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Fairness and Formation Rules of Coalitions

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Fairness and Formation Rules of Coalitions

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Abstract
In this paper, we study the problem of a fair redistribution of resources among agents of an exchange economy and how certain limitations imposed on coalition formation may impact the set of allocations judged fair. The study is conducted in atomless economies as well as in the so-called mixed markets.

JEL classification: D51, D63, D71, C02.

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1 Introduction

In this paper, we study the problem of a fair redistribution of resources among agents of an exchange economy and how restrictions on coalition formation rules may impact the set of allocations judged fair. The first notion of fairness is due to Foley (1967), according to which an allocation is said to be *fair* if it is *efficient*, in the sense of Pareto optimal, and *envy-free*, meaning that none prefers or envies the bundle of anybody else. Whenever the total initial endowment is equally divided among the agents, the resulting competitive allocation, called *equal-income competitive allocation*, fulfils efficiency and envy-freeness. The converse is not true either in large economies unless one strengthens the equity criterion as done, among others, by Zhou (1992) (see also Varian (1974), Gabszewicz (1975), Shitovitz (1992) and Basile, Graziano, and Pesce (2014) for an extension to asymmetric information economies).[1] Zhou (1992) proposes the notion of strict envy-freeness for which no agent envies the average bundle of any possible coalition by extending the object of potential envy from individuals (as in Foley (1967) and Varian (1974)) to coalitions. He shows that, in large economies, the equal-income Walrasian equilibria are the only strictly fair allocations. On the other hand in general, a *core* allocation, defined as an allocation that no coalition can improve upon (or block) by redistributing among its members their initial endowment and making everybody better off, may be not strictly envy-free even though so is the initial endowment (see Pazner and Schmeidler (1978)). Nevertheless, in atomless economies, the three concepts form the same set of allocations. This originates by combining the seminal Core-Walras Equivalence Theorem of Aumann (1964) with the identity, proved by Zhou (1992), between the set of equal-income competitive allocations and the set of strictly fair allocations.

These identities implicity impose no restriction on the set of possible coalitions. However, the formation of coalitions is often subject to very specific rules that allow only certain groups to be considered. This is the case, for instance, of international environmental agreements that, with the aim to reduce ozone depletion, climate change and marine pollution, involve only coalitions of a certain dimension (see Cabon-Dhersin and Ramani (2006)). In addition to possible restrictions on the size of formable coalitions, the rules may also provide for the exclusion or inclusion of pre-established coalitions. For example, international agreements may impose the exclusion of a certain State which has previously violated some common laws, or a research project may exclude groups with homogeneous competencies because its achievement needs to involve a team with a variety of skills. The set of possible coalitions may also be the result of a protocol providing for an enlargement of the pre-existing coalition in the agreement. Some international treaties may require the non-exclusion of certain States or if a company wants to expand without laying off any employees, possible new work groups have necessarily to include the old ones.

In this paper, we consider different kinds of limitations imposed on the coalition formation rules. In particular, we analyze norms forcing restrictions on the size of groups or on the inclusion/exclusion

[1] A different approach consists in imposing strong assumptions on preferences, like the existence of a continuum of tastes (not just a continuum of agents) as in Varian (1976).

of pre-established coalitions. We investigate the impact of these rules on the set of envied coalitions and, consequently, on the set of strictly fair allocations. Our analysis is conducted in atomless economies as well as in the so-called mixed markets in which a continuum of negligible individuals interact with few influential traders.

For the core, the seminal papers Schmeidler (1972), Grodal (1972) and Vind (1972) show that, in atomless economies, nothing changes if restrictions on the measure of blocking coalitions are imposed. Specifically, Schmeidler (1972) proved that an allocation is in the core if and only if it is not blocked by arbitrarily *"small"* coalitions. This characterization was further extended by Grodal (1972), who showed that to get the core, and hence the set of competitive allocations, it is enough to consider those *"small"* coalitions represented as a union of finitely many disjoint subgroups, each of which is arbitrarily *"small"* and consists of arbitrarily *"similar"* agents. Similarity derives from a pseudometric defined on the agents' measure space and it can be expressed in terms of agents' characteristics, that is preferences and initial endowments. Vind (1972) completed the analysis by showing that allocations outside the core can be blocked by arbitrarily *"large"* coalitions, or even better, by coalitions of any size. If the society is partitioned into a sufficiently large number of groups, then Okuda and Shitovitz (1985) show that, in atomless economies, an allocation belongs to the core if and only if it can not be blocked by any coalition that excludes or includes at least one element of the partition.

We aim to conduct a similar investigation for the set of envied coalitions and to see how the restriction on the formation of coalitions affects the set of strictly fair allocations. We prove that the set of strictly fair allocations keeps the same if agents are allowed to compare their bundles only with the average bundle of coalitions with measures below a certain threshold. Furthermore, nothing changes if potentially envied coalitions are restricted to those partitioned into finitely many subcoalitions with measures and diameters arbitrarily small. In light of the equivalence due to Zhou (1992), further characterizations of equal-income competitive equilibria are deduced. Namely, we reformulate and prove Schmeidler (1972) and Grodal (1972) theorems for the strict fairness notion in atomless economies (Proposition 4.2) and give a weaker formulation in mixed economies (Proposition 4.5). On the contrary, in general, Vind (1972)'s characterization can not be adapted to our context. We, indeed, show that if the set of potentially envied coalitions limits to those of measure above a certain threshold $\varepsilon > 0$, the set of allocations judgeable fair might enlarge so much that it does not coincide anymore with the set of equal-income competitive equilibria (Remark 5.3). The positive number ε can be interpreted as a tolerance threshold, in the sense that the envy of an agent towards small enough coalitions, that is of measure below ε, can be tolerated and neglected. From this perspective, the failure of Vind (1972)'s theorem allows the existence of an envy tolerance threshold. Finally, we provide a necessary and sufficient condition for an envious agent to envy coalitions of any size. We show that it is impossible to establish a tolerance threshold if and only if the allocation is not individually rational (Proposition 4.4). In other words, an envious agent envies a coalition of arbitrary measures if and only if she values the allocation unacceptable since she prefers to not trade and to keep her initial endowment.

We formalize an economy whose set of agents is decomposed into several groups by considering a countable (finite or infinite) covering of the set of agents T, that is a family $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$ of possible coalitions whose union gives back T. In Donnini and Pesce (2021a) we allow each agent to compare her own bundle only with the average bundle of any coalition contained in a group she belongs to and we impose absence of envy just within each C_i of the covering $\mathcal{R}$. We identify a class of coverings - the *connected coverings*[2] - for which equity within each C_i is sufficient to characterize equal-income competitive allocations. In this paper, we consider also non-connected coverings, that generalize the concept of partitions, and we adapt to strictly fair allocations the exclusion-inclusion rules introduced by Okuda and Shitovitz (1985) for the core. We prove that if $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$ is a non-connected covering with $|I| > 2$, an allocation is an equal-income competitive allocation if and only if it is efficient and no agent envies coalitions that exclude at least one element of the covering. This characterization fails under the inclusion rule, that is if only coalitions including at least one element of the covering are formable (Theorems 4.7 and 4.8).

This paper aims at contributing to the literature that studies how a limitation on the set of coalitions can modify a certain class of allocations. Several papers, extending the seminal results of Schmeidler (1972), Grodal (1972) and Vind (1972) to different frameworks and for different equilibrium concepts, belong to this study branch. For example, Khan (1974) considers the core of a finite economy, Hervés-Beloso, Moreno-García, Núñez, and Pascoa (2000), Evren and Hüsseinov (2008) study economies with infinite-dimensional commodity space, Hervés-Beloso, Moreno-García, and Yannelis (2005b), Hervés-Beloso, Meo, and Moreno-García (2014) examine economies with asymmetric information, Hervés-Beloso, Moreno-García, and Yannelis (2005a), Bhowmik and Cao (2012) combine asymmetric information and infinite-dimensional commodity space; Gilles (2019) considers production economies and Basile, Gilles, Graziano, and Pesce (2020) allow also the presence of collective goods. Shimomura (2022) and Hervés-Estévez and Moreno-García (2018) study, instead, how the restriction on the formation of coalitions affects the bargaining set defined as a weakening of the core. Okuda and Shitovitz (1985) analyze the core under inclusion/exclusion coalition formation rules, Bimonte and Graziano (2009) examine economies with asymmetric information, Basile and Graziano (2001) consider a coalitional approach, while Basile, Donnini, and Graziano (2010) combine asymmetric information and coalitional approach. This paper, instead, regards the notion of strict fairness rather than the core, and it considers economies with finitely many private goods, no uncertainty or asymmetric information. It would be worthwhile to seek if the results presented here can be extended to the frameworks analyzed in the papers cited above.

The paper is organized as follows. In Section 2 we introduce the model and the main definitions. Some preliminary results are shown in Section 3. Section 4 contains our main results. All the proofs are collected in the Appendix also containing some examples that underline the role of each assumption in the proofs of our results.

[2] See Section 4.2 for the definition of connected coverings.

2 The model and the basic notions

We consider a pure exchange economy $\mathcal{E} = \{\mathbb{R}^\ell_+, (T, \Sigma, \mu), (e(t), u_t)_{t\in T}\}$ with a finite number ℓ of different commodities, in which

- $\mathbb{R}^\ell_+$ is the non-negative orthant of the Euclidean space $\mathbb{R}^\ell$ and it denotes the commodity space;
- (T, Σ, μ) is a σ-additive, complete, probability space standing for the space of agents. Elements in Σ with positive measure are called *coalitions*;
- each agent $t \in T$ is characterized by the pair $(e(t), u_t)$, where $e(t) \in \mathbb{R}^\ell_+$ is t's initial endowment and $u_t : \mathbb{R}^\ell_+ \to \mathbb{R}$ is t's utility function representing her preferences. Two agents $t, s \in T$ are said to be *identical* or *of the same type* if they own the same economic characteristics, i.e. $(e(t), u_t) = (e(s), u_s)$.

Throughout the paper, we assume that

$(H1)$ for any $t \in T$, the function $u_t : \mathbb{R}^\ell_+ \to \mathbb{R}$ is continuous, strictly monotone and quasi-concave, and the map $u : (t, x) \to u_t(x)$ is $\Sigma\otimes\mathcal{B}(\mathbb{R}^\ell_+)$–measurable, where $\mathcal{B}(\mathbb{R}^\ell_+)$ is the σ-field of Borel subsets of $\mathbb{R}^\ell_+$;

$(H2)$ the function $e : T \to \mathbb{R}^\ell_+$, assigning to each agent $t \in T$ her initial endowment $e(t) \in \mathbb{R}^\ell_+$, is μ-integrable and $\int_T e(t)d\mu(t) \gg 0$ is the total initial endowment of the economy $\mathcal{E}$ that we denote, with abuse of notation, by e, i.e. $e = \int_T e(t)d\mu(t) \in \mathbb{R}^\ell_{++}$.

We allow the presence of μ-atoms, which are coalitions in Σ with no subset of smaller positive measure. Formally, an *atom* of the measure space (T, Σ, μ) is a set $A \in \Sigma$ with a positive measure such that $\mu(B) = \mu(A)$ or $\mu(B) = 0$ for every other $B \subseteq A$ and it represents a non-negligible agent in the market. Being $\mu(T) = 1 < \infty$, according to the atomless-atomic decomposition of measures, T can be partitioned into an *atomless* component representative of an ocean of negligible traders that we denote by T_0, and the atomic component $T_1 := T \setminus T_0$, which is the union of an at most countable family $\{A_1, A_2, \dots, A_k, \dots\}$ of disjoint atoms. This allows us to view as a special case both atomless economies (once T_1 is empty, $T = T_0$ and μ is the Lebesgue measure) and finite economies (when T_0 is null, T_1 is finite and μ is the counting measure), whereas if $T = T_0 \cup T_1$ has both components, i.e. $\mu(T_0)\mu(T_1) > 0$, the economy $\mathcal{E}$ is called a *mixed economy* or a *mixed market*. Since any atom is treated as a single trader, with abuse of notation we still denote by T_1 the collection $\{A_1, A_2, \dots, A_k, \dots\}$ and we write $A \in T_1$ instead of $A \subseteq T_1$. A measurable mapping is almost everywhere constant on an atom then, being $e : T \to \mathbb{R}^\ell_+$ and $u : T \times \mathbb{R}^\ell_+ \to \mathbb{R}$ measurable, for every $A \in T_1$ and $t, s \in A$ we have that $(e(t), u_t) = (e(s), u_s)$, that is t and s are of the same type. This is consistent with the interpretation of a μ-atom as representative of a group of individuals deciding to act only together. Given an atom $A \in T_1$, we denote by $S_A := \{t \in T_0 : (e(t), u_t) = (e(A), u_A)\}$ the set in T_0 whose members are of the same type of the atom A. If $\mu(S_A) > 0$, then S_A is called *atomless fringe of* A, and it can be interpreted, roughly speaking, as an *atomless copy* in T_0 of the atom A.

