CSEF
Centre for Studies in Economics and Finance

WORKING PAPER NO. 639

Multi-Leader-Common-Follower games with pessimistic leaders: approximate and viscosity solutions

M. Beatrice Lignola and Jacqueline Morgan

March 2022

University of Naples Federico II **University of Salerno** **Bocconi University, Milan**

CSEF - Centre for Studies in Economics and Finance
DEPARTMENT OF ECONOMICS AND STATISTICS – UNIVERSITY OF NAPLES FEDERICO II
80126 NAPLES - ITALY
Tel. and fax +39 081 675372 – e-mail: csef@unina.it
ISSN: 2240-9696

Working Paper no. 639

Multi-Leader-Common-Follower games with pessimistic leaders: approximate and viscosity solutions

M. Beatrice Lignola* **and Jacqueline Morgan†**

Abstract

We consider a two-stage game with *k* leaders having pessimistic attitude and one follower common to all leaders. Such a game, called CF game, may fail to have *pessimistic solutions*, even if the leader payoffs are linear and the optimal reaction of the follower to the leaders strategies is unique. So, we introduce two classes of games, called weighted value-potential and weighted potential CF games, and we illustrate their inherent difficulties and properties. For the more tractable class of weighted potential CF games, suitable *approximate and viscosity solutions* are introduced and are proven to exist under appropriate conditions, in line with what done for one-leader-one-follower games.

Mathematics Subject Classification: 91A10, 91A14, 49J45, 49J53

Keywords: Two-stage game, Weighted potential game, Pessimistic behavior, Viscosity solution, Lower semicontinuous set-valued map

* Università di Napoli Federico II. E-mail: lignola@unina.it

† Università di Napoli Federico II and CSEF. E-mail: morgan@unina.it

Table of contents

In this note, we consider Multi-Leader-Common-Follower games (see, for example, [17]), CF games for short, that is two-stage games with a k-players non-cooperative game at the first stage and a parametric one-player game at the second stage, a particular case of the so-called Multi-Leader-Multi-Follower games [23]. Their hierarchical nature leads to introduce different concepts of solution depending on the behavior of the leader, in particular on the extreme attitudes optimistic or pessimistic [16], in line with what proposed for One-Leader-One-Follower two-stage games (see [8] for an overview). However, the existence of such solutions is hard to be obtained, even if the set of the optimal solutions in the second stage is a singleton and the leaders payoffs are linear. Therefore, to our knowledge, only a few number of papers is concerned by existence results for CF games and, except the review [5], they can be divided into three groups:

- ▷ papers regarding a specific situation, derived from the real-world, which is solved by explicitly computing the solutions set [2], [3], [4];
- ▷ papers considering the mathematical problems associated to these concepts as special cases of equilibrium problems with equilibrium constraints [17], [12], [13];
- ▷ papers considering classes of CF games satisfying specific conditions [14], [15].

We aim to investigate the case of non-unique solution at the second stage, considered in [17], [14], [15], [5], assuming a pessimistic behavior of the leaders. In fact, in Section 2, a pessimistic solution concept for CF games is presented and, in Section 3, existence of such solutions is established for the class of *Weighted Value-Potential CF games*. The limit of considering this class being that the relative results cannot be formulated in terms of explicit conditions on the data, we also consider the less general class of *Weighted Potential CF games*, which anyway enlarges the ones considered in [14] and [15] in the case of one common follower. Then, we investigate the results which can be obtained in these new classes together with their difficulties, which are in line with those arisen when investigating One-Leader-One-Follower two-stage games with a pessimistic leader (also called weak Stackelberg games [19],[8]). Indeed, pessimistic solutions to weighted potential games may fail to exist even for nice data, so, suitable regularizations and related approximate solutions for CF games with pessimistic leaders are introduced and investigated. This leads to the introduction of suitable *approximate and viscosity* solutions, in line with what we did for One-Leader-Multi-Follower games in [18] and for One-Leader-One-Follower games in [20].

1 Setting, Preliminaries and Basic Results

Let E be a real Banach space, let V_1,...,V_k, $k \geq 1$, be real Banach spaces and let $V = V_1 \times ... \times V_k$. If H_i is a nonempty closed subset of V_i, we consider $H = \prod_{i=1,..,k} H_i$ and, given $\bar{\boldsymbol{x}} = (\bar{x}_1, ..., \bar{x}_k) \in H$, we denote by $\bar{\boldsymbol{x}}_{-i}$ the point $(\bar{x}_1, ..., \bar{x}_{i-1}, \bar{x}_{i+1}, ..., \bar{x}_k) \in H_{-i} = \prod_{j\neq i} H_j$ and by $(x_i, \bar{\boldsymbol{x}}_{-i})$ the point $(\bar{x}_1, ,.., \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, ..., \bar{x}_k) \in H$.

