_x(k,x_{k,\mathrm{L}},\bar{x})\right] \end{array}\right\} = \\ &u_x(k_\mathrm{B},x_\mathrm{B},\bar{x}) + \frac{1}{\sum_{k,l\in \mathrm{B}} f_{k,l}}\left\{\begin{array}{l} \sum_{k,l\in \mathrm{B}:k,l>_\mathrm{L} k_\mathrm{B},\mathrm{L}} f_{k,l}\left[u_x(k,x_{k,l},\bar{x}) - u_x(k_\mathrm{B},x_\mathrm{B},\bar{x})\right] \\ -\sum_{k,l>_\mathrm{L} k_\mathrm{B},\mathrm{L}} f_{k,l}\left[u_x(k_\mathrm{B}+1,x_{k,l},\bar{x}) - u_x(k_\mathrm{B},x_{k,\mathrm{L}},\bar{x})\right] \\ -\sum_{k,\mathrm{L}\in \mathrm{B}\setminus\{k_\mathrm{B},\mathrm{L}\}}(1-\mathrm{F}_k)\left[u_x(k+1,x_{k,l},\bar{x}) - u_x(k,x_{k,\mathrm{L}},\bar{x})\right] \end{array}\right\} = \\ &u_x(k_\mathrm{B},x_\mathrm{B},\bar{x}) + \frac{1}{\sum_{k,l\in \mathrm{B}} f_{k,l}}\left\{\begin{array}{l} \sum_{k,l\in \mathrm{B}:k,l>_\mathrm{L} k_\mathrm{B}+1,\mathrm{L}} f_{k,l}\left[u_x(k,x_{k,l},\bar{x}) - u_x(k_\mathrm{B}+1,x_\mathrm{B},\bar{x})\right] \\ -\sum_{k,\mathrm{L}\in \mathrm{B}\setminus\{k_\mathrm{B},\mathrm{L}\}}(1-\mathrm{F}_k)\left[u_x(k+1,x_{k,l},\bar{x}) - u_x(k,x_{k,\mathrm{L}},\bar{x})\right] \end{array}\right\} \underset{\text{(induction)}}{=} \\ &\qquad u_x(k_\mathrm{B},x_\mathrm{B},\bar{x}) - \frac{\sum_{k,l>_\mathrm{L}\mathrm{B}} f_{k,l}}{\sum_{k,l\in \mathrm{B}} f_{k,l}}\left[u_x\left(k^\mathrm{B}+1,x_\mathrm{B},\bar{x}\right) - u_x\left(k^\mathrm{B},x_\mathrm{B},\bar{x}\right)\right] \end{aligned}$$

which establishes the first order condition in the text. By Toikka (2011), the solution to the FOC solves the monotonicity constrained problem. □

Proof of Proposition 6: Part 1: By single crossing, $\mathrm{J}_x(x,\mathrm{K},\mathrm{L},\bar{x})$ dominates all other types and hence there is no bunching and no distortion (in the weak sense) at the top.

Part 2: Consider a nontrivial cell that does not contain a type k,L. Then, this cell is contained in one $k\times\mathcal{L}$ slice. Within such a slice, except at the switching type, however,

$$\mathrm{J}_x = u_x(k,x_{k,l},\bar{x}) + \zeta v_x(x_{k,l},l)$$

is increasing in l and decreasing in x, so no ironing is required, a contradiction.

Part 3: Notice that $u_x(x,k,\bar{x}) < 0 \iff x > x_k^\mathrm{D}(\bar{x})$. Suppose towards a contradiction that $x_k^\mathrm{D}(\bar{x}) \geq x_{k,l} > x_{k,l-1}$. Then, we know that the lex-monotonicity constraint is slack at $x_{k,l-1}$ and hence

$$\frac{\partial\mathcal{L}}{\partial x_{k,l-1}} = f_{k,l-1}\mathrm{J}_x(x_{k,l-1},k,l-1,\bar{x},\zeta) > 0$$

since $\check{x}_{k,l-1} > x_k^\mathrm{D}(\bar{x})$. This contradicts the optimality of $x_{k,l-1}$ given $\bar{x},\zeta$. □

