CSEF

Centre for Studies in Economics and Finance

WORKING PAPER NO. 617

Core and Stable Sets of Exchange Economies with Externalities

Maria Gabriella Graziano, Claudia Meo and Nicholas C. Yannelis

June 2021

University of Naples Federico II | University of Salerno | Bocconi University, Milan

CSEF - Centre for Studies in Economics and Finance
DEPARTMENT OF ECONOMICS – UNIVERSITY OF NAPLES
80126 NAPLES - ITALY
Tel. and fax +39 081 675372 – e-mail: csef@unina.it
ISSN: 2240-9696

WORKING PAPER NO. 617

Core and Stable Sets of Exchange Economies with Externalities

Maria Gabriella Graziano[*], Claudia Meo[†] and Nicholas C. Yannelis[‡]

Abstract

It is known that the core of an economy with externalities may be empty. We consider two concepts of dominance that allow us to prove that the set formed by individually rational, Pareto optimal allocations is stable and coincides with the core that, consequently, is non-empty.

Keywords: Other-regarding Preferences; Externalities; Stable Sets; Core.

JEL Classification: C71, D51, D70.

[*] Università di Napoli Federico II and CSEF. E-mail: mgrazian@unina.it

[†] Università di Napoli Federico II. E-mail: claudia.meo@unina.it

[‡] University of Iowa. E-mail: nicholas-yannelis@uiowa.edu

Table of Contents

1 Introduction

We investigate the relationship between stable sets, core and Pareto optimal allocations in the presence of externalities. When there is an externality in consumption, the utility function of each trader depends not only on his own consumption bundle but also on the consumption profile of some or all of the remaining consumers in the economy. As a consequence, several dominance relations can be considered according to how the counter–coalition $N \setminus S$ reacts to a deviation by S. In Graziano et al. (2017), we introduce the notion of stable sets with externalities for economies with a finite number of commodities and a finite set of agents and prove their existence and uniqueness for different classes of dominance relations. Precisely, we adopt a taxonomy based on the viewpoint of the blocking coalition S towards the reaction of the counter–coalition $N \setminus S$[1].

We complement the analysis presented in Graziano et al. (2017) by considering two other notions of dominance where we take a view which is close in the spirit to the so-called strong Nash equilibrium (see, Aumann 1961). We call them the γ–dominance and the δ–dominance[2]. They differ from those used in Graziano et al. (2017) in that agents outside of a blocking coalition S do not react at all and stick to the status quo allocation x that is blocked by S. Moreover, they differ each other in the quantity that traders in the coalition S are supposed to redistribute among themselves in order to be better off; this quantity is $\sum_{i\in S} e_i$ in the γ-dominance, where e_i denotes the initial endowment of trader i, while it is $\sum_{i\in S} x_i$ in the δ-dominance relation, where x_i is the commodity bundle of trader i under the status quo allocation x. The difference in resources which are redistributed among traders of a blocking coalition S allows us to interpret the γ-dominance as a static dominance relation and the δ-dominance as its dynamic counterpart.

Our main finding in this paper is that the set of all individually rational, Pareto optimal allocations is stable with respect to both dominance relations and equals the associated core. Since the core is frequently empty when there are externalities (see Holly (1994), Dufwenberg et al. (2011)), this equivalence result is appealing also because it provides conditions for the non-emptyness of the core. For the γ-dominance, the assumption that

[1] We distinguish between a pessimistic (or conservative) and an optimistic (or non-conservative) attitude of the deviating coalition S. This distinction results into three different notions of dominance relations that we name α, α_1 and α_2 dominance.

[2] In connection with the adopted terminology, we remark that Dufwenberg et al. (2011) study the γ-core as a cooperative solution for a pure exchange economy with other-regarding preferences. Borglin (1973) introduces the δ-dominance and studies the core associated with it.

agents in the counter-coalition do not change their consumption when a coalition S deviates has two relevant consequences on stable sets. First, allocations in a stable set are not necessarily Pareto optimal; this is in contrast with both the framework with no externalities (Einy and Shitovitz (2003)) and the cases with externalities analyzed in Graziano et al. (2017). Second, the internal stability of Pareto optimal allocations cannot be reached if one works with types; this is so because no market clearing condition is met by the consumption profile that is obtained after γ-blocking. We compensate for the market non-feasibility produced by γ-blocking by introducing the δ-dominance. The δ-blocking has a dynamic nature; indeed, a status quo allocation x is blocked at the time of consumption, i.e. in a new economy where the allocation x itself represents the initial endowment. In this dynamic framework, the crucial assumption that each type has an initial corner on a market commodity, that we use for the γ-dominance, becomes meaningless. For this reason, in the case of the δ-dominance, we prove the existence and uniqueness of stable sets and the equivalence with the core under different assumptions; precisely, we assume that agents are non-benevolent and preferences are separable.

The structure of the paper is as follows. In Section 2 we outline the economic model, along with the two dominance relations and all the definitions of the solution concepts that are needed in the paper. Section 3 is divided in two subsections which illustrate separately the results for each dominance relation. The last section also includes some concluding remarks.

2 The economic model

We consider an exchange economy E with a finite set of traders, $N = \{1, \ldots, n\}$, and a finite number l of commodities. $\mathbb{R}^l$ is the commodity space and every subset of N is referred to as a coalition. The consumption set of agent $i \in N$ is $X_i \equiv \mathbb{R}^l_+$ and the initial endowment for agent $i \in N$ is represented by a vector in X_i, denoted by e_i.

