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Abstract
We consider pure exchange economies with finitely many private goods including also non-Samuelsonian public goods. For this type of economies, the notion of competitive equilibrium called *cost share equilibrium* is founded on individual payments for public goods varying according to individual benefits. This situation naturally arises when a level of provision is interpreted as a whole configuration of public policies or when cost share functions are interpreted as voluntary contributions instead of predetermined tax systems (Mas Colell (1980)). We establish the equivalence of cost share equilibria with cooperative and non-cooperative game-theoretic solutions. In particular: 1. we characterize cost share equilibria as those allocations which cannot be improved upon by the society; 2. we characterize cost share equilibria as Nash equilibria of a game with two players. Then we discuss the existence of cost share equilibria in economies with public projects satisfying standard assumptions and provide a condition for the set of cost share equilibria to be non-empty. Our analysis of cooperative solutions is based on contribution schemes which capture the fraction of the total cost of collective goods that each coalition of agents is expected to cover.

JEL classification: D49, D51, C72.

Keywords: Non-Samuelsonian public goods, Cost share equilibrium, Aubin core, Nash equilibrium.

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1 Introduction

In this paper we study the existence of competitive equilibria and their equivalence with cooperative and non-cooperative game-theoretic solutions in economies in which agents' decisions are affected by non-market variables, called public goods or public projects. Examples include the public goods provision (transportation, health, education, international public goods like global climate), the regulation of private economic activities (regarding quality standard, safety of labour conditions, trade institutions), and social rules (laws, property rights).

To encompass these different non-market variables, we adopt the general mathematical framework proposed by Mas-Colell (1980) to represent the public sector and we do not limit the choice of a public good to a set with an Euclidean structure. Instead, we allow public projects to be drawn from a set without any mathematical structure, a general setting that includes a finite set of projects and clearly does not exclude an Euclidean structure. The absence of a linear structure on the set of public projects allows the analysis of public goods for which there is no reason to assume a commonly accepted order by traders. This would be the case of public goods for which different individuals might have different perceptions and hence assign different rankings. Moreover, when public projects are interpreted as public environments, i.e. collections of variables common to all agents but determined outside market mechanisms, one obtains a general framework incorporating many different economic problems[1].

In the paper we build on a concept of competitive equilibrium introduced in Basile, Graziano, and Pesce (2016) and founded on cost distributions of the total provision cost of the selected collective good configuration. This concept of equilibrium, called *cost share equilibrium*, generalizes the linear cost share equilibrium concepts explored in Mas-Colell (1980), Mas-Colell and Silvestre (1989), Gilles and Diamantaras (1998), Graziano and Romaniello (2012). In a linear cost share equilibrium, everyone has an equal provision, but consumers might pay different contributions depending on given cost share functions. Agents maximize their utility over their budget sets taking into account the share of the project cost and changes of the price of private commodities deriving from changes in the public project. Compared with previous literature, we do not assume that the individual contribution is the same for each public project, so that cost share equilibrium allows individual payments to vary according to individual benefits. Accordingly, on the cooperative side, the veto mechanism leading to equilibrium solutions involves a (contribution) measure which varies across public projects and is defined on the set of all coalitions.

In the first part of the paper, we provide cooperative and non-cooperative characterizations of cost

[1] Hammond and Villar (1998) and Hammond and Villar (1999) propose this interpretation of the Mas-Colell approach. In these contributions non-market variables include legal systems (such as the assignment of property rights), tax and benefits systems, as well as private goods provided by the public sector. For further interpretations of the non-Samuelsonian collective goods represented as elements of an unstructured set, see the discussion in Diamantaras and Gilles (1996), Gilles and Diamantaras (1998), Diamantaras, Gilles, and Scotchmer (1996), Gilles and Scotchmer (1997), Basile, De Simone, and Graziano (2005), Graziano (2007), Basile, Graziano, and Pesce (2016), Basile, Gilles, Graziano, and Pesce (2021).

share equilibria. Our two characterizations are related to each other and rest on the core of the economy and the Nash equilibria of a game with two players, respectively.

Regarding the core characterization, it is well known that for a finite exchange economy with private goods, the intersection of the cores of the sequence of the replications coincides with the set of competitive equilibrium allocations (Debreu and Scarf (1963)). The classical Debreu-Scarf veto system applied to replica economies is equivalent to the approach introduced in Aubin (1979), which also leads to a core that coincides with the competitive equilibria. The Aubin veto mechanism extends the notion of coalition and the ordinary veto since it allows agents to participate with a fraction of their endowments when forming a blocking coalition. In economies with public goods, both characterizations may fail. Replica economies require a large number of agents and the per capita cost for a public good is decreasing. This weakens the influence of small coalitions and makes the core larger. For this reason, we consider a veto mechanism which differs from the classical one due to Foley (1970). For each individual cost share function, we define a measure on the set of all coalitions to fix the contribution that each blocking coalition is expected to cover. This contribution measure depends explicitly on the public projects. Moreover, in line with the Aubin approach, we allow agents to block an allocation with a fraction of their endowments. In other words, we focus on the notion of Aubin core introduced by Graziano and Romaniello (2012) in economies with public goods which turns out to be equivalent to the set of cost share equilibria (see also Basile, Graziano, and Pesce (2016)).

In our first characterization of cost share equilibria, we prove a refinement of the core equivalence theorem by exploiting the veto power of the grand coalition. In particular, we prove that for a cost share function and the corresponding contribution measure, cost share equilibria coincide with allocations that cannot be blocked by a coalition in which each agent participates with a non zero fraction of her initial endowment. We interpret a blocking coalition with full support as the *society* and prove the result with a direct approach which does not rely on Vind's theorem[2]. The characterization of cost share equilibria in terms of the blocking power of the society is key to prove the second result of the paper in which cost share equilibria are connected to Nash equilibria of a two-player game. This equivalence extends to public goods economies results proved by Hervés-Beloso and Moreno-García (2009a) and Hervés-Beloso and Moreno-García (2009b), and is related to the literature on non-cooperative market games. We show the equivalence between cost share equilibria and Nash equilibria of a suitable society game with an implicit use of core equivalence results. The game is played by the society in two different roles: as player one, the society tries to achieve Pareto improvements; as player two, it chooses an allocation which is feasible in Aubin's sense. The game does not involve money and prices, but only the share of the cost of the project that each coalition is expected to cover. Our result contributes to the literature on strategic approaches to competitive equilibria in markets with public goods.

[2]The veto power of the grand coalition has been exploited for pure exchange economies by Hervés-Beloso and Moreno-García (2001) using the Vind's theorem on the measure of a blocking coalition in a continuum economy. It is well known that for economies with public projects Vind's theorem does not hold (see Basile, Graziano, and Pesce (2016))

The second part of the paper analyzes the existence of cost share equilibria. In the literature, there is no general existence theorem of cost share equilibrium, but Gilles and Diamantaras (1998) provide several examples of economies in which specific linear cost share equilibria exist. Since cost share equilibria are more general than linear cost share equilibria, these examples make not vacuous also the notion of cost share equilibrium. We propose examples of a well-behaved economy with public projects that, despite satsfying standard assumptions, exhibits an empty set of cost share equilibria and, consequently, an empty set of linear cost share equilibria. This leads us to explore conditions ensuring the existence of a cost share equilibrium. For any public project y, we consider the associated exchange economy $\mathcal{E}(y)$ with only private goods, in which agents are not faced with the problem of the unanimously agreed choice of a public project to be realized. This fictitious economy, obtained by modifying suitably agents' initial endowments, satisfies the hypotheses for the existence of a competitive equilibrium. We exhibit a sufficient condition for the existence of a cost share equilibrium involving competitive equilibrium allocations of any economy $\mathcal{E}(y)$ associated to any public project y.

The paper is organized as follows. In Section 2, we introduce the economic model, preliminary definitions and assumptions. Section 3 contains the main equivalence results and Section 4 focuses on the existence of cost share equilibria. Finally, we notice that the Aubin core-cost share equilibrium equivalence theorems joint with a direct proof of the existence of Aubin core allocations might open a room for proving cost share equilibrium existence. In Section 5 we discuss this issue, how our results can be extended to generalized cost share equilibria recently introduced in Basile, Gilles, Graziano, and Pesce (2021), and future research. The Appendix contains all the proofs.

2 The Economic Model

We study an exchange economy $\mathcal{E}$ with a finite number of consumers and private goods. We denote by $I = \{1, 2, \dots, n\}$ the *set of n agents* and we consider the nonnegative orthant of the m-dimensional Euclidean space, i.e. $\mathbb{R}^m_+$, as the *consumption space*.[3] We assume the presence of *public projects*, represented as elements of an abstrat set $\mathcal{Y}$ devoid of any mathematical structure. Mas-Colell (1980) first consider an abstract set of public goods to generalize Samuelson's notion of collective good (see also Diamantaras and Gilles (1996)). The outlay of any public good is expressed in terms of private goods only, through the so-called *cost function* $c : \mathcal{Y} \to \mathbb{R}^m_+$. Every agent $i \in I$ is endowed with an initial endowment of private goods, denoted by $\omega_i \in \mathbb{R}^m_+$, and an utility function representing her consumption preferences, denoted by $u_i : \mathbb{R}^m_+ \times \mathcal{Y} \to \mathbb{R}_+$. Public goods cause widespread externalities, as agents' utility functions depend, not only on the private goods bundle $x_i \in \mathbb{R}^m_+$, but also on the public project $y \in \mathcal{Y}$. Throughout the paper we assume that

(A1) Each agent owns a positive initial endowment, $\omega_i > 0$ for all $i \in I$, and that each private commodity is present on the market regardless of the cost of the realized project, i.e. $\omega \gg$

[3] We follow the standard vector inequality notation: $x \geqslant x'$ if $x_h \geqslant x'_h$ for all commodities $h = 1, \dots, m$; $x > x'$ if $x \geqslant x'$ and $x \neq x'$; and $x \gg x'$ if $x_h > x'_h$ for all commodities $h = 1, \dots, m$.

$c(y)$ for all $y \in \mathcal{Y}$, where ω denotes the total initial endowment in the economy $\mathcal{E}$ (i.e. $\omega = \sum_{i \in I} \omega_i$).

(A2) For any $i \in I$ and any $y \in \mathcal{Y}$, the restriction $u_i(\cdot, y) : \mathbb{R}^m_+ \to \mathbb{R}_+$ is continuous, strictly monotone and quasi-concave.