Proof of Theorem 2: Let $\bar{x}^{\mathrm{T}}, \zeta^{\mathrm{T}}$ be a fixed point of Γ and denote $x^{\mathrm{T}} = x^{\mathrm{FS}}(\bar{x}^{\mathrm{T}}, \zeta^{\mathrm{T}})$. Then, since x^{T} solves $\max_{x \in \mathrm{M}} \mathcal{L}(x, \bar{x}^{\mathrm{T}}, \zeta^{\mathrm{T}})$ and this problem is concave-convex (since the objective is concave (by Lemma 7) and M is convex), we have $\nabla_x \mathcal{L}(x^{\mathrm{T}}, \bar{x}^{\mathrm{T}}, \zeta^{\mathrm{T}}) \in \mathrm{N_M}(x^{\mathrm{T}})$ where $\mathrm{N_M}(x)$ denotes the outward normal cone to M at x.[24]

Consider now the plugin problem and denote its objective by $\hat{\mathcal{L}}$. By assumption, this is a concave-convex problem and therefore a vector x solves this problem if and only if $\nabla_x \hat{\mathcal{L}}(x) \in \mathrm{N_M}(x)$. We have

$$\begin{aligned}
\nabla_x \hat{\mathcal{L}}(x^{\mathrm{T}}) &= f \odot \Big(\nabla_x \left(u(x^{\mathrm{T}}, \bar{x}^{\mathrm{T}}) + \Delta(x^{\mathrm{T}}, \bar{x}^{\mathrm{T}}) \right) + \mathbb{E}[u_{\bar{x}}(x^{\mathrm{T}}, \bar{x}^{\mathrm{T}}) + \Delta_x(x^{\mathrm{T}}, \bar{x}^{\mathrm{T}}) + \kappa'(\bar{x})] \nabla_x v(x^{\mathrm{T}}, \bar{x}^{\mathrm{T}}) \Big) \\
&= f \odot \Big(\nabla_x \left(u(x^{\mathrm{T}}, \bar{x}^{\mathrm{T}}) + \Delta(x^{\mathrm{T}}, \bar{x}^{\mathrm{T}}) \right) + \zeta^{\mathrm{T}} \nabla_x v(x^{\mathrm{T}}, \bar{x}^{\mathrm{T}}) \Big) \\
&= \nabla_x \mathcal{L}(x^{\mathrm{T}}, \bar{x}^{\mathrm{T}}, \zeta^{\mathrm{T}}) \in \mathrm{N_M}(x^{\mathrm{T}})
\end{aligned}$$

Hence, the solution to the fixed point problem induces a solution to the plugin problem. By uniqueness of this solution, it is unique.

Conversely, let x^{FS} be the solution to $\nabla_x \hat{\mathcal{L}}(x^{\mathrm{FS}}) \in \mathrm{N_M}(x^{\mathrm{FS}})$ and $\bar{x}^{\mathrm{FS}} = \sum f_{k,l} v(x^{\mathrm{FS}}_{k,l}, l)$ and

$$\zeta^{\mathrm{FS}} = \sum f_{k,l} \left(u_{\bar{x}}(x^{\mathrm{FS}}_{k,l}, k, \bar{x}^{\mathrm{FS}}) - \chi_{l=\mathrm{L}} \frac{1 - \mathrm{F}_k}{f_{kl}} \int_k^{k+1} u_{k\bar{x}} \left(j, x^{\mathrm{FS}}_{k,l} \bar{x}^{\mathrm{FS}} \right) \mathrm{d}j \right) + \kappa'(\bar{x}^{\mathrm{FS}})$$