In our model agents care about others: their preferences can be represented by *other–regarding* utility functions, i.e. utility functions that depend not only on their own consumption bundle, but also on the consumption of the other traders in the economy E; that is, the utility function of agent i is defined by:

$$U_i : \mathbb{R}^{l \cdot n}_+ \longrightarrow \mathbb{R}.$$

This way of modeling preferences has been analyzed, among others, in Dufwenberg et al. (2011), Hervés-Beloso and Moreno-García (2021), Ok and Kockesen (2000) and Sobel

(2005).
Hereafter, the situation in which the utility of agent i only depends on his own consumption bundle $x_i \in \mathbb{R}^l_+$ will be referred to as the *selfish* case.
The exchange economy with externalities E is thus formalized by the following collection:

$$E = \{N;\ (X_i, U_i, e_i)_{i \in N}\}.$$

Definition 2.1 (Assignment and allocation) Given a coalition $S \subseteq N$, an **assignment for** S is a vector $y^S = (y_i)_{i \in S}$ such that:

i) $y_i \in X_i$, for every $i \in S$ (*consumption set feasibility*);

ii) $\sum_{i \in S} y_i = \sum_{i \in S} e_i$ (*physical feasibility*).

An **allocation** for the economy E is an assignment $x = (x_i)_{i \in N} \in \mathbb{R}^{l \cdot n}_+$ for the grand coalition N.

For every $x \in \mathbb{R}^{l \cdot n}$, the restriction of x to the components corresponding to traders of S, for $S \subseteq N$, will be denoted by x^S. Moreover, for a coalition of agents S, the notation $x = (y^S, z^{N \setminus S})$ will be used to denote the vector of $\mathbb{R}^{l \cdot n}_+$ whose components are equal to y_i if $i \in S$ and z_i if $i \in N \setminus S$. Given an allocation x, the symbol $E(x)$ denotes the economy with externalities obtained from E by replacing the original initial endowment e with x; that is:

$$E(x) = \{N;\ (X_i, U_i, x_i)_{i \in N}\}$$

We consider the two following notions of dominance.

Definition 2.2 (γ-Dominance) Let x and y be allocations of the economy E. We say that x γ–dominates y, denoted by $x \succ_\gamma y$, if there exists a non empty coalition S such that:

a. x is an assignment for S in E, that is: $\displaystyle\sum_{i \in S} x_i = \sum_{i \in S} e_i$;

b. $U_i(x^S, y^{N \setminus S}) > U_i(y)$, for all $i \in S$.

Definition 2.3 (δ-Dominance) Let x and y be allocations of the economy E. We say that x δ–dominates y, denoted by $x \succ_\delta y$, if there exists a non empty coalition S such that:

a. x is an assignment for S in $E(y)$, that is: $\sum_{i \in S} x_i = \sum_{i \in S} y_i$;

b. $U_i(x^S, y^{N \setminus S}) > U_i(y)$, for all $i \in S$.

In the sequel, the equivalent expression "the coalition S blocks the allocation y through x" will be also used and S will be referred to as a "blocking coalition". The notations $x \succ_\gamma^S y$ and $x \succ_\delta^S y$ will be used when the blocking coalition S needs to be explicitly mentioned.

The two dominance relations differ in the resources that members of S redistribute among themselves in order to be better off. According to such difference, the γ-dominance and the δ-dominance can be interpreted as *static* and *dynamic*, respectively. Precisely:

- in the γ-dominance, when a coalition S forms in order to block an allocation y, the complementary coalition $N \setminus S$ does not react to the deviation; that is, all traders that do not belong to S stick to allocation y[3].

- In the δ-dominance, an allocation y is distributed among agents. At the time of consumption, the coalition S deviates by redistributing resources received by its members under the status quo allocation y. As in the previous case, the dominance relation is characterized by an optimistic attitude of the deviating coalition S: indeed, it is assumed that traders in $N \setminus S$ do not react.

Notice that, in the γ-dominance, there is no market clearing; in fact, the distribution of commodities reached after that blocking takes place, that is, $t = (x^S, y^{N \setminus S})$, is not necessarily physically feasible for the grand coalition N. Also note that none of the dominance relations that we have introduced is transitive. Contrary to the dominance relations analyzed in Graziano at al. (2017), transitivity does not even occur when the coalition which blocks in the second domination is the grand coalition N. That is, for $\beta \in \{\gamma, \delta\}$:

$$\left.\begin{array}{l} x \succ_\beta^S y \\ y \succ_\beta^N t \end{array}\right\} \not\Rightarrow x \succ_\beta^S t$$

The simple reason is that in both cases the utility level of each trader in S is affected by the dominated allocation itself.

[3]This is the solution concept which corresponds to the Nash strong equilibria for normal games (see, Aumann 1961); for a general equilibrium framework, it has been recently analyzed in Dufwenberg et al. (2011), Di Pietro et al. (2020).

The notions of individual rationality, Pareto optimality, core and stable set can now be defined. For the γ–dominance, we adopt the following definition of individual rationality (see Le Van et al., 2001).

Definition 2.4 (Individual rationality in the γ-dominance) An allocation x is **individually rational** if for all $i \in N$ it holds true that:

$$U_i(x) \geq U_i(e_i, x^{N\setminus\{i\}}).$$

I^γ and I^δ will denote the set of all the individually rational allocations for the economy E under the γ and δ dominance, respectively.

Note that, if x is not individually rational, then it is γ-dominated by the initial endowment e through a singleton. Moreover, for the case of δ–dominance, all the allocations are individually rational. Indeed, if the allocation x cannot be δ–dominated by coalitions formed by single traders, then $U_i(x) \geq U_i(x_i, x^{N\setminus\{i\}})$ for each $i \in N$.