Let Y be a nonempty subset of E and let $L_1, ..., L_k$ be real-valued functions defined on $H \times Y$. Given $\boldsymbol{x} = (x_1, ..., x_k) \in H$, we assume that, for any $i = 1, ..., k$, x_i corresponds to a strategy profile of k leaders playing first non-cooperatively, and that the follower response $y \in Y$ is common to all leaders. Then, we consider a model in which the follower aims to minimize with respect to y his objective function $F : (\boldsymbol{x}, y) \in H \times Y \to F(\boldsymbol{x}, y) \in \mathbb{R} \cup \{+\infty\}$, knowing that y is constrained in $K(\boldsymbol{x})$, where $K : \boldsymbol{x} \in V \rightrightarrows K(\boldsymbol{x}) \subseteq Y$ is a set-valued map with nonempty values, so that he solves, for any $\boldsymbol{x} \in H$, the problem

$$P(\boldsymbol{x}) \qquad \text{find } \ \bar{y} \in K(\boldsymbol{x}) \text{ such that } F(\boldsymbol{x}, \bar{y}) \leq F(\boldsymbol{x}, y) \ \ \forall \, y \in K(\boldsymbol{x}).$$

The argmin map

$$\mathcal{M} : \boldsymbol{x} \in H \rightrightarrows \mathcal{M}(\boldsymbol{x}) = \{y \in K(\boldsymbol{x}) : F(\boldsymbol{x}, y) \leq F(\boldsymbol{x}, z) \ \forall \, z \in K(\boldsymbol{x})\}$$

comes to be defined and, in general, could be set-valued.
The case where the follower aims to maximize with respect to y his objective function F can be easily analized observing that $\max_{y \in K(\boldsymbol{x})} F(\boldsymbol{x}, y) = - \min_{y \in K(\boldsymbol{x})} -F(\boldsymbol{x}, y)$.
All leaders, when prepared for the worst, consider the functions

$$\mathcal{P}_i : \boldsymbol{x} \in H \to \mathcal{P}_i(\boldsymbol{x}) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} L_i(\boldsymbol{x}, y), \ \text{for } i = 1, ..., k \tag{1}$$

and modelize the following classical Nash Equilibrium Problem [22]

$$\text{find } \bar{\boldsymbol{x}} \in H \text{ such that } \ \mathcal{P}_i(\bar{\boldsymbol{x}}) \ = \ \inf_{x_i \in H_i} \mathcal{P}_i(x_i, \bar{\boldsymbol{x}}_{-i}) \ \text{ for } i = 1, .., k.$$

This leads to formulate the *Pessimistic Multi-Leader-Common-Follower problem*, pessimistic CF problem in short:

(PCF) find $\bar{\boldsymbol{x}} \in H$ such that

$$\sup_{y \in \mathcal{M}(\bar{\boldsymbol{x}})} L_i(\bar{\boldsymbol{x}}, y) = \inf_{x_i \in H_i} \sup_{y \in \mathcal{M}(x_i, \bar{\boldsymbol{x}}_{-i})} L_i(x_i, \bar{\boldsymbol{x}}_{-i}, y) \ \ \forall \, i = 1, .., k \tag{2}$$

A solution to (PCF) is called a *pessimistic solution to the CF game* .

When the leaders have an optimistic attitude, the *optimistic* version of (PCF)

(OCF) find $\bar{\boldsymbol{x}} \in H$ such that

$$\inf_{y \in \mathcal{M}(\bar{\boldsymbol{x}})} L_i(\bar{\boldsymbol{x}}, \bar{y}) = \inf_{x_i \in H_i} \inf_{y \in \mathcal{M}(x_i, \bar{\boldsymbol{x}}_{-i})} L_i(x_i, \bar{\boldsymbol{x}}_{-i}, y) \ \ \forall \, i = 1, .., k$$

has been considered and has been more investigated (see, for example: [17], [12], [13], [5]). In this case, the leaders consider a Nash equilibrium problem with payoffs given by

$$\mathcal{O}_i : \boldsymbol{x} \in H \to \mathcal{O}_i(\boldsymbol{x}) = \inf_{y \in \mathcal{M}(\boldsymbol{x})} L_i(\boldsymbol{x}, y), \ \text{for } i = 1, ..., k \tag{3}$$

A solution to (OCF) is called an *optimistic solution to the CF game.*

The next example, considered by Pang and Fukushima in [23], shows that CF games may fail to have optimistic or pessimistic solutions even if the map $\mathcal{M}$ is single-valued and the leaders' payoffs are linear.