Then, following the above chain of equalities backwards it is easy to see that $\nabla_x \mathcal{L}(x^{\mathrm{FS}}, \bar{x}^{\mathrm{FS}}, \zeta^{\mathrm{FS}}) \in \mathrm{N_M}(x^{\mathrm{FS}})$. Hence, $x^{\mathrm{FS}}(\bar{x}^{\mathrm{FS}}, \zeta^{\mathrm{FS}}) = x^{\mathrm{FS}}$ and $\Gamma(\bar{x}^{\mathrm{FS}}, \zeta^{\mathrm{FS}}) = (\bar{x}^{\mathrm{FS}}, \zeta^{\mathrm{FS}})$ by construction. □

Proofs for Observable Influence

Proof of Proposition 7: We can rewrite the problem in utility space, noting that $u_{kl} = (1 + \gamma \bar{x} k) x_{k,l} - \frac{1}{2} x^2_{k,l} - p_{k,l}$ or equivalently $p_{kl} = (1 + \gamma \bar{x} k) x_{k,l} - \frac{1}{2} x^2_{k,l} - u_{kl}$. Then, P is equivalent to $u_{k_0 l} = 0$ where equality follows from by the usual argument. IC is equivalent to $u_{k,l} \geq u_{k'l} + \gamma \bar{x}(k - k') x_{k',l}$. Again, by the usual arguments, local downward IC and monotonicity are sufficient and IC are binding, hence $u_{k,l} = \gamma \bar{x} \sum_{i=k_0}^{k-1} x_{i,l}$. Plugging this into the objective and applying summation by parts to the double sum, we arrive at the Proposition. □

Proof of Proposition 8: Note that $\zeta = \gamma \sum f_{k,l} \left(k - \frac{\mathrm{F}_l(\mathrm{K}) - \mathrm{F}_l(k)}{f_{kl}} \right) x_{k,l}(\zeta, \bar{x})$. Solving further yields

$$\begin{aligned}
\bar{x} &= \sum f_{k,l} \frac{l}{\mathbb{E}[l]} \left(1 + \gamma \bar{x} \left(k - \frac{\mathrm{F}_l(\mathrm{K}) - \mathrm{F}_l(k)}{f_{kl}} \right) + \frac{l}{\mathbb{E}[l]} \zeta \right) \\
&= 1 + \gamma \bar{x} \frac{\mathbb{E}[kl]}{\mathbb{E}[l]} - \gamma \bar{x} \frac{\mathbb{E}[kl - k_0 l]}{\mathbb{E}[l]} + \zeta \frac{\mathbb{E}[l^2]}{\mathbb{E}[l]^2} \\
&= 1 + \gamma \bar{x} k_0 + \zeta \frac{\mathbb{E}[l^2]}{\mathbb{E}[l]^2}
\end{aligned}$$

where we use that $\sum_{k,l} l \left(\mathrm{F}_l(\mathrm{K}) - \mathrm{F}_l(k) \right) = \sum_l l \mathrm{F}_l(\mathrm{K}) \sum_k (1 - \mathrm{F}(k|l)) = \sum_l l \mathrm{F}_l(\mathrm{K}) \mathbb{E}[k - k_0 | l] = \mathbb{E}[kl - k_0 l]$

$$\begin{aligned}
\zeta &= \gamma \sum f_{k,l} \left(k - \frac{\mathrm{F}_l(\mathrm{K}) - \mathrm{F}_l(k)}{f_{kl}} \right) \left(1 + \gamma \bar{x} \left(k - \frac{\mathrm{F}_l(\mathrm{K}) - \mathrm{F}_l(k)}{f_{kl}} \right) + \frac{l}{\mathbb{E}[l]} \zeta \right) \\
&= \gamma \left(k_0 + \gamma k_0 \zeta + \gamma \bar{x} \left(\mathbb{E}[k] - k_0^2 - 2k_0 + \mathbb{E}[\left(\frac{\mathrm{F}_l(\mathrm{K}) - \mathrm{F}_l(k)}{f_{kl}} \right)^2] \right) \right)
\end{aligned}$$

[24] Formally, $z \in \mathrm{N_M}(x)$ if $< z, m - x > \leq 0$ for all $m \in \mathrm{M}$.

