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Two characterizations of Cost Share Equilibria

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Abstract

We consider pure exchange economies with a finite number of private goods and the choice of non-Samuelsonian public goods. For this type of economies, Basile, Graziano, and Pesce (2016) proposed the notion of *cost share equilibrium* with individual payments for public goods varying according to individual benefits. This situation arises naturally when a level of provision is interpreted as a whole configuration of public policies or when cost share functions are interpreted as voluntary contributions rather than predetermined tax systems (see Mas- Colell (1980)). We establish the equivalence of cost share equilibria with cooperative and noncooperative game-theoretic solutions: 1. we characterize cost share equilibria as those allocations which cannot be improved upon by society; 2. we characterize cost share equilibria as the Nash equilibria of a game with two players. The cooperative solutions analyzed in the paper are defined via a contribution scheme which captures the fraction of the total cost of collective goods that each coalition of agents is expected to cover.

JEL classification: D49, D51, C72.

Keywords: Non-Samuelsonian public goods, Cost share equilibrium, Aubin core, Nash equilibrium.

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1 Introduction

In this paper we study competitive equilibria and their equivalence with cooperative and noncooperative game-theoretic solutions in economies where agents' private decisions are influenced by public goods or public projects. Examples include public goods provision (transport, health, education, and international public goods such as the global climate), regulation of private economic activities (regulation of quality standard, safe working conditions, trade institutions), social rules (laws, property rights) among the others. To account for different situations, we adopt the general mathematical framework proposed by Mas-Colell (1980) to represent the public sector of the economy. We do not limit choice of a public good to a set with Euclidean structure; rather, we allow public projects to be drawn from a set with no mathematical structure, i.e. a general setting which includes the case of a finite set of projects and clearly does not exclude Euclidean structure. The absence of a linear structure on the set of public projects allows treatment in particular of those public goods for which there is no reason to assume a commonly accepted order by traders. This applies to public goods that may be perceived differently by different individual, and hence may be ranked differently. Also, if public projects are understood as public environments, i.e. collections of variables common to all the agents but determined outside of market mechanisms, we end up with a general framework incorporating many different economic problems¹.

In the context described above, we focus on the concept of competitive equilibrium proposed by Basile, Graziano, and Pesce (2016) which is founded on the distribution of the total cost for the provision of the selected collective good configuration. This type of *cost share equilibrium* is more general than the linear cost share equilibrium notions explored by Mas-Colell (1980), Mas-Colell and Silvestre (1989), Gilles and Diamantaras (1998), Graziano and Romaniello (2012). In a linear cost share equilibrium, all the consumers face an equal provision, but they may pay a different contribution according to a given cost share function. Agents maximize their utility over their budget sets taking into account their share of the cost of the project and changes in the price of private commodities deriving from changes to the public project. Unlike the previous literature, we do not assume the same individual contribution for each public project and cost share equilibrium allows individual payments to vary according to individual benefits. Accordingly, on the cooperative side, the veto mechanism leading to equilibrium solutions involves a (contribution) measure, defined on the set of all coalitions, which varies across public projects.

In this paper, we provide cooperative and non-cooperative characterizations of cost share equilibria. Our two characterizations are related to each other and respectively rely on the core of the economy and the Nash equilibria of a game with two players.

In the case of the core, it is well known that in a finite exchange economy with private goods,

¹This interpretation of the Mas-Colell approach was proposed by Hammond and Villar (1998) and Hammond and Villar (1999) who consider that non-market variables include legal systems (such as assignment of property rights), tax and benefits systems, and also public sector provided private goods. For further interpretations of non-Samuelsonian collective goods represented as elements of an unstructured set, see the discussions in Diamantaras and Gilles (1996), Gilles and Diamantaras (1998), Diamantaras, Gilles, and Scotchmer (1996), Gilles and Scotchmer (1997), Basile, De Simone, and Graziano (2005), Graziano (2007), Basile, Graziano, and Pesce (2016), Gilles, Pesce, and Diamantaras (2020), Basile, Gilles, Graziano, and Pesce (2016).

the intersection of the cores of the sequence of the replications coincides with the set of competitive equilibrium allocations (Debreu and Scarf, 1963). The classical Debreu-Scarf veto system applied to replica economies is equivalent to the approach introduced in Aubin (1979), which also leads to a core that coincides with the competitive equilibria. The Aubin veto mechanism extends the notion of coalition and the ordinary veto since to form a blocking coalition it allows participation of agents with a fraction of their endowments.² In economies with public goods, both formulation of core equivalence may fail: replica economies require a large number of agents and the per capita cost of a public good is decreasing which weakens the influence of small coalitions and makes the core larger. For this reason, we consider a different veto mechanism from the classical one presented in Foley (1970). That is, for each individual cost share function, we define a measure on the set of all coalitions to fix the contribution which each blocking coalition is expected to cover. This contribution measure depends explicitly on the public project. Also, in line with Aubin's approach, we allow agents to block an allocation with a fraction of their endowments. In other words, we focus on the Aubin core notion proposed by Graziano and Romaniello (2012) for economies with public goods which turns out to be equivalent to the set of cost share equilibria (see also Basile, Graziano, and Pesce (2016)).

Our first characterization of cost share equilibria provides a refinement of the core equivalence theorem. This is achieved by exploiting the veto power of the grand coalition. Specifically, we prove that for a cost share function and the corresponding contribution measure, the cost share equilibria are precisely those allocations that cannot be blocked by a coalition in which each agent participates with a non zero fraction of her initial endowment. However, we show that the contribution of each member of a blocking coalition can not be chosen arbitrarily close to the total participation, contrary to the case of economies with no public goods (Hervés-Beloso and Moreno-García (2001)). A blocking coalition with full support is interpreted as the *society* and the result is proved using a direct approach that does not rely on Vind's theorem on the measure of a blocking coalition in a continuum economy (Vind (1972)).³ The characterization of cost share equilibria in terms of the society's blocking power is key to proving the second characterization in this paper in which cost share equilibria are connected to Nash equilibria of a two-player game. This equivalence extends to public goods economies results proved by Hervés-Beloso and Moreno-García (2009a) and Hervés-Beloso and Moreno-García (2009b), and is related to work on non-cooperative market games. It shows the equivalence between cost share equilibria and Nash equilibria of a suitable society game with implicit use of core equivalence results. The game is played by the society based on two different roles: as player 1, the society tries to achieve Pareto improvements; as player 2, it chooses an allocation which is feasible in the Aubin sense. Rather than involving money and prices, the game involves only the share of the project cost which each coalition is expected to cover. Our

²The Aubin veto has also been recently used to extend the notion of bargaining set and characterize the competitive allocations (see Liu (2017), Hervés-Beloso, Hervés-Estévez, and Moreno-García (2018), Hervés-Estévez and Moreno-García (2018b), Hervés-Estévez and Moreno-García (2018a), Graziano, Pesce, and Urbinati (2020) among others).

³Vind (1972) shows that, in a continuum economy, an allocation outside the core can be blocked by coalitions of arbitrary measure. The veto power of the grand coalition was exploited for pure exchange economies by Hervés-Beloso and Moreno-García (2001); Hervés-Beloso and Moreno-García (2008) using Vind's theorem on the measure of a blocking coalition in a continuum economy. It has been proved that Vind's theorem does not hold for economies with public projects (see Basile, Graziano, and Pesce (2016), Basile, Gilles, Graziano, and Pesce (2021)).

result contributes to work on strategic approaches to competitive equilibria in markets with public goods (see Faias, Moreno-García, and Wooders (2014), Hervés-Beloso and Moreno-García (2020) for a complete presentation). The generality of the adopted model, which imposes no mathematical structure on the set of public goods, leads to non-existence results that are discussed in Section 4.

The paper is organized as follows: Section 2 introduces the economic model, and the preliminary definitions and assumptions; Section 3 presents the main equivalence results; Section 4 discusses the existence of cost share equilibria, possible extensions of our results and suggests directions for future research. All the proofs are provided in the Appendix.

2 The economic model

We study an exchange economy \mathcal{E} with finite numbers of consumers and private goods. $I = \{1, 2, ..., n\}$ is the *set of n agents* and we take the non-negative orthant of the *m*-dimensional Euclidean space, i.e. \mathbb{R}^m_+ , as the *consumption space*.⁴ We assume the presence of *public projects*, represented as elements of an abstract set \mathcal{Y} devoid of any mathematical structure. Mas-Colell (1980) first considers an abstract set of public goods to generalize Samuelson's notion of collective good (see also Diamantaras and Gilles (1996)). The outlay of any public good is expressed in terms only of private goods, through the so-called *cost function* $c : \mathcal{Y} \to \mathbb{R}^m_+$. Every agent $i \in I$ has an initial endowment of private goods, denoted $\omega_i \in \mathbb{R}^m_+$, and a utility function which represents her consumption preferences and is denoted $u_i : \mathbb{R}^m_+ \times \mathcal{Y} \to \mathbb{R}$. Public goods cause widespread externalities since agents' utility functions depend not only on the bundle of private goods $x_i \in \mathbb{R}^m_+$, but also on the public project $y \in \mathcal{Y}$. Throughout the paper we assume that

- (A1) Each agent owns a positive initial endowment, $\omega_i > 0$ for all $i \in I$, and that each private commodity is present on the market regardless of the cost of the realized project, i.e. $\omega \gg c(y)$ for all $y \in \mathcal{Y}$, where ω denotes the total initial endowment in the economy \mathcal{E} (i.e. $\omega = \sum_{i \in I} \omega_i$).
- (A2) For any $i \in I$ and any $y \in \mathcal{Y}$, the restriction $u_i(\cdot, y) : \mathbb{R}^m_+ \to \mathbb{R}$ is continuous, strictly monotone and quasi-concave.

An **allocation** for the economy \mathcal{E} is the overall amount of the private goods assigned to each agent and the particular public project. Formally, an allocation is a pair (x, y), with $x = (x_1, \ldots, x_n) \in \mathbb{R}^{mn}_+$, where $x_i \in \mathbb{R}^m_+$ is the bundle of private commodities of agent *i*, and $y \in \mathcal{Y}$ is the public project. An allocation (x, y) is **feasible** if

$$\sum_{i=1}^n x_i + c(y) \le \sum_{i=1}^n \omega_i.$$

This means that the initial endowment is used to cover the costs of the realized project and is redistributed among the agents.

⁴We follow the standard vector inequality notation: $x \ge x'$ if $x_h \ge x'_h$ for all commodities h = 1, ..., m; x > x' if $x \ge x'$ and $x \ne x'$; and $x \gg x'$ if $x_h > x'_h$ for all commodities h = 1, ..., m.

The **cost distribution** is a function $\varphi : I \times \mathcal{Y} \to \mathbb{R}_+$ such that $\sum_{i \in I} \varphi(i, y) = 1$ for all $y \in \mathcal{Y}$, where $\varphi(i, y)$ describes how much an economic agent *i* must contribute to the cost of public project *y*. We use Φ to denote the class of all cost distribution functions.