An *allocation* is a μ-integrable function $x : T \longrightarrow \mathbb{R}^\ell_+$ that assigns to each agent $t \in T$ her bundle $x(t) \in \mathbb{R}^\ell_+$. The set of allocations is denoted by $\mathcal{A}$ and, by assumption $(H2)$, $e \in \mathcal{A}$. An allocation $x \in \mathcal{A}$ is said to be *feasible* if its aggregate equals the total initial endowment, i.e. $\int_T x(t)d\mu(t) = e$. An allocation $x \in \mathcal{A}$ is *individually rational* if $u_t(x(t)) \geqslant u_t(e(t))$ for almost all $t \in T$, that is if $\mu(R_x) = 0$, where $R_x := \{t \in T : u_t(e(t)) > u_t(x(t))\}$ defines the set of individuals that prefer keeping their initial endowment rather than trading and consuming the bundle x. A coalition S *blocks* or *improves upon* an allocation x via $y \in \mathcal{A}$, if $u_t(y(t)) > u_t(x(t))$ for almost all $t \in S$ and $\int_S y(t)d\mu(t) = \int_S e(t)d\mu(t)$. The core of the economy $\mathcal{E}$, denoted by C, is the set of feasible allocations not blocked by any coalition. A feasible allocation x is *efficient* or *Pareto optimal* if it is not blocked by the set of all agents T and it is called *competitive* or *Walrasian* if there exists a price vector $p \in \mathbb{R}^\ell_+ \setminus \{0\}$ such that for almost all agent $t \in T$, $p \cdot x(t) \leqslant p \cdot e(t)$ and $p \cdot y > p \cdot e(t)$ whenever $u_t(y) > u_t(x(t))$. If $p \cdot e(t) = p \cdot e$ for almost all $t \in T$, x is said to be an *equal-income competitive allocation*. We denote by W and W_{ei} respectively the set of competitive allocations and of equal-income competitive allocations of $\mathcal{E}$.
It is well known, since Aumann (1964), that in atomless economies competitive equilibria are the only allocations in the core, i.e. $W = C$. This identity is known as the *Core-Walras Equivalence Theorem*, which has been extended to mixed markets if there is enough competition among large traders. This is possible under the assumption that

$(A1)$ there are at least two atoms and all atoms are of the same type,

as in Theorem B of Shitovitz (1973), or under the assumption that

$(A2)$ any atom has an atomless fringe

as in Gabszewicz and Mertens (1971) (see also Basile, Graziano, and Pesce (2016), Donnini and Pesce (2020), Graziano, Pesce, and Urbinati (2023) for further generalizations).

3 Strict fairness and some preliminary results

In this section, we recall the main fairness notions and show some preliminary results needed for our analysis.

A first notion of fairness based on the absence of envy is due to Foley (1967), according to which, given an allocation $x \in \mathcal{A}$ and two agents $t, s \in T$, t *envies* s at x if t prefers s's bundle to her own, i.e. $u_t(x(s)) > u_t(x(t))$. An allocation in which almost no agent envies any other is known as *envy-free* and it is said to be *fair* if it is both efficient and envy-free. Any equal-income competitive allocation is fair, because it is efficient by the First Theorem of Welfare Economics, and it is envy-free inasmuch agents face the same budget set. Conversely, there are fair allocations which are non-supported by a competitive equilibrium price. Zhou (1992) characterizes the set W_{ei} using a stronger notion of fairness, called *strict fairness*, for which the supposed envy object of an individual is a coalition rather than a single agent.

Definition 3.1 (Strict fairness of Zhou (1992)) *Given an allocation $x \in \mathcal{A}$, a coalition $S \in \Sigma$ and an agent t with $t \notin S$, t **envies** S at x if $u_t(\bar{x}(S)) > u_t(x(t))$, where $\bar{x}(S) = \frac{1}{\mu(S)} \int_S x(s) d\mu(s)$.*[3] *The allocation x is called **strictly envy-free** if the set of envious agents at x, denoted by*

$$I_x := \{t \in T : u_t(\bar{x}(S)) > u_t(x(t)) \text{ for some } S \in \Sigma, \text{ with } t \notin S \text{ and } \mu(S) > 0\},$$

*has null measure (i.e. if $\mu(I_x) = 0$). The allocation x is **strictly fair** if it is both strictly envy-free and efficient. We denote by SF the set of strictly fair allocations of $\mathcal{E}$.*

Zhou (1992) shows that in atomless economies, under the assumption that

$(A3)$ the consumption set is R^ℓ_{++} and $u_t : \mathbb{R}^\ell_{++} \to \mathbb{R}$ is differentiable for all $t \in T$,

$\mathcal{W}_{ei} = SF$. This equivalence has been extended to mixed economies by Donnini and Pesce (2020).

This fairness notion satisfies a natural fundamental principle of any equity concept, that is *equals are treated equally.* Indeed, a strictly envy-free allocation assigns the same bundle to identical agents with the same strictly quasi-concave utility function u (see Lemma 3.5 in Donnini and Pesce (2020)). Whereas, if u is quasi-concave only, as stated in $(H1)$, agents of the same type get possibly different bundles lying on the same indifference curve. We denote by $\mathcal{A}_e$ the set of allocations assigning the same bundle to identical agents and we say that an allocation in $\mathcal{A}_e$ satisfies the *equal-bundle property*.

We show below that, given $x \in \mathcal{A}_e$, if any atom has an atomless fringe $(A2)$, any envious agent at x envies an atomless coalition. The proof is illustrated in the Appendix.

Proposition 3.2 *Suppose that $(A2)$ holds and let $x \in \mathcal{A}_e$. Then, any envious agent at x envies an atomless coalition.*

As a consequence of Proposition 3.2 we get that efficient allocations are strictly fair if and only if no atomless coalition is envied. More is true: under $(A2)$, even the set I_x of envious agents at x contains an *atomless* coalition, i.e. $\mu(I_x \cap T_0) > 0$. Thus, under the assumptions of Proposition 3.2, in order to test whether x is strictly envy-free it is enough to check if envy arises within the *atomless* sector T_0, disregarding the atomic part T_1 (see Section 4.2 for a further interpretation of Proposition 3.2). For this, the *equal-bundle property* (i.e. $x \in \mathcal{A}_e$) and the assumption $(A2)$ are crucial, as shown in the Appendix respectively via Examples 5.5 and 5.6, because, in a sense, they allow to move the object of envy from a coalition containing atoms to an *identical* atomless coalition. In particular, Example 5.5 illustrates an economy satisfying $(A2)$ and an allocation x assigning to agents of the same type different bundles lying on the same indifference curve (i.e. $x \notin \mathcal{A}_e$), such that no atomless coalition is envied even though $x \notin SF$. Similarly, Example 5.6 shows an economy with infinitely countably many atoms with no atomless fringe (i.e. the assumption $(A2)$ does not hold) in which an allocation

[3] Basically, t envies the possibility to join the coalition S, because she prefers what she would get on average being a member of S (i.e. $\bar{x}(S)$) rather than what she gets being alone (i.e. $x(t)$). Clearly, the condition that $t \notin S$ is irrelevant in atomless economies.

$x \in \mathcal{A}_e$ is not strictly fair but no atomless coalition is envied.

The *equal-bundle property* is also needed to prove that, in mixed markets, strictly fair allocations are individually rational. This is always true in atomless economies. Indeed, assume that the total initial endowment is equally divided among agents, i.e. $e(t) = e$ for almost all $t \in T$ - we call it "*equal-endowment assumption*" - then, given a feasible allocation x, any agent t preferring her initial endowment to $x(t)$ envies the coalition T at x, that is $R_x \subseteq I_x$. This inclusion might be strict (see Example 5.7 in the Appendix), and it is no longer valid in mixed markets in which, instead, the additional hypotheses $(A1)$ or $(A2)$, used for the Core-Walras equivalence Theorem, are needed. The proof of the following Proposition is given in the Appendix.

Proposition 3.3 *Let $\mathcal{E}$ be a mixed economy satisfying the equal-endowment assumption and let x be a feasible allocation with the equal-bundle property (i.e. $x \in \mathcal{A}_e$). If $(A1)$ or $(A2)$ holds, then $R_x \subseteq I_x$ and hence any strictly envy-free allocation is individually rational.*

Even for the above proposition, we can not dispense of the *equal-bundle property*, because if an allocation x assigns to identical agents different bundles although lying on the same indifference curve, nor $(A1)$ neither $(A2)$ ensures the inclusion $R_x \subseteq I_x$ as illustrated respectively via Examples 5.8 and 5.5 in the Appendix.

Finally, the *equal-bundle property* allows us to prove the following theorem that completes the analysis conducted by Zhou (1992) in mixed markets. Zhou (1992) proves that negligible agents have equal income at any strictly fair allocations when goods are valued at the supporting price (see Proposition 4.1 in Zhou (1992)) and this common income is no more than the income of any atom (see Proposition 4.2 in Zhou (1992)). We now observe that, if atoms have an atomless fringe $(A2)$, everybody gets the same income at a strictly fair allocation. Conversely, if x is an efficient allocation which is not strictly fair (i.e. $x \notin SF$), for any envious agent t there exists an atomless coalition S_t envied by t at x and the value of the average bundle of S_t at the supporting price is greater than the value of $x(t)$. The proof of the next theorem is shown in the Appendix.

Theorem 3.4 *Let $\mathcal{E}$ be an economy satisfying the assumptions $(A2)$ and $(A3)$. Assume that u_t is strictly quasi-concave for all $t \in T_1$. Let x be an efficient allocation in $\mathcal{A}_e$ and let p be its supporting price. Then,*

(1) *if $x \in SF \Rightarrow p \cdot x(t) = p \cdot \bar{x}(T_0)$, for almost all $t \in T$. Conversely,*

(2) *if $x \notin SF \Rightarrow$ for any $t \in I_x$, there exists an atomless coalition $F_t \subseteq T_0$ such that t envies F_t and $p \cdot x(t) < p \cdot \bar{x}(F_t)$.*

Remark 3.5 The definition of strict fairness strengthens the notion of *average fairness (A-fairness)* of Thomson (1982) according to which each individual weakly prefers her own bundle to the average of what all the others receive, i.e. $S = T \setminus \{t\}$ is the only coalition to be looked at in Definition 3.1 (see also Thomson (1988)). In a two-agent economy the fairness concepts of Foley (1967), Thomson (1982) and Zhou (1992) are equivalent but, in general, strict fairness (Definition 3.1) is the strongest one, whereas A-fairness of Thomson (1982) and the fairness notion of Foley (1967) are not comparable (see Proposition 1 in Thomson (1982) and also Thomson (2011) for an excellent survey on the fair allocation rules). In atomless economies, instead, A-fairness coincides with the notion of *per-capita-fairness* due to Pazner (1977), requiring that none prefers the average bundle of the entire economy to her own bundle. An allocation x is said to be *per-capita-envy-free* if $u_t(x(t)) \geqslant u_t(\bar{x}(T))$ for almost all $t \in T$. A feasible allocation x is said to be *per-capita-fair* if it is efficient and per-capita-envy-free. It is proved that, under the *equal-endowment assumption*, any individually rational allocation is A-envy-free, and hence also per-capita-envy-free (see Proposition 2 in Thomson (1982)). In atomless economies, the converse implication also holds and individual rational allocations are the only A-envy-free (and per-capita-envy-free) allocations. Examples 5.5 and 5.8 show that in mixed economies, even under $(A1)$ and $(A2)$, there might exist agents that are per-capita-envious but not A-envious neither strictly-envious.