An **allocation** for the economy $\mathcal{E}$ is a specification of the amount of private goods assigned to each agent and of a public project chosen to be realized. Formally, an allocation is a pair (x, y), with $x = (x_1, \dots, x_n) \in \mathbb{R}^{mn}_+$, where $x_i \in \mathbb{R}^m_+$ is the bundle of private commodities of agent i, and $y \in \mathcal{Y}$ is a public project. An allocation (x, y) is **feasible** if

$$\sum_{i=1}^{n} x_i + c(y) \leqslant \sum_{i=1}^{n} \omega_i.$$

This means that the initial endowment is used for covering the cost of the realized project and re-distributed among the agents.

A **cost distribution** is a function $\varphi : I \times \mathcal{Y} \to \mathbb{R}_+$ such that $\sum_{i \in I} \varphi(i, y) = 1$ for all $y \in \mathcal{Y}$, where $\varphi(i, y)$ describes how much an economic agent i must contribute to the establishment of the provision level of the configuration of public goods y. We denote by Φ the class of all cost distribution functions.

Let Δ be the simplex of $\mathbb{R}^m_+$, i.e. the set $\Delta = \left\{ p \in \mathbb{R}^m_+ | \sum_{h=1}^{m} p_h = 1 \right\}$.

Definition 2.1 *A feasible allocation (x, y) is a* ***cost share equilibrium*** *in $\mathcal{E}$ if there exists a price system $p : \mathcal{Y} \to \Delta$ and a cost distribution function $\varphi \in \Phi$ such that for every $i \in I$,*

(i) $p(y) \cdot x_i + \varphi(i, y) p(y) \cdot c(y) \leqslant p(y) \cdot \omega_i$, *and*

(ii) (x_i, y) *maximizes* u_i *on the budget set*

$$B_i(p, \varphi) = \left\{ (h, z) \in \mathbb{R}^m_+ \times \mathcal{Y} \,|\, p(z) \cdot h + \varphi(i, z) p(z) \cdot c(z) \leqslant p(z) \cdot \omega_i \right\}.$$

We denote by $CSE_\varphi(\mathcal{E})$ the set of all cost share equilibria for the cost distribution function φ and by $CSE(\mathcal{E})$ the set of all cost share equilibria in the economy $\mathcal{E}$, that is $CSE(\mathcal{E}) = \bigcup_{\varphi \in \Phi} CSE_\varphi(\mathcal{E})$.

Definition 2.1 is introduced in Basile, Graziano, and Pesce (2016) as a generalization of the notion of **linear cost share equilibrium** developed by Mas-Colell (1980) for economies with a single private good and extended by Diamantaras and Gilles (1996) to the case of multiple private commodities (see also Basile, Gilles, Graziano, and Pesce (2021) for a further generalization). The notion of linear cost share equilibrium is obtained whenever $\varphi(i, y) = \varphi(i)$ for all $i \in I$ and all $y \in \mathcal{Y}$. A special case is the **equal cost share equilibrium**, for which the cost distribution function φ is constantly equal to $\frac{1}{n}$. Thus, denoted by $ECE(\mathcal{E})$ and $LCE(\mathcal{E})$ respectively the set of *equal* and *linear* cost share equilibria, we have

$$ECE(\mathcal{E}) \subseteq LCE(\mathcal{E}) \subseteq CSE(\mathcal{E}). \tag{1}$$

It is worthwhile to note that a price system p depends on the public good as it incorporates possible variations in the private sector due to variations in the public good choice. The price specification for each possible public good must be known even though, in equilibrium, just one public project will be realized.

The core notion in the context of economies with public goods is introduced by Gilles and Diamantaras (1998) and further extended by Basile, Graziano, and Pesce (2016) to mixed markets and by Basile, Gilles, Graziano, and Pesce (2021) to production economies with an endogenous social division of labor. This notion rests on the concept of **contribution measure**. Denoted by $\mathcal{P}(I)$ the power set of I, which contains all possible coalitions, a **contribution measure** is a function $\sigma : \mathcal{P}(I) \times \mathcal{Y} \to [0,1]$ such that for each $y \in \mathcal{Y}$, $\sigma(\cdot, y)$ is additive on $\mathcal{P}(I)$; $\sigma(\emptyset, y) = 0$ and $\sigma(I, y) = 1$. Given a coalition $S \in \mathcal{P}(I)$ and public good $y \in \mathcal{Y}$, the vector $\sigma(S, y) \in \mathbb{R}^m_+$ indicates the total quantities of private commodities needed to provide y by members of S. Note that the contribution that each coalition S is required to cover for the realization of a public project y does not necessarily depend on its size.

Definition 2.2 *Given a contribution measure σ, a feasible allocation (x, y) is σ-blocked by a coalition S if there exists an alternative allocation (x', y') such that*

$$(i) \quad u_i(x'_i, y') > u_i(x_i, y), \quad for\ all\ i \in S\ and$$

$$(ii) \quad \sum_{i \in S} x'_i + \sigma(S, y')c(y') \leqslant \sum_{i \in S} \omega_i.$$

With this notion of blocking, the members of S are able to improve their own welfare using their initial endowments and proposing an alternative project y'. Notice that the feasibility over the coalition S, required in the blocking condition (ii), implies that $\sigma(S, y')c(y') \leqslant \sum_{i \in S} \omega_i$, thus the project y' represents an admissible choice for the coalition S because its members are able to cover the cost of y' using their own resources. The σ-**core** of the economy, denoted by $C_\sigma(\mathcal{E})$, is the set of all the feasible allocations which are not σ-blocked by any coalition of agents.

A feasible allocation is said to be **efficient** if it is not Pareto blocked by the coalition I of all agents. Clearly, any σ-core allocation is efficient. However, since $\sigma(I, y') = 1$ for each project $y' \in \mathcal{Y}$, the efficiency notion does not depend on the contribution measure.
There is a one-to-one relationship between the cost distribution functions and the contribution measures. In details,

- given a cost distribution function φ, there is a unique contribution measure associated to φ defined as the function $\sigma_\varphi : \mathcal{P}(I) \times \mathcal{Y} \to [0,1]$ such that

$$\sigma_\varphi(S, y) = \sum_{i \in S} \varphi(i, y), \quad \text{for all } S \in \mathcal{P}(I) \text{ and all } y \in \mathcal{Y}.$$

- Conversely, given a contribution measure σ, there is a unique cost distribution function φ_σ given by

$$\varphi_\sigma(i,y) = \sigma(\{i\},y), \quad \text{for all } i \in I \text{ and all } y \in \mathcal{Y}.$$

Remark 2.3 It is possible to show, with standard arguments, that any cost share equilibrium (x, y) with cost distribution function φ belongs to the σ_φ-core and a fortiori it is efficient, i.e. $CSE_\varphi(\mathcal{E}) \subseteq C_{\sigma_\varphi}(\mathcal{E})$. The inclusion may be strict, as illustrated in the next example.

Example 2.4 Consider an economy with two public goods, $\mathcal{Y} = \{y, z\}$, such that $c(y) = (2, 2)$ and $c(z) = (0, 1)$. Suppose there are two private goods, $\mathbb{R}^m_+ = \mathbb{R}^2_+$, and two agents, denoted by A and B, whose characterists are given as follows: $\omega_A = (5, 1)$, $\omega_B = (1, 5)$, and

$$u_A(f^1, f^2, y) = u_B(f^1, f^2, y) = \sqrt{f^1} + \sqrt{f^2};\ u_A(f^1, f^2, z) = u_B(f^1, f^2, z) = \sqrt{f^1} + \sqrt{f^2} - 2.$$

Consider the cost distribution $\varphi(A, t) = \varphi(B, t) = \frac{1}{2}$ for any $t \in \mathcal{Y}$, and the feasible allocation (h, y), with $h_A = (1, 1)$ and $h_B = (3, 3)$. With easy computations, it can be proved that (h, y) belongs to the σ_φ-core of $\mathcal{E}$, where σ_φ is the contribution measure associated to φ. On the other hand, (h, y) is not a φ-cost share equilibrium. Indeed, for any price system $p(y) = (p, q)$, the pair (g, y), where

$$g = (g_1, g_2) = \begin{cases} \left(\frac{4q}{p+q}, \frac{4p^2}{q(p+q)}\right), & \text{if } q \neq 0 \\ (4, k), & \text{otherwise,} \end{cases}$$

belongs to the budget set $B_A(p, \varphi)$ and it is such that $u_A(g, y) > u_A(h_A, y)$. Thus, (h, y) is not a φ-cost share equilibrium, that is $(h, y) \in C_{\sigma_\varphi}(\mathcal{E}) \setminus CSE_\varphi(\mathcal{E})$.

♦

3 Characterizations of cost share equilibria

In this section we prove two characterizations of cost share equilibria: a cooperative characterization in terms of Aubin core allocations and a non-cooperative equivalence of cost share equilibria as Nash equilibria of a game with two players.

3.1 Cost share equilibria and σ-Aubin core

We consider a weakening of the σ-core, by modifying the veto mechanism and enlarging the set of coalitions. We follow Aubin (1979)'s approach to allow agents to join a coalition with only a fraction of their resources. The idea is to assign to each individual i a real number $\gamma(i) \in [0, 1]$ representing her personal portion of endowment she wants to invest into a coalition. Formally, a **generalized** or **fuzzy** coalition is any couple (γ, S) where $\gamma : I \longrightarrow [0, 1]$ is a non-null function and S is its support, i.e. the set $\{i \in I : \gamma(i) > 0\}$. We denote by $\mathcal{F}$ the set of all generalized coalitions. A generalized coalition $(\gamma, S) \in \mathcal{F}$ is said to have **full support** if $\gamma(i) > 0$ for all $i \in I$, that is if $S = I$. We denote by $\tilde{\mathcal{F}}$ the set of generalized coalitions with full support. Any coalition S, called throughout the paper standard or crisp coalition, can be viewed as the generalized coalition (S, χ_S), where χ_S is its correspondent characteristic function, i.e. $\chi_S : I \longrightarrow \{0,1\}$ such that $\chi_S(i) = 1$ if $i \in S$ and $\chi_S(i) = 0$ otherwise. In this way, we enlarge the class of standard coalitions and, consequently, we reduce the set of unblocked allocations. Precisely, we define the notion σ-**Aubin core** for an economy with public projects as follows (see also Graziano and Romaniello (2012), Florenzano (1990), Noguchi (2000), Basile, Graziano, and Pesce (2016)).