Remark 2.1 It is worth doing a comparison between the definition by Le Van et al (2001) and the one adopted in Yannelis (1991), which sounds as follows: x is individually rational if for all $i \in N$ it holds that $U_i(x) \geq U_i(e)$.
First of all, for the concept of individual rationality due to Le Van et al. (2001), the following implication, which is natural for the selfish framework as well as for the definition adopted by Yannelis (1991) for the other-regarding utility context, does not hold:
If x is an individually rational allocation and y is such that $U_i(y) \geq U_i(x), \forall i \in N$, then y is individually rational.
That is, the property of being individually rational is not inherited through the dominance via the grand coalition N. Indeed, consider the following example with three agents and just one commodity. The initial endowment is given by $e = (1, 2, 1)$ and the interdependent utility functions of each trader are described as follows:

$$U_1(x) = x_1 + 2x_2\,;$$
$$U_2(x) = 2x_1 + x_3\,;$$
$$U_3(x) = x_3 + x_1\,.$$

Consider the allocations $x = \left(2, \frac{1}{2}, \frac{3}{2}\right)$ and $y = \left(3, \frac{1}{2}, \frac{1}{2}\right)$. The allocation x is individually rational; indeed:

$$U_1(x) = 3 \geq U_1(e_1, x_2, x_3) = 2\,;$$
$$U_2(x) = 4 + \frac{3}{2} \geq U_2(x_1, e_2, x_3) = 4 + \frac{3}{2}\,;$$
$$U_3(x) = \frac{7}{2} \geq U_3(x_1, x_2, e_3) = 3\,.$$

Moreover, $U_i(y) \geq U_i(x), \forall\, i = 1,2,3$. However, the allocation y is not individually rational since for trader 3 it holds that $U_3(y_1, y_2, e_3) = 4 \not\leq \frac{7}{2} = U_3(y)$.
Secondly, the two notions of individual rationality cannot be compared. In fact, on one hand, the foregoing example shows that the allocation y is individually rational according to the definition due to Yannelis (1991) but not according to Definition 2.4. On the other hand, let us consider the following framework with one good and three agents with initial endowment $e = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$ and utility functions $U_1(x_1, x_2, x_3) = x_3 + x_2$, $U_2(x_1, x_2, x_3) = x_3$ and $U_3(x_1, x_2, x_3) = x_2$, respectively. The allocation $x = \left(1, \frac{1}{3}, \frac{2}{3}\right)$ is individually rational according to Definition 2.4 but not according to the definition by Yannelis (1991).

The notion of Pareto optimality is the same for both dominance relations.

Definition 2.5 (Pareto optimality) An allocation x is **Pareto optimal** if it cannot be blocked by the grand coalition N through another allocation y. That is, there does not exist another allocation y such that:

$$U_i(y) > U_i(x), \text{ for every } i \in N\,.$$

We will denote by PO the set of the Pareto optimal allocations for the economy E.

Finally, the next two definitions formalize the notion of core in the static setting and in the dynamic setting, respectively.

Definition 2.6 (γ-Core) An allocation x is a **γ-core allocation** for the economy E if it cannot be γ–dominated by any coalition.
We will denote by C^γ the set of the γ–core allocations for the economy E.

Definition 2.7 (δ-Core) An allocation x is a **δ-core allocation** for the economy E if it cannot be δ–dominated by any coalition.
We will denote by C^δ the set of the δ–core allocations for the economy E.

The γ-core and the δ-core cannot be compared since the resources that are redistributed in a blocking coalition are different. However, by definition, an allocation x is in the δ-core of the economy E if and only if it belongs to the γ-core of the economy $E(x)$ where x itself represents the initial endowment[4].

When stable sets of pure exchange economies are investigated, the set of allocations which are individually rational and Pareto optimal becomes of interest (see Einy and Shitovitz (2003)).

[4]Allocations in the δ-core are called stable in Borglin (1973). The same notion of stability is adopted in a model with indivisible goods by Roth and Postlewaite (1977).

Definition 2.8 (EXTREME CORE) An allocation x is an **extreme core allocation** if it is individually rational and Pareto optimal.
The set formed by all the extreme core allocations of E will be called the extreme core of the economy. It will be denoted by C_e^γ and C_e^δ for the γ-dominance and the δ-dominance, respectively.

It holds that $C_e^\gamma = I^\gamma \cap PO$ and $C_e^\delta = PO$. Each allocation in the γ-core is Pareto optimal and individually rational, i.e. $C^\gamma \subseteq C_e^\gamma$. Moreover, the γ-core may be empty, as shown in the next example that is a variant of the one provided by Holly (1994) for an economy without externalities.

Example 2.1 Consider an economy with three agents, $N = \{1, 2, 3\}$, and one commodity. The initial endowment allocation is $e = (1, 0, 0)$ and the utility functions of each trader are as follows:

$$U_1(x) = x_2 + 2x_3\,;$$

$$U_2(x) = x_1 + 2x_2\,;$$

$$U_3(x) = 2x_1 + x_3\,.$$

Every feasible allocation $x = (x_1, x_2, x_3)$ with $x_2 > 0$ can be blocked by coalition $S = \{1, 3\}$ through the allocation y^S defined by:

$$y^S = (x_1 + \frac{1}{3}x_2; x_3 + \frac{2}{3}x_2)$$

On the other hand, the coalition $S' = \{1, 2\}$ blocks every allocation $x = (x_1, 0, x_3)$ (with $x_1 + x_3 = 1$) through the allocation $y^{S'}$ defined by:

$$y^{S'} = (0.9; 0.1)$$

We can thus conclude that the γ-core of this economy is empty.

As concerns the δ-core, the following inclusion holds true: $C^\delta \subseteq C_e^\delta = PO$. In the selfish case, under continuity and monotonicity assumptions, it the inverse inclusion also holds, that is, the δ-core equals the set of Pareto optimal allocation[5]. When there are externalities, we shall see in Section 3 that a Pareto optimal allocation may fail to be a δ-core allocation.