where we used[25]

$$\sum f_{k,l}\left(k-\frac{F_l(K)-F_l(k)}{f_{kl}}\right)^2 = \mathbb{E}[k^2]-2\sum k\left(F_l(K)-F_l(k)\right)+\mathbb{E}[\left(\frac{F_l(K)-F_l(k)}{f_{kl}}\right)^2]$$
$$=\mathbb{E}[k^2]-\left(\mathbb{E}\left[k^2\right]-\mathbb{E}\left[k\right]+k_0^2+2k_0\right)+\mathbb{E}[\left(\frac{F_l(K)-F_l(k)}{f_{kl}}\right)^2]$$
$$=\mathbb{E}\left[k\right]-k_0^2-2k_0+\mathbb{E}[\left(\frac{F_l(K)-F_l(k)}{f_{kl}}\right)^2]$$

Now, we can solve

$$\bar{x}=1+\gamma\bar{x}k_0+\zeta\frac{\mathbb{E}[l^2]}{\mathbb{E}[l]^2}$$
$$\zeta=\gamma\left(k_0+\gamma k_0\zeta+\gamma\bar{x}\left(\mathbb{E}\left[k\right]-k_0^2-2k_0+\mathbb{E}[\left(\frac{F_l(K)-F_l(k)}{f_{kl}}\right)^2]\right)\right)$$
$$=\frac{\gamma}{1-\gamma k_0}\left(k_0+\gamma\bar{x}\left(\mathbb{E}\left[k\right]-k_0^2-2k_0+\mathbb{E}[\left(\frac{F_l(K)-F_l(k)}{f_{kl}}\right)^2]\right)\right)$$

Solving this equation, we get

$$\frac{\zeta}{\gamma\bar{x}}=\Xi=\frac{k_0+\gamma\left(\mathbb{E}[k]-2k_0\left(1+k_0\right)+\mathbb{E}[\left(\frac{F_l(K)-F_l(k)}{f_{kl}}\right)^2]\right)}{1-\left(1-\frac{\mathbb{E}[l^2]}{\mathbb{E}[l]^2}\right)\gamma k_0}$$

then $x_{\cdot,l}$ is increasing in l if

$$\frac{l}{\mathbb{E}[l]}\zeta-\gamma\bar{x}\frac{F_K^l-F_k^l}{f_{kl}}\propto\frac{l}{\mathbb{E}[l]}\Xi-\frac{F_l(K)-F_l(k)}{f_{kl}}=\frac{l}{\mathbb{E}[l]}\Xi-\frac{1-F(k|l)}{f(k|l)}$$

is increasing in l. Furthermore, rents are simply $\gamma\bar{x}\sum_{j=0}^{k-1}x_{j,l}$, so the fact that $x_{\cdot,l}$ is increasing

[25]Where we used

$$\sum_k (k-k_0)\cdot(1-F_k)=\frac{1}{2}\left[\mathbb{E}\left[(k-k_0)^2\right]-\mathbb{E}\left[k-k_0\right]\right]$$
$$\sum_k k\cdot(1-F_k)=\sum_k (k-k_0)\cdot(1-F_k)+k_0\mathbb{E}[k-k_0]$$
$$=\frac{1}{2}\left[\mathbb{E}\left[(k-k_0)^2\right]-\mathbb{E}\left[k-k_0\right]\right]+k_0\mathbb{E}[k-k_0]$$
$$=\frac{1}{2}\left[\mathbb{E}\left[k^2\right]-\mathbb{E}\left[k\right]-2k_0\mathbb{E}\left[k\right]+k_0^2+k_0\right]+k_0\mathbb{E}[k-k_0]$$
$$=\frac{1}{2}\left[\mathbb{E}\left[k^2\right]-\mathbb{E}\left[k\right]\right]+\frac{1}{2}k_0^2+k_0$$

and

$$\sum k\left(F_l(K)-F_l(k)\right)=\sum_l F_l(K)\sum_k k(1-F(k|l))$$
$$=\sum_l F_l(K)\left[\frac{1}{2}\left[\mathbb{E}\left[k^2|l\right]-\mathbb{E}\left[k|l\right]\right]+\frac{1}{2}k_0^2+k_0\right]$$
$$=\frac{1}{2}\left[\mathbb{E}\left[k^2\right]-\mathbb{E}\left[k\right]\right]+\frac{1}{2}k_0^2+k_0$$