Definition 3.1 *Given a contribution function φ and its corresponding contribution measure σ_φ, a feasible allocation (x, y) is said to be σ_φ-blocked by a generalized coalition $(\gamma, S) \in \mathcal{F}$ if there exists an alternative allocation (x', y') such that*

$$\begin{array}{ll} (i) & u_i(x'_i, y') > u_i(x_i, y), \quad \textit{for all } i \in S \textit{ and} \\ (ii) & \displaystyle\sum_{i \in I} \gamma(i) x'_i + \sum_{i \in I} \gamma(i)\varphi(i, y')c(y') \leqslant \sum_{i \in I} \gamma(i)\omega_i. \end{array}$$

*The σ-**Aubin core** of the economy $\mathcal{E}$, denoted by $C^A_\sigma(\mathcal{E})$, is the set of all feasible allocations unblocked by any generalized coalition.*

We explicitly remark that, differently from the classical Aubin core, the feasibility condition involves also the term $\sum_{i \in I} \gamma(i)\varphi(i, y')c(y')$, representing the contribution of the generalized coalition (γ, S) to the realization of the project $y' \in \mathcal{Y}$. This contribution takes into account the *rate of participation* of each agent in the coalition. Precisely: given a cost distribution function φ, coherently with the fact that agent i actually participates in the blocking coalition using only the share $\gamma(i)$ of the initial endowment, we assume that she pays for the realization of the project according to the same share. We also remark that the feasibility constraint of Definition 2.2 refers to the contribution measure σ, whereas the one contained in Definition 3.1 refers to cost distribution function φ. Actually, one could consider the extension of σ from the set of ordinary coalitions $\mathcal{P}(I)$ to the class of generalized coalitions $\mathcal{F}$ and present the feasibility constraint in a similar form in both definitions. This extension is defined for each project $z \in \mathcal{Y}$ as $\widetilde{\sigma}(\gamma, S, z) = \displaystyle\sum_{i=1}^{n} \gamma(i)\sigma(\{i\}, z) = \sum_{i=1}^{n} \gamma(i)\varphi(i, z)$.

Example 2.4 shows that, in general, the core based on some contribution measure σ properly contains the set of cost share equilibria in which the individual cost shares are determined by Radon-Nikodym derivative of σ. Whereas, in economies with an atomless space of agents, under certain regularity conditions, the two sets coincide (see Basile, Graziano, and Pesce (2016), Theorem 1). Extending the idea of a contribution scheme to coalitions of more general type (generalized coalitions), a core-equivalence result can be obtained also for the case of finite economies. This is shown in the next theorem, which is also the base for further characterizations of cost share equilibria as well as a tool to derive their existence in some simple situations. Its proof is based on the same idea presented in Graziano and Romaniello (2012) for the case of linear cost share equilibria, i.e. the case in which contribution measures and cost distribution functions do not depend on public projects. The proof of the equivalence requires the following assumption, known in the literature as **the essentiality condition** (see Diamantaras and Gilles (1996)), which ensures that any variation in the public goods provision can be compensated for any agent by a suitable quantity of private goods.

$(A3)$ *Essentiality condition:* for any $x \in \mathbb{R}^m_+$, y, y' in $\mathcal{Y}$ and $i \in I$ there exists $x' \in \mathbb{R}^m_+$ such that $u_i(x', y') \geqslant u_i(x, y)$.[4]

Theorem 3.2 *Let φ be a cost distribution function and σ_φ be the corresponding contribution measure. If $(A1) - (A3)$ are satisfied, then $CSE_\varphi(\mathcal{E}) = C^A_{\sigma_\varphi}(\mathcal{E})$.*

This equivalence result, proved in Basile, Graziano, and Pesce (2016) (Theorem 4), explicitly depends on the cost distribution function and the corresponding contribution measure. It includes, as a particular case, the equivalence between the equal cost share equilibria and the proportional core. Moreover it also generalizes Theorem 4.1 of Graziano and Romaniello (2012) using an approach dispensing with the so-called *second essentiality condition* and the assumption of integrable utilities.[5] If, for some reasons, the set of generalized coalitions $\mathcal{F}$ limits to the class of generalized coalitions with full support $\tilde{\mathcal{F}}$, the Aubin core in general enlarges, since $\tilde{\mathcal{F}} \subseteq \mathcal{F}$, that is $C^A_\sigma(\mathcal{E}) \subseteq C^{Af}_\sigma(\mathcal{E})$, where $C^{Af}_\sigma(\mathcal{E})$ denotes the set of feasible allocations that cannot be σ_φ-blocked by generalized coalition with full support. In what follows, we prove that, actually, the Aubin core keeps unchanged and still coincides with the set of cost share equilibra if a stronger version of assumption $(A1)$ holds, that is if the initial endowment allows each agent to cover the cost of any public project, regardless of the cost distribution function, and to save a positive amount of each good for consumption, i.e.

$$(A1^*) \quad \omega_i - \varphi(i, y)c(y) \gg 0 \text{ for all } i \in I, \text{ for all } y \in \mathcal{Y} \text{ and for all } \varphi \in \Phi.$$

Theorem 3.3 *Let φ be a cost distribution function and σ_φ be the corresponding contribution measure. If $(A1^*) - (A3)$ are satisfied, then $CSE_\varphi(\mathcal{E}) = C^{Af}_{\sigma_\varphi}(\mathcal{E})$.*

Proof. See in the Appendix. ∎

Theorems 3.2 and 3.3 together imply that, given a contribution measure σ, $C^A_\sigma(\mathcal{E}) = C^{Af}_\sigma(\mathcal{E})$ because both coincide with the set of cost share equilibria. Theorem 3.3 uses assumptions quite weaker than those used in Graziano and Romaniello (2012) for linear cost share equilibria. It also generalises the equivalence result in Hervés-Beloso and Moreno-García (2001) for economies with no public goods and it will be used in the next subsection to characterize the cost share equilibria as the Nash equilibria of an associated game suitably defined.

3.2 Cost share equilibria as Nash equilibria of a game

For an economy without public projects, Hervés-Beloso and Moreno-García (2009a), Hervés-Beloso and Moreno-García (2009b) give a characterization of Walrasian equilibria in terms of Nash equilibria of an associated two-player game, named the **society game**, in which each player represents a

[4] A condition similar to $(A3)$ is used in Hammond and Villar (1998), which imposes a suitable restriction on $\mathcal{Y}$ to exclude those public projects which are so bad for some agent that no compensation is possible.

[5] Revisiting the proof of Theorem 4.1 in Graziano and Romaniello (2012), one notices that the assumption of integrable utilities, borrowed from Gilles and Diamantaras (1998), can be removed since the relevant allocations in the associated continuum economy are step functions and essentiality condition $(A3)$ is satisfied.

role of the society; the outcomes are given by the strategies themselves and prices appear neither in the strategy sets nor in the payoff functions. It is established that, independently of the number of consumers and commodities, Walrasian equilibrium is implementable as a strong Nash equilibrium of the associated game. We extend this result to economies with an abstract set of public projects. Using Theorem 3.3 and along the line of this characterization, we define a two-player game associated to the economy and we prove that cost share equilibria are exactly Nash equilibria of the associated game. As far as we know, the result represents a first attempt to give a game theoretical interpretation of cost share equilibria. It also implies the analogous characterization for linear and equal cost share equilibria as defined by Gilles and Diamantaras (1998) in the general setting of Mas-Colell (1980).

Throughout this section, we consider only economies $\mathcal{E}$ and cost distribution functions $\varphi \in \Phi$ satisfying assumptions $(A1^*)$, $(A2)$ and $(A3)$. Given a contribution measure σ, we construct a game G_σ, associated to the economy $\mathcal{E}$, with only two players $N = \{1, 2\}$. The stategy set of player 1, denoted by S_1, is given by the set of feasible allocations in $\mathcal{E}$, i.e.

$$S_1 = \left\{ (x, y) = (x_1, \ldots, x_n, y) \in \mathbb{R}^{nm}_+ \times \mathcal{Y}, \text{ such that } \sum_{i=1}^{n} x_i + c(y) \leqslant \sum_{i=1}^{n} \omega_i \right\}.$$

Assumption $(A1)$, in particular $\omega \gg c(y)$ for all $y \in \mathcal{Y}$, ensures that S_1 is non-empty since it contains, for instance, the feasible allocation $\left(\frac{1}{n}\left(\omega - c(y)\right), y\right) \in S_1$.
Given a cost distribution function φ and a real number $\alpha \in (0, 1)$, the strategy set of player 2, denoted by S_2, is given by

$$S_2 = \left\{ (\gamma, x, y) = (\gamma_1, \ldots, \gamma_n, x_1, \ldots, x_n, y) \in [\alpha, 1]^n \times \mathbb{R}^{nm}_+ \times \mathcal{Y} \text{ s.t. } \sum_{i=1}^{n} \gamma_i x_i + \sum_{i=1}^{n} \gamma_i \varphi(i, y) c(y) \leqslant \sum_{i=1}^{n} \gamma_i \omega_i \right\}.$$

The strategy set for player 2 allows to define an allocation $(x, y) \in \mathbb{R}^{nm}_+ \times \mathcal{Y}$ which satisfies the feasibility (ii) of Definition 3.1 with a participation rate greater or equal to α for every member. This impies that $\gamma_i > 0$ for all $i \in I$ and hence it also allows to define a generalized coalition (γ, T) with full support. Observe that S_2 is a non-empty set since $\left(\mathbf{1}, \frac{1}{n}\left(\omega - c(y)\right), y\right) \in S_2$, where $\mathbf{1}$ is the vector in $\mathbb{R}^n$ whose coordinates are constant and equal to 1. Notice also that the contribution measure is involved only in the strategy set S_2 and that the society, in the two different roles, may choose a different project to be realized.

Denoted by S the product set $S_1 \times S_2$, a strategy profile is any $s = (x, y, \gamma, g, z) \in S$, where $(x, y) \in S_1$ is a strategy for player 1 and $(\gamma, g, z) \in S_2$ is a strategy for player 2.
Given a strategy profile $s = (x, y, \gamma, g, z) \in S$, the payoff functions, F_1 and F_2, for players 1 and 2 respectively, are defined as follows

$$F_1(x, y, \gamma, g, z) = min_{i=1,\ldots,n} \left\{ f(\gamma_i)(u_i(x_i, y) - u_i(g_i, z)) \right\}$$

and

$$F_2(x, y, \gamma, g, z) = min_{i=1,\ldots,n} \{\gamma_i \left(u_i(g_i, z) - u_i(x_i, y)\right)\},$$

where f is a positive differentiable function defined in $[\alpha, 1]$ and such that $f'(x)x > f(x)$. Observe that $\frac{f(x)}{x}$ is a positive and strictly increasing function and, therefore, max $\left\{\frac{f(x)}{x}\right\} = f(1)$.

The associated game G_σ is then defined by $G_\sigma \equiv \{S_1, S_2, F_1, F_2\}$ and the notion of Nash equilibrium as follows.

Definition 3.4 *A strategy profile $s^* = (x^*, y^*, \gamma^*, g^*, z^*) \in S$ is a Nash equilibrium for G_σ if*

$$\begin{aligned} F_1(s^*) &\geqslant F_1(x, y, \gamma^*, g^*, z^*), \textit{ for every } (x, y) \in S_1, \textit{ and} \\ F_2(s^*) &\geqslant F_2(x^*, y^*, \gamma, g, z), \textit{ for every } (\gamma, g, z) \in S_2. \end{aligned}$$

We denote by $NE(G_\sigma)$ the set of Nash equilibria for the game G_σ.