Let Δ be the simplex of \mathbb{R}^m_+ , i.e. the set $\Delta = \left\{ p \in \mathbb{R}^m_+ | \sum_{h=1}^m p_h = 1 \right\}.$

Definition 2.1 A feasible allocation (x, y) is a **cost share equilibrium** in \mathcal{E} if there exist a price system $p : \mathcal{Y} \to \Delta$ and a cost distribution function $\varphi \in \Phi$ such that for every $i \in I$, (x_i, y) maximizes u_i on the budget set

$$B_i(p,\varphi) = \left\{ (h,z) \in \mathbb{R}^m_+ \times \mathcal{Y} \mid p(z) \cdot h + \varphi(i,z)p(z) \cdot c(z) \le p(z) \cdot \omega_i \right\}.$$

We denote by $CSE_{\varphi}(\mathcal{E})$ the set of all cost share equilibria for the cost distribution function φ and by $CSE(\mathcal{E})$ the set of all cost share equilibria in the economy \mathcal{E} , that is $CSE(\mathcal{E}) = \bigcup_{\varphi \in \Phi} CSE_{\varphi}(\mathcal{E})$.

Definition 2.1 was proposed by Basile, Graziano, and Pesce (2016) as a generalization of the notion of **linear cost share equilibrium** developed by Mas-Colell (1980) for economies with a single private good later extended to the case of multiple private commodities by Diamantaras and Gilles (1996) (see also Basile, Gilles, Graziano, and Pesce (2021) for a further generalization). A linear cost share equilibrium is obtained whenever $\varphi(i, y) = \varphi(i)$ for all $i \in I$ and all $y \in \mathcal{Y}$. The **equal cost share equilibrium** is a special case in which the cost distribution function φ is constantly equal to $\frac{1}{n}$. Thus, using $ECE(\mathcal{E})$ and $LCE(\mathcal{E})$ to denote the sets of *equal* and *linear* cost share equilibria respectively, we have

$$ECE(\mathcal{E}) \subseteq LCE(\mathcal{E}) \subseteq CSE(\mathcal{E}).$$
 (1)

It should be noted that a price system *p* depends on the public good since it incorporates possible variations in the private sector due to variations in the public good choice. The price specification for each possible public good must be known although, in equilibrium, only one public project will be realized.

The core notion in the context of economies with public goods was proposed initially by Gilles and Diamantaras (1998) and extended by Basile, Graziano, and Pesce (2016) to include mixed markets and by Basile, Gilles, Graziano, and Pesce (2021) to include production economies with endogenous social division of labor. The core notion relies on the idea of **contribution measure**. If $\mathcal{P}(I)$ denotes the power set of *I* which contains all possible coalitions, a **contribution measure** is a function $\sigma : \mathcal{P}(I) \times \mathcal{Y} \rightarrow [0, 1]$ such that for each $y \in \mathcal{Y}$, $\sigma(\cdot, y)$ is additive on $\mathcal{P}(I)$; $\sigma(\emptyset, y) = 0$ and $\sigma(I, y) = 1$. Given a coalition $S \in \mathcal{P}(I)$ and a public good $y \in \mathcal{Y}$, the vector $\sigma(S, y)c(y) \in \mathbb{R}^m_+$ indicates the total quantities of the private commodities which must be contributed by the members of *S* to provide *y*. Note that the contribution made by each coalition *S* to realize the public project *y* is not necessarily related to the size of the coalition.

Definition 2.2 Given a contribution measure σ , a feasible allocation (x, y) is σ -blocked by a coalition

S if there exists an alternative allocation (x', y') such that

(i)
$$u_i(x'_i, y') > u_i(x_i, y)$$
, for all $i \in S$ and
(ii) $\sum_{i \in S} x'_i + \sigma(S, y')c(y') \le \sum_{i \in S} \omega_i$.

The notion of blocking allows the members of *S* to improve their individual welfare using their initial endowments and proposing an alternative project y'. Notice that the feasibility over the coalition *S*, required in the blocking condition (*ii*), implies that $\sigma(S, y')c(y') \leq \sum_{i \in S} \omega_i$, thus the project y' is an admissible choice for the coalition *S* because its members are able to cover the cost of y' using their own resources. The σ -core of the economy, denoted by $C_{\sigma}(\mathcal{E})$, is the set of all the feasible allocations which are not σ -blocked by any coalition of agents.

A feasible allocation is said to be **efficient** if it is not Pareto blocked by the coalition I of all agents. Clearly, any σ -core allocation is efficient. However, since $\sigma(I, y') = 1$ for each project $y' \in \mathcal{Y}$, the efficiency notion does not depend on the contribution measure.

There is a one-to-one relationship between the cost distribution functions and the contribution measures. In details,

- given a cost distribution function φ , there is a unique contribution measure associated to φ defined as the function $\sigma_{\varphi} : \mathcal{P}(I) \times \mathcal{Y} \to [0, 1]$ such that

$$\sigma_{\varphi}(S, y) = \sum_{i \in S} \varphi(i, y), \quad \text{for all } S \in \mathcal{P}(I) \text{ and all } y \in \mathcal{Y};$$

- conversely, given a contribution measure σ , there is a unique cost distribution function φ_{σ} given by

$$\varphi_{\sigma}(i, y) = \sigma(\{i\}, y), \text{ for all } i \in I \text{ and all } y \in \mathcal{Y}.$$

Remark 2.3 The standard arguments make it possible to show that any cost share equilibrium (x, y) with a cost distribution function φ belongs to the σ_{φ} -core, i.e. $CSE_{\varphi}(\mathcal{E}) \subseteq C_{\sigma_{\varphi}}(\mathcal{E})$, and a fortiori is efficient. This inclusion may be strict, as illustrated in the next example.

Example 2.4 Consider an economy with two public goods, $\mathcal{Y} = \{y, z\}$, such that c(y) = (2, 2) and c(z) = (0, 1). Suppose there are two private goods, $\mathbb{R}^m_+ = \mathbb{R}^2_+$, and two agents, *A* and *B*, with characteristics: $\omega_A = (5, 1)$, $\omega_B = (1, 5)$, and

$$u_A(f^1, f^2, y) = u_B(f^1, f^2, y) = \sqrt{f^1} + \sqrt{f^2};$$
$$u_A(f^1, f^2, z) = u_B(f^1, f^2, z) = \sqrt{f^1} + \sqrt{f^2} - 2.$$

Consider the cost distribution $\varphi(A, t) = \varphi(B, t) = \frac{1}{2}$ for any $t \in \mathcal{Y}$, and the feasible allocation (h, y), with $h_A = (1, 1)$ and $h_B = (3, 3)$. Using computations, it can be proved easily that (h, y) belongs to the σ_{φ} -core of \mathcal{E} , where σ_{φ} is the contribution measure associated to φ . However, (h, y)

is not a φ -cost share equilibrium. Indeed, for any price system $p(y) = (1 - q, q) \in \Delta$, with $q \in [0, 1]$, the pair (g, y) defined as

$$g = (g_1, g_2) = \begin{cases} \left(4q, \frac{4(1-q)^2}{q}\right), & \text{if } q \in (0, 1) \\ (4, k), & \text{if } q = 0 \\ (4k, 0), & \text{if } q = 1 \end{cases}$$

with k > 1, belongs to the budget set $B_A(p, \varphi)$ and it is such that $u_A(g, y) > u_A(h_A, y)$. Thus, (h, y) is not a φ -cost share equilibrium, i.e. $(h, y) \in C_{\sigma_{\varphi}}(\mathcal{E}) \setminus CSE_{\varphi}(\mathcal{E})$.

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3 Cost share equilibria characterizations

In this section we prove two characterizations of cost share equilibria: a cooperative characterization in terms of Aubin core allocations and a non-cooperative equivalence of cost share equilibria as Nash equilibria of a two-player game.

3.1 Cost share equilibria and σ -Aubin core

We consider a weakening of the σ -core, by modifying the veto mechanism and enlarging the set of coalitions. We follow Aubin (1979)'s approach and allow agents to join a coalition contributing only with a fraction of their resources. The idea is to assign to each individual *i* a real number $\gamma(i) \in [0, 1]$ representing her personal portion of endowment she wants to invest in the coalition. Formally, a **generalized** or **fuzzy** coalition is any couple (γ, S) where $\gamma : I \rightarrow [0, 1]$ is a non-null function and *S* is its support, i.e. the set $\{i \in I : \gamma(i) > 0\}$. \mathcal{F} denotes the set of all generalized coalitions. A generalized coalition $(\gamma, S) \in \mathcal{F}$ is said to have **full support** if $\gamma(i) > 0$ for all $i \in I$, that is if S = I. We use $\tilde{\mathcal{F}}$ to denote the set of generalized coalitions with full support. Any coalition *S*, which in the remainder of the paper we describe as a standard or crisp coalition, can be viewed as the generalized coalition (χ_S, S) , where χ_S is its corresponding characteristic function, i.e. $\chi_S : I \rightarrow \{0, 1\}$ such that $\chi_S(i) = 1$ if $i \in S$ and $\chi_S(i) = 0$ otherwise. This allows us to enlarge the class of standard coalitions and therefore to reduce the set of unblocked allocations. We define the σ_{φ} -Aubin core notion for an economy with public projects as follows (see also Graziano and Romaniello (2012), Florenzano (1990), Noguchi (2000), Basile, Graziano, and Pesce (2016)).

Definition 3.1 Given a contribution function φ and its corresponding contribution measure σ_{φ} , a feasible allocation (x, y) is said to be σ_{φ} -blocked by a generalized coalition $(\gamma, S) \in \mathcal{F}$ if there exists an alternative allocation (x', y') such that

(i)
$$u_i(x'_i, y') > u_i(x_i, y)$$
, for all $i \in S$ and
(ii) $\sum_{i \in I} \gamma(i)x'_i + \sum_{i \in I} \gamma(i)\varphi(i, y')c(y') \le \sum_{i \in I} \gamma(i)\omega_i$.

The σ_{φ} -Aubin core of the economy \mathcal{E} , denoted by $C^{A}_{\sigma_{\varphi}}(\mathcal{E})$, is the set of all feasible allocations unblocked by any generalized coalition.

We should point out that in contrast to the classical Aubin core, the feasibility condition also involves the term $\sum_{i \in I} \gamma(i)\varphi(i, y')c(y')$, representing the contribution of the generalized coalition (γ, S) to the realization of the project $y' \in \mathcal{Y}$. This contribution takes account of the *rate of participation* of each member of the coalition. More specifically: given a cost distribution function φ , coherently with the fact that agent *i* actually participates in the blocking coalition using only the share $\gamma(i)$ of initial endowment, we assume that this agent pays towards the realization of the project in the same proportion. We note also that the feasibility constraint in Definition 2.2 refers to the contribution measure σ , while in Definition 3.1 it refers to cost distribution function φ . In fact, an extension of σ from the set of ordinary coalitions $\mathcal{P}(I)$ to the class of generalized coalitions \mathcal{F} would allow us to present the feasibility constraint in both definitions. This extension is defined for each project $z \in \mathcal{Y}$ as $\widetilde{\sigma}(\gamma, S, z) = \sum_{i=1}^{n} \gamma(i)\sigma(\{i\}, z) = \sum_{i=1}^{n} \gamma(i)\varphi(i, z)$.

Example 2.4 shows that in general, the core based on some contribution measure σ contains the set of cost share equilibria in which the individual cost shares are determined by the Radon-Nikodym derivative of σ whereas, in economies with an infinite number of agents, under certain regularity conditions, the two sets coincide (see Basile, Graziano, and Pesce (2016), Theorem 1). Extending the idea of a contribution scheme to coalitions of a more general type (generalized coalitions), a core-equivalence result can be obtained also for the case of finite economies. This leads to further characterizations of cost share equilibria and it is based on the same idea proposed by Graziano and Romaniello (2012) for linear cost share equilibria, i.e. the case in which contribution measures and cost distribution functions do not depend on public projects. Proving equivalence requires the following **essentiality condition** (see Diamantaras and Gilles (1996)), which ensures that any variation in the provision of public goods can be compensated for any agent by a suitable quantity of private goods.