4 Coalition formation rules and strict fairness

The notion of strict fairness introduced by Zhou (1992) implicity assumes that all coalitions in Σ are potentially formable. However, agents' grouping is often forced by precise rules or it is due to well-defined agreements deemed necessary by agents themselves for the achievement of specific goals. This leads to the possibility of considering as formable only elements of subfamilies of Σ.
The aim of this section is to investigate the fairness property when the set of potentially envied coalitions is restricted for some reason. To this end, given an allocation $x \in \mathcal{A}$ and a family of coalitions $\mathcal{S} \subseteq \Sigma$, we denote by

$$I_x(\mathcal{S}) := \{t \in T \,:\, u_t(\bar{x}(S)) > u_t(x(t)) \text{ for some } S \in \mathcal{S}, \text{ with } t \notin S \text{ and } \mu(S) > 0\} \tag{1}$$

the set of agents envying some coalition S in $\mathcal{S}$, and by $\mathcal{S} - SF$ the set of efficient allocations for which $\mu(I_x(\mathcal{S})) = 0$. Under this kind of limitation, since the class of potentially envied coalitions is reduced, the set of strictly fair allocations may enlarge (i.e. $SF \subseteq \mathcal{S} - SF$). Consequently, the equivalence with the set of equal-income competitive allocations, $W_{ei} = SF$, proved by Zhou (1992) might fail. In what follows, we investigate specific coalition formation rules and the possibility to get the equivalence $\mathcal{W}_{ei} = \mathcal{S} - SF$.

4.1 The size of envied coalitions

This section deals with coalition formation rules that impose restrictions on the measure of acceptable coalitions.

The seminal papers of Schmeidler (1972), Grodal (1972) and Vind (1972) show that, in atomless economies, under constraints involving the measure of the blocking coalition, nothing really changes for the core and the Core-Walras Equivalence Theorem still holds. Specifically, Schmeidler (1972) observes that, for arbitrary $\varepsilon \in (0, 1]$, the core C still coincides with W if only coalitions with a measure less than ε are allowed to form. This equivalence is generalized by Grodal (1972) by further restricting the set of potentially blocking coalitions to those that can be written as the union of at most ℓ subgroups, each of which has a measure and diameter less than ε. Vind (1972) completes the analysis by proving that any allocation outside the core can be blocked by an arbitrarily large coalition. We try to conduct a similar investigation in terms of envied coalitions and find out if the identity $W_{ei} = SF$ proved in Zhou (1992) persists. To this end, in the spirit of Schmeidler (1972) and Vind (1972) for any $\varepsilon \in (0, 1]$, we define the following subfamilies of Σ, $\mathcal{S}_{\varepsilon^-} := \{S \in \Sigma : 0 < \mu(S) \leqslant \varepsilon\}$, $\mathcal{S}_{\varepsilon^+} := \{S \in \Sigma : \mu(S) \geqslant \varepsilon\}$ and $\mathcal{S}_{\varepsilon} := \{S \in \Sigma : \mu(S) = \varepsilon\}$. According to (1), given an allocation $x \in \mathcal{A}$, agents in $I_x(\mathcal{S}_{\varepsilon^-})$ look at only coalitions with a measure not greater than ε; agents in $I_x(\mathcal{S}_{\varepsilon^+})$ consider only coalitions with a measure not less than ε, whereas $I_x(\mathcal{S}_{\varepsilon})$ contains agents that envy only coalitions of measure ε. Since $\mu(T) = 1$, it is straightforward to note that for any $x \in \mathcal{A}$ and $\varepsilon \in (0, 1]$[4]

(a) $I_x(\mathcal{S}_{\varepsilon}) = I_x(\mathcal{S}_{\varepsilon^-}) \cap I_x(\mathcal{S}_{\varepsilon^+})$;

(b) $I_x = I_x(\mathcal{S}_{\varepsilon^-}) \cup I_x(\mathcal{S}_{\varepsilon^+})$;

(c) $I_x(\mathcal{S}_{1^+}) = I_x(\mathcal{S}_1) \subseteq I_x = I_x(\mathcal{S}_{1^-})$;

(d) $\varepsilon_1 \leqslant \varepsilon_2 \Rightarrow I_x(\mathcal{S}_{\varepsilon_1^-}) \subseteq I_x(\mathcal{S}_{\varepsilon_2^-})$ and $I_x(\mathcal{S}_{\varepsilon_2^+}) \subseteq I_x(\mathcal{S}_{\varepsilon_1^+})$.

In atomless economies, as a mere consequence of Lyapunov's convexity theorem, Definition 3.1 is equivalent to the stronger notion of fairness according to which agents are envious if they prefer the average bundle of coalitions of measure no more than a certain threshold ε (see Zhou (1992), footnote 3 p. 167). Hence, for any $\varepsilon \in (0, 1]$, $\mathcal{S}_{\varepsilon^-} - SF = SF$, which can be viewed as a reformulation of Schmeidler (1972)'s theorem in terms of the envied coalition. Actually, the seminal theorem due to Schmeidler (1972) proves something more, that is: if x is blocked by a coalition S via $y \in \mathcal{A}$ then, for any $\varepsilon \in (0, \mu(S)]$, there exists $S' \subseteq S$ with $\mu(S') = \varepsilon$ blocking x via the same alternative allocation y. In order to restate this theorem in our context, we first observe that, by the Lemma in Garcia-Cutrin and Herves-Beloso (1993),[5] Definition 3.1 can be equivalently rewritten as follows (see also Remark 3.1 in Donnini and Pesce (2020)).

Definition 4.1 *An agent t envies a coalition S at an allocation x, if it is possible to redistribute among members of S the aggregate bundle $\int_S x(t)\,d\mu(t)$ in such a way that t prefers the bundle of almost every member of S to her own, that is if there exists an allocation y such that*

$$(i) \qquad u_t(y(s)) > u_t(x(t)) \text{ for almost all } s \in S, \text{ and}$$

$$(ii) \qquad \int_S y(t)\,d\mu(t) = \int_S x(t)\,d\mu(t).$$

A feasible allocation is strictly envy-free if the set of envious agents has measure zero, and it is strictly fair if it is both efficient and strictly envy-free.

[4] Similar considerations have been done by Hervés-Beloso and Moreno-García (2001) for blocking coalitions.

[5] See also Lemma 7.1 in ? and Lemma 2.4 in Donnini and Pesce (2021b).

In the spirit of Grodal (1972), the equivalence $\mathcal{S}_{\varepsilon^-} - SF = SF$ can be strengthened by further restricting the set of potentially envied coalitions to those partitioned into at most ℓ arbitrarily small subgroups, each of them containing arbitrarily *close* agents. Formally, we assume the existence of a measurable pseudometric d defined on T, which explains the term *close* agents. We consider, for any $C \subseteq T$, $diam(C) := sup\{d(r,s) : r,s \in C\}$, and for any $\varepsilon \in (0,1]$, $\mathcal{S}^m_{\varepsilon^-} := \{S \in \Sigma : S = \bigcup_{i=1}^m S_i$, with $m \leq \ell,\ 0 < \mu(S_i) \leqslant \varepsilon$ and $diam(S_i) \leqslant \varepsilon$, for each $i\}$. Grodal (1972) shows that if an allocation is blocked by a coalition S, then for every $\varepsilon \in (0, \mu(S)]$, there exists a subcoalition $S' \subseteq S$ which blocks x such that $S' = \bigcup_{i=1}^m S_i$ with $m \leq \ell, \mu(S_i) \leqslant \varepsilon$ and $diam(S_i) \leqslant \varepsilon$ for each i. We aim to prove a similar result in terms of strictly fair allocations.

In the light of Definition 4.1, an adaptation of Schmeidler (1972)'s and Grodal (1972)'s theorems to envied coalitions can be stated in the following way. The proof is shown in the Appendix.

Proposition 4.2 *Let $T = T_0$. If t envies a coalition S at x via y then, for any $\varepsilon \in (0, \mu(S)]$,*

i) t envies a subcoalition S' of S with $\mu(S') = \varepsilon$ via the same alternative allocation y;

ii) defined a measurable pseudometric d on T such that T is separable in the corresponding topology, there exists a subcoalition D of S envied by t at x such that $D = \bigcup_{i=1}^m D_i$, with $m \leqslant \ell$, and for every $i \in \{1, \dots, m\}$, $\mu(D_i) \leqslant \varepsilon$ and $diam\, D_i \leqslant \varepsilon$.

Proposition 4.2 implies that, for any $x \in \mathcal{A}$ and $\varepsilon \in (0,1]$, $I_x = I_x(\mathcal{S}_{\varepsilon^-}) = I_x(\mathcal{S}^m_{\varepsilon^-})$ and $I_x(\mathcal{S}_\varepsilon) = I_x(\mathcal{S}_{\varepsilon^+})$. Moreover, under $(A3)$ since $W_{ei} = SF$, we get that $W_{ei} = SF = \mathcal{S}_{\varepsilon^-} - SF = \mathcal{S}^m_{\varepsilon^-} - SF$. Example 5.7 in the Appendix shows that the inclusion $I_x(\mathcal{S}_\varepsilon) \subseteq I_x(\mathcal{S}_{\varepsilon^-})$ might be strict, which means that Proposition 4.2 (i) can not be reformulated in terms of enlargements of the measure of envied coalitions and it just holds for arbitrarily threshold ε below the measure of the envied coalition.

Vind (1972) proves that feasible allocations are outside the core if and only if they can be blocked by a coalition with arbitrary measure. Our next goal consists in establishing if a similar characterization holds for strictly fair allocations, that is if given an arbitrary threshold $\varepsilon > 0$, $\mathcal{S}_{\varepsilon^+} - SF = SF$. A reformulation of Vind (1972)'s theorem in our context is: given $x \in \mathcal{A}$ and $\varepsilon \in (0,1]$,

$$\mu(I_x) > 0 \Leftrightarrow \mu(I_x(\mathcal{S}_{\varepsilon^+})) > 0. \qquad (2)$$

Notice that one implication always holds, whereas if $0 = \mu(I_x(\mathcal{S}_{\varepsilon^+})) < \mu(I_x)$, then ε can be interpreted as a tolerance threshold, meaning that an agent can be considered *envious* only if she envies a coalition whose measure exceeds ε. In other words, if t envies only coalitions of measure below the threshold ε, t's envy can be neglected and an allocation is defined as *ε-tolerable envy-free* if $\mu(I_x(\mathcal{S}_{\varepsilon^+})) = 0$. In a sense, (2) describes the impossibility to fix a tolerance threshold, because whatever the threshold ε is, there exists an envied coalition whose measure exceeds it.
By means of the next example, we prove that, actually, (2) in general fails and hence, an adaptation of Vind's theorem to envied coalitions can not be proved.

Example 4.3 *Consider an atomless economy, where $T = (0,1)$ is the set of agents, $\mathbb{R}^2_{++}$ is the consumption set, $e = \left(\frac{5}{2}, \frac{3}{2}\right)$ is the total initial endowment which is equally divided among agents, whose utility functions are given by*

$$u_t(x_1, x_2) = \begin{cases} x_1^3 x_2, & \text{if } t \in \left(0, \frac{3}{4}\right) \\ x_1 x_2^2, & \text{if } t \in \left[\frac{3}{4}, 1\right) \end{cases}$$

The feasible allocation $x : T \longrightarrow \mathbb{R}^\ell_{++}$

$$x(t) = (x_1(t), x_2(t)) = \begin{cases} (3,1), & \text{if } t \in \left(0, \frac{3}{4}\right) \\ (1,3), & \text{if } t \in \left[\frac{3}{4}, 1\right) \end{cases}$$

is ε-tolerable envy-free, with $\varepsilon = \frac{\sqrt{13}+1}{12}$. Indeed, it can be proved that no agent in $\left(0, \frac{3}{4}\right)$ is envious, whereas agents in $\left[\frac{3}{4}, 1\right)$ can only envy coalitions of measure smaller than $\frac{\sqrt{13}+1}{12}$. Therefore, $0 = \mu(I_x(\mathcal{S}_{\tilde{\varepsilon}^+})) < \mu(I_x)$, for any $\tilde{\varepsilon} \geqslant \frac{\sqrt{13}+1}{12}$, and ε can be viewed as a tolerance envy threshold. △

The next proposition shows that there exists an envy-tolerance threshold only for individually rational allocations, providing a necessary and sufficient condition for envious agents to envy coalitions of any size. The proof is in the Appendix.

Proposition 4.4 *Let $\mathcal{E}$ be an atomless economy (i.e. $T = T_0$) satisfying the equal-endowment assumption. If x is a feasible non strictly envy-free allocation, then $\mu(I_x(\mathcal{S}_\varepsilon)) > 0$ for any $\varepsilon \in (0,1] \Leftrightarrow \mu(R_x) > 0$.*

Proposition 4.4 states that for any individually rational allocation x, which is not strictly envy-free, there exists an envy-tolerance threshold, that is $\mu(R_x) = 0 < \mu(I_x)$ implies the existence of an $\varepsilon \in (0,1)$ such that $\mu(I_x(\mathcal{S}_{\varepsilon^+})) = 0$. This also emerges from Example 4.3 above.