Example 1.1 [23, Example 4] Let $V = \mathbb{R}^2$, $E = \mathbb{R}$, $Y = [0,+\infty[$, $H_1 = H_2 = [0,1]$, $H = [0,1]^2$, and consider the real-valued functions:

$$L_1 : (x_1,x_2,y) = (\boldsymbol{x},y) \in H \times Y \longrightarrow \frac{1}{2}x_1 + y, \qquad L_2 : (x_1,x_2,y) = (\boldsymbol{x},y) \in H \times Y \longrightarrow -\left(\frac{1}{2}x_2 + y\right),$$

$$F : (x_1,x_2,y) = (\boldsymbol{x},y) \in H \times Y \longrightarrow y(x_1 + x_2 - 1) + \frac{1}{2}y^2.$$

The argmin map $\mathcal{M}$ of the follower is single-valued:

$$\mathcal{M} : \boldsymbol{x} = (x_1,x_2) \in H \rightrightarrows \mathcal{M}(\boldsymbol{x}) = \{\max(0, 1 - x_1 - x_2)\} \subseteq Y,$$

so that $\mathcal{P}_i = \mathcal{O}_i$ and one can see that the normal form game $(\mathcal{P}_1, \mathcal{P}_2, H_1, H_2)$, where

$$\mathcal{P}_1(x_1,x_2) = \max\left(\frac{1}{2}x_1, 1 - \frac{1}{2}x_1 - x_2\right) \quad \mathcal{P}_2(x_1,x_2) = \min\left(-\frac{1}{2}x_2, -1 + x_1 + \frac{1}{2}x_2\right),$$

does not have any Nash equilibrium and the CF game does not have any pessimistic nor optimistic solution.

In this paper, in order to get our results easy to use in concrete applications, we assume that:
- $E = \mathbb{R}^h$, $h \in \mathbb{N}$;
- $V_i = \mathbb{R}^{m_i}$ for $i = 1, ..., k$;
- $m = \sum_{i=1}^{k} m_i$ and $H = \prod_{i=1,..,k} H_i \subseteq \mathbb{R}^m$;
- $\mathcal{M}(\boldsymbol{x}) \neq \emptyset$, $\forall\, \boldsymbol{x} \in H$.

Nevertheless, our results could be naturally extended to infinite dimensional Banach spaces by appropriately balancing the use of strong and weak convergence in the hypotheses, as already done in [20].
Now, we review some basic definitions of set-valued analysis that we will use in the following section.
If $(C_n)_n$ is a sequence of nonempty subsets of $\mathbb{R}^p$, the *Painlevé-Kuratowski upper and lower limits* [1] of the sequence $(C_n)_n$ are defined by :
- $z \in \limsup_n C_n$ if there exists $(z_k)_k$ converging to z such that, for a subsequence $(C_{n_k})_k$ of $(C_n)_n$, $z_k \in C_{n_k}$ for any $k \in \mathbb{N}$;
- $z \in \liminf_n C_n$ if there exists $(z_n)_n$ converging to z such that $z_n \in C_n$ for n sufficiently large.

A set-valued map T from $H \subseteq \mathbb{R}^m$ to $Y \subseteq \mathbb{R}^h$ is:
- *lower semicontinuous* over H if for every $\boldsymbol{x} \in H$ and every sequence $(\boldsymbol{x}_n)_n$ converging to $\boldsymbol{x}$ in H and every $y \in T(\boldsymbol{x})$ there exists a sequence $(y_n)_n$ converging to y such that $y_n \in T(\boldsymbol{x}_n)$ for n sufficiently large, i.e.
$$T(\boldsymbol{x}) \subseteq \liminf_n T(\boldsymbol{x}_n);$$
- *closed* over H if for any $\boldsymbol{x} \in H$ and any $(\boldsymbol{x}_n)_n$ converging to $\boldsymbol{x}$ in H, if $(y_k)_k$ converges to y in Y and $y_k \in T(\boldsymbol{x}_{n_k})$ for any $k \in \mathbb{N}$, then we have that $y \in T(\boldsymbol{x})$, i.e.:
$$\limsup_n T(\boldsymbol{x}_n) \subseteq T(\boldsymbol{x});$$

- *concave* over H if the set H is convex and for any $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in H and any $\lambda \in [0,1]$ one has:
$$T(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \subseteq \lambda T(\boldsymbol{x}_1) + (1-\lambda)T(\boldsymbol{x}_2).$$

Here we present a *simple* version of the existence theorem of Nash equilibria for normal form games, but, as well known, several results allow to use less restrictive assumptions.