(A3) Essentiality condition: for any $x \in \mathbb{R}^m_+$, y, y' in \mathcal{Y} and $i \in I$ there exists $x' \in \mathbb{R}^m_+$ such that $u_i(x', y') \ge u_i(x, y)$.⁵

Theorem 4 of Basile, Graziano, and Pesce (2016): Let φ be a cost distribution function and σ_{φ} be the corresponding contribution measure. If (A1) - (A3) are satisfied, then $CSE_{\varphi}(\mathcal{E}) = C^{A}_{\sigma_{\varphi}}(\mathcal{E})$.

This equivalence result depends explicitly on the cost distribution function and the corresponding contribution measure. As a particular case, it includes the equivalence between the equal cost share equilibria and the proportional core. It also generalizes Theorem 4.1 in Graziano and Romaniello (2012) using an approach which dispenses with the so-called *second essentiality condition* and the assumption of integrable utilities.⁶

If, for some reasons, the set of generalized coalitions $\mathcal F$ limits to the class of generalized coali-

⁵A condition similar to (A3) is assumed in Hammond and Villar (1998) and imposes a suitable restriction on \mathcal{Y} which excludes those public projects that are so bad for some agent to make compensation impossible.

⁶Revisiting the proof of Theorem 4.1 in Graziano and Romaniello (2012), we observe that there is no need for the assumption of integrable utilities borrowed from Gilles and Diamantaras (1998), since the relevant allocations in the associated continuum economy are step functions and the essentiality condition (*A*3) is satisfied.

tions with full support $\tilde{\mathcal{F}}$, the Aubin core tends to increase since $\tilde{\mathcal{F}} \subseteq \mathcal{F}$, i.e. $C^A_{\sigma_{\varphi}}(\mathcal{E}) \subseteq C^{Af}_{\sigma_{\varphi}}(\mathcal{E})$, where $C^{Af}_{\sigma_{\varphi}}(\mathcal{E})$ is the set of feasible allocations that cannot be σ_{φ} -blocked by generalized coalition with full support. In what follows, we prove that the Aubin core actually remains unchanged and still coincides with the set of cost share equilibria if a stronger version of assumption (A1) holds, i.e. if the initial endowment allows each agent to cover the cost of any public project, regardless of the cost distribution function, and to save a positive amount of each good for consumption, i.e.

$$(A1^*)$$
 $\omega_i \gg c(y)$ for all $i \in I$, and all $y \in \mathcal{Y}$.

Notice that, since for every $i \in I$ and every $y \in \mathcal{Y}$, there exists $\tilde{\varphi} \in \Phi$ with $\tilde{\varphi}(i, y) = 1$, assumption (A1^{*}) is equivalent to $\omega_i - \varphi(i, y)c(y) \gg 0$ for all $i \in I$, all $y \in \mathcal{Y}$ and all $\varphi \in \Phi$.

Theorem 3.2 Let φ be a cost distribution function and σ_{φ} be the corresponding contribution measure. If $(A1^*) - (A3)$ are satisfied, then $CSE_{\varphi}(\mathcal{E}) = C_{\sigma_{\varphi}}^{Af}(\mathcal{E})$.

Proof. See Appendix.

Theorem 4 of Basile, Graziano, and Pesce (2016) and Theorem 3.2 taken together imply that given a cost distribution function φ , $C_{\sigma_{\varphi}}^{A}(\mathcal{E}) = C_{\sigma_{\varphi}}^{Af}(\mathcal{E})$ because both coincide with the set of cost share equilibria. Theorem 3.2 rests on less strong assumptions than those in Graziano and Romaniello (2012) for the linear cost share equilibria case. It also generalizes the equivalence result in Hervés-Beloso and Moreno-García (2001) for economies with no public goods and is used in Subsection 3.2 to characterize the cost share equilibria as the Nash equilibria of a suitably defined associated game. In economies with no public goods more is true: a feasible allocation is a competitive equilibrium allocation if and only if it is not blocked by the society in the sense of Aubin with a contribution of each member arbitrarily close to the total participation. This rests on Vind's theorem on the measure of a blocking coalition in an associated continuum economy and, in particular, implies that for any feasible allocation blocked by a generalized coalition (γ , I) with full support, the participation rates $\{\gamma(i)\}_{i\in I}$ can be chosen arbitrarily close to one (see Hervés-Beloso, Moreno-García, and Yannelis (2005)). In what follows, we show that this equivalence cannot be extended to economies with public goods. The reason resides in the fact that Vind's theorem may not hold in our framework (see Basile, Graziano, and Pesce (2016), Basile, Gilles, Graziano, and Pesce (2021)).

Example 3.3 Consider an economy with one private good, i.e. m = 1, two public goods, i.e. $\mathcal{Y} = \{y, z\}$ with $c(y) = \frac{1}{2}$ and $c(z) = \frac{3}{4}$; and three agents, i.e. $I = \{A, B, C\}$. For all $i \in I$, let $\omega_i = 1$ and $u_i(f, t) = f$ for all $t \in \mathcal{Y}$. Consider the following cost distribution function

$$\varphi(i,y) = \begin{cases} 1, & if & i = A \\ 0, & if & i \in \{B,C\} \end{cases} \text{ and } \varphi(i,z) = \begin{cases} 1/9, & if & i = A \\ 6/9, & if & i = B \\ 2/9, & if & i = C, \end{cases}$$

and the allocation (f, y), with $f_i = \begin{cases} 0.5, & \text{if } i = A \\ 1, & \text{if } i \in \{B, C\}. \end{cases}$

Note that (f, y) is feasible and it is blocked by the following generalized coalition (γ, I) with full support

$$\gamma(i) = \begin{cases} 1, & if \ i \in \{A, C\} \\ \frac{1}{14}, & if \ i = B, \end{cases} \text{ via the allocation } (g, z), \text{ with } g_i = \begin{cases} 0.6, & if \ i = A \\ 1.2, & if \ i = B \\ 1.1, & if \ i = C. \end{cases}$$

On the other hand, we now show that, for any $\alpha \ge \frac{5}{8}$, (f, y) cannot be blocked in the Aubin sense by a generalized coalition (γ, I) with a participation rate of each member $\gamma(i) \ge \alpha$. Indeed, assume to the contrary that this is possible via an alternative allocation (g, t) with t = y. Then, in particular,

$$g_A > 0.5 = f_A$$

$$g_B > 1 = f_B$$

$$g_C > 1 = f_C$$

$$\gamma(A)g_A + \gamma(B)g_B + \gamma(C)g_C + \frac{1}{2}\gamma(A) \le 1(\gamma(A) + \gamma(B) + \gamma(C)),$$

which implies this contraddiction

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$$\frac{1}{2}\gamma(A)+\gamma(B)+\gamma(C)+\frac{1}{2}\gamma(A)<\gamma(A)+\gamma(B)+\gamma(C).$$

Similarly, if t = z then, in particular,

$$\begin{split} g_A &> 0.5 = f_A \\ g_B &> 1 = f_B \\ g_C &> 1 = f_C \\ \gamma(A)g_A + \gamma(B)g_B + \gamma(C)g_C + \frac{3}{4}(\frac{1}{9}\gamma(A) + \frac{6}{9}\gamma(B) + \frac{2}{9}\gamma(C)) \leq 1(\gamma(A) + \gamma(B) + \gamma(C)), \end{split}$$

which implies this contraddiction

$$\frac{24}{36}\alpha \leq \left(\frac{18}{36}\gamma(B) + \frac{6}{36}\gamma(C)\right) < \frac{15}{36}\gamma(A) \Rightarrow \gamma(A) > \frac{24\alpha}{15} \geq 1.$$

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3.2 Cost share equilibria as Nash equilibria of a game

For an economy with no public projects, Hervés-Beloso and Moreno-García (2009a), Hervés-Beloso and Moreno-García (2009b) provide a characterization of Walrasian equilibria in terms of the Nash equilibria in an associated two-player game, the **society game**, in which each player represents a role of the society, the outcomes are given by the strategies and neither the strategy sets nor the payoff functions contain prices. It has been shown that, independently of the number of consumers and commodities, a Walrasian equilibrium is implementable as a strong Nash equilibrium in the associated game. We extend this to economies with an abstract set of public projects. Using Theorem 3.2 and similar characterization, we define a two-player game associated to the economy and we prove that the cost share equilibria are exactly the Nash equilibria of the associated game. To our knowledge, this is the first attempt to provide a game theoretical interpretation of cost share equilibria. It implies characterizations for the linear and equal cost share equilibria defined by Gilles and Diamantaras (1998) in the general setting of Mas-Colell (1980).

Throughout this section, we consider only economies \mathcal{E} and cost distribution functions $\varphi \in \Phi$ which satisfy assumptions (A1^{*}), (A2) and (A3). We construct a game G_{φ} associated to \mathcal{E} , with two players $N = \{1, 2\}$. The strategy set of player 1 denoted S_1 , is given by the set of feasible allocations in \mathcal{E} , i.e.

$$S_1 = \left\{ (x, y) = (x_1, \dots, x_n, y) \in \mathbb{R}^{nm}_+ \times \mathcal{Y}, \text{ such that } \sum_{i=1}^n x_i + c(y) \le \sum_{i=1}^n \omega_i \right\}.$$

Assumption (A1), in particular $\omega \gg c(y)$ for all $y \in \mathcal{Y}$, ensures that S_1 is non-empty since it contains, for instance, the feasible allocation $\left(\frac{1}{n}(\omega - c(y)), y\right) \in S_1$.

The strategy set of player 2, denoted S_2 , is given by

$$S_2 = \left\{ (\gamma, x, y) = (\gamma_1, \dots, \gamma_n, x_1, \dots, x_n, y) \in (0, 1]^n \times \mathbb{R}^{nm}_+ \times \mathcal{Y} \text{ s.t. } \sum_{i=1}^n \gamma_i \varphi(i, y) c(y) \le \sum_{i=1}^n \gamma_i \omega_i \right\}.$$

The strategy set for player 2 allows an allocation $(x, y) \in \mathbb{R}^{nm}_+ \times \mathcal{Y}$ to be defined which satisfies the feasibility (*ii*) of Definition 3.1 with a participation rate γ_i for every member $i \in I$. This implies $\gamma_i > 0$ for all $i \in I$ and hence also allows definition of a generalized coalition (γ, I) with full support. Observe that S_2 is a non-empty set since $(\mathbf{1}, \frac{1}{n} (\omega - c(y)), y) \in S_2$, where **1** is the vector in \mathbb{R}^n whose coordinates are constant and equal to 1. Notice also that the society in its two different roles may choose a different project.

Using *S* for the product set $S_1 \times S_2$, a strategy profile is any $s = (x, y, \gamma, g, z) \in S$, where $(x, y) \in S_1$ is the player 1 strategy and $(\gamma, g, z) \in S_2$ is the player 2 strategy.