Notice that for any $(x, y) \in S_1$, $(\mathbf{1}, x, y) \in S_2$ and $F_1(x, y, \gamma, x, y) = F_2(x, y, \gamma, x, y) = 0$ for any $\gamma \in [\alpha, 1]^n$ such that $(\gamma, x, y) \in S_2$. Furthermore, the following interesting properties listed in the next proposition hold.

Proposition 3.5 *Given $s \in S$,*

(1) $F_n(s) > 0 \Rightarrow F_m(s) < 0$ *with* $n \neq m$.

(2) $F_2(s^*) \geqslant 0$ *for any* $s^* \in NE(G_\sigma)$.

(3) If (x, y) is a feasible not Pareto optimal allocation in $\mathcal{E}$, there exists $(x', y') \in S_1$ such that $F_1(x', y', \gamma, g, z) > F_1(x, y, \gamma, g, z)$ for all $(\gamma, g, z) \in S_2$. Vice versa, if (x, y) is Pareto optimal in $\mathcal{E}$ and $(\gamma, x, y) \in S_2$ for some $\gamma \in [\alpha, 1]^n$, then $F_1(x', y', \gamma, x, y) \leqslant F_1(x, y, \gamma, x, y) = 0$ for all $(x', y') \in S_1$.

(4) $(x, y) \in C_\sigma^{Af}(\mathcal{E}) \iff F_2(x, y, \gamma, g, z) \leqslant 0$ *for all* $(\gamma, g, z) \in S_2$*;*

(5) $(x, y) \in CSE_{\varphi_\sigma}(\mathcal{E}) \Rightarrow (x, y, \mathbf{1}, x, y) \in NE(G_\sigma)$.

Proof. See in the Appendix. ■

Condition (1) means that the payoffs for both players can not be simultaneously positive whereas, by (2), the payoff of player 2 is non-negative at any Nash equilibrium. For (3), if (x, y) is not Pareto optimal, player 1 can improve upon her playoff, whereas if player 2 select $(\gamma, x, y) \in S_2$ and (x, y) is Pareto optimal for the economy $\mathcal{E}$, then the best reply of player 1 is the same efficient allocation (x, y). Theorem 3.3 together with (4) ensures that if (x, y) is a cost share equilibria, player 2 gets non-positive playoff regardless of her strategy. Finally, condition (5) states a first relation between the set of cost share of the economy $\mathcal{E}$ and the Nash equilibria of the associated game G_σ.

The next proposition shows that at a Nash equilibrium both players achieve the same zero playoff even though they might choose different public projects.

Proposition 3.6 *Assume that $u_i(x,y) > 0 \Rightarrow x > 0$ and $u_i(\cdot,y)$ is concave for all $i \in I$ and $y \in \mathcal{Y}$. If s^* is a Nash equilibrium for the game G_σ, then $F_1(s^*) = F_2(s^*) = 0$.*

Proof. See in the Appendix. ■

We are now ready to prove the characterization of φ-cost share equilibria of the economy $\mathcal{E}$ as Nash equilibria of the associated game G_σ.

Theorem 3.7 *Let $\mathcal{E}$ be an economy with public goods, σ be a contribution measure and φ_σ be its corresponding cost distribtion function. Then,*
$s^* = (x^*, y^*, \gamma^*, g^*, z^*) \in NE(G_\sigma) \quad \Rightarrow \quad (x^*, y^*) \in CSE_{\varphi_\sigma}(\mathcal{E})$.
Reciprocally,
$(x^*, y^*) \in CSE_{\varphi_\sigma}(\mathcal{E}) \;\Rightarrow\; s^* = (x^*, y^*, \gamma^*, g^*, z^*) \in NE(G_\sigma)$, *for any* $s^* = (x^*, y^*, \gamma^*, g^*, z^*) \in S$ *with* $u_i(g_i^*, z^*) = u_i(x_i^*, y^*)$ *for every* $i \in I$.
In particular,
$(x^*, y^*) \in CSE_{\varphi_\sigma}(\mathcal{E}) \iff (x^*, y^*, \gamma^*, x^*, y^*) \in NE(G_\sigma)$, *with* $\gamma_i^* = \gamma^*$ *for every* $i \in I$.

Proof. See in the Appendix. ■

As a consequence of Proposition 3.6 and Proposition 3.5 (1) we derive that, since for each contribution measure σ, the game G_σ has only two players and the coalition formed by both players has no incentive to deviate, any Nash equilibrium is a strong Nash equilibrium of the game G_σ. Therefore, in the light of Theorem 3.7, cost share equilibria are implementable as a Nash equilibrium of the society game.

4 Existence of cost share equilibria

In the literature, there is no general existence theorem of cost share equilibria, but Gilles and Diamantaras (1998) analyze several examples of economies in which specific linear cost share equilibria exist. In what follows, we prove that the set of cost share equilibria might be empty even in economies satisfying standard assumptions.

We first examine a relation between a cost share equilibrium of an economy with public goods and a competitive equilibrium of a pure exchange economy without pubblic goods suitably constructed. Precisely, given an economy $\mathcal{E} = \{I, \mathbb{R}^m_+, \mathcal{Y}, c, (u_i, \omega_i)_{i\in I}\}$ and a cost distribution function $\varphi \in \Phi$ satisfying assumption $(A1^*)$, $(A2)$ and $(A3)$, for each given $y \in \mathcal{Y}$, we define the economy $\mathcal{E}(y, \varphi) = \{I, \mathbb{R}^m_+, (u_i(\cdot, y), \omega_i^y)_{i\in I}\}$, devoid of public goods, with the same set of agents I, the same commodity space for private consumption $\mathbb{R}^m_+$, and such that, for each agent $i \in I$, $\omega_i^y = \omega_i - \phi(i, y)c(y)$ and $\omega_i^y \gg 0$ because of $(A1^*)$. A **competitive equilibrium** in the economy $\mathcal{E}(y, \varphi)$ consists of a pair (x, p) where x is a feasible allocation, i.e. $\sum_{i\in I} x_i \leqslant \sum_{i\in I} \omega_i^y$, and $p \in \Delta$ is a price vector such that $p \cdot x_i \leqslant p \cdot \omega_i^y$ for all $i \in I$ and $u_i(k, y) > u_i(x_i, y) \Rightarrow p \cdot k > p \cdot \omega_i^y$.

Proposition 4.1 *If $(x^*, y^*) \in CSE_\varphi(\mathcal{E})$ with price function $p^* : \mathcal{Y} \to \Delta$, then x^* is a competitive equilibrium with price $p^*(y^*)$ in the economy $\mathcal{E}(y^*, \varphi)$.*
Conversely if, for each $y \in \mathcal{Y}$, (x_y, p_y) is a competitive equilibrium in the economy $\mathcal{E}(y, \varphi)$ and if there exists $y^* \in \mathcal{Y}$ such that $u_i(x_{iy^*}, y^*) \geqslant u_i(x_{iz}, z)$ for all $i \in I$ and all $z \in \mathcal{Y}$, then (x_{y^*}, y^*) is a cost share equilibrium for the economy $\mathcal{E}$ with the equilibrium price $p^* : \mathcal{Y} \to \mathbb{R}^m_+$ given by $p^*(z) := p_z$.

Proof. See in the Appendix ■

A cost share equilibrium (x^*, y^*) is a competitive equilibrium in the economy without public goods $\mathcal{E}(y^*, \varphi)$, constructed on the base of the public project y^* arisen in equilibrium. Conversely, if there is a competitive equilibrium preferred by everybody with respect to all the other competitive equilibria (that Pareto dominates), then this constitutes a cost share equilibrium in the economy $\mathcal{E}$ with public projects. The mere existence of a competitive equilibrium in each economy $\mathcal{E}(y, \varphi)$, defined above, does not ensure the existence of a cost share equilibrium in the economy $\mathcal{E}$ with public goods, because a competitive equilibrium that Pareto dominates all the others might not exist. This is the case shown in the next proposition.

Proposition 4.2 *The set of cost share equilibria may be empty, i.e. $CSE(\mathcal{E}) = \bigcup_{\varphi \in \Phi} CSE_\varphi(\mathcal{E}) = \emptyset$.*

Proof. Consider an economy with two public goods, i.e. $\mathcal{Y} = \{y, z\}$, two agents, i.e. $I = \{A, B\}$, and two private goods, i.e. $\mathbb{R}^m_+ = \mathbb{R}^2_+$. The primitives of the economy are given as follows:

$$u_A(f^1, f^2, y) = u_B(f^1, f^2, z) = \sqrt{f^1} + \sqrt{f^2}$$

$$u_A(f^1, f^2, z) = u_B(f^1, f^2, y) = \sqrt{f^1} + \sqrt{f^2} + 5;$$

$$c = c(y) = c(z) = \left(\frac{2}{3}, \frac{2}{3}\right);$$

$$\omega_A = (1, 3) \;\; \omega_B = (3, 1).$$

We first observe that for any $(\varphi_A(\cdot), \varphi_B(\cdot)) \geqslant 0$ such that $\varphi_A(\cdot) + \varphi_B(\cdot) = 1$,

$$\begin{aligned} \omega_A - \varphi_A(\cdot)c &= (1, 3) - \varphi_A(\cdot)\left(\frac{2}{3}, \frac{2}{3}\right) = \left(1 - \frac{2}{3}\varphi_A(\cdot), 3 - \frac{2}{3}\varphi(A)\right) \gg 0 \\ \omega_B - \varphi_B(\cdot)c &= (3, 1) - \varphi_B(\cdot)\left(\frac{2}{3}, \frac{2}{3}\right) = \left(3 - \frac{2}{3}\varphi_B(\cdot), 1 - \frac{2}{3}\varphi_B(\cdot)\right) \gg 0. \end{aligned}$$