Theorem 1.1 [22] *Let G_i, $i = 1, ..., k$, be real-valued functions defined in $H \subseteq \mathbb{R}^m$ and assume that:*
N_1) the set H_i is compact and convex for any $i = 1, ..., k$;
N_2) the function $G_i(\cdot, \boldsymbol{x}_{-i})$ is quasi-convex over H_i, for any $i = 1, ..., k$ and any $\boldsymbol{x}_{-i} \in H_{-i}$;
N_3) the function $G_i(x_i, \cdot)$ is upper semicontinuous over H_{-i}, for any $i = 1, ..., k$ and any $x_i \in H_i$;
N_4) the function G_i is lower semicontinuous over H, for any $i = 1, ..., k$.
Then, there exists a Nash equilibrium point for the normal form game $(G_1, ..., G_k, H_1, ..., H_k)$.

Therefore, bearing in mind the formulation of problem (PCF), it is clear that semicontinuity and quasi-convexity properties of the so called *marginal functions* of the sup-type are crucial for proving the existence of pessimistic solutions to CF games. Classical conditions for achieving such properties are briefly recalled below.

Proposition 1.1 *Let g be a real-valued function defined in $U \times W \subseteq \mathbb{R}^m \times \mathbb{R}^h$, where U and W are nonempty closed sets, and let T be a set-valued map from U to W.*
1. If we assume that:
S_1) the set-valued map T is lower semicontinuous over U;
S_2) the function g is lower semicontinuous over $U \times W$;
then, the marginal function
$$s : u \in U \to s(u) = \sup_{w \in T(u)} g(u,w) \tag{4}$$
is lower semicontinuous over U.
2. If we assume that:
S_3) the set W is compact and the set-valued map T is closed over U;
S_4) the function g is upper semicontinuous over $U \times W$;
then, the marginal function s is upper semicontinuous over U.
3. If we assume that:
S_5) the sets U and W are convex and the set-valued map T is concave over U;
S_6) the function g is quasi-convex over $U \times W$;
then, the marginal function s is quasi-convex over U.

Proof The proof of points 1. and 2. can be found in [7] or in [1], the proof of 3. can be found in [11]. □

However, it is well known that conditions S_1) and S_5) can be not satisfied when the constraints map is the argmin map $\mathcal{M}$. Thus, in Section 3, we consider particular classes of CF games, having in mind that the search of a Nash equilibrium for a normal form game $(G_1, ..., G_k, H_1, ..., H_k)$ can be reduced to the search of a minimum point whenever we deal

with a *weighted potential game in the sense of Monderer and Shapley* [21].
We recall that in [10, Theorem 2.1] it has been proven that the game $(G_1, ..., G_k, H_1, ..., H_k)$ is weighted potential if and only if there exist a function P, defined on H and called weighted potential of the normal form game, a vector $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_k) \in \mathbb{R}^k_{++} = \{(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k : \alpha_i > 0 \ for \ i = 1, ..., k\}$ and k real-valued functions p_i, defined on H_{-i}, such that

$$G_i(\boldsymbol{x}) \ = \ \alpha_i P(\boldsymbol{x}) \ + \ p_i(\boldsymbol{x}_{-i}), \quad \forall \ i \ = \ 1, ..., k.$$

Weighted potential normal form games are tractable since the set of their Nash equilibria contains the set of the minimum points for the weighted potential function, so that existence of Nash equilibria is guaranteed by the lower semicontinuity of the function P and no convexity condition on the payoffs is needed; note that new existence and uniqueness results of Nash equilibria for such games can be found in [9]. Then, exploiting this feature of weighted potential normal form games, one is induced to consider suitable classes of CF games in order to overcome the lack of pessimistic solutions, which may fail to exist even for linear payoffs (see Example 2.1). First investigations in this direction can be found in [14], [15]. In the next section, two classes of CF games satisfying a potentiality-like property are considered and their peculiarities are investigated.

2 Weighted Value-Potentials and Weighted Potentials for a CF Game

With same notations as in Section 2, we start this section by introducing the concept of weighted value-potential for a CF game.

Definition 2.1 *A CF game is said to be Weighted Value-Potential if there exists a real-valued function $\mathcal{P}$ (called weighted value-potential of the CF game) defined on H and, for any $i = 1, ..., k$, there exist $\alpha_i \in \mathbb{R}_{++}$ and a real-valued function h_i defined on H_{-i}, such that, for all $\boldsymbol{x} \in H$, one has:*

$$\mathcal{P}_i(\boldsymbol{x}) \ = \ \sup_{y \in \mathcal{M}(\boldsymbol{x})} L_i(\boldsymbol{x}, y) \ = \ \alpha_i \mathcal{P}(\boldsymbol{x}) \ + \ h_i(\boldsymbol{x}_{-i}),$$

i.e. the game $(\mathcal{P}_1, .., \mathcal{P}_k, H_1, .., H_k)$ is a weighted potential normal form game as defined in the end of Section 2.

For what concerning the pessimistic solutions to weighted value-potential CF games, the next existence result holds.