Next we define the notion of stable set. We consider a dominance relation β where $\beta \in \{\gamma, \delta\}$ and provide a unique definition that fits for both cases.

[5]This equivalence in the selfish preferences case follows from the equivalence between the weak and strong dominance. Notice that, in our model, both dominance relations are defined in a strong sense and generate weak core notions.

Definition 2.9 (Stable set) Let $\succ^\beta$ be a dominance relation. A non-empty set $V^\beta \subset I^\beta$ of individually rational allocations is said to be:

- β–**internally stable** if the following condition holds:

$$\text{if } x \in V^\beta, \text{ then there is no } y \in V^\beta \text{ such that } y \succ_\beta x;$$

- β–**externally stable** if the following condition holds:

$$\text{if } x \in I^\beta \setminus V^\beta, \text{ then there is } y \in V^\beta \text{ such that } y \succ_\beta x;$$

- a (Von Neumann–Morgenstern) β**-stable set** if it is both β–internally stable and β–externally stable.

Some remarks are in order. First of all, no relations between the two notions of γ-stability and δ-stability hold true. Second, by the internal stability of the core, the inclusion $C^\gamma \subseteq V^\gamma$ holds true for each γ-stable set V^γ. The same remark holds for the δ-dominance, i.e. it is true that $C^\delta \subseteq V^\delta$ for each δ-stable set V^δ.

3 Results

In this section we investigate the existence and uniqueness of stable sets for both the dominance relations, γ and δ. Since the sets of assumptions needed in the two cases are rather different, we will divide the analysis in two separated subsections.

3.1 Stable sets for the γ-dominance

The following assumptions will be used throughout.

(A.1) (Social monotonicity) For every consumption profiles x and y with $x \geq y$, it holds that $U_i(x) \geq U_i(y)$, $\forall\, i \in N$.

(A.2) For every $i \in N$, U_i is continuous.

(A.3) (Boundary equivalence) Let x be a consumption profile and let Z be the set defined by $Z = \{j \in N : x_j \text{ has a zero component}\}$; then, for every $i \in N$:

$$U_i(x) = U_i(0^Z, x^{N \setminus Z}).$$

(A.4) For all $i \in N$ there exists $k_i \in \{1, \dots, l\}$ such that for every $j \neq i$, $e_j^{k_i} = 0$ (where $e_j^{k_i}$ denotes the k_i–th component of the vector e_j).

(A.5) $\sum_{i \in N} e_i \gg 0$.

Assumptions (A.1) requires that utility functions are increasing in the consumption of all traders; this permits other–regarding utilities to exhibit altruism but not envy. Assumption (A.2) is standard while assumption (A.3) states that the utility levels that trader i gets from an allocation x and from an allocation where the boundary components of x are substituted for zero are the same. For this reason, this assumption is referred to as the *boundary equivalence assumption*. Assumption $(A.4)$ means that each trader has a corner on some commodity, although each commodity is present in the market as a consequence of (A.5). This assumption is sometimes referred to as the *glove market assumption* on the initial endowments and is frequently encountered when studying stable sets.

Next we focus on the connections among the notions of core, stable set and extreme core for the γ-dominance relation. The first result is about some basic inclusions that hold true with no need of additional assumptions.

Proposition 3.1 *The following inclusions hold true:*

$$C^\gamma \subseteq C_e^\gamma \ \text{ and } \ C^\gamma \subseteq V^\gamma,$$

where V^γ is any γ–stable set. For a two person economy the previous inclusions are in fact equalities.

Our aim is now to provide conditions under which the equivalences hold true whatever the number of traders is. The equivalences are relevant because imply the existence and uniqueness of stable sets and the non emptiness of the core for an economy with externalities.

The next two results are preliminary to prove that the extreme core C_e^γ is stable. The first one, whose proof is trivial, has however a crucial role in proving both Lemma 3.2 and the external stability of the set C_e^γ; indeed, it allows to bypass the fact that the individual rationality is not inherited by allocations via dominance through the grand coalition N. The second result is key to prove the external stability of C_e^γ.

Lemma 3.1 *Under Assumptions (A.1), (A.3) and (A.4), every allocation is individually rational.*

Proof. Assume by contradiction that an allocation x is not individually rational. Then, for an agent i it is true that $U_i(e_i, x^{N\setminus\{i\}}) > U_i(x)$ and then, by assumptions (A.1), (A.3) and (A.4) we obtain $U_i(0, x^{N\setminus\{i\}}) = U_i(e_i, x^{N\setminus\{i\}}) > U_i(x) \geq U_i(0, x^{N\setminus\{i\}})$, which is a contradiction. $\square$

Lemma 3.2 *Under Assumption (A.2), every allocation that is not Pareto optimal can be γ–dominated by a Pareto optimal allocation through the grand coalition.*

Proof. Let z be an allocation which is not Pareto optimal and δ a positive real number. Denote by $\mathcal{A}$ the set of all the allocations for the economy E and consider the following set:

$$A = \{x \in \mathcal{A} : U_i(x) \geq U_i(z) + \delta, \forall\, i \in N\}.$$

The set A is nonempty; indeed, since z is not Pareto optimal, there exists an allocation t such that $U_i(t) > U_i(z), \forall\, i \in N$. Moreover, the set A is compact.
Define the function $\tilde{U}$ as follows:

$$\tilde{U}(x_1, \dots, x_n) = \sum_{i \in N} U_i(x).$$

By assumption (A.2), it is continuous on A.
Hence, $\tilde{U}$ has a maximal element on the set A. Let us denote it by g. It holds that $U_i(g) > U_i(z), \forall i \in N$.
We want to prove that g is also Pareto optimal.
By way of contradiction, let us suppose that g is not Pareto optimal. Then, there exists an allocation t such that:

$$U_i(t) > U_i(g), \forall i \in N.$$

The allocation t belongs to A. However, it holds that:

$$\tilde{U}(t) = \sum_{i \in N} U_i(t) > \sum_{i \in N} U_i(g) = \tilde{U}(g)\,,$$

and this contradicts the fact that g is a maximal element for the function $\tilde{U}$ on the set A. Hence, we conclude that g is Pareto optimal and this concludes the proof. $\square$

The following result focuses on the relation between stability and Pareto optimality.