Moreover,

$$\omega_A + \omega_B = (4, 4) \gg \left(\frac{2}{3}, \frac{2}{3}\right) = c.$$

We want to show that there is no cost share equilibrium. We proceed by the way of contradiction. Let (x, t) be a cost share equilibrium with price function $p^* : \mathcal{Y} \to \Delta$ and $\varphi^* : I \times \mathcal{Y} \to \mathbb{R}_+$, where $t \in \{y, z\}$. Thanks to Proposition 4.1, we have that $x_i \gg 0$ for any $i \in \{A, B\}$, and we can limit our attention on the economies

$$\mathcal{E}(y, \varphi^*) = \{\{A, B\}, (u_i(\cdot, y), \omega_i^{y, \varphi^*})_{i \in \{A, B\}}\} \text{ and } \mathcal{E}(z, \varphi^*) = \{\{A, B\}, (u_i(\cdot, z), \omega_i^{z, \varphi^*})_{i \in \{A, B\}}\},$$

where $\omega_i^{y,\varphi^*} = \omega_i - \varphi_i^*(y)c$ and $\omega_i^{z,\varphi^*} = \omega_i - \varphi_i^*(z)c$, for any $i \in \{A, B\}$. In the economy $\mathcal{E}(t, \varphi^*)$ described above, regardless of $t \in \{y, z\}$, for any price (p_1, p_2) agents' demand functions are

$$\begin{aligned}
(f_A^1, f_A^2) &= \left(\frac{p_2}{p_1(p_1+p_2)}\left[p_1 + 3p_2 - \frac{2}{3}\varphi_A^*(t)(p_1+p_2)\right]; \frac{p_1}{p_2(p_1+p_2)}\left[p_1 + 3p_2 - \frac{2}{3}\varphi_A^*(t)(p_1+p_2)\right]\right) \\
(f_B^1, f_B^2) &= \left(\frac{p_2}{p_1(p_1+p_2)}\left[3p_1 + p_2 - \frac{2}{3}\varphi_B^*(t)(p_1+p_2)\right]; \frac{p_1}{p_2(p_1+p_2)}\left[3p_1 + p_2 - \frac{2}{3}\varphi_B^*(t)(p_1+p_2)\right]\right).
\end{aligned}$$

Then, being

$$\omega_A^t + \omega_B^t = (1,3) - \varphi_A^*(t)\left(\frac{2}{3}, \frac{2}{3}\right) + (3,1) - \varphi^*(B,t)\left(\frac{2}{3}, \frac{2}{3}\right) = (4,4) - \left(\frac{2}{3}, \frac{2}{3}\right) = \left(\frac{10}{3}, \frac{10}{3}\right),$$

the aggregate excess demand of both economies $\mathcal{E}(t, \varphi^*)$ is

$$z(p_1, p_2) = [f_A(p_1,p_2) + f_B(p_1,p_2)] - [\omega_A^t + \omega_B^t] = \left(\frac{10}{3}\frac{p_2}{p_1} - \frac{10}{3}; \frac{10}{3}\frac{p_1}{p_2} - \frac{10}{3}\right), \tag{2}$$

which satisfies the so-called gross substitute (GS) property (see Definition 17.F.2 in Mas-Colell, Whinston, and Green (1995)) since

$$\frac{\partial z^1}{\partial p_2} = \frac{10}{3p_1} > 0 \qquad \frac{\partial z^2}{\partial p_1} = \frac{10}{3p_2} > 0.$$

Then, Proposition 17.F.3 in Mas-Colell, Whinston, and Green (1995) ensures the uniqueness of the equilibrium price[6] and therefore the uniqueness of the equilibrium allocation.
If $t = y$, the unique competitive equilibrium in the economy $\mathcal{E}(y, \phi^*)$ is

$$\begin{aligned}
(f_A^{*1}, f_A^{*2}) &= \left(2 - \frac{2}{3}\varphi_A^*(y), 2 - \frac{2}{3}\varphi_A^*(y)\right), \\
(f_B^{*1}, f_B^{*2}) &= \left(2 - \frac{2}{3}\varphi_B^*(y), 2 - \frac{2}{3}\varphi_B^*(y)\right), \text{ with} \\
p_1^*(y) &= p_2^*(y),
\end{aligned}$$

and, by Proposition 4.1, $(x, t) = (f^*, y)$. Consider, the allocation (g, z) such that

$$(g_A^1, g_A^2, z) = \left(1 - \frac{2}{3}\varphi_A^*(z), 3 - \frac{2}{3}\varphi_A^*(z), z\right),$$

and notice that

$$\begin{aligned}
(g_A^1, g_A^2) &\gg (0,0), \\
u_A(g_A^1, g_A^2, z) &= \sqrt{1 - \frac{2}{3}\varphi_A^*(z)} + \sqrt{3 - \frac{2}{3}\varphi_A^*(z)} + 5 > 5 > 2\sqrt{2} \geq 2\sqrt{2 - \frac{2}{3}\varphi_A^*(y)} = \\
&= u_A(f_A^{*1}, f_A^{*2}, y) = u_A(x_A^1, x_A^2, y),
\end{aligned}$$

and for any $(p_1, p_2) \in \mathbb{R}_+^2 \setminus \{0\}$, $p_1 g_A^1 + p_2 g_A^2 + \varphi_A^*(z)(p_1+p_2)c \leqslant p_1\omega_A^1 + p_2\omega_A^2$.

[6] If there were two prices, then they must be collinear, but if $p(y) \in \triangle$, i.e., it is normalized, it must be unique.

In particular, it holds for $(p_1^*(z), p_2^*(z))$ whatever this pair is. This contradicts the fact that (x, y) is a cost share equilibrium.

On the other hand, if $t = z$ the unique competitive equilibrium in the economy $\mathcal{E}(z, \phi^*)$ is given as follows:

$$\begin{aligned}
(f_A^{*1}, f_A^{*2}) &= \left(2 - \frac{2}{3}\varphi_A^*(z), 2 - \frac{2}{3}\varphi_A^*(z)\right), \\
(f_B^{*1}, f_B^{*2}) &= \left(2 - \frac{2}{3}\varphi_B^*(z), 2 - \frac{2}{3}\varphi_B^*(z)\right), \text{ with} \\
p_1^*(z) &= p_2^*(z),
\end{aligned}$$

and, by Proposition 4.1, $(x, t) = (f^*, z)$. Notice that the allocation (g, y) such that

$$(g_B^1, g_B^2, y) = \left(3 - \frac{2}{3}\varphi_B^*(y), 1 - \frac{2}{3}\varphi_B^*(y), y\right),$$

is such that

$$\begin{aligned}
(g_B^1, g_B^2) &\gg (0, 0) \\
u_B(g_B^1, g_B^2, y) &= \sqrt{3 - \frac{2}{3}\varphi_B^*(y)} + \sqrt{1 - \frac{2}{3}\varphi_B^*(y)} + 5 > 5 > 2\sqrt{2} \geq 2\sqrt{2 - \frac{2}{3}\varphi_B^*(z)} = \\
&= u_B(f_B^{*1}, f_B^{*2}, z) = u_B(x_B^1, x_B^2, z),
\end{aligned}$$

and for any $(p_1, p_2) \in \mathbb{R}_+^2 \setminus \{0\}$, $p_1 g_B^1 + p_2 g_B^2 + \varphi_B^*(y)(p_1 + p_2)c \leqslant p_1 \omega_B^1 + p_2 \omega_B^2$.

In particular, it holds for $(p_1^*(y), p_2^*(y))$ whatever this pair is, and this is a contradiction. Therefore, there is no cost share equilibrium in this economy. ■

Remark 4.3 We have shown above an economy satisfying "standard" assumptions which has an empty set of cost share equilibria. With similar computations, one may exhibit another example with two public goods and different costs, as $c(y) = (1, 1)$ and $c(z) = \left(\frac{2}{3}, \frac{2}{3}\right)$.

Remark 4.4 In the above example, we have constructed only two economies without public goods $\mathcal{E}(y, \varphi^*)$ and $\mathcal{E}(z, \varphi^*)$, getting a competitive equilibrium in each of them. These two competitive equilibrium allocations differ only for φ and, since $\varphi_A(y) > \varphi_A(z) \Rightarrow \varphi_B(y) = 1 - \varphi_A(y) < 1 - \varphi_A(z) = \varphi_B(z)$, none of the two Pareto dominates the other. This is consistent with Proposition 4.1.

Gilles and Scotchmer (1997) provide a necessary and sufficient condition for the existence of an equilibrium (see Theorem 2) in terms of *efficient scale*. Their existence result is driven by an Edgeworth like equivalence theorem in replica economies and assumes that the cost of projects is described as a multifunction which cannot be reduced to a function for the assumptions they made. We will present below a stronger assumption guaranteeing the existence of a cost share equilibrium.

Assumption 4.5 *Let $\mathcal{E}$ be an economy and $\varphi \in \Phi$ such that for any agent $i \in I$,*

(1) *for any allocation* (x,y), $u_i(x,y) = v_i^1(x) + v^2(y)$, *where* $v_i^1 : \mathbb{R}_+^m \to \mathbb{R}$ *and* $v^2 : \mathcal{Y} \to \mathbb{R}$;

(2) $argmax_{y \in Y} v^2(y) \neq \emptyset$,

(3) $\omega_i - \varphi(i,y)c(y) = \omega_i - \varphi(i,z)c(z)$ *for any* $y, z \in \mathcal{Y}$.

By Assumption 4.5 (1), agents' utility functions are separable between the utility achieved from the consumption of private goods and from the realization of a public project. The second component v^2 is common to everybody. Although this assumption is restrictive, it is satisfied in most of the examples described in the literature and it is also often used in the so-called *other regarding preferences* literature which involves different widespread externalities (see Dufwenberg, Heidhues, Kirchsteiger, Riedel, and Sobel (2011) among the others). Assumption 4.5 (2) imposes the existence of at least a public project that maximizes the common component of the utility related to public goods v^2. In a sense, it imposes an order among public goods commonly accepted by agents, embedding $\mathcal{Y}$ with a mathematical structure and v^2 with some topological properties; however it trivially holds if $\mathcal{Y}$ is finite. Assumption 4.5 (3) requires that the ammount of goods available for consumption is the same after the contribution to the establishment of any public good. This assumption allows us to construct an economy $\mathcal{E}^*$ with only private goods and show that any competitive equilibrium of $\mathcal{E}^*$ defines a cost share equilibria for $\mathcal{E}$. The economy $\mathcal{E}^*$ is defined as $\mathcal{E}^* = \{I, \mathbb{R}_+^m, (v_i^1, \omega_i^*)_{i \in I}\}$, where $\omega_i^* = \omega_i - \varphi(i, \cdot)c(\cdot)$.

Theorem 4.6 *Let* $\mathcal{E}$ *be an economy and* φ *a cost distribution function satisfying Assumption 4.5. There exists a cost share equilibrium of* $\mathcal{E}$, *i.e.* $CSE_\varphi(\mathcal{E}) \neq \emptyset$.