Given a strategy profile $s = (x, y, \gamma, g, z) \in S$, the respective payoff functions, F_1 and F_2 , for players 1 and 2, are defined as follows

$$F_1(x, y, \gamma, g, z) = \min_{i=1,...,n} \{ f(\gamma_i) (u_i(x_i, y) - u_i(g_i, z)) \}$$

and

$$F_2(x, y, \gamma, g, z) = \min_{i=1,...,n} \{ \gamma_i (u_i(g_i, z) - u_i(x_i, y)) \},\$$

where *f* is a positive differentiable function defined in (0, 1] and such that f'(x)x > f(x). Observe that $\frac{f(x)}{x}$ is a positive and strictly increasing function and, therefore, max $\left\{\frac{f(x)}{x}\right\} = f(1)$.

The associated game G_{φ} then is defined by $G_{\varphi} \equiv \{S_1, S_2, F_1, F_2\}$ and the Nash equilibrium is defined as follows.

Definition 3.4 A strategy profile $s^* = (x^*, y^*, \gamma^*, g^*, z^*) \in S$ is a Nash equilibrium for G_{φ} if

$$F_1(s^*) \geq F_1(x, y, \gamma^*, g^*, z^*), \text{ for every } (x, y) \in S_1, \text{ and}$$

$$F_2(s^*) \geq F_2(x^*, y^*, \gamma, g, z), \text{ for every } (\gamma, g, z) \in S_2.$$

 $NE(G_{\varphi})$ is the set of Nash equilibria for the game G_{φ} .

Notice that for any $(x, y) \in S_1$, $(\mathbf{1}, x, y) \in S_2$ and $F_1(x, y, \gamma, x, y) = F_2(x, y, \gamma, x, y) = 0$ for any $\gamma \in (0, 1]^n$ such that $(\gamma, x, y) \in S_2$. In addition, the following interesting properties listed in the next proposition hold.

Proposition 3.5 Given $s \in S$,

- (1) $F_n(s) > 0 \Rightarrow F_m(s) < 0$ with $n \neq m$.
- (2) $F_2(s^*) \ge 0$ for any $s^* \in NE(G_{\varphi})$.
- (3) If (x, y) is a feasible not Pareto optimal allocation in E, there exists (x', y') ∈ S₁ such that F₁(x', y', γ, g, z) > F₁(x, y, γ, g, z) for all (γ, g, z) ∈ S₂. Vice versa, if (x, y) is Pareto optimal in E and (γ, x, y) ∈ S₂ for some γ ∈ (0, 1]ⁿ, then F₁(x', y', γ, x, y) ≤ F₁(x, y, γ, x, y) = 0 for all (x', y') ∈ S₁.

(4)
$$(x,y) \in C^{Af}_{\sigma_{\varphi}}(\mathcal{E}) \iff F_2(x,y,\gamma,g,z) \le 0 \text{ for all } (\gamma,g,z) \in S_2.$$

(5)
$$(x, y) \in CSE_{\varphi}(\mathcal{E}) \Rightarrow (x, y, \mathbf{1}, x, y) \in NE(G_{\varphi}).$$

Proof. See Appendix.

Condition (1) means that the payoffs for both players can not be simultaneously positive whereas condition (2) implies that player 2's payoff is non-negative at any Nash equilibrium. It follows from (3) that if (x, y) is not Pareto optimal, player 1 can improve upon her payoff whereas if player 2 selects $(\gamma, x, y) \in S_2$ and (x, y) is Pareto optimal for the economy \mathcal{E} , then player 1's best response will be the same efficient allocation (x, y). Theorem 3.2 together with condition (4) ensures that if (x, y) is a cost share equilibria, player 2 gets a non-positive payoff regardless of her strategy. Finally, condition (5) gives a first relation between the set of the cost share for \mathcal{E} and the Nash equilibria of the associated game G_{φ} .

The next proposition shows that at a Nash equilibrium both players achieve the same zero payoff although their chosen public projects might be different.

Proposition 3.6 Assume that for all $i \in I$, $u_i(0, \cdot) : \mathcal{Y} \to \mathbb{R}$ is a constant function and that for all $i \in I$ and for all $y \in \mathcal{Y}$, $u_i(\cdot, y)$ is concave. If s^* is a Nash equilibrium for the game G_{φ} , then $F_1(s^*) = F_2(s^*) = 0$.

Proof. See Appendix.

We can now characterize the φ -cost share equilibria for economy \mathcal{E} as the Nash equilibria of the associated game G_{φ} .

Theorem 3.7 Let \mathcal{E} be an economy with public goods and φ be a cost distribution function. Assume that for all $i \in I$, $u_i(0, \cdot) : \mathcal{Y} \to \mathbb{R}$ is a constant function and that for all $i \in I$ and for all $y \in \mathcal{Y}$, $u_i(\cdot, y)$ is concave. Then, $s^* = (x^*, y^*, \gamma^*, g^*, z^*) \in NE(G_{\varphi}) \implies (x^*, y^*) \in CSE_{\varphi}(\mathcal{E}).$ Reciprocally, $(x^*, y^*) \in CSE_{\varphi}(\mathcal{E}) \implies s^* = (x^*, y^*, \gamma^*, g^*, z^*) \in NE(G_{\varphi}), \text{ for any}$ $s^* = (x^*, y^*, \gamma^*, g^*, z^*) \in S$ with $u_i(g_i^*, z^*) = u_i(x_i^*, y^*)$ for every $i \in I$. In particular, $(x^*, y^*) \in CSE_{\varphi}(\mathcal{E}) \iff (x^*, y^*, \gamma^*, x^*, y^*) \in NE(G_{\varphi}), \text{ with } \gamma_i^* = \gamma^* \text{ for every } i \in I.$

Proof. See Appendix.

Propositions 3.5 (1) and 3.6 show that, since for each cost distribution function φ , the game G_{φ} has only two players and there are no incentives for the coalition between these two players to deviate, any Nash equilibrium will be a strong Nash equilibrium in the game G_{φ} . Therefore, Theorem 3.7 allows the cost share equilibria to be implemented as a strong Nash equilibrium of the society game.

4 Extensions and final results

The cost share equilibrium for economies with non-Samuelsonian collective goods studied in our paper originates from the linear cost share equilibrium proposed by Gilles and Diamantaras (1998). In a linear cost share equilibrium, all agents optimize given a certain share of the cost of the provision of public goods in the economy. Each agent pays a fraction of the total costs of the public goods. In a cost share equilibrium, we also assume that the cost shares depend on the public goods configuration. This implies that for a certain contribution scheme φ , the fraction contributed by agent *i* under the project $z \in \mathcal{Y}$ is defined as $\varphi(i, z)p(z) \cdot c(z)$, where p(z) is the conjectural price system and c(z) the cost. Then, for a certain contribution scheme φ , the corresponding cost share equilibria are equivalent to Aubin σ_{φ} -core allocations. The contribution measure σ_{φ} defines the contribution made by the coalition as the sum of the individual cost shares weighted by the share of participation of the agents in the coalition. We also provided a characterization of more general cost share equilibria in terms of Nash equilibria in a two-player society game.

Below we discuss implications and extensions of our results.

4.1 Existence of cost share equilibria

The literature does not provide a general existence theorem of cost share equilibrium; however Gilles and Diamantaras (1998) analyze several examples of economies in which specific linear cost share equilibria exist. Since cost share equilibria are more general than linear cost share equilibria, these examples also support the notion of cost share equilibrium. Here, we provide examples of a well-behaved economy with public projects which, despite the standard assumptions being satisfied, exhibits an empty set of cost share equilibria and, consequently, an empty set of linear cost share equilibria.

Proposition 4.1 The set of cost share equilibria may be empty, i.e. $CSE(\mathcal{E}) = \bigcup_{\varphi \in \Phi} CSE_{\varphi}(\mathcal{E}) = \emptyset$.

Proof. See Appendix.

This prompts exploration and provision of conditions that ensure the existence of a cost share equilibrium. We notice that the Aubin core-cost share equilibrium equivalence theorem combined with a direct proof of the existence of Aubin core allocations might allow proof of the existence of cost share equilibrium. Gilles and Scotchmer (1997) suggest a necessary and sufficient condition for the existence of an equilibrium (see Theorem 2) in terms of *efficient scale*. Their existence result is driven by an Edgeworth like equivalence theorem for replica economies and assumes that the cost of the projects takes the form of a multifunction which cannot be reduced to a function for the assumptions they make.

Allouch and Predtetchinski (2008) provide an elementary proof of non-emptyness of the Aubin core in a pure exchange economy. Unlike the Debreu and Scarf (1963) result and its numerous extensions, their proof does not require any asymptotic intersection and instead of allowing the economy to become large through replication, they enlarge the set of feasible payoffs for the economy in the utility space. Hence, the result is established directly using Fan's coincidence theorem. We believe that the elementary arguments related to this approach can be adapted and extended to economies with non-Samuelsonian collective goods. A direct proof of the existence of the Aubin core is relevant in our framework since it would allow us to exploit the core-equilibrium equivalence in Theorem 4 of Basile, Graziano, and Pesce (2016) to prove the existence of cost share equilibria. We leave this issue to future research.

4.2 The case of possibly non-linear cost distribution

In Basile, Gilles, Graziano, and Pesce (2021), the notion of cost share equilibrium and the corresponding core are extended by replacing the scaling cost φ and the corresponding contribution σ with a multi-dimensional cost and a multi-dimensional contribution measure. This allows for highly nonlinear scaling of the costs of individual and coalitional provision, which extends the theory. The resulting equilibrium notion, which here we describe as generalized, is defined as follows.

A multi-dimensional cost distribution is a function $\varphi: I \times \mathcal{Y} \to \mathbb{R}^m_+$ such that

$$\sum_{i\in I} \varphi(i,z) = c(z), \text{ for each } z \in \mathcal{Y}.$$

Similarly, a multi-dimensional contribution measure is defined as an additive function $\sigma : \mathcal{P}(I) \times \mathcal{Y} \to \mathbb{R}^m_+$ such that $\sigma(I, z) = c(z)$, for each $z \in \mathcal{Y}$. Moreover, as in the scalar case, there is a one-to-one relationship between the cost distribution functions and the contribution measures.

Definition 4.2 A feasible allocation $(x_1, ..., x_n, y)$ is a generalized cost share equilibrium in \mathcal{E} if there exist a price system $p : \mathcal{Y} \to \Delta$ and a multi-dimensional cost distribution function φ such that for every $i \in I$, (x_i, y) maximizes u_i on the budget set

$$B_i(p,\varphi) = \left\{ (h,z) \in \mathbb{R}^m_+ \times \mathcal{Y} \mid p(z) \cdot h + p(z) \cdot \varphi(i,z) \le p(z) \cdot \omega_i \right\}.$$

 $GCS(\mathcal{E})$ is the collection of all generalized cost share equilibria in the economy \mathcal{E} and, for a fixed cost distribution function φ , $GCE_{\varphi}(\mathcal{E})$ the set of generalized cost share equilibria with associated cost distribution φ . Then it becomes clear that the inclusions $ECE(\mathcal{E}) \subseteq LCE(\mathcal{E}) \subseteq CSE(\mathcal{E}) \subseteq$ $GCE(\mathcal{E})$ hold true since each scalar cost distribution φ generates a multi-dimensional cost distribution based on the product $\varphi(i, z)c(z)$.