Summing up, in atomless economies, if t envies a coalition S at x, the measure of the envied coalition S can be arbitrarily *"reduced"* (Proposition 4.2 (i)) and, without loss of generality, S can be partitionable into at most ℓ arbitrarily small subgroups of arbitrarily close agents (Proposition 4.2 (ii)) but, on the other hand, the size of S cannot be arbitrarily *"enlarged"* (Example 4.3, see also Examples 5.7 in the Appendix). Therefore, under $(A3)$, we get the following chain of relationships:

$$W_{ei} = SF = \mathcal{S}_{\varepsilon^-} - SF = \mathcal{S}^m_{\varepsilon^-} - SF \subsetneq \mathcal{S}_{\varepsilon^+} - SF.$$

Proposition 3.2 allows getting a weaker formulation of these results for mixed markets. Precisely, in the presence of large traders, the measure of an envied coalition can be reduced only below a certain threshold $\bar{\alpha}$ and not arbitrarily as in the case of atomless economies. The proof of the next proposition is in the Appendix.

Proposition 4.5 *Let $\mathcal{E}$ be a mixed economy satisfying the assumption $(A2)$ and let $x \in \mathcal{A}_e$. If $t \in I_x(\mathcal{S}_\varepsilon)$, for some $\varepsilon > 0$, then* (*i*) *there exists* $\bar{\alpha} \in (0, \varepsilon]$ *such that* $t \in I_x(\mathcal{S}_\alpha)$ *for any* α *in* $(0, \bar{\alpha}]$*;*

(*ii*) *given a measurable pseudometric* d *defined on* T_0 *such that* T_0 *is separable in the corresponding topology, there exist* $\bar{\alpha} \in (0, \varepsilon]$ *and a coalition* D *envied by* t *at* x *such that* $D = \bigcup_{i=1}^{m} D_i$*, with* $m \leqslant \ell$ *and for every* $i \in \{1, \ldots, m\}$*,* $\mu(D_i) \leqslant \bar{\alpha}$ *and* $\operatorname{diam} D_i \leqslant \bar{\alpha}$*.*

4.2 The inclusion or exclusion structures

In Donnini and Pesce (2021a) we propose a local notion of strict fairness by imposing the absence of envy only among people that are "related" or "connected" in some way. To formalize the concept of connection among individuals, we assume that the society is made up of different groups so that two individuals are related if they are members of the same coalition. Formally, we consider a countable (finite or infinite) covering of the set of agents T, that is a family $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$ of possible coalitions such that $\bigcup_{i \in I} C_i = T$, and we impose the absence of envy only within each C_i of $\mathcal{R}$.

Definition 4.6 (**$\mathcal{R}$-Strict fairness of Donnini and Pesce (2021a)**) *Let* $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$ *be a covering of* T*. An allocation* x *is said to be* $\mathcal{R}$-**strictly envy-free** *if for any* C_i *in* $\mathcal{R}$ *and for almost every* t *in* C_i*, there does not exist a coalition* $S \subseteq C_i$ *such that* $u_t(\bar{x}(S)) > u_t(x(t))$*. The allocation* x *is* $\mathcal{R}$-**strictly fair** *if it is both efficient and* $\mathcal{R}$*-strictly envy-free. The set of* $\mathcal{R}$*-strictly fair allocations is denoted by* $\mathcal{R}SF$*.*

Definition 4.6 generalizes both the notions of Zhou (1992) and Cato (2010) (see also Cato (2012) for production economy). Borrowing the terminology used in the social network literature, we can construct a one-to-one correspondence between a covering $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$ of T and a social network, in which the nodes are the elements C_i of $\mathcal{R}$. An undirected edge connects two nodes C_i and C_j if and only if $\mu(C_i \cap C_j) > 0$. In this case, C_i and C_j are said to be connected. A *path* is a sequence of elements of $\mathcal{R}$ that are connected to each other, and a covering is said to be *connected* if for every pair of its elements, there exists a path linking up them. Each covering $\mathcal{R}$ designs a network and vice versa[6]. The covering $\mathcal{R}$ is exogenous because for our analysis the reasons that induce a certain composition of the society are irrelevant. It is not excluded that $\mathcal{R}$ is obtained by a metric on the space of agents' characteristics, as in Grodal (1972) and in Basile, Gilles, Graziano, and Pesce (2020) among others, so that for instance each set C_i contains agents with similar tastes; or by a metric on the space of agents as in Cato (2010) and interpret C_i as the set of neighbours in the spatial sense[7].

In Donnini and Pesce (2021a) it is proved that, under $(A3)$, if $\mathcal{R}$ is a connected covering of T, $\mathcal{R}$-strict fairness characterizes equal-income competitive allocations, i.e. $W_{ei} = SF = \mathcal{R}SF$, meaning that once the society is structured in such a way that no group is isolated, it is enough to avoid envy

[6] A partition of T, for instance, corresponds to an edgeless graph, that is a graph with isolated nodes. Vice versa, a network with isolated nodes defines a partition of T.

[7] In these cases $\mathcal{R}SF$ can be viewed as a local version of the strict fairness and thus, even though in principle it is more general, we use the term "local" to refer to it.

locally to ensure fairness globally. Here, we complete the analysis by considering also the case of non-connected coverings.

Following the idea described in Okuda and Shitovitz (1985), we now consider coalition formation rules involving inclusion or exclusion operations. Okuda and Shitovitz (1985) prove that given any partition of T, $\{C_1, ..., C_k\}$, with k at least equal to one unit more than the number of commodities in the market (i.e. $k \geq \ell + 1$), an allocation belongs to the core if and only if it cannot be improved upon by any coalition that includes at least one C_i. Furthermore, they prove that the core coincides with the set of allocations that cannot be blocked by any coalition that excludes at least one C_i.

In what follows we conduct a similar investigation in terms of envied coalitions. Given a covering, we require each agent to analyze as potentially enviable only coalitions that exclude or include at least one element of the covering. Formally, given a covering $\mathcal{R} = \{C_i\}_{i \in I \subseteq \mathbb{N}}$, we define

$$\begin{aligned} \mathcal{R}_e &:= \{S \in \Sigma \,:\, \mu(S \cap C_i) = 0 \text{ for almost one } C_i \in \mathcal{R} \text{ and } \mu(S) > 0\}, \text{ and} \\ \mathcal{R}_i &:= \{S \in \Sigma \,:\, S \supseteq C_i \text{ for almost one } C_i \in \mathcal{R}\}. \end{aligned}$$

An allocation is $\mathcal{R}_e$-strictly fair if it is efficient and almost every agent does not envy any coalition in $\mathcal{R}_e$. Similarly, an allocation is $\mathcal{R}_i$-strictly fair if it is efficient and almost every agent does not envy any coalition in $\mathcal{R}_i$. We denote by $\mathcal{R}_e SF$ and $\mathcal{R}_i SF$ respectively the set of *$\mathcal{R}_e$-strictly fair* and *$\mathcal{R}_i$-strictly fair allocations.*

In mixed markets, a natural covering of T is $\mathcal{R} = \{T_0, T_1\}$. In this case, Proposition 3.2 returns an equivalence in terms of exclusion coalition structure. Indeed, define $\mathcal{S}_{T_1}$ as the family of coalitions excluding atoms, i.e. $\mathcal{S}_{T_1} := \{S \in \Sigma \,:\, \mu(S \cap T_1) = 0 \text{ and } \mu(S) > 0\}$, and consider only allocations with the *equal-bundle property* (i.e. in $\mathcal{A}_e$) then, under $(A2)$, $SF \cap \mathcal{A}_e = \mathcal{S}_{T_1} - SF \cap \mathcal{A}_e$. If, in addition, $(A3)$ holds, we get that $\mathcal{W}_{ei} = \mathcal{S}_{T_1} - SF \cap \mathcal{A}_e$.

In an atomless economy $\mathcal{E}$ with a finite number of agents' types, instead, a natural covering of T is given by $\mathcal{R} = \{T_1, \ldots, T_n\}$, where each T_j consists of agents that share a given utility function u_j and a given initial endowment bundle e_j, that is $T_j = \{t \in T : u_t = u_j \text{ and } e(t) = e_j\}$ contains agents of the same type.[8] It is well known that from the atomless economy $\mathcal{E}$ it is possible to suitably define a finite economy $\mathcal{E}_n$, with n agents each of them is a representative of a certain type in $\mathcal{E}$, that is the economic characteristics of each $j \in I := \{1, \ldots, n\}$ are u_j and e_j. Given an allocation x of the atomless economy $\mathcal{E}$, one can define a corresponding allocation $x^* = (x_1^*, \ldots, x_n^*)$ in $\mathcal{E}_n$ that assigns to any $j \in I$ the average bundle of T_j, i.e. $x_j^* := \frac{1}{\mu(T_j)} \int_{T_j} x(t) d\mu(t)$. Conversely, given an allocation x^* in $\mathcal{E}_n$, one can define a corresponding allocation x in $\mathcal{E}$ assigning to any agent of type j the bundle x_j^*. Such an allocation satisfies the *equal-bundle property* by definition. It can be proved that given an allocation x in $\mathcal{E}$ with the *equal-bundle property* (i.e. $x \in \mathcal{A}_e$), then x is $\mathcal{R}_e$-fair if and only

[8] T is partitioned as $T = \bigcup_{j=1}^{n} T_j$, where each set T_j can be viewed as a class in the quotient $T/\sim$ with respect to a suitably defined equivalence relation $\sim$ on T.

if the associated allocation x^* in $\mathcal{E}_n$ is strictly fair[9].

We now study the impact that inclusion or exclusion coalition formation structure can have on the set of strictly fair allocations. We analyse separately atomless economies and mixed markets. In both cases, we distinguish between connected and non-connected coverings.

In atomless economies with connected coverings, local and global strict fairness coincide. This is shown in Donnini and Pesce (2021a). However, we prove here that imposing an inclusion or exclusion restriction on the set of potentially envied coalitions enlarges the set of strictly fair allocations. This is consistent with the result of Okuda and Shitovitz (1985) because a partition is a non-connected covering of T. Furthermore, we show that $\mathcal{R}_iSF$ and $\mathcal{R}_eSF$ are not comparable, in the sense that both sets $\mathcal{R}_iSF \setminus \mathcal{R}_eSF$ and $\mathcal{R}_eSF \setminus \mathcal{R}_iSF$ are non-empty. On the other hand, when the covering is non-connected and it contains more than two elements, then $\mathcal{R}_eSF$ collapse to the set of strictly fair allocations SF. The same does not hold for $\mathcal{R}_iSF$ and $\mathcal{R}SF$ which are actually non-comparable notions. Finally, in the case of partitions with just two elements, no equivalence holds true, because $SF \subseteq \mathcal{R}_eSF \subseteq \mathcal{R}SF$ and the inclusions might be strict. The proof of the following theorem is provided in the Appendix.

Theorem 4.7 *Let $\mathcal{E}$ be an atomless economy (i.e. $T = T_0$) satisfying the assumption $(A3)$ and let $\mathcal{R} = \{C_i\}_{i \in I}$ be a covering of T.*

(1) If $\mathcal{R}$ is connected, then $SF = \mathcal{R}SF$. Whereas, it might be that $SF = \mathcal{R}SF \subsetneq \mathcal{R}_iSF$ and $SF = \mathcal{R}SF \subsetneq \mathcal{R}_eSF$; moreover $\mathcal{R}_iSF$ and $\mathcal{R}_eSF$ might be non-comparable.

(2) If $\mathcal{R}$ is non-connected and $|I| > 2$, then $SF = \mathcal{R}_eSF$. Whereas, it might be that $SF = \mathcal{R}_eSF \subsetneq \mathcal{R}_iSF$ and $SF = \mathcal{R}_eSF \subsetneq \mathcal{R}SF$; moreover $\mathcal{R}_iSF$ and $\mathcal{R}SF$ might be non-comparable.

(3) If $\mathcal{R}$ is non-connected and $|I| = 2$, then $SF \subseteq \mathcal{R}_eSF \subseteq \mathcal{R}SF$ and the inclusions might be strict; moreover $\mathcal{R}_eSF$ and $\mathcal{R}_iSF$ might be non-comparable.