Proof. Let x^* be a competitive equilibrium for the economy $\mathcal{E}^*$ with respect to the price vector p^*, and let $y^* \in argmax_{y \in Y} v^2(y)$ which exists by Assumption 4.5 (2). We now show that $(x^*, y^*) \in CSE_\varphi(\mathcal{E})$ with respect to the price function $p^* : \mathcal{Y} \to \mathbb{R}_+^m$ defined as $p^*(z) = p^*$ for any $z \in \mathcal{Y}$. First note that for all $i \in I$,

$$p^* \cdot x_i^* \leqslant p^* \cdot \omega_i^* \Rightarrow p^*(y^*) \cdot x_i^* + p^*(y^*) \cdot \varphi(i, y^*)c(y^*) \leqslant p^*(y^*) \cdot \omega_i.$$

If (g, z) is such that $u_i(g_i, z) > u_i(x_i^*, y^*)$ for some agent i and $p^*(z) \cdot g_i + p^*(z) \cdot \varphi(i,z)c(z) \leqslant p^*(z) \cdot \omega_i$, then

$$u_i(g_i, z) > u_i(x_i^*, y^*) \Rightarrow v_i^1(g_i) + v^2(z) > v_i^1(x_i^*) + v^2(y^*) \geqslant v_i^1(x_i^*) + v^2(z) \Rightarrow v_i^1(g_i) > v_i^1(x_i^*).$$

Being x^* a competitive equilibrium in $\mathcal{E}^*$, $p^* \cdot g_i > p^* \cdot \omega_i^*$, which implies that $p^*(z) \cdot g_i + p^*(z) \cdot \varphi(i,z)c(z) > p^*(z) \cdot \omega_i$, a contradiction. ∎

5 Extensions and Final Remarks

The cost share equilibrium for economies with non-Samuelsonian collective goods studied in our paper originates from linear cost sharing equilibrium introduced in Gilles and Diamantaras (1998). In a linear cost share equilibrium, all agents optimize given a certain cost share to be contributed towards the provision of public goods in the economy. Each agent pays a fraction of the total costs of public goods. In a cost share equilibrium we assume additionally that the cost shares depend on the public goods configuration. This implies that for a certain contribution scheme φ, the fraction contributed by agent i under the project $z \in \mathcal{Y}$ is defined as $\varphi(i,z)p(z) \cdot c(z)$, where $p(z)$ is the conjectural price system and $c(z)$ the cost. It turns out that, for a certain contribution scheme φ, the corresponding cost share equilibria are equivalent to Aubin σ-core allocations. The contribution measure σ defines the coalitional contribution as sum of the individual cost shares weighted by the share of participation of agents in the coalition. In the paper, we have also provided a characterization of this more general cost share equilibria in terms of Nash equilibria of an associated two-player game, referred to as the society game. Moreover we have established that one should not expect an existence result under very general assumptions.

We discuss below possible extensions of our results.

5.1 The case of possibly non-linear cost distribution

In Basile, Gilles, Graziano, and Pesce (2021), the notion of cost share equilibrium and the corresponding core are further extended by replacing the scaling cost φ and the corresponding contribution σ with a multi-dimensional cost and a multi-dimensional contribution measure. This allows the introduction of highly nonlinear scaling of the costs of individual and coalitional provision, extending the scope of the theory. The resulting equilibrium notion, which we will call generalized here, is defined as follows.

A multi-dimensional cost distribution is a function $\varphi : I \times \mathcal{Y} \to \mathbb{R}^m_+$ such that

$$\sum_{i \in I} \varphi(i,z) = c(z), \quad \text{for each } z \in \mathcal{Y}.$$

Similarly, a multi-dimensional contribution measure is defined as an additive function $\sigma : \mathcal{P}(I) \times \mathcal{Y} \to \mathbb{R}^m_+$ such that $\sigma(I,z) = c(z)$, for each $z \in \mathcal{Y}$. Moreover, as in the scalar case, there is a one-to-one relationship between the cost distribution functions and the contribution measures.

Definition 5.1 *A feasible allocation $(x_1, \dots, x_n, y)$ is a* ***generalized cost share equilibrium*** *in $\mathcal{E}$ if there exists a price system $p : \mathcal{Y} \to \Delta$ and a multi-dimensional cost distribution function φ such that for every $i \in I$, (x_i, y) maximizes u_i on the budget set*

$$B_i(p, \varphi(i)) = \left\{(h,z) \in \mathbb{R}^m_+ \times \mathcal{Y} \mid p(z) \cdot h + p(z) \cdot \varphi(i,z) \leqslant p(z) \cdot \omega_i\right\}.$$

We denote by $GCS(\mathcal{E})$ the collection of all generalized cost share equilibria in the economy $\mathcal{E}$ and, for a fixed cost distribution function φ, by $GCE_\varphi(\mathcal{E})$ the set of generalized cost share equilibria with associated cost distribution φ. Then clearly the inclusions $ECE(\mathcal{E}) \subseteq LCE(\mathcal{E}) \subseteq CSE(\mathcal{E}) \subseteq GCE(\mathcal{E})$ hold true since each scalar cost distribution φ generates a multi-dimensional cost distribution by means of the product $\varphi(i,z)c(z)$.

Definition 5.2 *Let σ be a multi-dimensional contribution measure and φ the relative cost distribution. A feasible allocation $(x_1, \ldots, x_n, y)$ is said to be σ-Aubin blocked if it is possible to find an Aubin coalition $(\gamma, S) \in \mathcal{F}$ and an allocation (g, z) such that*

$$\sum_{i=1}^{n} \gamma_i g_i + \sum_{i=1}^{n} \gamma_i \varphi(i, z) \leqslant \sum_{i=1}^{n} \gamma_i \omega_i$$

$$u_i(g_i, z) > u_i(x_i, y), \quad \forall i \in S.$$

The generalized σ-Aubin core of the economy, denoted by $GC^A_\sigma(\mathcal{E})$ is defined accordingly.
It is easy to verify that all the results proved in Section 3 and in Section 4 remain true for generalized cost share equilibria. In particular, given a multi-dimensional cost distribution φ and the corresponding measure σ, the equivalence $GCE_\varphi(\mathcal{E}) = GC^A_\sigma(\mathcal{E})$ holds true. The proof of the equivalence follows the same argument of Theorem 3.2 making use of the equivalence between generalized σ-core allocations and cost share equilibria proved in the case of an atomless economy with public goods in Basile, Gilles, Graziano, and Pesce (2021).
What is important to point out in this Section is that, although the set of equilibria is larger under possibly non-linear cost contributions, the existence is still not guaranteed. We show this point below by adapting Proposition 4.2 to generalized cost share equilibria.

Proposition 5.3 *The set of generalized cost share equilibria may be empty, i.e. $GCE(\mathcal{E}) = \bigcup_{\varphi \in \Phi} GCE_\varphi(\mathcal{E}) = \emptyset$.*

Proof. Consider the same economy described in Proposition 4.2 and observe that for any $\varphi : I \times \mathcal{Y} \longrightarrow \mathbb{R}^2_+$ we have that

$$\varphi(A, \cdot) + \varphi(B, \cdot) = c(\cdot) = \left(\frac{2}{3}, \frac{2}{3}\right) \iff 0 \leq \varphi(\cdot, \cdot) \leq \left(\frac{2}{3}, \frac{2}{3}\right).$$

Therefore

$$\begin{aligned} \omega_A - \varphi(A, \cdot) &\geq (1, 3) - \left(\frac{2}{3}, \frac{2}{3}\right) \gg 0 \\ \omega_B - \varphi(B, \cdot) &\geq (3, 1) - \left(\frac{2}{3}, \frac{2}{3}\right) \gg 0. \end{aligned}$$

Moreover,

$$\omega_A + \omega_B = (4, 4) \gg \left(\frac{2}{3}, \frac{2}{3}\right) = c.$$

We want to show that there is no generalized cost share equilibrium. We proceed by the way of contradiction. Thus, let (x, t) be a generalized cost share equilibrium with price function $p^* : \mathcal{Y} \longrightarrow \triangle$ and $\varphi^* : I \times \mathcal{Y} \longrightarrow \mathbb{R}^m_+$, where $t \in \{y, z\}$. As before, $x_i \gg 0$ for any $i \in \{A, B\}$ and because of Proposition 4.1, we can fix our attention on the economies $\mathcal{E}(y, \varphi^*)$ and $\mathcal{E}(z, \varphi^*)$ which have a unique competitive equilibrium given respectively by

$$\begin{aligned}(f_A^{*1}, f_A^{*2}) &= \frac{4-\varphi_1^*(A,y)-\varphi_2^*(A,y)}{2}\cdot(1,1),\\ (f_B^{*1}, f_B^{*2}) &= \frac{4-\varphi_1^*(B,y)-\varphi_2^*(B,y)}{2}\cdot(1,1),\ \text{with}\\ p_1^*(y) &= p_2^*(y),\end{aligned}$$

and

$$\begin{aligned}(f_A^{*1}, f_A^{*2}) &= \frac{4-\varphi_1^*(A,z)-\varphi_2^*(A,z)}{2}\cdot(1,1),\\ (f_B^{*1}, f_B^{*2}) &= \frac{4-\varphi_1^*(B,z)-\varphi_2^*(B,z)}{2}\cdot(1,1),\ \text{with}\\ p_1^*(z) &= p_2^*(z).\end{aligned}$$

Now, if $t=y$, then $(x,t)=(f^*,y)$. Consider, the allocation (g,z) defined as

$$(g_A^1, g_A^2, z) = \left(1-\varphi_1^*(A,z), 3-\varphi_2^*(A,z), z\right),$$

and notice that

$$\begin{aligned}&(g_A^1, g_A^2) \gg (0,0),\\ u_A(g_A^1, g_A^2, z) &= \sqrt{1-\varphi_1^*(A,z)}+\sqrt{3-\varphi_2^*(A,z)}+5>5>2\sqrt{2}\geq\\ &\geq 2\sqrt{\frac{4-\varphi_1^*(A,z)-\varphi_2^*(A,z)}{2}} = u_A(f_A^{*1}, f_A^{*2}, y) = u_A(x_A^1, x_A^2, y),\end{aligned}$$

and for any $(p_1,p_2)\in\mathbb{R}_+^2\setminus\{0\}$

$$p_1g_A^1+p_2g_A^2+p_1\varphi_1^*(A,z)+p_2\varphi_2^*(A,z) = p_1+3p_2 = p_1\omega_A^1+p_2\omega_A^2.$$

In particular, it holds for $(p_1^*(z), p_2^*(z))$ whatever this pair is. This contradicts the fact that (x,y) is a generalized cost share equilibrium. On the other hand, if $t=z$, then $(x,t)=(f^*,z)$. Then with the same argument, by considering the allocation (g,y) such that

$$(g_B^1, g_B^2, y) = \left(3-\varphi_1^*(B,y), 1-\varphi_2^*(B,y), y\right),$$

also in this case we can obtain a contradiction. Therefore, there is no cost share equilibrium in this economy. ■

Finally, as in Remark 4.3, one may exhibit an example with different costs in which the set of generalized cost share equilibria is empty.