Definition 4.3 Let φ be a multi-dimensional cost distribution function and σ_{φ} be the relative multi-dimensional contribution measure. A feasible allocation (x_1, \dots, x_n, y) is said to be σ_{φ} -Aubin blocked if it is possible to find a fuzzy coalition $(\gamma, S) \in \mathcal{F}$ and an allocation (g, z) such that

$$\sum_{i=1}^{n} \gamma_{i} g_{i} + \sum_{i=1}^{n} \gamma_{i} \varphi(i, z) \leq \sum_{i=1}^{n} \gamma_{i} \omega_{i}$$

 $u_i(g_i,z) > u_i(x_i,y), \quad \forall i \in S.$

The generalized σ_{φ} -Aubin core of the economy, denoted $GC^{A}_{\sigma_{\varphi}}(\mathcal{E})$ is defined accordingly.

It is easy to verify that all the results proved in Sections 3 and 4.1 hold also for the generalized cost share equilibria. In particular, given a multi-dimensional cost distribution φ and the corresponding measure σ_{φ} , the equivalence $GCE_{\varphi}(\mathcal{E}) = GC^{A}_{\sigma_{\varphi}}(\mathcal{E})$ holds true. The proof of the equivalence depends on the same argument in Theorem 4 of Basile, Graziano, and Pesce (2016) and makes use of the equivalence between generalized σ -core allocations and cost share equilibria proved in the case of an atomless economy with public goods in Basile, Gilles, Graziano, and Pesce (2021).

Important to note here is that although the set of equilibria is larger under possibly non-linear cost contributions, existence is still not guaranteed. We demonstrate this below by adapting Proposition 4.1 to generalized cost share equilibria.

Proposition 4.4 The set of generalized cost share equilibria may be empty, i.e. $GCE(\mathcal{E}) = \bigcup_{\varphi \in \Phi} GCE_{\varphi}(\mathcal{E}) = \emptyset.$

Proof. See Appendix.

4.3 The case of mixed markets

Basile, Graziano, and Pesce (2016) considers models of economies that involve both small and large traders and the choice of a public project. The main elements in their *mixed markets* are an atomless sector of consumers representing the ocean of negligible traders, a set of atoms representing the influential agents, and a contribution scheme which specifies the cost of the provision of the public good for both individual agents and coalitions. Negligible and influential agents are defined with respect to the size measure in the agents' space. Small and large contributors are defined similarly with respect to the measure underlying the contribution scheme. They do not assume any mathematical structure for the set of public projects.

Within this framework and using the Aubin approach to cooperation, they establish that the set of φ -cost share equilibria coincides with the σ -core (in the case of finite economies, our Theorem 4 of Basile, Graziano, and Pesce (2016)). The equivalence between Walrasian equilibria of a

pure exchange economy and Nash equilibria of a society game was proved by Hervés-Beloso and Moreno-García (2009a) in the case also of mixed markets. Similarly, we expect that our characterization in terms of Nash equilibria can be extended to cost share equilibria of mixed economies with an abstract set of public projects.

Finally, since cost share equilibria of a mixed markets with finitely many atoms are in a oneto-one correspondence with cost share equilibria of finite economies, our results show also that we cannot expect a general existence theorem for cost share equilibria in mixed markets.

4.4 The case of infinitely many commodities

Graziano (2007)'s paper deals with the two fundamental theorems of welfare economics for production economies with a finite set of agents, infinitely many private goods, and a set of public projects. The problem of efficiency and decentralization is addressed using very general assumptions and the set of public projects does not have a mathematical structure. In particular, the welfare theorems impose interiority assumptions on the commodity space or classical properness assumptions on preferences and production sets, assuming that the commodity space enjoys a Riesz space structure. This requirement combined with the lattice structure of the price space seems to be indispensable to carry over lattice theoretical arguments connected with properness conditions.

The notion of cost share equilibria can be formulated at the same level of economic generality, not only on the public goods sector of the model but also on its private goods counterpart. This level of generality allows among other things investigation of infinite-horizon economies, asset pricing models, differentiated commodity models, and allocation problems. Aubin core equivalence results for pure exchange economies with infinitely many commodities are not novel in the literature (see Noguchi (2000), Khan and Sagara (2022) among others). Hence we expect that the optimality properties of cost share equilibria will hold true in this more general setting. Clearly in this context the non-existence problems that emerge are even more severe.

5 Appendix

5.1 **Proofs of section 3**

Proof of Theorem 3.2. The inclusion $CSE_{\varphi}(\mathcal{E}) \subseteq C_{\sigma_{\varphi}}^{Af}(\mathcal{E})$ is always met. In the opposite case, let (x, y) be an allocation that can not be σ_{φ} -blocked by a generalized coalition with full support and let us prove that it belongs to $C_{\sigma_{\varphi}}^{A}(\mathcal{E})$. The conclusion will follow from Theorem 4 of Basile, Graziano, and Pesce (2016). Assume, by contradiction, that there exist an allocation (g, z) and a generalized coalition (y, S), with $I \setminus S \neq \emptyset$, such that

(i)
$$\sum_{i=1}^{n} \gamma(i)g_i + \sum_{i=1}^{n} \gamma(i)\varphi(i,z)c(z) \le \sum_{i=1}^{n} \gamma(i)\omega_i$$

(ii)
$$u_i(g_i,z) > u_i(x_i,y), \quad \forall i \in S.$$

Notice that, since $S = \{i \in I : \gamma(i) > 0\}$, (*i*) is equivalent to

(i)
$$\sum_{i\in S} \gamma(i)g_i \leq \sum_{i\in S} \gamma(i)\omega_i^*$$

where $\omega_i^* = \omega_i - \varphi(i, z)c(z)$ and $\omega_i^* \gg 0$ by (A1^{*}). Then, using the standard arguments, without loss of generality (*i*) can be rewritten as

$$\sum_{i\in\mathcal{S}}\gamma(i)g_i\ll\sum_{i\in\mathcal{S}}\gamma(i)\omega_i^*.$$
(2)

By (A3), for each $i \in I \setminus S$ there exists $x'_i \in \mathbb{R}^m_+$ such that $u_i(x'_i, z) \ge u_i(x_i, y)$ and hence, by monotonicity (A2), given a positive vector $K \in \mathbb{R}^m_{++}$, $u_i(x'_i + K, z) > u_i(x_i, y)$ for all $i \in I \setminus S$.

Let $v = \sum_{i \in S} \gamma(i)\omega_i - \sum_{i \in S} \gamma(i)\varphi(i,z)c(z) - \sum_{i \in S} \gamma(i)g_i = \sum_{i \in S} \gamma(i)\omega_i^* - \sum_{i \in S} \gamma(i)g_i \gg 0$ and $t = \sum_{i \in I \setminus S} [x'_i + K + \varphi(i,z)c(z) - \omega_i]$. Let $\varepsilon \in (0,1)$ be such that $\varepsilon t \leq v$, whose existence follows from the fact that $v \gg 0$ (see (2)). Define $\tilde{g} = g\chi_S + (x' + K)\chi_{I\setminus S}$ and the generalized coalition with full support $(\tilde{\gamma}, I)$, where $\tilde{\gamma} = \gamma\chi_S + \varepsilon\chi_{I\setminus S}$. Then, $u_i(\tilde{g}_i, z) > u_i(x_i, y)$ for all $i \in I$, and

$$\sum_{i\in I} \tilde{\gamma}(i)\tilde{g}_i + \sum_{i\in I} \tilde{\gamma}(i)\varphi(i,z)c(z) - \sum_{i\in I} \tilde{\gamma}(i)\omega_i = -v + \varepsilon t \le 0.$$

Therefore, (x, y) is blocked by the generalized coalition with full support $(\tilde{\gamma}, I)$ via the alternative pair (\tilde{g}, z) , and this is a contradiction.

Proof of Proposition 3.5. Condition (1) follows directly from the definition of players' payoff functions. Let us prove (2). Let $s^* = (x^*, y^*, \gamma^*, g^*, z^*) \in S$ be a Nash equilibrium for the game G_{φ} . By Definition 3.4, $F_2(s^*) \geq F_2(x^*, y^*, \gamma, g, z)$ for all $(\gamma, g, z) \in S_2$, and in particular $F_2(s^*) \geq S_2$ $F_2(x^*, y^*, \mathbf{1}, x^*, y^*) = 0$. To prove (3), let (x, y) be a feasible allocation, i.e. $(x, y) \in S_1$, which is not Pareto optimal in the economy \mathcal{E} , then there exists an alternative feasible allocation (x', y')such that $u_i(x'_i, y') > u_i(x_i, y)$ for all $i \in I$. This means that there exists $(x', y') \in S_1$ such that $u_i(x'_i, y') - u_i(g_i, z) > u_i(x_i, y) - u_i(g_i, z)$ for all $i \in I$ and for all $(\gamma, g, z) \in S_2$. Hence, since f is a positive function, we have $F_1(x', y', \gamma, g, z) > F_1(x, y, \gamma, g, z)$ for all $(\gamma, g, z) \in S_2$. Conversely, if (x, y) is Pareto optimal in \mathcal{E} and $(\gamma, x, y) \in S_2$ for some $\gamma \in (0, 1]^n$, then $(x, y) \in S_1$ and 0 = $F_1(x, y, y, x, y) \ge F_1(x', y', y, x, y)$ for all $(x', y') \in S_1$, ortherwise we contradict the efficiency of (x, y). This concludes the proof of (3). Condition (4) follows from Definition 3.1 and the players' payoff functions. Finally, to prove (5), let (x, y) be a cost share equilibrium. Then (x, y) is feasible and hence $(x, y) \in S_1$ and $(1, x, y) \in S_2$. Since (x, y) is Pareto optimal, it follows from (3) that $F_1(x', y', \mathbf{1}, x, y) \le F_1(x, y, \mathbf{1}, x, y) = 0$ for all $(x', y') \in S_1$. On the other hand, Theorem 3.2 implies⁷ that $(x, y) \in C^{Af}_{\sigma_{\varphi}}(\mathcal{E})$ and from (4) that $F_2(x, y, \gamma, g, z,) \leq F_2(x, y, \mathbf{1}, x, y) = 0$ for all $(\gamma, g, z) \in S_2$. Therefore, $(x, y, \mathbf{1}, x, y)$ is a Nash equilibrium for the game G_{φ} .

The proof of Proposition 3.6 needs the following lemma which exploits the same arguments used for Lemma 4.2 in Hervés-Beloso and Moreno-García (2009b).

⁷Recall that throughout Section 3.2 we assume that (A1^{*}), (A2) and (A3) hold.