By combining Theorem 4.7 above and Proposition 3.4 of Zhou (1992), we provide sufficient conditions for further characterizations of the equal-income competitive equilibria in atomless economies.

In mixed markets with non-connected coverings, no equivalence holds. For connected coverings, instead, we distinguish three different situations. If some element in the covering contains only atoms then the excluded fairness coincides with the strict fairness provided that only allocations with *equal-bundle property* matter. If each element C_i of the covering $\mathcal{R}$, which includes an atom, also contains a non-negligible piece of its atomless fringe (i.e. $A \in C_i \Rightarrow \mu(S_A \cap C_i) > 0$), then local and global fairness coincide. Finally, in all the other cases, $\mathcal{R}_eSF$ and $\mathcal{R}SF$ are non-comparable notions. The proof of the following theorem is shown in the Appendix.

[9] A similar one-to-one correspondence between the economies $\mathcal{E}$ and $\mathcal{E}_n$ can be proved in terms of *average-fair* allocations.

Theorem 4.8 *Let $\mathcal{E}$ be a mixed economy (i.e. $\mu(T_0)\mu(T_1) > 0$) satisfying the assumption $(A3)$ and let $\mathcal{R} = \{C_i\}_{i\in I}$ be a covering of T.*

(1) If $\mathcal{R}$ is non-connected, then $SF \subseteq \mathcal{R}_eSF \subseteq \mathcal{R}SF$ and the inclusions might be strict.

(2) Let $\mathcal{R}$ be a connected covering of T and assume that $(A2)$ holds.

(2.1) If, for some $C_i \in \mathcal{R}$, $\mu(T_0 \cap C_i) = 0$, then $SF \cap \mathcal{A}_e = \mathcal{R}_eSF \cap \mathcal{A}_e$. Furthermore, it might be that $\mathcal{R}_eSF \subsetneq \mathcal{R}SF$.

(2.2) If u_t is strictly quasi-concave for almost all $t \in T$ and if, given $A \in T_1$ and $C_i \in \mathcal{R}$, $A \in C_i \implies \mu(S_A \cap C_i) > 0$, then $SF = \mathcal{R}SF$. Furthermore, it might be that $SF = \mathcal{R}SF \subsetneq \mathcal{R}_eSF \cap \mathcal{A}_e$.

(2.3) If

(a) $\mu(T_0 \cap C_i) > 0$ for every $C_i \in \mathcal{R}$, and

(b) there exist $A \in T_1$ and $C_i \in \mathcal{R}$, such that $A \in C_i$ and $\mu(S_A \cap C_i) = 0$,

then $\mathcal{R}_eSF$ and $\mathcal{R}SF$ might be non-comparable.

Remark 4.9 By combining (2.1) and (2.2) of Theorem 4.8 above, we provide sufficient conditions for the equivalence $\mathcal{W}_{ei} = SF = \mathcal{R}SF = \mathcal{R}_eSF \cap \mathcal{A}_e$.

5 Appendix

5.1 Proofs

Proof of Proposition 3.2. [10] Let x be an allocation with the *equal-bundle property*, i.e. $x \in \mathcal{A}_e$. Let t be an envious agent at x, i.e. $t \in I_x$, and S be a coalition envied by t at x. Assume that $\mu(S \cap T_1) > 0$, otherwise the proof is already concluded. From Remark 3.2 of Donnini and Pesce (2020)[11], without loss of generality, the set $J := \{n \in \mathbb{N} : \mu(A_n \cap S) > 0\}$ is finite. For any $n \in J$, let $D_n := \{s \in T : u_s = u_{A_n}\}$. By assumption $(A2)$, for any $n \in J$, $\mu(D_n \cap T_0) > 0$ and, since $x \in \mathcal{A}_e$, $x(s) = x_n$ for any $s \in D_n$. Define $D := \bigcup_{n\in J} D_n$. By Definition 4.1, let $y \in \mathcal{A}$ be such that $\bar{y}(S) = \bar{x}(S)$ and $u_t(y(s)) > u_t(x(t))$ for almost all s in S. Without loss of generality, we can suppose that $y(s) = y_n$ for any $s \in D_n$[12]. Thus, we get that

$$0 = \int_S [y(s) - x(s)]d\mu(s) = \int_{S\setminus D} [y(s) - x(s)]d\mu(s) + \sum_{n\in J} [y_n - x_n]\mu(D_n \cap S). \tag{3}$$

[10] The proof of Proposition 3.2 contains some arguments used in the demonstration of Theorem 3.6 in Donnini and Pesce (2020).

[11] From Remark 3.2 of Donnini and Pesce (2020), if an agent t envies a coalition S containing infinitely countably many atoms, t also envies a subcoalition B of S with finitely many atoms only. A similar result is obtained by Greenberg and Shitovitz (1994) for blocking coalitions and the core of a mixed economy.

[12] Just define the allocation $\tilde{y} := y\chi_{T\setminus D} + \sum_{n\in J} \bar{y}(D_n \cap S)\chi_{D_n}$, where χ_A denotes the characteristic function of a set A, and apply Lemma in Garcia-Cutrin and Herves-Beloso (1993).

Denote $J_1 := \{n \in J : \mu(D_n \cap S) > \mu(D_n \cap T_0)\}$. If $J_1 \neq \emptyset$, define for any $n \in J_1$, $\alpha_n := \frac{\mu(D_n \cap T_0)}{\mu(D_n \cap S)}$ and $\alpha := \min_{n \in J_1} \alpha_n$; if $J_1 = \emptyset$ define $\alpha = 1$. By (3),

$$\alpha \int_{S \setminus D} [y(s) - x(s)] d\mu(s) + \alpha \sum_{n \in J} [y_n - x_n] \mu(D_n \cap S) = 0; \tag{4}$$

moreover, for all $n \in J$, $\alpha\mu(D_n \cap S) \leqslant \mu(D_n \cap T_0)$. Hence, for any $n \in J$, there exists $B_n \subseteq D_n \cap T_0$ such that $\mu(B_n) = \alpha\mu(D_n \cap S)$. Furthermore, being $S \setminus D \subseteq T_0$ and $\alpha \leqslant 1$, by Lyapunov's convexity theorem, there exists $B \subseteq S \setminus D$ such that $\mu(B) = \alpha\mu(S \setminus D)$ and $\int_B [y(s) - x(s)] d\mu(s) = \alpha \int_{S \setminus D} [y(s) - x(s)] d\mu(s)$. Consider the atomless coalition $G := \left(\bigcup_{n \in J} B_n\right) \cup B$ and note that by (4),

$$\begin{aligned} \int_G [y(s) - x(s)] d\mu(s) &= \sum_{n \in J} [y_n - x_n] \mu(B_n) + \int_B [y(s) - x(s)] d\mu(s) \\ &= \alpha \sum_{n \in J} [y_n - x_n] \mu(D_n \cap S) + \alpha \int_{S \setminus D} [y(s) - x(s)] d\mu(s) = 0. \end{aligned}$$

This means that t envies the atomless coalition G via the allocation $z = y\chi_B + \sum_{n \in J} y_n \chi_{B_n} + x\chi_{T \setminus G}$.
■

Proof of Proposition 3.3. Let x be a feasible allocation with the *equal-bundle property* and t be in R_x. Since the economy $\mathcal{E}$ satisfies the *equal-endowment assumption*, this means that

$$u_t(e(t)) = u_t(e(T)) = u_t(\bar{x}(T)) > u_t(x(t)). \tag{5}$$

If $t \in T_0$ then (5) implies that t envies $T \setminus \{t\}$ and then, $t \in I_x$. Whereas, if $t \in T_1$ and $(A2)$ holds, by using the same arguments of the proof of Proposition 3.2, we get that t envies an atomless coalition at x. Hence, $t \in I_x$. Finally, if $t \in T_1$ and $(A1)$ holds, then pick a different atom $B \in T_1$, whose existence is ensured by $(A1)$, and denote $\alpha = \frac{\mu(B)}{\mu(T_1)} \in (0, 1)$. Since $x \in \mathcal{A}_e$ and the equal-endowment assumption holds, by $(A1)$, it follows that $x(s) - e(s) = \frac{1}{\mu(T_1)} \int_{T_1} [x(k) - e(k)] d\mu(k)$ for any $s \in T_1$. Moreover, by Lyapunov's convexity theorem, there exists $S \subseteq T_0$ such that $\mu(S) = \alpha\mu(T_0)$ and $\int_S [x(k) - e(k)] d\mu(k) = \alpha \int_{T_0} [x(k) - e(k)] d\mu(k)$. Therefore, define $D := S \cup B$ and note that $t \notin D$, moreover

$$\begin{aligned} \int_D [x(k) - e(k)] d\mu(k) &= \int_S [x(k) - e(k)] d\mu(k) + [x(B) - e(B)] \mu(B) = \\ &= \alpha \int_{T_0} [x(k) - e(k)] d\mu(k) + \alpha \int_{T_1} [x(k) - e(k)] d\mu(k) = \\ &= \alpha \int_T [x(k) - e(k)] d\mu(k) = 0. \end{aligned}$$

Finally, by (5), $u_t(e(s)) > u_t(x(t))$ for almost all $s \in D$. This, by Definition 4.1, means that t envies the coalition D at x and hence $t \in I_x$. ■

Proof of Theorem 3.4. Let x be a strictly fair allocation. By Proposition 4.1 in Zhou (1992), $p \cdot x(t) = p \cdot \bar{x}(T_0)$, for almost all $t \in T_0$. For any $t \in T_1$,[13] let S_t be its atomless fringe whose existence is ensured

[13] Proposition 4.2 in Zhou (1992) ensures that, under the assumption that for each atom there exists another non-

by $(A2)$. Since $x \in \mathcal{A}_e$, it follows that for almost all $s \in S_t$, $p \cdot x(t) = p \cdot \bar{x}(S_t) = p \cdot x(s) = p \cdot \bar{x}(T_0)$. Conversely, if x is not strictly envy-free, by Proposition 3.2, for any $t \in I_x$, there exists an atomless coalition F_t, $F_t \subseteq T_0$, such that t envies F_t, i.e. $u_t(\bar{x}(F_t)) > u_t(x(t))$. Then, being p a supporting price of x, we have that $p \cdot x(t) < p \cdot \bar{x}(F_t)$. This concludes the proof. ■

Proof of Proposition 4.2. According to Definition 4.1, t envies a coalition S at x via y if

(1) $u_t(y(s)) > u_t(x(t))$ for almost all $s \in S$, and

(2) $\int_S y(t)\, d\mu(t) = \int_S x(t)\, d\mu(t)$.

i) Define the vector-valued atomless measure ν on Σ, restricted to S, by $\nu(S') := \left(\int_{S'} [y(s) - x(s)] d\mu(s); \mu(S')\right) \in \mathbb{R}^{\ell+1}$, with $S' \subseteq S$. Since $\nu(\emptyset) = (0, \ldots, 0, 0)$ and $\nu(S) = (0, \ldots, 0, \mu(S))$, for any $\varepsilon \in (0, \mu(S)]$, by Lyapunov's convexity theorem, there exists a subcoalition S' of S such that $\nu(S') = \frac{\varepsilon}{\mu(S)}\nu(S)$. Then, being $S' \subseteq S$, from (1) we get, in particular, that $u_t(y(s)) > u_t(x(t))$ for almost all $s \in S'$ and from (2) that

$$\int_{S'} [y(t) - x(t)]\, d\mu(t) = \frac{\varepsilon}{\mu(S)} \int_S [y(t) - x(t)]\, d\mu(t) = 0.$$

Hence, t envies S' at x via the same allocation y and $\mu(S') = \frac{\varepsilon}{\mu(S)}\mu(S) = \varepsilon$. This conclude the proof of *i)*.

ii) Let $\{s_i\}_{i\in\mathbb{N}}$ be a dense subset in S on the pseudometric d. Then, T can be written as $T = \bigcup_{i=1}^{\infty} B\left(s_i, \frac{\varepsilon}{2}\right)$, where each $B\left(s_i, \frac{\varepsilon}{2}\right)$ denotes the ball centered in s_i with radius $\frac{\varepsilon}{2}$.
Define $S_1 := S \cap B\left(s_1, \frac{\varepsilon}{2}\right)$ and, for each $i > 1$, $S_i := \left(S \cap B\left(s_i, \frac{\varepsilon}{2}\right)\right) \setminus \bigcup_{j=1}^{i-1} S_j$. Consider the family $\{S_i\}_{i\in J}$, where $J = \{i \in \mathbb{N} : \mu(S_i) > 0\}$, which is composed by disjoint subcoalitions of S such that $\mu\left(\bigcup_{i\in J} S_i\right) = \mu(S)$. For every $i \in J$ define $a_i := \int_{S_i} [y(s) - x(s)] d\mu(s)$ and $\tilde{C} := conv\{a_i, i \in J\}$[14].