in l is sufficient for this to hold. A necessary condition has all cumulative sums increasing. Whenever $l_1 > l_2$ for every k,

$$\sum_{j=0}^{k}\left(\frac{l_1}{\mathbb{E}[l]}\Xi - \frac{1-\mathrm{F}(k_j|l_1)}{f(k_j|l_1)}\right) > \sum_{j=0}^{k}\left(\frac{l_2}{\mathbb{E}[l]}\Xi - \frac{1-\mathrm{F}(k_j|l_2)}{f(k_j|l_2)}\right) =$$

$$k\frac{l_1-l_2}{\mathbb{E}[l]}\Xi + \sum_{j=0}^{k}\frac{1-\mathrm{F}(k_j|l_2)}{f(k_j|l_2)} - \frac{1-\mathrm{F}(k_j|l_1)}{f(k_j|l_1)}$$

In particular, if $k_0 = 0$, we have $\Xi = \gamma\left(\mathbb{E}[k] + \mathbb{E}[\left(\frac{\mathrm{F}_l(\mathrm{K})-\mathrm{F}_l(k)}{f_{kl}}\right)^2]\right)$.

For expected rents from influence, we require that the following is increasing in l

$$\sum_k f(k|l)\left(\frac{l}{\mathbb{E}[l]}\Xi - \frac{1-\mathrm{F}(k|l)}{f(k|l)}\right) = \frac{l}{\mathbb{E}[l]}\Xi - \mathbb{E}[k-k_0|l]$$

□

B Examples

B.1 Network Good: Decentralized vs Efficient

For the decentralized case, note that $x_k = 1 + \gamma\bar{x}^{\mathrm{D}}k$ and hence

$$\bar{x}^{\mathrm{D}} = \frac{1}{1-\gamma\frac{\mathbb{E}[kl]}{\mathbb{E}[l]}}.$$

For the first best, plugging $x^\star_{k,l}$ into the definition of $\bar{x}^\star$ and $\zeta^\star$, we arrive at

$$\zeta = \sum_{k,l} f_{kl}\gamma k\left(1+\gamma\bar{x}k + \frac{l}{\mathbb{E}[l]}\zeta\right) = \gamma\mathbb{E}[k] + \gamma^2\mathbb{E}[k^2]\bar{x} + \zeta\gamma\frac{\mathbb{E}[kl]}{\mathbb{E}[l]}$$
$$\bar{x} = \sum_{k,l} f_{kl}\frac{l}{\mathbb{E}[l]}\left(1+\gamma\bar{x}k + \frac{l}{\mathbb{E}[l]}\zeta\right) = 1 + \gamma\bar{x}\frac{\mathbb{E}[kl]}{\mathbb{E}[l]} + \zeta\frac{\mathbb{E}[l^2]}{\mathbb{E}[l]^2}$$

Solving this 2x2 linear system gives

$$\bar{x}^\star = \frac{1 + \frac{\mathbb{E}[l^2]}{\mathbb{E}[l]^2\left[1-\gamma\frac{\mathbb{E}[kl]}{\mathbb{E}[l]}\right]}\gamma\mathbb{E}[k]}{1-\gamma\frac{\mathbb{E}[kl]}{\mathbb{E}[l]} - \frac{\gamma^2\mathbb{E}[l^2]\mathbb{E}[k^2]}{\mathbb{E}[l]^2\left[1-\gamma\frac{\mathbb{E}[kl]}{\mathbb{E}[l]}\right]}}.$$

B.2 Network Good: 2x2

Consider the setting of Section (2.2.1) with $\mathcal{K} = \mathcal{L} = \{0,1\}$.