Lemma 5.1 Assume that for all $i \in I$, $u_i(0, \cdot) : \mathcal{Y} \to \mathbb{R}$ is a constant function and that for all $i \in I$ and for all $y \in \mathcal{Y}$, $u_i(\cdot, y)$ is concave. If $s^* = (x^*, y^*, \gamma^*, g^*, z^*)$ is a Nash equilibrium of the game G_{φ} , then

$$\gamma_i^*(u_i(g_i^*, z^*) - u_i(x_i^*, y^*)) = \gamma_j^*(u_j(g_j^*, z^*) - u_j(x_j^*, y^*)), \text{ for every } i, j \in I.$$

Proof. Let $s^* = (x^*, y^*, \gamma^*, g^*, z^*)$ be a Nash equilibrium. By (2) of Proposition 3.5, $F_2(s^*) \ge 0$. Define the non-empty set

$$B(s^*) = \left\{ i \in I : \gamma_i^*(u_i(g_i^*, z^*) - u_i(x_i^*, y^*)) = \min_{k \in I} \{\gamma_k^*(u_i(g_k^*, z^*) - u_k(x_k^*, y^*))\} = F_2(s^*) \right\}.$$

We need to show that $B(s^*) = I$. Suppose to the contrary that there exists an agent $j \in I \setminus B(s^*)$, for which $\gamma_j^*(u_j(g_j^*, z^*) - u_j(x_j^*, y^*)) > F_2(s^*) \ge 0$. Then, by monotonicity of $u_j(\cdot, z^*)$, we have that $u_j(g_j^*, z^*) > u_j(x_j^*, y^*) \ge u_j(0, y^*) = u_j(0, z^*) \Rightarrow g_j^* > 0$, whereas, by continuity of $u_j(\cdot, z^*)$, there exists some $\varepsilon \in (0, 1)$ such that $\gamma_j^*(u_j(\varepsilon g_j^*, z^*) - u_j(x_j^*, y^*)) > F_2(s^*)$. Define $\delta = \frac{\gamma_j^*(1-\varepsilon)g_j^*}{\sum_{i\in B(s^*)} \gamma_i^*} > 0$ and g' as

$$g'_{i} = \begin{cases} \varepsilon g_{j}^{*}, & \text{if } i = j \\ g_{i}^{*} + \delta, & \text{if } i \in B(s^{*}) \\ g_{i}^{*}, & \text{otherwise.} \end{cases}$$

It can easily be proved that $(\gamma^*, g', z^*) \in S_2$ and by monotonicity $F_2(x^*, y^*, \gamma^*, g', z^*) > F_2(s^*)$ which is an absurd since s^* is a Nash equilibrium. Thus, $B(s^*) = I$.

We are now ready to prove Proposition 3.6.

Proof of Proposition 3.6. Let $s^* = (x^*, y^*, \gamma^*, g^*, z^*)$ be a Nash equilibrium for the game G_{φ} which is, by Lemma 5.1, such that

$$F_2(s^*) = \gamma_i^* (u_i(g_i^*, z^*) - u_i(x_i^*, y^*)) = C \text{ for all } i \in I,$$
(3)

where $C \ge 0$ according to Proposition 3.5 (2). Then we need to prove that C = 0. We proceed by the way of contradiction and assume that C > 0. Note that $\tilde{\gamma} := \max_i \{\gamma_i^*\} = 1$, otherwise $F_2\left(x^*, y^*, \frac{\gamma^*}{\tilde{\gamma}}, g^*, z^*\right) > F_2(s^*)$. Since $\frac{f(\gamma)}{\gamma}$ is a positive and strictly increasing function, by (3), we get that

$$F_1(s^*) = \min_i \{ f(\gamma_i^*) (u_i(x_i^*, y^*) - u_i(g_i^*, z^*)) \} = \min_i \left\{ f(\gamma_i^*) \frac{-C}{\gamma_i^*} \right\} = -Cf(1) < 0,$$
(4)

i.e. $F_1(s^*) < 0 < F_2(s^*)$. Notice that there are $i \in I$ such that $\gamma_i^* < 1$, otherwise $(g^*, z^*) \in S_1$ and $0 = F_1(g^*, z^*, \gamma^*, g^*, z^*) \le F_1(s^*) < 0$, which is a contradiction. Furthermore,

$$\gamma_i^* < 1 \Longrightarrow f(\gamma_i^*)(u_i(x_i^*, y^*) - u_i(g_i^*, z^*)) > F_1(s^*).$$
(5)

We now can show that for each $\gamma_i^* < 1 \implies x_i^* \neq 0$. Suppose to the contrary that the set $A := \{i \in I : \gamma_i^* < 1 \text{ and } x_i^* = 0\}$ is not empty, and let $\varepsilon \in (0, 1)$ be such that $\frac{\gamma_i^*}{\varepsilon} \le 1$ for all $i \in A$. Notice that, since $u_i(\cdot, z)$ is concave, we have that for all $i \in A$,

$$u_{i}(\varepsilon g_{i}^{*}, z^{*}) = u_{i}(\varepsilon g_{i}^{*} + (1 - \varepsilon)x_{i}^{*}, z^{*}) \ge \varepsilon u_{i}(g_{i}^{*}, z^{*}) + (1 - \varepsilon)u_{i}(x_{i}^{*}, z^{*}) =$$

$$= \varepsilon u_{i}(g_{i}^{*}, z^{*}) + (1 - \varepsilon)u_{i}(0, z^{*}).$$
(6)

Let $K = \frac{1-\varepsilon}{2\varepsilon} \sum_{i \in A} \gamma_i^*(\omega_i - \varphi(i, z^*)c(z^*))$ which is positive because of (A1*). Define

$$\gamma'_{i} = \begin{cases} \frac{\gamma_{i}^{*}}{\varepsilon}, & \text{if } i \in A, \\ \gamma_{i}^{*}, & \text{otherwise,} \end{cases} \qquad \qquad g'_{i} = \begin{cases} \varepsilon g_{i}^{*} + \frac{\varepsilon K}{\sum_{i \in A} \gamma_{i}^{*}}, & \text{if } i \in A, \\ g_{i}^{*} + \frac{K}{\sum_{i \notin A} \gamma_{i}^{*}}, & \text{otherwise,} \end{cases}$$

and note that $(\gamma', g', z^*) \in S_2$. Indeed, since $(\gamma^*, g^*, z^*) \in S_2$ we obtain

$$\begin{split} \sum_{i=1}^{n} \gamma_i' g_i' + \sum_{i=1}^{n} \gamma_i' \varphi(i, z^*) c(z^*) - \sum_{i=1}^{n} \gamma_i' \omega_i &= \sum_{i \in A} \gamma_i^* g_i^* + K + \sum_{i \notin A} \gamma_i^* \varphi(i, z^*) c(z^*) + \sum_{i \in A} \frac{\gamma_i^*}{\varepsilon} \varphi(i, z^*) c(z^*) + \sum_{i \in A} \gamma_i^* \varphi(i, z^*) c(z^*) + \sum_{i \notin A} \gamma_i^* \varphi(i, z^*) c(z^*) - \sum_{i \notin A} \frac{\gamma_i^*}{\varepsilon} \omega_i - \sum_{i \notin A} \gamma_i^* \varphi(i, z^*) c(z^*) + \sum_{i \notin A} \gamma_i^* \varphi(i, z^*) c(z^*) - \sum_{i \notin A} \frac{\gamma_i^*}{\varepsilon} \omega_i - \sum_{i \notin A} \gamma_i^* \omega_i &= \sum_{i=1}^{n} \gamma_i^* g_i^* + \sum_{i=1}^{n} \gamma_i^* \varphi(i, z^*) c(z^*) + \sum_{i \notin A} \gamma_i^* \omega_i &= \sum_{i=1}^{n} \gamma_i^* g_i^* + \sum_{i=1}^{n} \gamma_i^* \varphi(i, z^*) c(z^*) + \sum_{i \notin A} \gamma_i^* \omega_i &= \sum_{i=1}^{n} \gamma_i^* g_i^* + \sum_{i=1}^{n} \gamma_i^* \varphi(i, z^*) c(z^*) + \sum_{i \notin A} \gamma_i^* \omega_i &= \sum_{i=1}^{n} \gamma_i^* g_i^* + \sum_{i=1}^{n} \gamma_i^* \varphi(i, z^*) c(z^*) + \sum_{i \notin A} \gamma_i^* \omega_i &= \sum_{i=1}^{n} \gamma_i^* g_i^* + \sum_{i=1}^{n} \gamma_i^* \varphi(i, z^*) c(z^*) + \sum_{i \notin A} \gamma_i^* \omega_i &= \sum_{i=1}^{n} \gamma_i^* g_i^* + \sum_{i=1}^{n} \gamma_i^* \varphi(i, z^*) c(z^*) + \sum_{i \notin A} \sum_{i \notin A} \gamma_i^* \omega_i &= \sum_{i=1}^{n} \gamma_i^* \varphi(i, z^*) c(z^*) + \sum_{i \notin A} \sum_{i \notin A$$

Furthermore, by monotonicity and (6), we have that for $i \in A$,

$$\gamma_i'[u_i(g_i', z^*) - u_i(x_i^*, y^*)] = \frac{\gamma_i^*}{\varepsilon} \left[u_i \left(\varepsilon g_i^* + \frac{\varepsilon K}{\sum_{i \in A} \gamma_i^*}, z^* \right) - u_i(x_i^*, y^*) \right] >$$

$$> \frac{\gamma_i^*}{\varepsilon} [u_i(\varepsilon g_i^*, z^*) - u_i(x_i^*, y^*)] = \frac{\gamma_i^*}{\varepsilon} [u_i(\varepsilon g_i^*, z^*) - u_i(0, z^*)] \ge \frac{\gamma_i^*}{\varepsilon} [\varepsilon u_i(g_i^*, z^*) + (1 - \varepsilon)u_i(0, z^*) - u_i(0, z^*)] = \gamma_i^* [u_i(g_i^*, z^*) - u_i(0, z^*)] = \gamma_i^* [u_i(g_i^*, z^*) - u_i(0, y^*)] = \gamma_i^* [u_i(g_i^*, z^*) - u_i(x_i^*, y^*)],$$

and by monotonicity, we have that for all $i \notin A$

$$\gamma'_{i}[u_{i}(g'_{i}, z^{*}) - u_{i}(x^{*}_{i}, y^{*})] > \gamma^{*}_{i}[u_{i}(g^{*}_{i}, z) - u_{i}(x^{*}_{i}, y^{*})].$$

Hence, $F_2(x^*, y^*, \gamma', g', z^*) > F_2(s^*)$ which is an absurd, being s^* a Nash equilibrium. Therefore, denoted by A' the set $A' := \{i \in I : \gamma_i^* < 1\}$, then we have for all $i \in A', x_i^* > 0$ and, by (5), $f(\gamma_i^*)(u_i(x_i^*, y^*) - u_i(g_i^*, z^*)) > F_1(s^*)$. Let $\delta \in (0, 1)$ be such that $f(\gamma_i^*)(u_i(\delta x_i^*, y^*) - u_i(g_i^*, z^*)) > F_1(s^*)$.

 $F_1(s^*)$ for all $i \in A'$ and consider the pair $(x', y^*) \in S_1$, given by

$$x_i' = \begin{cases} \delta x_i^*, & \text{if } i \in A' \\ x_i^* + \frac{1-\delta}{|I \setminus A'|} \sum_{j \in A'} x_j^* > x_i^*, & \text{if } i \notin A', \end{cases}$$

which, based on monotonicity, gives a higher payoff to player1, i.e. $F_1(x', y^*, \gamma^*, g^*, z^*) > F_1(s^*)$. This contradicts the assumption that s^* is a Nash equilibrium and hence C = 0, which concludes the proof.

Proof of Theorem 3.7. Let $s^* = (x^*, y^*, \gamma^*, g^*, z^*)$ be a Nash equilibrium for the game G_{φ} and assume that (x^*, y^*) is not a φ -cost share equilibrium allocation. Then, by Theorem 3.2 there exists $(\gamma, g, z) \in S_2$ such that $F_2(x^*, y^*, \gamma, g, z) > 0$ and, by Proposition 3.6, $F_2(x^*, y^*, \gamma, g, z) > 0 = F_2(s^*)$, which is impossible by Definition 3.4.

For the converse, let (x^*, y^*) be a φ -cost share equilibrium and assume that (x^*, y^*, γ, g, z) is not a Nash equilibrium, where $(\gamma, g, z) \in S_2$ is such that $u_i(x_i^*, y^*) = u_i(g_i, z)$ for every $i \in I$. Then,

- (*I*) there exists $(x, y) \in S_1$ such that $F_1(x, y, \gamma, g, z) > F_1(x^*, y^*, \gamma, g, z) = 0$; or
- (II) there exists $(\gamma', g', z') \in S_2$ such that $F_2(x^*, y^*, \gamma', g', z') > F_2(x^*, y^*, \gamma, g, z) = 0$.