Observe that by (2)

$$\sum_{i\in J} a_i = \sum_{i\in J} \int_{S_i} [y(s) - x(s)] d\mu(s) = \int_S [y(s) - x(s)] d\mu(s) = 0. \tag{6}$$

Let H be the smallest affine subspace containing $\tilde{C}$ and denote by $int_H\tilde{C}$ the interior of $\tilde{C}$ relative to H, we now show that $0 \in int_H\tilde{C}$. Assume to the contrary that $0 \notin int_H\tilde{C}$. Then, there exists $p \in H$, with $p \neq 0$, such that $p \cdot a_i \geqslant 0$ for every $i \in J$. From (6) we get that for every $i \in J$

$$0 \leqslant p \cdot a_i = p \cdot \left(\sum_{j\in J} a_j\right) - p \cdot \left(\sum_{j\in J\setminus\{i\}} a_j\right) = -p \cdot \left(\sum_{j\in J\setminus\{i\}} a_j\right) \leqslant 0,$$

implying that $p \cdot a_i = 0$ for every $i \in J$. Therefore, $H' = \{a \in H : p \cdot a = 0\}$ is an affine space containing $\tilde{C}$ and it is smaller than H. This contradicts the definition of H.

negligible trader of the same type, $p \cdot x(t) \geqslant p \cdot \bar{x}(T_0)$ for all $t \in T_1$.

[14] $convX$ denotes the convex hull of X.

Let $dimH = m \leqslant \ell$. By Caratheodory's theorem there exist $m+1$ elements of $\{a_i\}_{i\in J}$ such that 0 can be written as their convex combination. With abuse of notation we still denote them $a_1, \dots, a_{m+1}$, then $0 = \sum_{i=1}^{m+1} \lambda_i a_i$ where $\lambda_1, \dots, \lambda_{m+1} \in [0,1]$ and $\sum_{i=1}^{m+1} \lambda_i = 1$. Since $0 \in conv\{a_i : i = 1, \dots, m+1\}$, there exists a boundary point of $conv\{a_i : i = 1, \dots, m+1\}$, $b \leqslant 0$, such that $b = \sum_{i=1}^{m} \alpha_i a_i$ with $\alpha_1, \dots, \alpha_m \in [0,1]$ and $\sum_{i=1}^{m} \alpha_i = 1$.

Define for every $i = 1, \dots, m$ the atomless measure $\nu_i : \Sigma_{|S_i} \to \mathbb{R}^{\ell+1}$ as $\nu_i(B) = \left(\mu(B), \int_B [y(s) - x(s)] d\mu(s)\right)$. By Lyapunov's convexity theorem, for every $i = 1, \dots, m$ there exists $D_i \subseteq S_i$ for which $\nu_i(D_i) = \alpha_i \nu_i(S_i)$. Let $D = \bigcup_{i=1}^{m} D_i$ and note that

$$\mu(D) = \sum_{i=1}^{m} \mu(D_i) = \sum_{i=1}^{m} \alpha_i \mu(S_i) \leqslant \varepsilon \sum_{i=1}^{m} \alpha_i = \varepsilon,$$

hence $\mu(D_i) \leqslant \varepsilon$ for every $i = 1, \dots, m$. Moreover, for every $i = 1, \dots, m$, $D_i \subseteq S_i \subseteq B\left(a_i, \frac{\varepsilon}{2}\right)$, then $diamD_i \leqslant \varepsilon$. Furthermore, $D \subseteq S$ implies $u_t(y(s)) > u_t(x(t))$ for almost all $s \in D$. Now, let the assignment z be defined on D as $z(s) := y(s) - \dfrac{b}{\mu(D)}$. By monotonicity and (1), since $z(s) \geqslant y(s)$, $u_t(z(s)) \geqslant u_t(y(s)) > u_t(x(t))$ for all $s \in D$. Moreover,

$$\int_D [z(s)-x(s)] d\mu(s) = \int_D [y(s)-x(s)] d\mu(s) - b = \sum_{i=1}^{m} \alpha_i \int_{S_i} [y(s)-x(s)] d\mu(s) - b = \sum_{i=1}^{m} \alpha_i a_i - b = 0.$$

Hence, t envies D at x.

■

As already observed, under the *equal-endowment assumption*, any strictly envy-free allocation x is individually rational, because $R_x \subseteq I_x$. Proposition 4.4 derives from the following lemma, which states that, under the *equal-endowment assumption*, an envious agent t envies at x a coalition of arbitrarily measure if and only if she values x unacceptable forasmuch as she would prefer her initial endowment to x.

Lemma 5.1 *Let x be a feasible non strictly envy-free allocation and assume that $e(t) = e$ for almost all $t \in T$. Then, $t \in I_{x,\varepsilon}$ for all $\varepsilon \in (0,1] \Leftrightarrow t \in I_{x,1} \Leftrightarrow t \in R_x$.*

Proof. If $t \in I_{x,\varepsilon}$ for all $\varepsilon \in (0,1]$, in particular it holds for $\varepsilon = 1$, which means that t envies a coalition S at x with $\mu(S) = \mu(T) = 1$. Being x feasible, $\bar{x}(S) = e$ and hence $u_t(e) = u_t(\bar{x}(S)) > u_t(x(t))$ implying that $t \in R_x$. Conversely, if $t \in R_x$ then t envies at x the coalition of all the agents T, that is $t \in I_{x,1}$. Proposition 4.2 ensures that for all $\varepsilon \in (0,1]$, $t \in I_{x,\varepsilon}$. ■

Proof of Proposition 4.4. It directly follows from Lemma 5.1. ■

Proof of Proposition 4.5. If $t \in I_x(\mathcal{S}_\varepsilon)$ for some $\varepsilon > 0$, there exists a coalition $S \in \mathcal{S}_\varepsilon$ such that t envies S at x and $\mu(S) = \varepsilon$. By Proposition 3.2, we get an atomless coalition $S' \subseteq T_0$ such that t envies S' at x. Let $\bar{\alpha}$ be the measure of S'. The proof of Proposition 3.2 returns that $\bar{\alpha} \leqslant \varepsilon$. Statements (i) and (ii) directly follow from Proposition 4.2 applied to S'.

■

We now illustrate the following example, obtained with suitable modifications of Example 4.1 of Donnini and Pesce (2021a)[15], that is useful to prove different statements of Theorem 4.7.

Example 5.2 *Let $\mathcal{E}$ be an atomless economy whose consumption set is $\mathbb{R}^2_{++}$, the set of agents is $T = (0,1)$, the total initial endowment is $e = (1,1)$ which is equally divided among agents whose utility functions are given by*

$$u_t(x_1,x_2) = \begin{cases} x_1 x_2 & \text{if } t \in D_1 = \left(0, \frac{1}{2}\right] \\ x_1^2 x_2 & \text{if } t \in D_2 = \left(\frac{1}{2}, 1\right). \end{cases}$$

Consider the following feasible allocation $x : T \longrightarrow \mathbb{R}^2_{++}$

$$x(t) = (x_1(t), x_2(t)) = \begin{cases} \left(\frac{4+2\sqrt{19}}{15}, \frac{2+2\sqrt{19}}{9}\right) & \text{if } t \in D_1 \\ \left(\frac{26-2\sqrt{19}}{15}, \frac{16-2\sqrt{19}}{9}\right) & \text{if } t \in D_2. \end{cases}$$

Following the computation of Donnini and Pesce (2021a) we get that x is efficient and no agent in D_2 is envious, whereas any agent in D_1 envies a coalition S if and only if $\frac{\mu(S \cap D_1)}{\mu(S)} \in \left(\frac{2}{3}, 1\right)$, that is if and only if $\mu(S \cap D_1) > 2\mu(S \cap D_2)$.

Remark 5.3 Note that the efficient allocation x, defined above, belongs to $\mathcal{S}_{\varepsilon^+} - SF$ with $\varepsilon = \frac{3}{4}$ and it is not strictly fair. Thus, Example 5.2 also shows that Proposition 4.2 can not be reformulated in terms of enlargement of the measure of envied coalition because the equivalence $SF = \mathcal{S}_{\varepsilon^+} - SF$ might fail.

Proof of Theorem 4.7.
(1) The equivalence $SF = \mathcal{R}SF$ in the case of a connected covering $\mathcal{R}$ is proved in Donnini and Pesce (2021a). Consider now, in the economy defined in Example 5.2 above, the connected covering $\mathcal{R} = \{C_1, C_2, C_3\}$ with $C_1 = \left(0, \frac{1}{4}\right] \cup \left(\frac{1}{2}, \frac{3}{4}\right)$, $C_2 = \left(\frac{1}{4}, \frac{1}{2}\right] \cup \left(\frac{5}{8}, \frac{7}{8}\right]$ and $C_3 = \left(\frac{1}{4}, 1\right)$. Observe that for any $i = 1,2,3$, and for every $S \supseteq C_i$, $\mu(S \cap D_1) \leqslant 2\mu(S \cap D_2)$, therefore no coalition $S \supseteq C_i$ is envied and then $x \in \mathcal{R}_i SF$. However, $x \notin \mathcal{R}SF$, because, for instance, every t in $\left(0, \frac{1}{4}\right] \subseteq C_1$ envies the coalition $S = \left(0, \frac{1}{4}\right] \cup \left(\frac{1}{2}, \frac{9}{16}\right) \subseteq C_1$. Hence,

$$x \in \mathcal{R}_i SF \setminus \mathcal{R}SF.$$

Moreover, observe that $x \notin \mathcal{R}_e SF$ because $\mu(S \cap C_2) = 0$. Therefore,

$$x \in \mathcal{R}_i SF \setminus \mathcal{R}_e SF.$$

Consider now, in the same economy of Example 5.2 above, the connected covering $\mathcal{R} = \{C_1, C_2\}$

[15] Example 4.1 of Donnini and Pesce (2021a) is used for a different purpose.

with $C_1 = \left(0, \frac{1}{2}\right]$ and $C_2 = \left(\frac{1}{4}, 1\right)$. It is easy to show that $x \in \mathcal{R}_e SF$. On the other hand, $x \notin \mathcal{R}_i SF$, because every t in $\left[\frac{1}{4}, \frac{1}{2}\right]$ envies the coalition $S = \left(0, \frac{9}{16}\right) \supseteq C_1$. This implies that

$$x \in \mathcal{R}_e SF \setminus \mathcal{R}_i SF.$$

Moreover, $x \notin \mathcal{R}SF$ because, for instance, every t in $\left(\frac{1}{4}, \frac{1}{2}\right] \subseteq C_2$ envies the coalition $S = \left(\frac{1}{4}, \frac{9}{16}\right) \subseteq C_2$ and hence

$$x \in \mathcal{R}_e SF \setminus \mathcal{R}SF.$$

(2) Let $\mathcal{R} = \{C_i\}_{i \in I}$ be a non-connected covering of T for which $|I| > 2$. Let us define an equivalence relation on $\mathcal{R}$ so that two sets C_i and C_j are equivalent if and only if there exists a path linking C_i and C_j. For every $i \in I$, let $[C_i]$ denote the class of sets equivalent to C_i. This equivalent relation defines, through equivalence classes, a partition of T such that

- (a) for every $i \in I$, $\bigcup_{j \notin J_i} C_j \subseteq T \setminus C_i$ where $J_i := \{j \in I : C_j \in [C_i]\}$; and
- (b) since $|I| > 2$, for every i and j in I, $(T \setminus C_i)$ and $(T \setminus C_j)$ are connected, i.e. $\mu((T \setminus C_i) \cap (T \setminus C_j)) > 0$.