Theorem 2.1 *Assume that a CF game with pessimistic leaders is weighted value-potential with $\mathcal{P}$ as a weighted value-potenital and that*
P_1) the set H_i is compact for any $i = 1, ..., k$;
P_2) the weighted value-potential $\mathcal{P}$ is lower semicontinuous over H.
Then, there exists a Nash equilibrium for the normal form game $(\mathcal{P}_1, ..., \mathcal{P}_k, H_1, ..., H_k)$, i.e. there exists a pessimistic solution to the CF game

Proof Assumptions P_1) and P_2) imply that the function $\mathcal{P}$ has a minimum point in H, which turns out to be a Nash equilibrium for the game $(\mathcal{P}_1, ..., \mathcal{P}_k, H_1, ..., H_k)$. $\square$

Although the property of being a weighted value-potential game is not very restrictive, we are aware that it presents inherent difficulties:
• the above class of games is not easy to be described since a constrained marginal function has to be explicitly computed;
• Theorem 3.1 does not contain direct assumptions on the payoffs of the leaders as well on the constraints and the payoff of the follower.

Thus, we consider a smaller but more tractable class of CF games.

Definition 2.2 *A CF game is said to be Weighted Potential if there exists a real-valued function π (called weighted potential of the CF game) defined on H and, for any $i = 1, ..., k$, there exists $\beta_i \in \mathbb{R}_{++}$ such that, for all $(\boldsymbol{x}, y) \in H \times Y$, one has:*

$$L_i(x_i, \boldsymbol{x}_{-i}, y) - L_i(x_i', \boldsymbol{x}_{-i}, y') = \beta_i \left[\pi(x_i, \boldsymbol{x}_{-i}, y) - \pi(x_i', \boldsymbol{x}_{-i}, y')\right] \ \ \forall\, (x_i', y') \in H_i \times Y. \quad (5)$$

The following proposition says that the weighted potential property for CF games can be equivalently expressed in a more handy way.

Proposition 2.1 *A CF game is weighted potential if and only if there exists a real-valued function Π defined on $H \times Y$, and, for any $i = 1, ..., k$, there exist $\beta_i \in \mathbb{R}_{++}$ and a real-valued function Φ_i, defined on H_{-i}, such that, for all $(\boldsymbol{x}, y) \in H \times Y$ one has:*

$$L_i(\boldsymbol{x}, y) \ = \ \beta_i \, \Pi(\boldsymbol{x}, y) \ + \ \Phi_i(\boldsymbol{x}_{-i}). \quad (6)$$

Moreover, the function Π is a weighted potential of the game CF.

Proof If the game is weighted potential, then, for any $i = 1, ..., k$, equality (5) implies that the difference function $L_i - \beta_i \pi$ depends only on $\boldsymbol{x}_{-i}$, so that condition (6) is satisfied by taking $\Pi = \pi$ and $\Phi_i = L_i - \beta_i \pi$.
Conversely, from (6) we get (5) by considering $\pi = \Pi$. $\square$

We now show that the class of weighted potential CF games is strictly included in the class of weighted value-potential CF games.

Proposition 2.2 *Any weighted potential CF game is also weighted value-potential. If Π is a weighted potential for CF then $\mathcal{P}(x) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$ is a weighted value-potential for CF.*

Proof If a CF game satisfies condition (6), then it satisfies Definition 3.1 by setting, for any $i = 1, ..., k$, $\alpha_i = \beta_i$, $\mathcal{P}(\boldsymbol{x}) = \mathcal{P}(x) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$ and $h_i = \Phi_i$. $\square$

The converse of the above result does not hold in general.

Example 2.1 Consider the CF game defined by:

$$E = V_1 = V_2 = \mathbb{R}, H_1 = H_2 = [0,1], Y = [0,1], F(x_1, x_2, y) = 0, K(x_1, x_2) = [0,1],$$

$$L_1(x_1, x_2, y) = -x_1 + y - yx_2, \qquad L_2(x_1, x_2, y) = -x_1 y + y - x_1.$$

It is clearly not weighted potential, but it is weighted value-potential since

$$\mathcal{P}_1(x_1, x_2) = 1 - x_1 - x_2, \quad \mathcal{P}_2(x_1, x_2) = 1 - 2\,x_1,$$

and we can take $\alpha_1 = 1$, $\alpha_2 = 1$, $\mathcal{P}(\boldsymbol{x}) = 1 - x_1$, $h_1(x_2) = -x_2$, $h_2(x_1) = -x_1$.

Remark 2.1 In [14, Definition 2.2] and in [15, Definition 3.2], potential and quasi-potential Multi-Leader-Multi-Follower games have been respectively defined and we remark that:
• the concept of weighted potential game generalizes both of them, when a single follower is common to all leaders;
• weighted value-potential games may be not quasi-potential, in general. Indeed, the game presented in Example 3.1 is not quasi-potential but it is weighted value-potential.