Proposition 3.2 *Let the economy E satisfy the Assumptions (A.1),(A.3),(A.4) and (A.5). If V^γ is a γ–stable set, then every allocation in V^γ is Pareto optimal.*

Proof. Assume, by contradiction, that there is an allocation $t \in V^\gamma$ that is not Pareto optimal. Then, there exists an allocation y such that $U_i(y) > U_i(t)$, for each trader $i \in N$. By the internal stability of V, y does not belong to V. Hence, by the external stability, there exists an allocation x in V such that $x \succ_\gamma y$. Let $S \subseteq N$ be a coalition such that:

1. x is an assignment for S;
2. $U_i(x^S, y^{N\setminus S}) > U_i(y)$, for every $i \in S$.

We show that there exists a trader $\underline{i} \in S$ such that $x_{\underline{i}} \gg 0$. Indeed, assume by way of contradiction that for every $i \in S$, x_i has a zero component. Then, by assumption (A.3), it follows that:

$$U_i(x^S, y^{N\setminus S}) = U_i(0^S, y^{N\setminus S}).$$

If, for every $i \in S$, y_i has a zero component, we reach a contradiction; indeed:

$$U_i(y) = U_i(y^S, y^{N\setminus S}) = U_i(0^S, y^{N\setminus S}) < U_i(x^S, y^{N\setminus S}) = U_i(0^S, y^{N\setminus S}).$$

On the contrary, if there exists $i \in S$ such that $y_i \gg 0$, consider the following two sets:

$$S_1 = \{j \in S : y_j \gg 0\};$$
$$S_2 = \{j \in S : y_j \text{ has a zero component}\}.$$

It holds that, for every $i \in S$:

$$U_i(y) = U_i(y^S, y^{N\setminus S}) = U_i(y^{S_1}, 0^{S_2}, y^{N\setminus S}) \geq$$
$$U_i(0^{S_1}, 0^{S_2}, y^{N\setminus S}) = U_i(0^S, y^{N\setminus S}) = U_i(x^S, y^{N\setminus S})$$

and we reach a contradiction.

We can thus conclude that there is at least one trader $i \in S$ which receives a strictly positive commodity bundle and, as a consequence, $\sum_{i\in S} x_i = \sum_{i\in S} e_i \gg 0$. By assumption (A.4), this implies that $S = N$. Hence, the inequality in 2. can be rewritten as:

$$U_i(x) > U_i(y), \text{ for all } i \in N.$$

Therefore, $x \succ_\gamma t$, which contradicts the internal stability of V. □

We can now prove that the extreme core C_e^γ is a stable set.

Theorem 3.1 *Let the economy E satisfy the Assumptions (A.1),(A.3),(A.4) and (A.5). Then, the set C_e^γ is γ–internally stable.*

Proof. By way of contradiction, we suppose that there exist two allocations x and y in C_e^γ such that $x \succ_\gamma y$; that is, there exists a coalition $S \subseteq N$ such that:

1. x is an assignment for S;
2. $U_i(x^S, y^{N \setminus S}) > U_i(y)$, for every $i \in S$.

As in the proof of Proposition 3.2, we can show that there exists a trader $\underline{i} \in S$ such that $x_{\underline{i}} \gg 0$. As a consequence, $\sum_{i \in S} x_i = \sum_{i \in S} e_i \gg 0$. By assumption (A.4), this implies that $S = N$. Therefore, the inequality in 2. can be rewritten as:

$$U_i(x) > U_i(y), \text{ for all } i \in N,$$

which contradicts the Pareto optimality of allocation y. □

Theorem 3.2 *Let the economy E satisfy the Assumptions (A.1)-(A.4). Then, the extreme core $C_e(E)$ is γ-externally stable.*

Proof. The result is an easy consequence of Lemma 3.2. □

The next theorem summarizes the results for the γ-dominance.

Theorem 3.3 (Existence and Uniqueness) *Let the economy E satisfy the Assumptions (A.1)-(A.5). Then the core C^γ and the extreme core C_e^γ coincide with the unique stable set of the economy under the γ-dominance.*

Proof. The inclusion $C^\gamma \subseteq C_e^\gamma$ follows from Proposition 3.1. By Lemma 3.1, the set C_e^γ is formed by all Pareto optimal allocations. Let y be an allocation in C_e^γ and assume by contradiction that it does not belong to C^γ. Then there exist a coalition $S \subseteq N$ and a redistribution x such that:

1. x is an assignment for S;
2. $U_i(x^S, y^{N \setminus S}) > U_i(y)$, for every $i \in S$.

Since the arguments of Proposition 3.2 do not depend on the feasibility of x, we may conclude that x γ-dominates y through N and this contradicts the fact that y is Pareto optimal. Hence $C^\gamma = C_e^\gamma$ and the conclusion follows from Theorem 3.1 and Theorem 3.2. □

3.2 Stable sets for the δ-dominance

This section focuses on the existence and uniqueness of stable sets when the δ-dominance is considered. As in Borglin (1973), a key assumption that we maintain is the *separability* of utility functions: for each trader i, $U_i(x_i, x^{N\setminus\{i\}}) \geq U_i(x_i', x^{N\setminus\{i\}})$ for some $x^{N\setminus\{i\}}$ implies that $U_i(x_i, z^{N\setminus\{i\}}) \geq U_i(x_i', z^{N\setminus\{i\}})$, for each $z^{N\setminus\{i\}} \in \mathbb{R}_+^{l\cdot(n-1)}$. The following assumptions will be used throughout this section.