5.2 The case of mixed markets

Basile, Graziano, and Pesce (2016) considers models of economies that involve both small and large traders as well as the choice of a public project. The main elements in their setup of *mixed markets* are an atomless sector of consumers representing the ocean of negligible traders, a set of atoms representing the influential agents, and a contribution scheme that specifies the cost for the provision of the public good for individual agents as well as for coalitions. Negligible and influential agents are defined with respect to the size measure in the agents' space. Small and large contributors are similarly defined with respect to the measure underlying the contribution scheme. They do not assume any mathematical structure for the set of public projects.

Within this framework and using the Aubin approach to cooperation, they establish that the set of φ-cost share equilibria coincides with the σ-core (in the case of finite economies, our Theorem 3.2). The equivalence between Walrasian equilibria of a pure exchange economy and Nash equilibria of a society game is proved by Hervés-Beloso and Moreno-García (2009a) also in the case of mixed markets. Similarly, we expect that our characterization in terms of Nash equilibria can be extended to cost share equilibria of mixed economies with an abstract set of public projects.

Finally, since cost share equilibria of a mixed markets with finitely many atoms are in a one to one correspondence with cost share equilibria of finite economies, the results of our work also say that one should not expect a general existence theorem for the cost share equilibria of mixed markets.

5.3 The case of infinitely many commodities

The paper Graziano (2007) deals with the two fundamental theorems of welfare economics for production economies with a finite set of agents, infinitely many private goods, and a set of public projects. The problem of efficiency and decentralization is addressed under very general assumptions and the set of public projects is without any mathematical structure. In particular, welfare theorems are proved imposing interiority assumptions on the commodity space or under classical properness assumptions on preferences and production sets and assuming that the commodity space enjoys a Riesz space structure. This requirement together with the lattice structure of the price space seems to be indispensable to carry over lattice theoretical arguments connected with properness conditions.

The notion of cost share equilibria can be formulated at the same level of economic generality, not only on the public goods sector of the model but also for its private goods counterpart. This level of generality permits to cover, among others, investigation of infinite-horizon economies, asset pricing models, differentiated commodity models, and allocation problems. Aubin core equivalence results for pure exchange economies with infinitely many commodities are not new in the literature (see, among the others, Noguchi 2000). Hence we expect that the optimality properties of cost share equilibria hold true also in this more general setting. Clearly in this context the non-existence problems emerge in an even more severe manner.

5.4 Aubin core existence

In Allouch and Predtetchinski (2008), an elementary proof of non-emptyness of the Aubin core of a pure exchange economy is provided. Unlike the Debreu and Scarf (1963) result and its numerous extensions, their proof does not require any asymptotic intersection and instead of allowing the economy to become large through replication, they enlarge the set of feasible payoffs for the economy in utility space. Hence, the result is established directly by using the Fan's coincidence theorem. We believe that the elementary arguments used under this approach can be adapted to be extended to economies with non-Samuelsonian collective goods. A direct proof of the existence of Aubin core is relevant in our framework since it opens a room for using core-equilibrium equivalence of Theorem 3.2 for proving the existence of cost share equilibria. We leave this theme to be developed as the subject of future research.

6 Appendix

Proof of Theorem 3.3. The inclusion $CSE_{\varphi}(\mathcal{E}) \subseteq C^{Af}_{\sigma_{\varphi}}(\mathcal{E})$ is always met. For the converse, let (x, y) be an allocation that can not be σ-blocked by a generalized coalition with full support and let us prove that it belongs to $C^{A}_{\sigma_{\varphi}}(\mathcal{E})$. The conclusion will follow from Theorem 3.2. Assume, by contradiction, that there exist an allocation (g, z) and a generalized coalition (γ, S), with $I \setminus S \neq \emptyset$, such that

$$\begin{array}{ll} (i) & \sum_{i=1}^{n} \gamma(i) g_i + \sum_{i=1}^{n} \gamma(i)\varphi(i, z)c(z) \leqslant \sum_{i=1}^{n} \gamma(i)\omega_i \\ (ii) & u_i(g_i, z) > u_i(x_i, y), \qquad \forall i \in S. \end{array}$$

Notice that, being $S = \{i \in I : \gamma(i) > 0\}$, (i) is equivalent to

$$(i) \quad \sum_{i \in S} \gamma(i) g_i \leqslant \sum_{i \in S} \gamma(i)\omega_i^*,$$

where $\omega_i^* = \omega_i - \varphi(i, z)c(z)$ and $\omega_i^* \gg 0$ by $(A1^*)$. Then, by using standard arguments, without loss of generality (i) can be rewritten as

$$\sum_{i \in S} \gamma(i) g_i \ll \sum_{i \in S} \gamma(i)\omega_i^*. \tag{3}$$

By $(A3)$, for each $i \in I \setminus S$ there exists $x_i' \in \mathbb{R}^m_+$ such that $u_i(x_i', z) \geqslant u_i(x_i, y)$ and hence, by monotonicity $(A2)$, given a positive vector $K \in \mathbb{R}^m_{++}$, $u_i(x_i' + K, z) > u_i(x_i, y)$ for all $i \in I \setminus S$.
Let $v = \sum_{i \in S} \gamma(i)\omega_i - \sum_{i \in S} \gamma(i)\varphi(i, z)c(z) - \sum_{i \in S} \gamma(i) g_i = \sum_{i \in S} \gamma(i)\omega_i^* - \sum_{i \in S} \gamma(i) g_i \gg 0$ and $t = \sum_{i \in I \setminus S} [x_i' + K + \varphi(i, z)c(z) - \omega_i]$. Let $\varepsilon \in (0, 1)$ be such that $\varepsilon t \leqslant v$, whose existence follows from the fact that $v \gg 0$ (see (3)). Define $\tilde{g} = g\chi_S + (x' + K)\chi_{I \setminus S}$ and the generalized coalition with full support $(\tilde{\gamma}, I)$, where $\tilde{\gamma} = \gamma\chi_S + \varepsilon\chi_{I \setminus S}$. Then, $u_i(\tilde{g}_i, z) > u_i(x_i, y)$ for all $i \in I$, and

$$\sum_{i \in I} \tilde{\gamma}(i)\tilde{g}_i + \sum_{i \in I} \tilde{\gamma}(i)\varphi(i, z)c(z) - \sum_{i \in I} \tilde{\gamma}(i)\omega_i = -v + \varepsilon t \leqslant 0.$$

Therefore, (x, y) is blocked by the generalized coalition with full support $(\tilde{\gamma}, I)$ via the alternative pair $(\tilde{g}, z)$, and this is a contradiction. ∎

Proof of Proposition 3.5. Condition (1) directly follows from the definition of players'payoff functions. Let us prove (2). Let $s^* = (x^*, y^*, \gamma^*, g^*, z^*) \in S$ be a Nash equilibrium for the game G_σ. By Definition 3.4, $F_2(s^*) \geqslant F_2(x^*, y^*, \mathbf{1}, g, z)$ for all $(\gamma, g, z) \in S_2$, and in particular $F_2(s^*) \geqslant F_2(x^*, y^*, \gamma, x^*, y^*) = 0$. To prove (3), let (x, y) be a feasible allocation, i.e. $(x, y) \in S_1$, which is not Pareto optimal in the economy $\mathcal{E}$, then there exists an alternative feasible allocation (x', y') such that $u_i(x_i', y') > u_i(x_i, y)$ for all $i \in I$. This means that there exists $(x', y') \in S_1$ such that $u_i(x_i', y') - u_i(g_i, z) > u_i(x_i, y) - u_i(g_i, z)$ for all $i \in I$ and for all $(\gamma, g, z) \in S_2$ and for all $i \in I$. Hence, being f a positive function, we have $F_1(x'y', \gamma, g, z) > F_1(x, y, \gamma, g, z)$ for all $(\gamma, g, z) \in S_2$. Conversely, if (x, y) is Pareto optimal in $\mathcal{E}$ and $(\gamma, x, y) \in S_2$ for some $\gamma \in [\alpha, 1]^n$, then $(x, y) \in S_1$ and $0 = F_1(x, y, \gamma, x, y) \geqslant F_1(x', y', \gamma, x, y)$ for all $(x', y') \in S_1$, orthерwise we contradict the efficiency of (x, y) (see Remark 2.3). This concludes the proof of (3). Condition (4) follows from Definition 3.1 and players' playoff functions. Finally, let us prove (5) and let (x, y) be a cost share equilibrium. In particular (x, y) is feasible and hence $(x, y) \in S_1$ and $(\mathbf{1}, x, y) \in S_2$. Since (x, y) is Pareto optimal, from (3) it follows that $F_1(x', y', \mathbf{1}, x, y) \leqslant F_1(x, y, \mathbf{1}, x, y) = 0$ for all $(x', y') \in S_1$. On the other hand, Theorem 3.3 implies that $(x, y) \in C_\sigma^{Af}(\mathcal{E})$ and from (4) that $F_2(x, y, \gamma, g, z,) \leqslant F_2(x, y, \mathbf{1}, x, y) = 0$ for all $(\gamma, g, z) \in S_2$. Therefore, $(x, y, \mathbf{1}, x, y)$ is a Nash equilibrium for the game G_σ. ∎

The proof of Proposition 3.6 needs the following lemma which bases on the same arguments used in Lemma 4.2 of Hervés-Beloso and Moreno-García (2009b).

Lemma 6.1 *Assume that* $u_i(x, y) > 0 \Rightarrow x > 0$ *for all* $i \in I$ *and* $y \in \mathcal{Y}$. *If* $s^* = (x^*, y^*, \gamma^*, g^*, z^*)$ *is a Nash equilibrium of the game* G_σ, *then*

$$\gamma_i^*(u_i(g_i^*, z^*) - u_i(x_i^*, y^*)) = \gamma_j^*(u_j(g_j^*, z^*) - u_j(x_j^*, y^*)), \text{ for every } i, j \in I.$$

Proof. Let $s^* = (x^*, y^*, \gamma^*, g^*, z^*)$ be a Nash equilibrium. By (2) of Proposition 3.5, $F_2(s^*) \geqslant 0$. Define the non-empty set

$$B(s^*) = \left\{ i \in I : \gamma_i^*(u_i(g_i^*, z^*) - u_i(x_i^*, y^*)) = \min_{k \in I}\{\gamma_k^*(u_i(g_k^*, z^*) - u_k(x_k^*, y^*))\} = F_2(s^*) \right\}.$$

We need to show that $B(s^*) = I$. Suppose to the contrary that there exists an agent $j \in I \setminus B(s^*)$, for which $\gamma_j^*(u_j(g_j^*, z^*) - u_j(x_j^*, y^*)) > F_2(s^*) \geqslant 0$. This means that $u_j(g_j^*, z^*) > u_j(x_j^*, y^*) \geqslant 0 \Rightarrow g_j^* > 0$. By continuity of $u_j(\cdot, z^*)$ there exists some $\varepsilon \in (0, 1)$ such that $\gamma_j^*(u_j(\varepsilon g_j^*, z^*) - u_j(x_j^*, y^*)) > F_2(s^*)$. Define $\delta = \frac{\gamma_j^*(1-\varepsilon)g_j^*}{\sum_{i \in B(s^*)} \gamma_i^*} > 0$ and g' as

$$g_i' = \begin{cases} \varepsilon g_j^*, & \text{if } i = j \\ g_i^* + \delta, & \text{if } i \in B(s^*) \\ g_i^*, & \text{otherwise.} \end{cases}$$

It can be easily proved that $(\gamma^*, g', z^*) \in S_2$ and by monotonicity $F_2(x^*, y^*, \gamma^*, g', z^*) > F_2(s^*)$ which is an absurd being s^* a Nash equilibrium. Thus, $B(s^*) = I$. ∎

We are now ready to prove Proposition 3.6.