Define $\mathcal{R}^* = \{(T \setminus C_i)\}_{i \in I}$ and notice that, by (a), $\mathcal{R}^*$ is a covering of T, and by (b), $\mathcal{R}^*$ is connected. Then, by Donnini and Pesce (2021a), we have that $\mathcal{R}^* SF = SF \subseteq \mathcal{R}_e SF$. On the other hand, by definition, $\mathcal{R}_e SF \subseteq \mathcal{R}^* SF$. Hence the conclusion.
Consider now in the economy described by Example 5.2 above the non-connected covering $\mathcal{R} = \{C_1, C_2, C_3\}$ with $C_1 = \left(0, \frac{1}{2}\right]$, $C_2 = \left(\frac{1}{2}, \frac{3}{4}\right]$, and $C_3 = \left(\frac{3}{4}, 1\right)$. Notice that $x \in \mathcal{R}SF$; on the other hand $x \notin \mathcal{R}_e SF$ because, for instance, every t in $\left(0, \frac{1}{2}\right]$ envies the coalition $S = \left(0, \frac{9}{16}\right) \subseteq T \setminus C_3$. Hence,

$$x \in \mathcal{R}SF \setminus \mathcal{R}_e SF.$$

Moreover, $x \notin \mathcal{R}_i SF$ because the coalition $S = \left(0, \frac{9}{16}\right)$ contains C_1, thence

$$x \in \mathcal{R}SF \setminus \mathcal{R}_i SF.$$

Consider now in the same economy $\mathcal{E}$ of Example 5.2 the non-connected covering $\mathcal{R} = \{C_1, C_2, C_3\}$ with $C_1 = \left(0, \frac{1}{8}\right) \cup \left(\frac{1}{2}, \frac{3}{4}\right)$, $C_2 = \left[\frac{1}{8}, \frac{1}{4}\right] \cup \left(\frac{1}{2}, \frac{3}{4}\right)$, and $C_3 = \left(\frac{1}{4}, \frac{1}{2}\right] \cup \left[\frac{3}{4}, 1\right)$. Notice that, given any C_i with $i = 1, 2, 3$, any coalition S containing C_i is such that $\mu(S \cap D_1) \leqslant 2\mu(S \cap D_2)$. This implies that $x \in \mathcal{R}_i SF$. On the other hand, $x \notin \mathcal{R}_e SF$ because, for instance, every t in $\left(0, \frac{1}{2}\right)$ envies the coalition $S = \left(0, \frac{1}{4}\right) \cup \left(\frac{1}{2}, \frac{9}{16}\right) \subseteq T \setminus C_3$, i.e.

$$x \in \mathcal{R}_i SF \setminus \mathcal{R}_e SF.$$

Moreover, $x \notin \mathcal{R}SF$ because every t in $\left(0, \frac{1}{8}\right) \subseteq C_1$ envies the coalition $S = \left(0, \frac{1}{8}\right) \cup \left(\frac{1}{2}, \frac{17}{32}\right) \subseteq C_1$, hence

$$x \in \mathcal{R}_i SF \setminus \mathcal{R}SF.$$

(3) Let $\mathcal{R} = \{C_1, C_2\}$ be a partition of T. If x is an allocation in $\mathcal{R}_e SF$ then, for every $C_i \in \mathcal{R}$, almost every agent envies no coalition $S \subseteq T \setminus C_i = C_j$, with $j \neq i$. This means that there is no envy inside any $C_i \in \mathcal{R}$ and then $x \in \mathcal{R}SF$. Hence, $SF \subseteq \mathcal{R}_e SF \subseteq \mathcal{R}SF$.

Consider in the economy $\mathcal{E}$ of Example 5.2 the partition $\mathcal{R} = \{C_1, C_2\}$ of T, with $C_1 = \left(0, \frac{1}{2}\right]$ and $C_2 = \left(\frac{1}{2}, 1\right)$. It is easy to show that $x \in \mathcal{R}_e SF$. Furthermore, since, for instance, every t in $\left(0, \frac{1}{2}\right]$ envies the coalition $S = \left(0, \frac{9}{16}\right)$ which contains C_1, the allocation x does not belong to $\mathcal{R}_i SF$ and a fortiori to SF. Then,

$$x \in \mathcal{R}_e SF \setminus \mathcal{R}_i SF$$

and the inclusion $SF \subseteq \mathcal{R}_e SF$ might be strict.

Let us consider in the same economy $\mathcal{E}$ of Example 5.2 the partition $\mathcal{R} = \{C_1, C_2\}$ of T, with $C_1 = \left(0, \frac{1}{4}\right] \cup \left(\frac{3}{4}, 1\right)$ and $C_2 = \left(\frac{1}{4}, \frac{3}{4}\right]$. Given any $C_i \in \mathcal{R}$, any coalition S containing C_i is such that $\mu(S \cap D_1) \leqslant 2\mu(A \cap D_2)$. Then, the allocation x belongs to $\mathcal{R}_i SF$. On the other hand, since every t in $\left(\frac{1}{4}, \frac{1}{2}\right] \subseteq C_2$ envies the coalition $S = \left(\frac{1}{4}, \frac{9}{16}\right) \subseteq C_2 = T \setminus C_1$, the allocation x does not belong to $\mathcal{R}SF$ and a fortiori nor to $\mathcal{R}_e SF$. Thus,

$$x \in \mathcal{R}_i SF \setminus \mathcal{R}_e SF.$$

Example 2.3 of Donnini and Pesce (2021a) illustrates an economy in which there exist a partition $\mathcal{R} = \{C_1, C_2\}$ of T and an efficient allocation x such that $x \in \mathcal{R}SF \setminus \mathcal{R}_e SF$. This completes the proof.

■

As for the case of atomless economies and Theorem 4.7, in order to prove different statements of Theorem 4.8, we introduce the following example obtained as a suitable modification of Example 5.2.

Example 5.4 *Let $\mathcal{E}$ be an economy where the consumption set is $\mathbb{R}^2_{++}$ and the set of agents is $T = T_0 \cup T_1$, where $T_0 = \left(0, \frac{3}{4}\right)$ and $T_1 = \{A_1, A_2\}$ with $\mu(A_1) = \mu(A_2) = \frac{1}{8}$. The total initial endowment is $e = (1, 1)$ which equally divided among agents whose utility functions are given by*

$$u_t(x_1, x_2) = \begin{cases} x_1 x_2 & \text{if } t \in D_1 = \left(0, \frac{1}{2}\right] \\ x_1^2 x_2 & \text{if } t \in D_2 = \left(\frac{1}{2}, \frac{3}{4}\right) \cup T_1. \end{cases}$$

Consider the following feasible allocation $x : T \longrightarrow \mathbb{R}^2_{++}$

$$x(t) = (x_1(t), x_2(t)) = \begin{cases} \left(\frac{4+2\sqrt{19}}{15}, \frac{2+2\sqrt{19}}{9}\right) & \text{if } t \in D_1 \\ \left(\frac{26-2\sqrt{19}}{15}, \frac{16-2\sqrt{19}}{9}\right) & \text{if } t \in D_2. \end{cases}$$

Notice that $x \in \mathcal{A}_e$ is efficient and every agent t in D_2 is not envious, whereas each agent in D_1 envies a coalition S if and only if $\mu(S \cap D_1) > 2\mu(S \cap D_2)$.

Proof of Theorem 4.8.

(1) Let $\mathcal{R} = \{C_i\}_{i\in I}$ be a non-connected covering of T. Note that any $C_i \in \mathcal{R}$ is contained in at least one $T \setminus C_j$ with $j \neq i$. Now, if $x \in \mathcal{R}_e SF$, then there is no envy in each $C_i \in \mathcal{R}$, otherwise a subcoalition of some $T \setminus C_j$ would be envied and this is impossible. Then, $x \in \mathcal{R}SF$ and hence, $SF \subseteq \mathcal{R}_e SF \subseteq \mathcal{R}SF$. In order to show that these inclusions are strict, consider in the economy described by Example 5.4 the non-connected covering $\mathcal{R} = \{C_1, C_2\}$, with $C_i = D_i$ for $i = 1, 2$. It is easy to show that $x \in \mathcal{R}_e SF \setminus SF$. Consider now, in the same economy $\mathcal{E}$ of Example 5.4, the non-connected covering $\mathcal{R} = \{C_1, C_2, C_3\}$ with $C_1 = \left(0, \frac{1}{2}\right]$, $C_2 = \left(\frac{1}{2}, \frac{3}{4}\right)$, and $C_3 = T_1$. It is easy to show that $x \in \mathcal{R}SF \setminus \mathcal{R}_e SF$ because every t in $\left(0, \frac{1}{2}\right]$ envies the coalition $\left(\frac{1}{4}, \frac{9}{16}\right)$ which is contained in $T \setminus C_3$.

(2.1) The inclusion $SF \cap \mathcal{A}_e \subseteq \mathcal{R}_e SF \cap \mathcal{A}_e$ holds by definition. For the converse, consider an allocation x in $\mathcal{R}_e SF \cap \mathcal{A}_e$ and assume to the contrary that it is not strictly fair. Proposition 3.2 ensures that there exists a non-negligible group of agents envying atomless coalitions. Let C_i be an element of $\mathcal{R}$ such that $\mu(T_0 \cap C_i) = 0$, which exists by assumption, and note that any atomless envied coalition S is such that $\mu(S \cap C_i) = 0$. This implies that $x \notin \mathcal{R}_e SF$ which is an absurd.[16]
In order to show that the inclusion $\mathcal{R}_e SF \subseteq \mathcal{R}SF$ might be strict, consider in the economy $\mathcal{E}$ of Example 5.4 the connected covering $\mathcal{R} = \{C_1, C_2, C_3, C_4\}$ with $C_1 = \left(0, \frac{1}{4}\right) \cup A_1$, $C_2 = \left[\frac{1}{4}, \frac{1}{2}\right] \cup A_1$, $C_3 = \left(\frac{1}{2}, \frac{3}{4}\right) \cup A_2$, and $C_4 = T_1$. Note that any coalition S included in some C_i with $i = 1, 2$ is such that $\mu(S \cap D_1) \leqslant 2\mu(S \cap D_2)$, whereas any C_i with $i = 3, 4$ contains only agents of the same type. This implies that $x \in \mathcal{R}SF$. On the other hand, every agent in $\left(0, \frac{1}{2}\right]$ envies the coalition $S = \left(\frac{1}{4}, \frac{9}{16}\right)$ which is such that $\mu(S \cap C_1) = 0$ and hence $x \notin \mathcal{R}_e SF$, that is $x \in \mathcal{R}SF \setminus \mathcal{R}_e SF$.

(2.2) The equivalence $SF = \mathcal{R}SF$ in the case of connected covering for which given $A \in T_1$ and $C_i \in \mathcal{R}, A \in C_i \Rightarrow \mu(S_A \cap C_i) > 0$ is proved in Donnini and Pesce (2021a). In order to show that the inclusion $\mathcal{R}SF \subseteq \mathcal{R}_e SF$ might be strict, consider in the economy $\mathcal{E}$ of Example 5.4 the connected covering $\mathcal{R} = \{C_1, C_2\}$ with $C_1 = \left(0, \frac{1}{2}\right]$ and $C_2 = \left(\frac{1}{4}, \frac{3}{4}\right) \cup A_1 \cup A_2$. Notice that $T \setminus C_1 = D_2$ and $T \setminus C_2 = \left(0, \frac{1}{4}\right]$ which is contained in D_1. This implies that $x \in \mathcal{R}_e SF \cap \mathcal{A}_e$. On the other hand, every agent in $\left(\frac{1}{4}, \frac{1}{2}\right] \subseteq C_2$ envies the coalition $S = \left(\frac{1}{4}, \frac{9}{16}\right) \subseteq C_2$, meaning that $x \notin \mathcal{R}SF$. Hence, $x \in \mathcal{R}_e SF \setminus \mathcal{R}SF$.