(B.1) $U_i(x_i, x^{N\setminus\{i\}}) = U_i(x_i', x^{N\setminus\{i\}})$ implies that $U_j(x_i, x^{N\setminus\{i\}}) = U_j(x_i', x^{N\setminus\{i\}})$, for $j \in N$.

(B.2) (Social Group Monotonicity) For every allocation x and every coalition S, if $z \geq \sum_S x_i$, then there is an allocation $(y^S, x^{N\setminus S})$ such that $U_i(y^S, x^{N\setminus S}) > U_i(x)$ for every $i \in S$ and $z = \sum_S y_i$.

(B.3) For every $i \in N$, U_i is continuous.

Under this set of assumptions, an individual (internal) utility function u_i can be defined for each trader i with the property that $U_i(x) = U_i(u_1(x_1), \cdots u_n(x_n))$ (see Lemma 1 in Borglin (1973)). The utility function u_i of trader i is defined only on his own consumption and allows to introduce the following additional assumption.

(B.4) (Non-benevolence) For every $i, j \in N$, $u_j(y_j) \geq u_j(x_j)$ implies that $U_i(x_j, x^{N\setminus j}) \geq U_i(y_j, x^{N\setminus j})$.

Assumption (B.1) is a weaker form of non-malevolence; Assumption (B.2), in turn, ensures that any excess of resources for a coalition can be redistributed among its members in such a way that each of them is better off (see Dufwenberg et al. (2011)). This assumption is usually adopted to extend the Second Welfare theorem in the presence of externalities. Assumption (B.4) expresses non-benevolence of i towards j and reflects the fact that U_i is non-increasing in its component u_j.

We already remarked that all allocations are individually rational and δ–core allocations are Pareto optimal. Moreover, the core C^δ is δ–internally stable and is included in every δ–stable set, as consequence of the external stability. Therefore, whenever the set C^δ is externally stable, it contains every stable set, otherwise the internal stability of the stable set would be contradicted. In such case, C^δ is the unique δ–stable set. The next proposition summarizes these basic inclusions.

Proposition 3.3 *Let V^δ be a δ–stable set of the economy E. The following inclusions hold true:*

$$C^\delta \subseteq PO \text{ and } C^\delta \subseteq V^\delta.$$

For a selfish economy E^s where the utility of each trader only depends on his own consumption, the equality $C^\delta = PO$ holds under continuity and monotonicity of preferences. The non trivial inclusion, $PO \subseteq C^\delta$, can be proved by two different arguments. It may follow from the fact that, under the aforementioned assumptions, the weak and strong dominance coincide. Hence, if $y^S \in \mathbb{R}_+^{l\cdot|S|}$ satisfies $\sum_S y_i = \sum_S x_i$ and $U_i(y_i) > U_i(x_i)$, for each $i \in S$, from the allocation $(y^S, x^{N\setminus S})$, which weakly blocks x through N, one can construct with standard arguments a new allocation which also strongly blocks x over the coalition N. An alternative proof of the inclusion $PO \subseteq C^\delta$ for a selfish economy can be provided through the Second Welfare theorem. Under the assumptions of continuity, monotonicity and, moreover, convexity of preferences, a Pareto optimal allocation x of a selfish economy is efficient (see, Borglin(1973), page 485). Let us denote by p the non-zero price supporting x. If x does not belong to C^δ, for a given y^S it is true that $U_i(y_i) > U_i(x_i)$, for each $i \in S$, and moreover $\sum_S y_i = \sum_S x_i$. Since $p \cdot y_i > p \cdot x_i$ for each $i \in S$, one easily obtains a contradiction.

When there are externalities, none of the previous arguments works. In particular, if a coalition S is able to redistribute its resources in such a way that $U_i(y^S, x^{N\setminus S}) > U_i(x^S, x^{N\setminus S})$ for each $i \in S$, the allocation $(y^S, x^{N\setminus S})$, which is favorable for S, can be unfavorable for the complementary coalition $N \setminus S$. Hence, the inclusion $PO \subseteq C^\delta$ does not hold under the usual assumptions of continuity and monotonicity (an example is presented in Borglin (1973), Section III). On the other hand, Pareto optimal allocations are not necessarily efficient and the Second Welfare theorem does not hold. Our aim in the rest of the section is to prove that, under our set of assumptions, the equality $C^\delta = PO$ holds true for an economy with externalities. In this case, the set C^δ is the unique stable set of the economy under the δ-dominance.

First of all, the following result can be proved by following the same argument as in Lemma 3.2.

Lemma 3.3 *Under Assumption (B.3), every allocation that is not Pareto optimal can be δ–dominated by a Pareto optimal allocation through the grand coalition N. Hence, the set $\mathcal{PO}$ is δ-externally stable.*

The next lemma, whose proof follows closely the one of Lemma 3 in Borglin (1973), enables to restore the inclusion of Pareto optimal allocations in the core C^δ in the presence of externalities.