B.2.1 Benchmark Allocations

The decentralized solution has

$$x_0^{\mathrm{D}} = 1, \qquad x_1^{\mathrm{D}} = 1 + \gamma\overline{x}$$

with

$$\overline{x} = \frac{f_{0,1} + f_{1,1}\,(1+\gamma\overline{x})}{f_{0,1} + f_{1,1}} \implies \overline{x} = \frac{f_{0,1} + f_{1,1}}{(f_{0,1} + f_{1,1}\,(1-\gamma))}$$

A decentralized equilibrium exists for every $\gamma < 1 + \frac{f_{0,1}}{f_{1,1}}$.

Computing the efficient allocation is straighforward but yields unwieldy expressions. However, it holds

$$x^{\star}_{0,1} - x^{\star}_{1,0} = \frac{\gamma\,(f_{1,0} - f_{0,1})}{1 - f_{0,0} - (1-\gamma)\,f_{1,0}}$$

Therefore, the first best is implementable if and only if $f_{0,1} \geq f_{1,0}$.

B.2.2 Full Screening

We fully solve for the optimal screening contract in a Mathematica notebook, available upon request. There, we go through all possible bunching scenarios. Only the following scenarios 1, 2, and 5 are possibly optimal.[26]

$$0 < x_{0,0} < x_{0,1} < x_{1,0} < x_{1,1} \tag{SC1}$$

$$0 < x_{0,0} = x_{0,1} < x_{1,0} < x_{1,1} \tag{SC2}$$

$$0 = x_{0,0} = x_{0,1} < x_{1,0} < x_{1,1} \tag{SC5}$$

It can be shown that the candidate solution is $\succ_{\mathrm{L}^+}$-monotonic if and only if $x_2 > x_1 \iff \gamma > 1$. This condition is inconsistent with existence (of the first best) for, say, a uniform distribution, but distributions where it is consistent and we have no bunching can be constructed.[27]

B.2.3 Observable Influence

We have

$$x^{\mathrm{OI}}_{0,0} = 1 - \frac{f_{1,0}}{f_{0,0}}\gamma\bar{x}, \qquad x^{\mathrm{OI}}_{0,1} = 1 - \frac{f_{1,1}}{f_{0,1}}\gamma\bar{x} + \zeta\frac{1}{f_{0,1} + f_{1,1}}$$

$$x^{\mathrm{OI}}_{1,0} = 1 + \gamma\bar{x}, \qquad x^{\mathrm{OI}}_{1,1} = 1 + \gamma\bar{x} + \zeta\frac{1}{f_{0,1} + f_{1,1}}$$

[26]The fact that always $x_{0,1} < x_{1,0} < x_{1,1}$ – in other words that only the information rent distortion induces bunching – is aconsequence of this particular example in which $\zeta \cdot \mathrm{L}$ is proportional to $\gamma\bar{x}$. If we had a generic network sale we could make the $\bar{l} \to \underline{l}$ jump large relative to the expectation and make this relevant.

Because of this bunching structure, the value of x_1 tells us everything: If = 1 we are in strictly monotonic allocation, else the $k = 0$ slice is bunched (sometimes with the 0 bound binding).

[27]Clearly, strict concavity of the first-best problem implies existence of the second best solution. The second-best can exist even if the FB does not but we have

$$\gamma < \frac{f_{0,1}}{f_{0,1}f_{1,0} - \sqrt{(1-f_{0,0})f_{0,1}(1-f_{0,0}-f_{1,0})(1-f_{0,0}-f_{0,1})}}$$

In particular, this is the case when $x^{\mathrm{FB}}_{0,1}$ diverges: The resulting divergence of information rents can reign in the second best value. We use this region to demonstrate that consumer surplus in the screening solution can exceed the decentralized surplus.