For what concerning weighted potential CF games, the next existence result holds.

Theorem 2.2 *Assume that a CF game is weighted potential with* Π *as a weighted potential and that*
P_1) *the set* H_i *is compact for any* $i = 1, ..., k$;
P_3) *the function* $\mathcal{P}(x) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$ *is lower semicontinuous over* H.
Then, there exists a Nash equilibrium point for the normal form game $(\mathcal{P}_1, ..., \mathcal{P}_k, H_1, ..., H_k)$, *i.e. there exists a pessimistic solution to the CF game.*

The proof of this theorem is straightforward and it is omitted.

However, even if the leader's payoffs are linear, assumption P_3) may fail to be satisfied and weighted potential games may fail to have equilibria, as in the next example.

Example 2.2 Consider the CF game defined by:

$$E = V_1 = V_2 = \mathbb{R}, H_1 = H_2 = [0,1], Y = [0,1], F(x_1, x_2, y) = x_1 y, K(x_1, x_2) = [0,1],$$

$$L_1(x_1, x_2, y) = x_1 + y - x_2, \qquad L_2(x_1, x_2, y) = \frac{1}{2}(x_1 + y).$$

Then, the CF game is weighted potential (so, it is weighted value-potential too) and a weighted potential is $\Pi(\boldsymbol{x}, y) = x_1 + y$ with $\beta_1 = 1$ and $\beta_2 = \frac{1}{2}$. However, the problem (PCF) does not have pessimistic solutions.
Indeed, the second stage argmin map $\mathcal{M}$ is defined by:

$$\mathcal{M}(x_1, x_2) = [0,1] \text{ when } x_1 = 0 \text{ and } \mathcal{M}(x_1, x_2) = \{0\} \text{ when } x_1 > 0 \qquad ,$$

so we have:

$$\mathcal{P}_1(x_1,x_2) = 1 - x_2, \qquad \mathcal{P}_2(x_1,x_2) = \frac{1}{2} \qquad \text{when } x_1 = 0,$$

$$\mathcal{P}_1(x_1,x_2) = x_1 - x_2, \qquad \mathcal{P}_2(x_1,x_2) = \frac{x_1}{2} \qquad \text{when } x_1 > 0.$$

The normal form game $(\mathcal{P}_1, \mathcal{P}_2, H_1, H_2)$ does not have Nash equilibria since, for any $(x_1, x_2) \in H$, $\mathcal{P}_1(x_1,x_2) > -x_2 = \inf_{x_1' \in H_1} \mathcal{P}_1(x_1', x_2)$ and the best reply of leader 1 is empty-valued.

Moreover, by taking $\alpha_1 = 1$, $\alpha_2 = \frac{1}{2}$, one can check that this game is weighted value-potential with weighted value-potential $\mathcal{P}(\boldsymbol{x}) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$ and that $\mathcal{P}$ does not have any minimum point in H since $\mathcal{P}(\boldsymbol{x}) = 1$ if $x_1 = 0$ and $\mathcal{P}(\boldsymbol{x}) = x_1$ if $x_1 \in]0,1]$.

In spite of the previous example, the following one shows that a weighted value-potential game may have pessimistic solutions even if the weighted value-potential $\mathcal{P}$ in Definition 3.1 does not have any optimal point.

Example 2.3 Consider the CF game defined by:

$$E = V_1 = V_2 = \mathbb{R}, H_1 = H_2 = [0,1], Y = [0,1], F(x_1,x_2,y) = x_1 x_2\, y, K(x_1,x_2) = [0,1],$$

$$L_1(x_1,x_2,y) = x_1x_2 + y + x_2, \qquad L_2(x_1,x_2,y) = x_1x_2 + y + x_1.$$

Then, the CF game is weighted potential with potential $\Pi(x_1,x_2,y) = x_1x_2 + y$, so, in light of Proposition 3.2, it is also weighted value-potential and a weighted value-potential is $\mathcal{P}(\boldsymbol{x}) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$.

The argmin map $\mathcal{M}$ is defined by $\mathcal{M}(x_1,x_2) = [0,1]$ when $x_1x_2 = 0$ and $\mathcal{M}(x_1,x_2) = \{0\}$ when $x_1x_2 \neq 0$, so we get that $\mathcal{P}(x_1,x_2) = 1$, when $x_1x_2 = 0$, $\mathcal{P}(x_1,x_2) = x_1x_2$, when $x_1x_2 \neq 0$ and $\mathcal{P}$ does not have any minimum point in H.