Lemma 3.4 *Suppose that Assumptions (B.1)-(B.4) hold true. Let x be a Pareto optimal allocation and let S be a coalition such that for a vector $y^S \in \mathbb{R}_+^{l\cdot|S|}$ it is true that $U_i(y^S, x^{N\setminus S}) \geq U_i(x^S, x^{N\setminus S})$, for each $i \in S$. Then, $u_i(y_i) \geq u_i(x_i)$, for each $i \in S$.*

Proof. Let S be a coalition such that for a vector $y^S \in \mathbb{R}_+^{l\cdot|S|}$ it is true that $U_i(y^S, x^{N\setminus S}) \geq U_i(x^S, x^{N\setminus S})$, for each $i \in S$. Suppose, by contradiction, that there exists a non-empty coalition $T \subseteq S$ such that

i) $u_i(y_i) < u_i(x_i)$, for each $i \in T$;

ii) $u_i(y_i) \geq u_i(x_i)$, for each $i \in S \setminus T$.

Assumption (B.4), joint with conditions i) and ii), gives, respectively:

$$U_i(y^T, x^{S\setminus T}, x^{N\setminus S}) \geq U_i(x^T, x^{S\setminus T}, x^{N\setminus S}), \text{ for each } i \in S \setminus T;$$

$$U_i(y^T, x^{S\setminus T}, x^{N\setminus S}) \geq U_i(y^T, y^{S\setminus T}, x^{N\setminus S}), \text{ for each } i \in T.$$

As a consequence, $U_i(y^T, x^{S\setminus T}, x^{N\setminus S}) \geq U_i(x^T, x^{S\setminus T}, x^{N\setminus S})$, for each $i \in S$; the same inequality also holds for $i \in N \setminus S$.

Consider now $z \in \mathbb{R}_+^{l\cdot n}$ such that $z^S < x^S$, $z^{N\setminus S} = x^{N\setminus S}$ and $U_i(z) = U_i(y^T, x^{S\setminus T}, x^{N\setminus S}) \geq U_i(x^T, x^{S\setminus T}, x^{N\setminus S})$, for each $i \in N$. Since $\sum_N z_i \leq \sum_N x_i$, by Social Group Monotonicity there exists $t \in \mathbb{R}_+^{l\cdot n}$ such that $\sum_N t_i = \sum_N x_i$ and $U_i(t) > U_i(z) \geq U_i(x)$, for each $i \in N$, which contradicts the Pareto optimality of allocation x. $\square$

Corollary 3.1 *Under the Assumptions (B.1)-(B.4), it holds true that $PO \subseteq C^\delta$.*

Proof. Let x be a Pareto optimal allocation and suppose that $x \notin C^\delta$. Then there exist a vector $y^S \in \mathbb{R}_+^{l\cdot|S|}$ and a non empty coalition S such that:

a. $\displaystyle\sum_{i\in S} y_i = \sum_{i\in S} x_i;$

b. $U_i(y^S, x^{N\setminus S}) > U_i(x)$, for all $i \in S$.

Let z be the feasible allocation defined by $z^S = y^S$ and $z^{N\setminus S} = x^{N\setminus S}$. Then, by Lemma 3.4, $u_i(z_i) \geq u_i(x_i)$, for each $i \in S$, $u_i(z_i) = u_i(x_i)$, for each $i \in N \setminus S$. It holds that $u_i(z_i) > u_i(x_i)$ for at least one agent $i \in S$ (otherwise, by repeatedly using (B.1), we would get $U_i(z) = U_i(x)$, for all $i \in N$, which contradicts condition b).

Consider now the selfish economy E^s which has the same characteristics of E and in which the utility function of trader i coincides with his internal utility u_i. Since x is Pareto optimal in E, by Lemma 2 in Borglin (1973)[6], x is Pareto optimal in E^s. Moreover, the previous arguments ensure that x is weakly dominated by z. Since the internal utility functions are continuous and monotonic in E^s, one can construct in a standard way a new allocation of E^s which strictly dominates x, contradicting its Pareto optimality. The contradiction implies our conclusion. $\square$

Remark 3.1 It is worth noting that the same arguments adopted in the proof of Corollary 3.1 also show that, under assumptions (B.1)-(B.4), (weakly) Pareto optimal allocations defined by means of the strict dominance coincide with strong Pareto optimal allocations defined by means of the weak dominance.

Next theorem summarizes all the results about the δ-dominance.

Theorem 3.4 (Existence and Uniqueness) *Let the economy E satisfy the Assumptions (B.1)-(B.4). Then, the core C^δ and the set of Pareto optimal allocations coincide with the unique stable set of the economy under the δ-dominance.*

Proof. The equivalence $C^\delta = PO$ follows from Proposition 3.3 and Corollary 3.1. Since $C^\delta = PO$ is internally stable by definition, and externally stable by Lemma 3.3, the set C^δ is a von Neumann-Morgenstern stable set. Finally, for any other stable set V^δ, Proposition 3.3 tells us that $C^\delta \subseteq V^\delta$. Since two stable sets cannot be a proper subset each of the other, $C^\delta = V^\delta$. $\square$

3.3 Concluding remarks

We end the paper with some further remarks.

Remark 3.2 The results that we have provided for the γ-core hold true when the γ–dominance is replaced by its pessimistic version that we call the γ_1-dominance. Given

[6] See also Theorem 3 in Dufwenberg et al. (2011).

two allocations x and y of the economy E, we say that x γ_1–dominates y, denoted by $x \succ_{\gamma_1} y$, if there exists a non empty coalition S such that:

a. x is an assignment for S;

b. $U_i(x^S, z^{N\setminus S}) > U_i(y)$, for all $i \in S$ and for all redistribution $z^{N\setminus S}$ on the complementary coalition such that $\sum_{i\in N\setminus S} z^{N\setminus S} = \sum_{i\in N\setminus S} y^{N\setminus S}$.