(2.3) Consider in the economy $\mathcal{E}$ of Example 5.4 the connected covering $\mathcal{R} = \{C_1, C_2, C_3\}$ with $C_1 = \left(0, \frac{1}{4}\right) \cup A_1$, $C_2 = \left[\frac{1}{4}, \frac{1}{2}\right] \cup A_2$, and $C_3 = \left(\frac{1}{2}, \frac{3}{4}\right) \cup A_1 \cup A_2$. Notice that the assumptions of Theorem 4.8(2.3) are fulfilled. Indeed, $\mu(C_i \cap T_0) > 0$ for every $C_i \in \mathcal{R}$ and, for instance, the atom A_1 belongs to $C_1 \in \mathcal{R}$, but C_1 contains no piece of its atomless fringe, i.e. $\mu(C_1 \cap S_{A_1}) = 0$.[17] Moreover, any subset S of some C_i with $i = 1, 2$ is such that $\mu(S \cap D_1) \leqslant 2\mu(S \cap D_2)$, and C_3 contains only agents of the same type. This implies that $x \in \mathcal{R}SF$. Note also that, any agent in $\left(0, \frac{1}{2}\right]$ envies the coalition $S = \left(\frac{1}{4}, \frac{9}{16}\right)$ which is such that $\mu(S \cap C_1) = 0$, meaning that $x \notin \mathcal{R}_e SF$ and hence $x \in \mathcal{R}SF \setminus \mathcal{R}_e SF$.
Consider now in the same economy $\mathcal{E}$ of Example 5.4 the connected covering $\mathcal{R} = \{C_1, C_2\}$ with $C_1 = \left(0, \frac{3}{4}\right)$ and $C_2 = \left(\frac{1}{2}, \frac{3}{4}\right) \cup A_1 \cup A_2$. Notice that the assumptions of Theorem 4.8(2.3) are fulfilled

[16] Notice that the non connection of the covering is not necessary for the proof of (2.1).
[17] The same holds true for the atom A_2 and C_2.

and since $T \setminus C_1 = A_1 \cup A_2$ and $T \setminus C_2 = \left(0, \frac{1}{2}\right]$, $x \in \mathcal{R}_e SF$. However, every agent in $\left(0, \frac{1}{2}\right] \subseteq C_1$ envies the coalition $S = \left(\frac{1}{4}, \frac{9}{16}\right)$ which is contained in C_1, and hence $x \notin \mathcal{R}SF$. Thus, $x \in \mathcal{R}_e SF \setminus \mathcal{R}SF$. This completes the proof.

■

5.2 Examples

The next example underlines the role of the *equal-bundle property* in Propositions 3.2 and 3.3. We, indeed, define an economy satisfying $(A2)$ and an allocation x, assigning to identical agents different bundles lying on the same indifference curve, at which it is impossible to move the object of envy from a coalition containing atoms to an atomless coalition and for which the inclusion $R_x \subseteq I_x$ fails.

Example 5.5 *Consider a mixed economy whose consumption set is $\mathbb{R}^2_{++}$ and the set of agents $T = T_0 \cup T_1$ is composed by $T_0 = \left(0, \frac{1}{2}\right)$ and $T_1 = \{A\}$, with $\mu(A) = \frac{1}{2}$. The total initial endowment is $e = (2, 2)$ and agents' utility functions are*

$$u_t(x_1, x_2) = \begin{cases} x_1 x_2, & \text{if } t = A \text{ or } t \in \left(0, \frac{1}{4}\right) \\ x_1^2 x_2, & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right) \end{cases}$$

Consider the following feasible allocation $x : T \longrightarrow \mathbb{R}^2_{++}$

$$x(t) = (x_1(t), x_2(t)) = \begin{cases} (3, 1), & \text{if } t = A \\ (1, 3), & \text{if } t \in T_0. \end{cases}$$

Notice that $(A2)$ is satisfied, while $x \notin \mathcal{A}_e$, since the atom and its fringe receive different bundles lying on the same indifference curve. Any agent $t \in \left[\frac{1}{4}, \frac{1}{2}\right)$ envies the coalition $T \setminus \{t\}$ at x, but no atomless coalition is envied. Moreover, since $u_A(x(A)) < u_A(e(A))$, x is not individually rational and A belongs to R_x but not to I_x. Thus, $A \in R_x \setminus I_x$ and $R_x \subsetneq I_x$. Hence, even under $(A2)$, Proposition 3.3 might fail.[18]

△

In the next example, obtained by suitably modifying Example 4.3, we want to stress that also the assumption $(A2)$ is crucial for Proposition 3.2.

Example 5.6 *Consider a mixed economy whose consumption set is $\mathbb{R}^2_{++}$. The set of agents is $T = T_0 \cup T_1$, where $T_0 = \left(0, \frac{1}{4}\right)$, $T_1 = \{A_n\}_{n \in \mathbb{N}}$ with $\mu(A_n) = \frac{3}{2^{n+2}}$. The total initial endowment is $e = \left(\frac{5}{2}, \frac{3}{2}\right)$ and agents' utility functions are given by*

$$u_t(x_1, x_2) = \begin{cases} x_1^3 x_2, & \text{if } t \in T_1 \\ x_1 x_2^2, & \text{if } t \in T_0 \end{cases}$$

[18] Note that agent A is also *per-capita* envious and not *average* envious nor *strictly* envious.

Consider the following feasible allocation $x : T \longrightarrow \mathbb{R}^2_{++}$,

$$x(t) = (x_1(t), x_2(t)) = \begin{cases} (3,1), & \text{if } t = T_1 \\ (1,3), & \text{if } t \in T_0. \end{cases}$$

Notice that $x \in \mathcal{A}_e$ *and that the economy satisfies the assumption*

$(A1^*)$ *there are infinitely countably many atoms and all atoms are of the same type,*

but not $(A2)$. *Observe that no atomless coalition is envied, while every agent t in* T_0 *envies, for instance,* $S = \left[0, \frac{1}{6}\right] \cup A_4$.
Then, even under the assumption $(A1^*)$ *used in Donnini and Pesce (2020) to restore the equivalence* $SF = W_{ei}$ *in mixed economies, Proposition 3.2 might fail.* △

The following example shows that, even in atomless economy, the inclusions $R_x \subseteq I_x$ and $I_x(\mathcal{S}_\varepsilon) \subseteq I_x(\mathcal{S}_{\varepsilon^-})$ might be strict.

Example 5.7 *Consider an atomless economy with two goods, in which* $T = (0,1)$, *the total initial endowment is the vector* $e = \left(\frac{37}{10}, \frac{12}{5}\right)$, *and agents' utility functions are given by*

$$u_t(x_1, x_2) = \begin{cases} x_1^2 x_2, & \text{if } t \in \left(0, \frac{4}{5}\right] \\ x_1 x_2^3, & \text{if } t \in \left(\frac{4}{5}, \frac{9}{10}\right] \\ x_1 x_2, & \text{if } t \in \left(\frac{9}{10}, 1\right). \end{cases}$$

Consider the following feasible allocation $x : T \longrightarrow \mathbb{R}^2_{++}$,

$$x(t) = \begin{cases} (4,2), & \text{if } t \in \left(0, \frac{4}{5}\right] \\ (2,5), & \text{if } t \in \left(\frac{4}{5}, \frac{9}{10}\right] \\ (3,3), & \text{if } t \in \left(\frac{9}{10}, 1\right), \end{cases}$$

Notice that any agent in $\left(\frac{9}{10}, 1\right)$ *belongs to* I_x, *since, for instance, she envies the coalition* $S = \left(\frac{4}{5}, \frac{9}{10}\right]$, *but she does not belongs to* R_x.
Moreover agents in $\left(\frac{9}{10}, 1\right)$ *can not envy a coalition of measure* $\frac{7}{8}$. *Indeed, any coalition S of measure* $\frac{7}{8}$ *can be written as the union of the coalitions* $S_1 := S \cap \left(0, \frac{4}{5}\right]$, $S_2 := S \cap \left(\frac{4}{5}, \frac{9}{10}\right]$ *and* $S_3 := S \cap \left(\frac{9}{10}, 1\right)$. *Defined* $\alpha := \frac{\mu(S_1)}{\mu(S)}$, $\beta := \frac{\mu(S_2)}{\mu(S)}$ *and* $\gamma := \frac{\mu(S_3)}{\mu(S)}$, *being* $\gamma = 1 - \alpha - \beta$, *we get that* $\alpha \in \left[\frac{27}{35}, \frac{32}{35}\right]$, $\beta \in \left[0, \frac{4}{35}\right]$, $\gamma \in \left[0, \frac{4}{35}\right]$ *and* $\alpha + \beta \in \left[\frac{31}{35}, 1\right]$. *By easy computation, it can be shown that*

$$\bar{x}(S) = \alpha(4,2) + \beta(2,5) + \gamma(3,3) = (4\alpha + 2\beta + 3(1-\alpha-\beta), 2\alpha + 5\beta + 3(1-\alpha-\beta)) = (\alpha - \beta + 3, -\alpha + 2\beta + 3)$$

and for all $t \in \left(\frac{9}{10}, 1\right)$

$$u_t(\bar{x}(S)) = (\alpha - \beta + 3)(-\alpha + 2\beta + 3) = -\alpha^2 + 3\alpha\beta + 3\beta - 2\beta^2 + 9 \leqslant 9 = u_t(x(t)),$$

meaning that no agent t *in* $\left(\frac{9}{10}, 1\right)$ *envies a coalition* S *of measure* $\frac{7}{8}$*. Hence,* $I_x(\mathcal{S}_\varepsilon) \subsetneqq I_x(\mathcal{S}_{\varepsilon^-})$*, with* $\varepsilon = \frac{7}{8}$*, because* $\left(\frac{9}{10}, 1\right) \subseteq I_x(\mathcal{S}_{\varepsilon^-}) \setminus I_x(\mathcal{S}_\varepsilon)$*.*
Notice that $\mu(I_x(\mathcal{S}_{\varepsilon^+})) > 0$ *with* $\varepsilon = \frac{7}{8}$*, since, every agent* $t \in \left(0, \frac{4}{5}\right]$ *envies, for instance,* $T \setminus \{t\}$*. Thus the allocation* x *can not be considered* ε*-tolerable envy-free, as the one illustrated in Example 4.3.* △

In what follows we show that, even in economies with countably many atoms all of the same type (i.e. under the assumption $(A1^*)$ stated before), the *equal-bundle property* is crucial for Proposition 3.3

Example 5.8 *Consider a mixed economy whose consumption set is* $\mathbb{R}^2_{++}$*,* $T = T_0 \cup T_1$ *with* $T_0 = \left(0, \frac{1}{2}\right)$ *and* $T_1 = \{A_n\}_{n \in \mathbb{N}}$*, with* $\mu(A_n) = \frac{1}{2^{n+1}}$*. The total initial endowment is* $e = \left(\frac{7}{4}, \frac{5}{4}\right)$ *and*

$$u_t(x_1, x_2) = \begin{cases} x_1 x_2, & \text{if } t \in T_1 \\ x_1^2 x_2, & \text{if } t \in T_0 \end{cases}$$

are agents' utility functions. Consider the following feasible allocation $x : T \longrightarrow \mathbb{R}^2_{++}$

$$x = (x_1(t), x_2(t)) = \begin{cases} (1, 2), & \text{if } t = A_1 \\ (2, 1), & \text{if } t \in T_0 \text{ or } t = A_n \text{ with } n \neq 1 \end{cases}$$

Notice that $(A1^*)$ *is satisfied, while* $x \notin \mathcal{A}_e$*, since* A_1 *receives a different bundle (but lying on the same indifference curve) than the other atoms.*
Notice that x *is not individually rational because* $u_{A_1}(e(A_1)) > u_{A_1}(x(A_1))$*, but* A_1 *is not envious. Thus,* $A_1 \in R_x \setminus I_x$ *and* $R_x \subsetneqq I_x$*.* [19]

△

The relevance of the *equal-endowment assumption* in Theorem 4.4 is now pointed out. In the next example we indeed show that, in atomless economies in which different agents own different initial resources, a strictly fair allocation x might lose the individual rationality property as the inclusion $R_x \subseteq I_x$ might not hold.

Example 5.9 *Consider an atomless economy with* $T = (0, 1)$ *as the set of agents and* $\mathbb{R}^2_{++}$ *as the consumption set. Each agent is characterized by the following initial endowment and utility function*

$$e(t) = \begin{cases} \left(\frac{31}{10}, 1\right), & \text{if } t \in \left(0, \frac{3}{4}\right) \\ \left(\frac{7}{10}, 3\right), & \text{if } t \in \left[\frac{3}{4}, 1\right) \end{cases} \qquad u_t(x_1, x_2) = \begin{cases} x_1^3 x_2, & \text{if } t \in \left(0, \frac{3}{4}\right) \\ x_1 x_2^3, & \text{if } t \in \left[\frac{3}{4}, 1\right). \end{cases}$$

[19] Note that the atom A_1 is *per-capita* envious and not *average* envious nor *strictly* envious.

Notice that the equal-endowment assumption *is not satisfied and the following feasible allocation* $x : T \longrightarrow \mathbb{R}^2_{++}$,

$$x(t) = (x_1(t), x_2(t)) = \begin{cases} (3,1), & \textit{if } t \in \left(0, \frac{3}{4}\right) \\ (1,3), & \textit{if } t \in \left[\frac{3}{4}, 1\right) \end{cases}$$

is strictly envy-free, since $I_x = \emptyset$*, whereas it is not individually rational, because any* $t \in \left(0, \frac{3}{4}\right)$ *belongs to* R_x*. Thus,* $R_x \nsubseteq I_x$*.*

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