However, since we have:

$$\mathcal{P}_1(x_1,x_2) = 1 + x_2, \qquad \mathcal{P}_2(x_1,x_2) = 1 + x_1 \qquad \text{when } x_1x_2 = 0$$

$$\mathcal{P}_1(x_1,x_2) = x_1x_2 + x_2, \qquad \mathcal{P}_2(x_1,x_2) = x_1x_2 + x_1 \qquad \text{when } x_1x_2 \neq 0,$$

the point $(0,0)$ is a Nash equilibrium for the normal form game $(\mathcal{P}_1, \mathcal{P}_2, H_1, H_2)$, i.e. the CF game has $(0,0)$ as a pessimistic solution.

Therefore, we can conclude that, for both classes of CF games previously considered, the existence of solutions is not necessarily guaranteed. Then, in line with what has been done for one-leader-multi-followers games [18] and for one-leader-one-follower games with pessimistic behavior of the leaders [20], we will face a weighted potential CF game with pessimistic behavior of the leaders through appropriate regularizations of the optimal response map of the common follower.

We start by illustrating this regularization method considering the CF game in Example 3.2. Let ε be a positive number smaller than $1/4$ and consider the ε-minimum map

$$\mathcal{M}^\varepsilon : \boldsymbol{x} \in H \rightrightarrows \mathcal{M}^\varepsilon(\boldsymbol{x}) = \left\{ y \in K(\boldsymbol{x}) : F(\boldsymbol{x}, y) \leq \inf_{z \in K(\boldsymbol{x})} F(\boldsymbol{x}, z) + \varepsilon \right\}.$$

With data of Example 3.2, we get:

$$\mathcal{M}^\varepsilon(x_1,x_2) \,=\, [0,1] \text{ if } x_1 \,\in\, [0,\varepsilon], \;\; \mathcal{M}^\varepsilon(x_1,x_2) \,=\, \left[0,\frac{\varepsilon}{x_1}\right] \text{ if } x_1 \in\, [\varepsilon,1].$$

Then, the marginal function of Π over this regularized map, denoted by $\mathcal{P}^\varepsilon$, is continuous on H since

$$\mathcal{P}^\varepsilon(\boldsymbol{x}) = \sup_{y\in\mathcal{M}^\varepsilon(\boldsymbol{x})} \Pi(\boldsymbol{x},y) = x_1{+}1 \text{ if } x_1 \,\in\, [0,\varepsilon], \;\; \mathcal{P}^\varepsilon(\boldsymbol{x}) = \sup_{y\in\mathcal{M}^\varepsilon(\boldsymbol{x})} \Pi(\boldsymbol{x},y) \,=\, x_1 + \frac{\varepsilon}{x_1} \text{ if } x_1 \,\in\, [\varepsilon,1].$$

One can check that, for any $\varepsilon \in]0,1/4[$, the minimum points of $\mathcal{P}^\varepsilon$ are $(\sqrt{\varepsilon},x_2)$ for any $x_2 \in [0,1]$; so, the minimum value of $\mathcal{P}^\varepsilon$ is $2\sqrt{\varepsilon}$.
Therefore, even if the considered CF game does not have pessimistic solutions, we can bypass this lack by considering the limits when ε tends to zero of the above approximate pessimistic solutions, i.e. points $(0,x_2)$, as reasonable solutions to the problem (PCF), since we have that:

$$\lim_{\varepsilon\to 0} \mathcal{P}^\varepsilon(\sqrt{\varepsilon},x_2) \,=\, \lim_{\varepsilon\to 0} \inf_{\boldsymbol{x}\in H} \sup_{y\in\mathcal{M}^\varepsilon(\boldsymbol{x})} \Pi(\boldsymbol{x},y) \,=\, 0 \,=\, \inf_{\boldsymbol{x}\in H} \sup_{y\in\mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x},y),$$

and points $(0,x_2)$ allow both the leaders to realize the security value of a new two-stage game, with only one leader, having as a strategy $\boldsymbol{x}\in H$, namely:

$$\text{find } \bar{\boldsymbol{x}} \in H \text{ such that } \sup_{y\in\mathcal{M}(\bar{\boldsymbol{x}})} \Pi(\bar{\boldsymbol{x}},y) \,=\, \inf_{\boldsymbol{x}\in H} \sup_{y\in\mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x},y).$$

Thus, in line with the terminology we adopted in One-Leader-Multi-Follower games [18] or in Bilevel Optimization problems [20], we can introduce the concept of *pessimistic viscosity solution* to (PCF) in the class of weighted CF potential games.