This dominance is characterized by a less optimistic attitude on behalf of the deviating coalition S: indeed, it is assumed that $N \setminus S$ can react by redistributing the quantity $\sum_{i\in N\setminus S} y_i$ among its members and S deviates as long as its members benefit from the final profile whatever redistribution takes place. The total amount $\sum_{i\in N\setminus S} y_i$ is allotted among the members in $N \setminus S$ and no market clearing condition is considered; indeed, the distribution of items reached after that blocking takes place, that is, $t = (x^S, z^{N\setminus S})$ in the γ_1–dominance, is not necessarily physically feasible for the grand coalition N. For any two allocations x and y, $x \succ_{\gamma_1} y$ implies $x \succ_\gamma y$. As a consequence, the inclusion $C^\gamma \subseteq C^{\gamma_1}$ holds true. Under the assumptions of Theorem 3.3, from the equivalence between the γ–core and the set of Pareto optimal allocations, it follows the set C^{γ_1} coincide with the unique stable set. In particular, there is no difference in terms of stability between the optimistic and pessimistic attitude of a blocking coalition S that characterizes the γ and the γ_1–dominance relations, respectively.

Remark 3.3 The equivalence between PO and C^δ, which follows from Corollary 3.1, can be proved in an alternative way based on a suitable version of the Second Welfare Theorem. The following convexity assumption is needed for this alternative approach:

(B.5) For every $i \in N$, U_i is convex over the component i [7].

Under assumptions (B.1)-(B.5), if x is a Pareto optimal allocation, the allocation x is efficient with a supporting price p, $p \neq 0$ (see, Theorem 3 in Borglin (1973)). That is, for any coalition S, $U_i(y^S, x^{N\setminus S}) > U_i(x^S, x^{N\setminus S})$ for each $i \in S$ implies that $p \cdot \sum_S y_i > p \cdot \sum_S x_i$. Using this form of efficiency, it is easy to show that a Pareto optimal allocation x belongs to C^δ.

[7]Notice that the convexity of U_i over allocations would be in general far more stringent (see Dufwenberg et al. (2011)). In particular, the convexity of U_i over the whole allocation would be not compatible with assumption (B.5).

Remark 3.4 As in Graziano et al. (2017), the assumption $X_i \equiv \mathbb{R}^l_+$, for each $i \in N$, can be relaxed requiring that the consumption set X_i of trader $i \in N$ is a subset of $\mathbb{R}^l_+$ that can vary according to the coalition that trader i joins. In this case, the notation $X_i(S)$ will be used to denote the consumption set of agent i when he takes part of coalition S. This way of modeling consumption sets is general enough to recover an asymmetric information framework with an exogenous rule that regulates information sharing among agents. In such a context, in fact, traders' consumption sets do not coincide with the positive orthant of the commodity space: due to the information constraints, they are smaller subsets of it, they differ from agent to agent depending on the initial information and, moreover, they can vary according to what coalition is joined and what are the opportunities of communication within the coalition. The results of Section 3 remain true assuming that $X_i(N)$ is closed, convex and $0 \in X_i(N)$, for every $i \in N$. Moreover, the following assumption on the initial endowment is needed: $e_i \in X_i(S)$, for each coalition S with $i \in S$. That is, the initial endowment is always available for trader i, irrespective of what coalitions he joins.

References

Aumann, R. J. (1961): The core of a cooperative game without side payments. Transactions of the American Mathematical Society 98 (3), 539–552.

Borglin, A. (1973): Price characterization of stable allocations in exchange economies with externalities. Journal of Economic Theory 6, 483–494.

Di Pietro, C., Graziano, M.G., Platino, V. (2020): Social loss with respect to the core of an economy with externalities. Economic Theory, https://doi.org/10.1007/s00199-020-01293-x.

Dufwenberg, M., Heidues, P., Kirchsteiger, G. Riedel, F., Sobel, J. (2011): Other-Regarding Preferences in General Equilibrium. Review of Economic Studies 78, 613–639.

Ehlers, L., (2007): Von Neumann-Morgenstern stable sets in matching problems. Journal of Economic Theory 134, 537–547.

Einy, E., Holzman,R., Monderer, D., Shitovitz, B. (1996): Core and Stable Sets of Large Games Arising in Economics. Journal of Economic Theory 68, 200–211.

Einy, E., Shitovitz, B. (2003): Symmetric von Neumann–Morgenstern stable sets in pure exchange economies. Games and Economic Behavior 43, 28–43.

Graziano, M.G., Meo, C., Yannelis, N.C. (2015): Stable sets for asymmetric information economies. International Journal of Economic Theory, 11, 137–154.

Graziano, M.G., Meo, C., Yannelis, N.C. (2017): Stable sets for exchange economies with interdependent preferences. Journal of Economic Behavior and Organization 140, 267–286.

Graziano, M.G., Meo, C., Yannelis, N.C. (2020): Shapley and Scarf housing markets with consumption externalities. Journal of Public Economic Theory 22, 1481–1514.

Hervés-Beloso, C., Moreno-García E., (2021): Revisiting the Coase theorem. Economic Theory, https://doi.org/10.1007/s00199-020-01330-9.

Holly, C. (1994): An exchange economy can have an empty α-core. Economic Theory 4, 453–461.

Le Van, C., Page, F.H., Wooders, M.H. (2001): Arbitrage and equilibrium in economies with externalities. Journal of Global Optimization 20, 309–321.

Ok, E., Kockesen, L., (2000): Negatively Interdependent Preferences. Social Choice and Welfare, 533–558.

Roth, A.E., Postlewaite, A. (1977): Weak versus strong domination in a market with indivisible goods. Journal of Mathematical Economics 4, 131–137.

Sobel, J. (2005): Interdependent preferences and reciprocity. Journal of Economic Literature, Vol. XVIII, 392–436.

Yannelis, N. C. (1991): The core of an economy without ordered preferences, in M. A. Khan and N. C. Yannelis (eds), Equilibrium theory in infinite dimensional spaces, New-York: Springer.