The aggregate variables are

$$\begin{aligned}\zeta^{\mathrm{OI}} &= \gamma\sum f_{k,l}\left(k - \frac{F_l(K) - F_l(k)}{f_{kl}}\right)x^{\mathrm{OI}}_{k,l}\\ &= \gamma\bar{x}^{\mathrm{OI}}\left[\gamma\left[f_{1,0} + f_{1,1} + f_{0,0}\left(\frac{f_{1,0}}{f_{0,0}}\right)^2 + f_{0,1}\left(\frac{f_{1,1}}{f_{0,1}}\right)^2\right]\right] = \gamma\bar{x}^{\mathrm{OI}}\Xi\\ \bar{x}^{\mathrm{OI}} &= 1 + \zeta^{\mathrm{OI}}\frac{1}{f_{0,1} + f_{1,1}}\end{aligned}$$

which we solve for

$$\bar{x}^{\mathrm{OI}} = \frac{1}{1 - \gamma\Xi\frac{1}{f_{0,1}+f_{1,1}}}, \qquad \zeta^{\mathrm{OI}} = \frac{\gamma\Xi}{1 - \gamma\Xi\frac{1}{f_{0,1}+f_{1,1}}} > 0.$$

By direct substitution of $f_{i,j} = \frac{1}{4} + (-1)^{\chi_{i\neq j}}\rho$, we see that there are rents from influence if $x^{\mathrm{OI}}_{0,0} < x^{\mathrm{OI}}_{0,1}$ which is the case if $\gamma - 8\rho + 16\gamma\rho^2 < 0$ or, equivalently for $\rho \in [-.25, .25]$ if $\rho < \bar{\rho} = \frac{1-\sqrt{1-\gamma^2}}{4\gamma}$.

B.3 Pollution 2×2

B.3.1 Benchmark Allocations

Since the aggregate externality does not affect firms' profits, we have $x^{\mathrm{D}}_0 = 1, x^{\mathrm{D}}_1 = 2$. The equations characterizing the first best are given by

$$0 = 1 - \frac{x_{k,l}}{k+1} - x_{k,l}(1-l)\zeta^{\star}(\bar{x})$$

where

$$0 = \sum f_{k,l}u_{\bar{x}}\left(k, x^{\star}_{k,l}, \overline{x}\right) + \kappa'(\bar{x}) - \zeta^{\star}(\bar{x}).$$

implies $\kappa = \zeta^{\star}(\bar{x})$ and by definition $\bar{x}^{\star} = -\frac{1}{2}\left[f_{0,0}x^2_{0,0} + f_{1,0}x^2_{1,0}\right]$. Plugging those conditions in the FOC delivers $\boldsymbol{x}^{\star} = \left[\frac{1}{1+\kappa}, 1, \frac{1}{\frac{1}{2}+\kappa}, 2\right]$.

B.3.2 Full Screening

The full screening problem reads

$$\max_{\boldsymbol{x}\in M}\sum f_{k,l}\left[u(k, x_{k,l}, \overline{x}) - \chi_{l=L}\left\{\frac{1-F_k}{f_{kl}}\int_k^{k+1} u_k(\kappa, x_{k,l}, \overline{x})\,d\kappa\right\} - \kappa\frac{(1-l)}{2}x^2_{k,l}\right]$$

$$\begin{aligned}\max_{\boldsymbol{x}\in M} f_{0,0}\left[x_{0,0} - \frac{1}{2}[1+\kappa]x^2_{0,0}\right] + f_{0,1}\left[x_{0,1} - \frac{1}{2}x^2_{0,1} - \left\{\frac{1 - f_{0,0} - f_{0,1}}{f_{0,1}}\left[\frac{1}{4}x^2_{0,1}\right]\right\}\right] +\\ + f_{1,0}\left[x_{1,0} - \frac{1}{2}\left[\frac{1}{2}+\kappa\right]x^2_{1,0}\right] + f_{1,1}\left[x_{1,1} - \frac{1}{4}x^2_{1,1}\right]\end{aligned}$$

First order conditions give the vector of candidate solutions $\check{\boldsymbol{x}} = \left[\frac{1}{1+\kappa}, \frac{2f_{0,1}}{1-f_{0,0}+f_{0,1}}, \frac{1}{\frac{1}{2}+\kappa}, 2\right]$. $\check{\boldsymbol{x}} \in \mathrm{M}$ if and only if

$$\frac{1}{1+\kappa} < \frac{2f_{0,1}}{1-f_{0,0}+f_{0,1}} < \frac{1}{\frac{1}{2}+\kappa}$$

which can be rearranged to deliver the conditions on κ given in the text.