In the first case, (x^*, y^*) is not efficient, because (x, y) is feasible and $u_i(x_i, y) > u_i(g_i, z) = u_i(x_i^*, y^*)$ for all $i \in I$, and this is a contradiction (see Remark 2.3).

In the second case, (x^*, y^*) is a σ_{φ} -dominated allocation in the sense of Aubin, because $u_i(g'_i, z') > u_i(x^*_i, y^*)$ for every $i \in I$ and $\sum_{i \in I} \gamma'_i g'_i + \sum_{i \in I} \gamma'_i \varphi(i, z') c(z') \leq \sum_{i \in I} \gamma'_i \omega_i$, with $\gamma'_i > 0$ for all *i*. This contradicts the fact that (x^*, y^*) is a φ_{σ} -cost share equilibrium (see Theorem 3.2).

5.2 **Proofs of section 4**

In what follows, we prove that the set of cost share equilibria might be empty even in economies that satisfy standard assumptions.

First we examine the relation between cost share equilibria in an economy with public goods and competitive equilibria of an appropriately constructed pure exchange economy without public goods. Specifically, given an economy

 $\mathcal{E} = \{I, \mathbb{R}^m_+, \mathcal{Y}, c, (u_i, \omega_i)_{i \in I}\}\$ and a cost distribution function $\varphi \in \Phi$ satisfying assumptions $(A1^*)$, (A2) and (A3), for each given $y \in \mathcal{Y}$, we define the economy $\mathcal{E}(y, \varphi) = \{I, \mathbb{R}^m_+, (u_i(\cdot, y), \omega_i^y)_{i \in I}\},\$ devoid of public goods, with the same set of agents *I*, the same commodity space for private consumption \mathbb{R}^m_+ , and such that, for each agent $i \in I$, $\omega_i^y = \omega_i - \varphi(i, y)c(y)$ and $\omega_i^y \gg 0$ as the result of (A1*). A **competitive equilibrium** in the economy $\mathcal{E}(y, \varphi)$ consists of a pair (x, p) where *x* is a feasible allocation, i.e. $\sum_{i \in I} x_i \leq \sum_{i \in I} \omega_i^y, p \in \Delta$ is a price vector such that $p \cdot x_i \leq p \cdot \omega_i^y$ for all $i \in I$ and $u_i(k, y) > u_i(x_i, y) \Rightarrow p \cdot k > p \cdot \omega_i^y$.

Proposition 5.2 If $(x^*, y^*) \in CSE_{\varphi}(\mathcal{E})$ with price function $p^* : \mathcal{Y} \to \Delta$, then x^* is a competitive equilibrium with price $p^*(y^*)$ in the economy $\mathcal{E}(y^*, \varphi)$. Conversely if, for each $y \in \mathcal{Y}$, (x_y, p_y) is a

competitive equilibrium in the economy $\mathcal{E}(y, \varphi)$ and if there exists $y^* \in \mathcal{Y}$ such that $u_i(x_{iy^*}, y^*) \ge u_i(x_{iz}, z)$ for all $i \in I$ and all $z \in \mathcal{Y}$, then (x_{y^*}, y^*) is a cost share equilibrium for the economy \mathcal{E} with the equilibrium price $p^* : \mathcal{Y} \to \mathbb{R}^m_+$ given by $p^*(z) := p_z$.

Proof. If $(x^*, y^*) \in CSE_{\varphi}(\mathcal{E})$, x^* is feasible and it satisfies the budget set in $\mathcal{E}(y^*, \varphi)$, i.e. $p^*(y^*) \cdot x_i^* \leq p^*(y^*) \cdot \omega_i^{y^*}$. To conclude, we show that it is also maximal in the budget set. Let g be an allocation such that $u_i(g_i, y^*) > u_i(x_i^*, y^*)$ and $p^*(y^*) \cdot g_i \leq p^*(y^*) \cdot \omega_i^{y^*}$ for some $i \in I$. This implies that $p^*(y^*) \cdot g_i + \varphi^*(i, y^*)p^*(y^*) \cdot c(y^*) \leq p^*(y^*) \cdot \omega_i$, which is a contradiction since (x^*, y^*) is a cost share equilibrium.

Conversely, let (x_y, p_y) be a competitive equilibrium in each economy $\mathcal{E}(y, \varphi)$ and $y^* \in \mathcal{Y}$ be such that $u_i(x_{iy^*}, y^*) \ge u_i(x_{iz}, z)$ for all $z \in \mathcal{Y}$. Assume to the contrary that (x_{y^*}, y^*) is not a cost share equilibrium for the economy \mathcal{E} with the equilibrium price $p^* : \mathcal{Y} \to \mathbb{R}^m_+$ given by $p^*(z) := p_z$. Since, (x_{y^*}, y^*) is feasible and satisfies the budget constraint in \mathcal{E} , it means that there exist an agent $i \in I$ and an alternative allocation (g_i, z) such that

(*i*)
$$u_i(g_i, z) > u_i(x_{iy^*}, y^*)$$
 and

(*ii*)
$$p^*(z) \cdot g_i + p^*(z) \cdot \varphi(i, z)c(z) \le p^*(z) \cdot \omega_i$$
.

From (*ii*) it follows that $p_z \cdot g_i \leq p_z \cdot \omega_i^z$, and hence, since (x_z, z) is a competitive equilibrium allocation for $\mathcal{E}(z, \varphi)$, it follows that $u_i(g_i, z) \leq u_i(x_{iz}, z) \Rightarrow u_i(x_{iy^*}, y^*) < u_i(g_i, z) \leq u_i(x_{iz}, z) \leq u_i(x_{iy^*}, y^*)$, a contradiction.

A cost share equilibrium (x^*, y^*) is a competitive equilibrium in the economy with no public goods $\mathcal{E}(y^*, \varphi)$, constructed on the basis of the public project y^* arisen in equilibrium. Furthermore, by the strict monotonicity assumption, $p^*(y^*) \gg 0$. Conversely, if there is a competitive equilibrium that is preferred by all the agents over all the other competitive equilibria (i.e. which Pareto dominates), then this constitutes a cost share equilibrium in the economy \mathcal{E} with public projects. The mere existence of a competitive equilibrium in each economy $\mathcal{E}(y, \varphi)$ as defined above, does not ensure the existence of a cost share equilibrium in the economy \mathcal{E} with public goods, because there may not be a competitive equilibrium which Pareto dominates all the other equilibria. This applies to the Proposition 4.1 whose proof is shown below.

Proof of Proposition 4.1. Consider an economy with two public goods, i.e. $\mathcal{Y} = \{y, z\}$, two agents, i.e. $I = \{A, B\}$, and two private goods, i.e. $\mathbb{R}^m_+ = \mathbb{R}^2_+$. The primitives of the economy are given by:

$$u_A(f^1, f^2, y) = u_B(f^1, f^2, z) = \sqrt{f^1} + \sqrt{f^2}$$
$$u_A(f^1, f^2, z) = u_B(f^1, f^2, y) = \sqrt{f^1} + \sqrt{f^2} + 5;$$
$$c = c(y) = c(z) = \left(\frac{2}{3}, \frac{2}{3}\right);$$
$$\omega_A = (1, 3) \ \omega_B = (3, 1).$$

First we observe that for any $(\varphi_A(\cdot), \varphi_B(\cdot)) \ge 0$ such that $\varphi_A(\cdot) + \varphi_B(\cdot) = 1$,

$$\omega_{A} - \varphi_{A}(\cdot)c = (1,3) - \varphi_{A}(\cdot)\left(\frac{2}{3}, \frac{2}{3}\right) = \left(1 - \frac{2}{3}\varphi_{A}(\cdot), 3 - \frac{2}{3}\varphi_{A}(\cdot)\right) \gg 0$$

$$\omega_{B} - \varphi_{B}(\cdot)c = (3,1) - \varphi_{B}(\cdot)\left(\frac{2}{3}, \frac{2}{3}\right) = \left(3 - \frac{2}{3}\varphi_{B}(\cdot), 1 - \frac{2}{3}\varphi_{B}(\cdot)\right) \gg 0.$$

Moreover,

$$\omega_A + \omega_B = (4, 4) \gg \left(\frac{2}{3}, \frac{2}{3}\right) = c.$$

We want to show that there is no cost share equilibrium. We proceed by the way of contradiction. Let (x, t) be a cost share equilibrium with price function $p^* : \mathcal{Y} \to \Delta$ and $\varphi^* : I \times \mathcal{Y} \to \mathbb{R}_+$, where $t \in \{y, z\}$. Thanks to Proposition 5.2, we have that $p^*(t) \gg 0$ and it can be shown that $x_i \gg 0$ for any $i \in \{A, B\}$, which allows us to focus on the economies

$$\mathcal{E}(y,\varphi^*) = \{\{A,B\}, (u_i(\cdot,y),\omega_i^{y,\varphi^*})_{i\in\{A,B\}}\} \text{ and } \mathcal{E}(z,\varphi^*) = \{\{A,B\}, (u_i(\cdot,z),\omega_i^{z,\varphi^*})_{i\in\{A,B\}}\},$$

where $\omega_i^{y,\varphi^*} = \omega_i - \varphi_i^*(y)c$ and $\omega_i^{z,\varphi^*} = \omega_i - \varphi_i^*(z)c$, for any $i \in \{A, B\}$. In the economy $\mathcal{E}(t,\varphi^*)$ described above, regardless of $t \in \{y, z\}$, for any price $(p_1, p_2) \gg 0$ the agents' demand functions are

$$(f_A^1, f_A^2) \left(\frac{p_2}{p_1(p_1 + p_2)} \left[p_1 + 3p_2 - \frac{2}{3} \varphi_A^*(t)(p_1 + p_2) \right]; \frac{p_1}{p_2(p_1 + p_2)} \left[p_1 + 3p_2 - \frac{2}{3} \varphi_A^*(t)(p_1 + p_2) \right] \right) \\ (f_B^1, f_B^2) \left(\frac{p_2}{p_1(p_1 + p_2)} \left[3p_1 + p_2 - \frac{2}{3} \varphi_B^*(t)(p_1 + p_2) \right]; \frac{p_1}{p_2(p_1 + p_2)} \left[3p_1 + p_2 - \frac{2}{3} \varphi_B^*(t)(p_1 + p_2) \right] \right) .$$

Then, being

$$\omega_A^t + \omega_B^t = (1,3) - \varphi_A^*(t) \left(\frac{2}{3}, \frac{2}{3}\right) + (3,1) - \varphi_B^*(t) \left(\frac{2}{3}, \frac{2}{3}\right) = (4,4) - \left(\frac{2}{3}, \frac{2}{3}\right) = \left(\frac{10}{3}, \frac{10}{3}\right),$$

the aggregate excess demand of both economies $\mathcal{E}(t, \varphi^*)$ is

$$z(p_1, p_2) = [f_A(p_1, p_2) + f_B(p_1, p_2)] - [\omega_A^t + \omega_B^t] = \left(\frac{10}{3}\frac{p_2}{p_1} - \frac{10}{3}; \frac{10}{3}\frac{p_1}{p_2} - \frac{10}{3}\right),\tag{7}$$

which satisfies the so-called gross substitute (GS) property (see Definition 17.F.2 in Mas-Colell, Whinston, and Green (1995)) since

$$\frac{\partial z^1}{\partial p_2} = \frac{10}{3p_1} > 0 \qquad \frac{\partial z^2}{\partial p_1} = \frac{10}{3p_2} > 0$$

Proposition 17.F.3 in Mas-Colell, Whinston, and Green (1995) ensures the uniqueness of the equilibrium price⁸ and therefore the uniqueness of the equilibrium allocation.