Proof of Proposition 3.6. Let $s^* = (x^*, y^*, \gamma^*, g^*, z^*)$ be a Nash equilibrium for the game G_σ which is, by Lemma 6.1, such that

$$F_2(s^*) = \gamma_i^*(u_i(g_i^*, z^*) - u_i(x_i^*, y^*)) = C \text{ for all } i \in I, \tag{4}$$

where $C \geqslant 0$ because of Proposition 3.5 (2). Note that $\tilde{\gamma} := \max\{\gamma_i\} = 1$, otherwise $F_2\left(x^*, y^*, \frac{\gamma^*}{\tilde{\gamma}}, g^*, z^*\right) > F_2(s^*)$. Since $\frac{f(\gamma)}{\gamma}$ is a positive and strictly increasing function, by (4), we get that

$$F_1(s^*) = \min\{f(\gamma_i^*)(u_i(x_i^*, y^*) - u_i(g_i^*, z^*))\} = \min\left\{f(\gamma_i^*)\frac{-C}{\gamma_i^*}\right\} = -Cf(1) \leqslant 0. \tag{5}$$

Our goal is, then, to prove that $C = 0$. We proceed by the way of contradiction and assume that $C > 0$, that is $F_1(s^*) < 0 < F_2(s^*)$. Notice that there are $i \in I$ such that $\gamma_i^* < 1$, otherwise $(g^*, z^*) \in S_1$ and $0 = F_1(g^*, z^*, \gamma^*, g^*, z^*) \leqslant F_1(s^*) < 0$, which is a contradiction. Furthermore,

$$\gamma_i^* < 1 \Rightarrow f(\gamma_i^*)(u_i(x_i^*, y^*) - u_i(g_i^*, z^*)) > F_1(s^*). \tag{6}$$

We now show that for each $\gamma_i^* < 1 \Rightarrow x_i \neq 0$. Suppose to the contrary that the set $A := \{i \in I : \gamma_i^* < 1 \text{ and } x_i = 0\}$ is not empty, and let $\varepsilon \in (0,1)$ be such that $\frac{\gamma_i^*}{\varepsilon} \leqslant 1$ for all $i \in A$. Notice that, being $u_i(\cdot, z)$ concave, we have that for all $i \in A$,

$$u_i(\varepsilon g_i^*, z^*) = u_i(\varepsilon g_i^* + (1-\varepsilon)x_i^*, z^*) \geqslant \varepsilon u_i(g_i^*, z) + (1-\varepsilon)u_i(x_i^*, z^*) = \varepsilon u_i(g_i^*, z) + (1-\varepsilon)u_i(0, z^*) = \varepsilon u_i(g_i^*, z),$$

that is

$$\frac{1}{\varepsilon}u_i(\varepsilon g_i^*, z) \geqslant u_i(g_i^*, z). \tag{7}$$

Let $K = \frac{1-\varepsilon}{2\varepsilon}\sum_{i\in A}\gamma_i^*(\omega_i - \sigma(\{i\}, z^*)c(z^*))$ which is positive because of $(A1^*)$. Define

$$\gamma_i' = \begin{cases} \frac{\gamma_i^*}{\varepsilon}, & if\ i \in A, \\ \gamma_i^*, & \text{otherwise,} \end{cases} \qquad g_i' = \begin{cases} \varepsilon g_i^* + \frac{\varepsilon K}{\sum_{i\in A}\gamma_i^*}, & if\ i \in A, \\ g_i^* + \frac{K}{\sum_{i\notin A}\gamma_i^*}, & \text{otherwise,} \end{cases}$$

and note that $(\gamma', g', z^*) \in S_2$. Furthermore, by monotonicity and (7), we have that for $i \in A$,

$$\gamma_i'[u_i(g_i', z^*) - u_i(x_i^*, y^*)] > \frac{\gamma_i^*}{\varepsilon}[u_i(\varepsilon g_i^*, z^*)] \geqslant \gamma_i^*[u_i(g_i^*, z) - u_i(x_i^*, y^*)],$$

and by monotonicity, we have that for all $i \notin A$

$$\gamma_i'[u_i(g_i', z^*) - u_i(x_i^*, y^*)] > \gamma_i^*[u_i(g_i^*, z) - u_i(x_i^*, y^*)].$$

Hence, $F_2(x^*, y^*, \gamma', g', z^*) > F_2(s^*)$ which is an absurd, being s^* a Nash equilibrium. Therefore, denoted by A' the set $A' := \{i \in I : \gamma(i) < 1\}$, we have that for all $i \in A'$, $x_i^* > 0$ and, by (6),

$f(\gamma_i^*)(u_i(x_i^*, y^*) - u_i(g_i^*, z^*)) > F_2(s^*)$. Let $\delta \in (0,1)$ be such that $f(\gamma_i^*)(u_i(\delta x_i^*, y^*) - u_i(g_i^*, z^*)) > F_2(s^*)$ for all $i \in A'$ and consider the pair $(x', y^*) \in S_1$, given by

$$x_i' = \begin{cases} \delta x_i^*, & if\ i \in A' \\ x_i^* + \frac{1-\delta}{|I \setminus A'|} \sum_{j \in A'} x_j^* > x_i^*, & if\ i \notin A', \end{cases}$$

which, by monotonicity, gives higher playoff to player 1, i.e. $F_1(x', y^*, \gamma^*, g^*, z^*) > F_1(s^*)$. This contradicts the assumption that s^* is a Nash equilibrium and hence $C = 0$, concluding the proof. ■

Proof of Theorem 3.7. Let $s^* = (x^*, y^*, \gamma^*, g^*, z^*)$ be a Nash equilibrium for the game G_σ and assume that (x^*, y^*) is not a φ_σ-cost share equilibrium allocation. Then, by Theorem 3.3 there exists $(\gamma, g, z) \in S_2$ such that $F_2(x^*, y^*, \gamma, g, z) > F_2(s^*)$, which is impossible by Definition 3.4.
For the converse, let (x^*, y^*) be a φ_σ-cost share equilibrium and assume that (x^*, y^*, γ, g, z) is not a Nash equilibrium, where $(\gamma, g, z) \in S_2$ is such that $u_i(x_i^*, y^*) = u_i(g_i, z)$ for every $i \in I$. Then,

- (I) there exists $(x, y) \in S_1$ such that $F_1(x, y, \gamma, g, z) > F_1(x^*, y^*, \gamma, g, z) = 0$; or
- (II) there exists $(\gamma', g', z') \in S_2$ such that $F_2(x^*, y^*, \gamma', g', z') > F_2(x^*, y^*, \gamma, g, z) = 0$.

In the first case, (x^*, y^*) is not an efficient, because (x, y) is a feasible and $u_i(x_i, y) > u_i(g_i, z) = u_i(x_i^*, y^*)$ for all $i \in I$, and this is a contradiction (see Remark 2.3).
In the second case, (x^*, y^*) is a σ-dominated allocation in the sense of Aubin, because $u_i(g_i', z') > u_i(x_i^*, y^*)$ for every $i \in I$ and $\sum_{i \in I} \gamma_i' g_i' + \sum_{i \in I} \gamma_i \sigma(\{i\}, z') c(z') \leqslant \sum_{i \in I} \gamma_i' \omega_i$, with $\gamma_i' > 0$ for all i. This contradicts the fact that (x^*, y^*) is a φ_σ-cost share equilibrium. ■

Proof of Proposition 4.1. If $(x^*, y^*) \in CSE_\varphi(\mathcal{E})$, x^* is feasible and it satisfies the budget set in $\mathcal{E}(y^*, \varphi)$, i.e. $p^*(y^*) \cdot x_i^* \leqslant p^*(y^*) \cdot \omega_i^{y^*}$. To conclude, show that it is also maximal in the budget set. Let g be an allocation such that $i \in I$, $u_i(g_i, y^*) > u_i(x_i^*, y^*)$ and $p^*(y^*) \cdot g_i \leqslant p^*(y^*) \cdot \omega_i^{y^*}$ for some i. This implies that $p^*(y^*) \cdot g_i + \varphi^*(i, y^*) p^*(y^*) \cdot c(y^*) \leqslant p^*(y^*) \cdot \omega_i$, which is a contradiction since (x^*, y^*) is a cost share equilibrium.
Conversely, let (x_y, p_y) be a competitive equilibrium in each economy $\mathcal{E}(y, \varphi)$ and $y^* \in \mathcal{Y}$ be such that $u_i(x_{y^*}, y^*) \geqslant u_i(x_z, z)$ for all $z \in \mathcal{Y}$. Assume, to the contrary that, (x_{y^*}, y^*) is not a cost share equilibrium for the economy $\mathcal{E}$ with the equilibrium price $p^* : \mathcal{Y} \to \mathbb{R}_+^m$ given by $p^*(z) := p_z$. Since, (x_{y^*}, y^*) is feasible and satisfies the budget constraint in $\mathcal{E}$, it means that there exist an agent $i \in I$ and an alternative allocation (g_i, z) such that

(i) $\quad u_i(g_i, z) > u_i(x_{iy^*}, y^*)$

(ii) $\quad p^*(z) \cdot g_i + p^*(z) \cdot \varphi(i, z) c(z) \leqslant p^*(z) \cdot \omega_i$.

From (ii) it follows that $p_z \cdot g_i \leqslant p_z \cdot \omega_i^z$, and hence, since (x_z, z) is a competitive equilibrium allocation for $\mathcal{E}(z, \varphi)$, it follows that $u_i(g_i, z) \leqslant u_i(x_z, z) \Rightarrow u_i(x_{iy^*}, y^*) < u_i(g_i, z) \leqslant u_i(x_z, z) \leqslant u_i(x_{iy^*}, y^*)$, a contradiction. ■

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