Now suppose $\check{x}_{0,1} = 1 < \frac{1}{1+\kappa} = \check{x}_{0,0}$ so low productivity sectors have to be bunched at level $x_{0,0} = x_{0,1} = x_0$. We know that $x_{1,\cdot} = \check{x}_{1,\cdot}$ from Lemma 3. Dropping constant terms, the objective reads

$$f_{0,0}\left[x_0 - \frac{1}{2}[1+\kappa]x_0^2\right] + f_{0,1}\left[x_0 - \frac{1}{2}x_0^2 - \left\{\frac{1-f_{0,0}-f_{0,1}}{4f_{0,1}}x_0^2\right\}\right]$$

that delivers

$$x_0 = \frac{f_{0,0}+f_{0,1}}{\left[\kappa f_{0,0} + \frac{1}{2}(1+f_{0,0}+f_{0,1})\right]}, \qquad x_{1,\cdot} = \check{x}_{1,\cdot}$$

Finally suppose $\check{x}_{0,1} = 1 > \frac{1}{\frac{1}{2}+\kappa} = \check{x}_{1,0}$ so low productivity green sector has to be bunched with the high productivity dirty sector at level $x_{0,1} = x_{1,0} = x_{\mathrm{B}}$ that maximizes

$$f_{0,1}\left[x_{\mathrm{B}} - \frac{1}{2}x_{\mathrm{B}}^2 - \left\{\frac{1-f_{0,0}-f_{0,1}}{4f_{0,1}}x_{\mathrm{B}}^2\right\}\right] + f_{1,0}\left[x_{\mathrm{B}} - \frac{1}{2}\left[\frac{1}{2}+\kappa\right]x_{\mathrm{B}}^2\right]$$

that delivers

$$x_{\mathrm{B}} = \frac{f_{0,1}+f_{1,0}}{\frac{1}{2}(1-f_{0,0}+f_{0,1}+f_{1,0}) + f_{1,0}\kappa}$$

B.3.3 Observable Influence

Notice that in this case we are effectively solving two independent screening problems since the coupling through $\overline{x}$ is missing as the latter does not enter utility. For $l = 1$ (green sector) we have a standard screening contract with 2 types that we omit, while for $l = 0$ we have a screening with externality with objective

$$\max_{x_{0,0},x_{0,1}} f_{0,0}\left[x_{0,0} - \frac{1}{2}x_{0,0}^2\right] + f_{1,0}\left[x_{1,0} - \frac{1}{4}x_{1,0}^2 - \frac{1}{4}x_{0,0}^2\right] - \kappa\frac{1}{2}\left[f_{0,0}(x_{0,0})^2 + f_{1,0}(x_{1,0})^2\right]$$

that gives

$$f_{1,0}\left[1 - \left(\frac{1}{2}+\kappa\right)x_{1,0}\right] = 0 \implies x_{1,0} = \frac{1}{\left(\frac{1}{2}+\kappa\right)} = x_{1,0}^\star$$

$$\frac{2f_{0,0}}{f_{1,0}+2f_{0,0}(1+\kappa)} = x_{0,0}$$

and

$$\overline{x} = -\frac{1}{2}\left[f_{0,0}\left(\frac{2f_{0,0}}{f_{1,0}+2f_{0,0}(1+\kappa)}\right)^2 + f_{1,0}\left(\frac{1}{\left(\frac{1}{2}+\kappa\right)}\right)^2\right] > \overline{x}^\star$$

there is suboptimally low pollution since the 0,1 agent produces its efficient level while the 0,0 agent is downward distorted.[28]

[28] Here we have solved assuming that the seller cares about pollution. If she doesn't then she will select (as if $\kappa = 0$ in the previous solution) $x_{1,0} = 2$ and $x_{0,0} = \frac{2f_{0,0}}{f_{1,0}+2f_{0,0}}$, inducing a level of the externality that can either exceed (for small κ) or fall short of $\overline{x}^\star$.