⁸If there were two prices, then they must be collinear, but if $p(y) \in \Delta$, i.e. it is normalized, it must be unique.

If t = y, the unique competitive equilibrium in the economy $\mathcal{E}(y, \varphi^*)$ is

$$(f_A^{*1}, f_A^{*2}) = \left(2 - \frac{2}{3}\varphi_A^*(y), 2 - \frac{2}{3}\varphi_A^*(y)\right),$$

$$(f_B^{*1}, f_B^{*2}) = \left(2 - \frac{2}{3}\varphi_B^*(y), 2 - \frac{2}{3}\varphi_B^*(y)\right), \text{ with } p_1^*(y) = p_2^*(y),$$

and, according to Proposition 5.2, $(x, t) = (f^*, y)$. Consider, the allocation (g, z) such that

$$(g_A^1, g_A^2, z) = \left(1 - \frac{2}{3}\varphi_A^*(z), 3 - \frac{2}{3}\varphi_A^*(z), z\right),$$

and notice that

$$(g_A^1, g_A^2) \gg (0, 0),$$

$$u_A(g_A^1, g_A^2, z) = \sqrt{1 - \frac{2}{3}\varphi_A^*(z)} + \sqrt{3 - \frac{2}{3}\varphi_A^*(z)} + 5 > 5 > 2\sqrt{2}$$

$$\geq 2\sqrt{2 - \frac{2}{3}\varphi_A^*(y)} = u_A(f_A^{*1}, f_A^{*2}, y) = u_A(x_A^1, x_A^2, y)$$

and for any $(p_1, p_2) \in \mathbb{R}^2_+ \setminus \{0\}, \ p_1 g_A^1 + p_2 g_A^2 + \varphi_A^*(z) (p_1 + p_2) c \le p_1 \omega_A^1 + p_2 \omega_A^2.$

In particular, it holds for $(p_1^*(z), p_2^*(z))$ whatever the pair. This is inconsistent with (x, y) being a cost share equilibrium.

On the other hand, if t = z the unique competitive equilibrium in the economy $\mathcal{E}(z, \varphi^*)$ is given as:

$$(f_A^{*1}, f_A^{*2}) = \left(2 - \frac{2}{3}\varphi_A^*(z), 2 - \frac{2}{3}\varphi_A^*(z)\right),$$

$$(f_B^{*1}, f_B^{*2}) = \left(2 - \frac{2}{3}\varphi_B^*(z), 2 - \frac{2}{3}\varphi_B^*(z)\right), \text{ with } p_1^*(z) = p_2^*(z),$$

and, based on Proposition 5.2, $(x, t) = (f^*, z)$. Notice that the allocation (g, y) given by

$$(g_B^1, g_B^2, y) = \left(3 - \frac{2}{3}\varphi_B^*(y), 1 - \frac{2}{3}\varphi_B^*(y), y\right),$$

is such that

$$\begin{array}{rcl} (g_B^1,g_B^2) & \gg & (0,0) \\ u_B(g_B^1,g_B^2,y) & = & \sqrt{3-\frac{2}{3}\varphi_B^*(y)} + \sqrt{1-\frac{2}{3}\varphi_B^*(y)} + 5 > 5 > 2\sqrt{2} \\ & \geq & 2\sqrt{2-\frac{2}{3}\varphi_B^*(z)} = u_B(f_B^{*1},f_B^{*2},z) = u_B(x_B^1,x_B^2,z), \end{array}$$

and for any $(p_1, p_2) \in \mathbb{R}^2_+ \setminus \{0\}, \ p_1g_B^1 + p_2g_B^2 + \varphi_B^*(y)(p_1 + p_2)c \le p_1\omega_B^1 + p_2\omega_B^2.$

In particular, it holds for $(p_1^*(y), p_2^*(y))$ whatever the pair, which is a contradiction. Therefore,

in this economy there is no cost share equilibrium.

Remark 5.3 Above we have described an economy that satisfies the "standard" assumptions and which has an empty set of cost share equilibria. Similar computations could provide another example with two public goods and different costs, as c(y) = (1, 1) and $c(z) = (\frac{2}{3}, \frac{2}{3})$.

Remark 5.4 The above example includes only two economies without public goods $\mathcal{E}(y, \varphi^*)$ and $\mathcal{E}(z, \varphi^*)$, and in both cases a competitive equilibrium. These two competitive equilibrium allocations differ only in φ and since $\varphi_A(y) > \varphi_A(z) \Rightarrow \varphi_B(y) = 1 - \varphi_A(y) < 1 - \varphi_A(z) = \varphi_B(z)$, neither of the two Pareto dominates the other. This is consistent with Proposition 5.2.

Remark 5.5 In the light of Proposition 4.1 and Theorem 3.2, for any given cost distribution function $\varphi \in \Phi$, the σ_{φ} -Aubin core may be empty too. In what follows, we illustrate a direct proof by using the same economy described in the proof of Proposition 4.1. We show that, given an arbitrarily cost distribution function $\varphi \in \Phi$, any feasible allocation is σ_{φ} -blocked by a generalized coalition $(\gamma, T) \in \tilde{\mathcal{F}}$ with full support, and hence the σ_{φ} -Aubin core is empty. Consider the economy described in the proof of Proposition 4.1 and note that any feasible allocation (f, t), with $t \in \mathcal{Y}$, is such that $f_A^j + f_B^j + \frac{2}{3} = 4$ for any $j \in \{1, 2\}$ and hence $f_i^j \leq \frac{10}{3}$ for any $i \in \{A, B\}$ and any $j \in \{1, 2\}$. Given $\varphi \in \Phi$, if t = y then (f, y) is σ_{φ} -blocked by $(\gamma, T) \in \tilde{\mathcal{F}}$, where $\gamma_A = \frac{K}{K+1}$ and $\gamma_B = \frac{1}{K+1}$ with K > 15, via the alternative allocation (g, z) defined as $g_A = (0, 0)$ and $g_B = (K(1 - \frac{2}{3}\varphi_A) + (3 - \frac{2}{3}\varphi_A); K(3 - \frac{2}{3}\varphi_A) + (1 - \frac{2}{3}\varphi_B))$. Similarly, if t = z then (f, z) is σ_{φ} -blocked by $(\gamma', T) \in \tilde{\mathcal{F}}$, where $\gamma'_A = \frac{1}{K+1}$ and $\gamma'_B = \frac{K}{K+1}$ with K > 15, via the alternative allocation (h, y), where $h_A = (K(3 - \frac{2}{3}\varphi_B) + (1 - \frac{2}{3}\varphi_B) + (3 - \frac{2}{3}\varphi_B))$ and $h_B = (0, 0)$.

Proof of Proposition 4.4. Consider the same economy described in the proof of Proposition 4.1 and observe that for any $\varphi : I \times \mathcal{Y} \to \mathbb{R}^2_+$ we have that

$$\varphi(A,\cdot) + \varphi(B,\cdot) = c(\cdot) = \left(\frac{2}{3}, \frac{2}{3}\right) \iff 0 \le \varphi(\cdot, \cdot) \le \left(\frac{2}{3}, \frac{2}{3}\right).$$

Therefore

$$\omega_A - \varphi(A, \cdot) \geq (1, 3) - \left(\frac{2}{3}, \frac{2}{3}\right) \gg 0$$

$$\omega_B - \varphi(B, \cdot) \geq (3, 1) - \left(\frac{2}{3}, \frac{2}{3}\right) \gg 0.$$

Moreover,

$$\omega_A + \omega_B = (4, 4) \gg \left(\frac{2}{3}, \frac{2}{3}\right) = c.$$

We want to show that there is no generalized cost share equilibrium. We proceed by the way of contradiction. Thus, let (x, t) be a generalized cost share equilibrium with price function $p^* : \mathcal{Y} \to \Delta$ and $\varphi^* : I \times \mathcal{Y} \to \mathbb{R}^m_+$, where $t \in \{y, z\}$. As before, $x_i \gg 0$ for any $i \in \{A, B\}$ and because of Proposition 5.2, we focus on the economies $\mathcal{E}(y, \varphi^*)$ and $\mathcal{E}(z, \varphi^*)$ which have a unique competitive

equilibrium given respectively by

$$\begin{array}{lll} (f_A^{*1}, f_A^{*2}) & = & \displaystyle \frac{4 - \varphi_1^*(A, y) - \varphi_2^*(A, y)}{2} \cdot (1, 1), \\ (f_B^{*1}, f_B^{*2}) & = & \displaystyle \frac{4 - \varphi_1^*(B, y) - \varphi_2^*(B, y)}{2} \cdot (1, 1), \text{ with } \\ p_1^*(y) & = & \displaystyle p_2^*(y), \end{array}$$

and

$$\begin{aligned} (f_A^{*1}, f_A^{*2}) &= \frac{4 - \varphi_1^*(A, z) - \varphi_2^*(A, z)}{2} \cdot (1, 1), \\ (f_B^{*1}, f_B^{*2}) &= \frac{4 - \varphi_1^*(B, z) - \varphi_2^*(B, z)}{2} \cdot (1, 1), \text{ with } \\ p_1^*(z) &= p_2^*(z). \end{aligned}$$

Now, if t = y, then $(x, t) = (f^*, y)$. Consider, the allocation (g, z) defined as

$$(g_A^1, g_A^2, z) = (1 - \varphi_1^*(A, z), 3 - \varphi_2^*(A, z), z),$$

and notice that

$$(g_A^1, g_A^2) \gg (0, 0),$$
$$u_A(g_A^1, g_A^2, z) = \sqrt{1 - \varphi_1^*(A, z)} + \sqrt{3 - \varphi_2^*(A, z)} + 5 > 5 > 2\sqrt{2} \ge$$
$$\ge 2\sqrt{\frac{4 - \varphi_1^*(A, z) - \varphi_2^*(A, z)}{2}} = u_A(f_A^{*1}, f_A^{*2}, y) = u_A(x_A^1, x_A^2, y),$$

and for any $(p_1, p_2) \in \mathbb{R}^2_+ \setminus \{0\}$

$$p_1g_A^1 + p_2g_A^2 + p_1\varphi_1^*(A, z) + p_2\varphi_2^*(A, z) = p_1 + 3p_2 = p_1\omega_A^1 + p_2\omega_A^2$$

In particular, it holds for $(p_1^*(z), p_2^*(z))$ whatever the pair. This contradicts the fact that (x, y) is a generalized cost share equilibrium. On the other hand, if t = z, then $(x, t) = (f^*, z)$. Then by the same argument, by considering the allocation (g, y) such that

$$(g_B^1, g_B^2, y) = (3 - \varphi_1^*(B, y), 1 - \varphi_2^*(B, y), y),$$

in this case also we obtain a contradiction. Therefore, there is no cost share equilibrium in this economy.

Finally, as in Remarks 5.3 and 5.5, there are examples with different costs in which the set of generalized cost share equilibria is empty and the generalized σ -Aubin core is empty too.

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