Definition 2.3 *Let the CF game be weighted potential with Π as weighted potential. A point $\bar{\boldsymbol{x}} \in H$ is a pessimistic viscosity solution to the problem (PCF) if for every sequence of positive numbers $(\varepsilon_n)_n$ decreasing to zero there exists a sequence $(\bar{\boldsymbol{x}}_n)_n$, $\bar{\boldsymbol{x}}_n \in H$ for any $n\in\mathbb{N}$, such that:*

V_1) *a subsequence $(\bar{\boldsymbol{x}}_{n_k})_k$ converges towards $\bar{\boldsymbol{x}}$;*

V_2) *for any $n\in\mathbb{N}$, $\bar{\boldsymbol{x}}_n$ is a minimum point for the function*

$$\mathcal{P}^{\varepsilon_n} : \boldsymbol{x}\in H \to \mathcal{P}^{\varepsilon_n}(\boldsymbol{x}) \,=\, \sup_{y\in\mathcal{M}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x},y)$$

i.e.

$$\mathcal{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \le \mathcal{P}^{\varepsilon_n}(\boldsymbol{x}) \;\; \forall\, \boldsymbol{x} \,\in H;$$

V_3)

$$\lim_n \mathcal{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \,=\, \inf_{\boldsymbol{x}\in H} \sup_{y\in\mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x},y).$$

Roughly speaking, a pessimistic viscosity solution to a weighted potential CF game is a cluster point of a sequence of minimum points of suitable regularizations $\mathcal{P}^{\varepsilon_n}$ of the function $\mathcal{P}$, as defined in Proposition 3.2, whose values approach the security value of a one-leader-one-follower two-stage game with an hypothetic pessimistic leader having the weighted potential

Π as payoff.
Therefore, in order to investigate such a concept, it is primarily interesting to determine conditions which guarantee that for any fixed positive number ε, the function $\mathcal{P}^\varepsilon$ defined by

$$\mathcal{P}^\varepsilon(\boldsymbol{x}) = \sup_{y \in \mathcal{M}^\varepsilon(\boldsymbol{x})} \Pi(\boldsymbol{x}, y),$$

and called *approximate weighted value-potential*, has a minimum point in H.

Theorem 2.3 *Assume that a CF game is weighted potential with* Π *as a weighted potential. If the following assumptions hold:*
i) the sets H_i *and* Y *are compact for any* $i = 1, .., k$*;*
ii) the set-valued map K *is closed, lower semicontinuous and convex-valued over* H*;*
iii) the function F *is continuous over* $H \times Y$*;*
iv) the function $F(\boldsymbol{x}, \cdot)$ *is strictly quasi-convex [6] over* $K(\boldsymbol{x})$ *for every* $\boldsymbol{x} \in H$ *;*
v) the weighted potential Π *is lower semicontinuous over* $H \times Y$*;*
then, for every $\varepsilon > 0$*, there exists a minimum point, called* ε*-pessimistic solution to the CF game, for the approximate weighted value-potential* $\mathcal{P}^\varepsilon$*.*

Proof Due to compactness of the set H, we have only to prove that the function $\mathcal{P}^\varepsilon$ is lower semicontinuous over H. Since also the set Y is assumed to be compact, assumptions ii)-iv) imply that the map $\mathcal{M}^\varepsilon$ is lower semicontinuous over H (see, for example, [20, Prop. 2.5]) and, by point 1. in Proposition 2.1, assumption v) guarantees that $\mathcal{P}^\varepsilon$ is lower semicontinuous on H. □

Finally, we present an existence result for pessimistic viscosity solutions of weighted potential CF games.

Theorem 2.4 *Assume that the CF game is weighted potential with* Π *as a weighted potential. If assumptions in Theorem 3.3 and the following hold:*
vi) for every $\boldsymbol{x} \in H$ *there exists a sequence* $(\boldsymbol{x}_n)_n$ *converging yo* $\boldsymbol{x}$ *in* H *such that for every* $y \in Y$ *and every sequence* $(y_n)_n$ *converging to* y *in* Y *one has*

$$\limsup_n \Pi(\boldsymbol{x}_n, y_n) \leq \Pi(\boldsymbol{x}, y);$$

then, there exists a pessimistic viscosity solution for the CF game.

Proof Let $(\varepsilon_n)_n$ be a sequence of positive numbers decreasing to zero and let $(\bar{\boldsymbol{x}}_n)_n$ be a sequence of points of H, existing by Theorem 3.3, such that $\bar{\boldsymbol{x}}_n$ is a minimum point of $\mathcal{P}^{\varepsilon_n}$ for any $n \in \mathbb{N}$. Such a sequence has a subsequence converging towards a point $\bar{\boldsymbol{x}} \in H$. In order to prove that $\bar{\boldsymbol{x}}$ is a pessimistic viscosity solution to the CF game, we need only to prove condition V_3), that is

$$\lim_n \mathcal{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) = \lim_n \inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y) = \inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y).$$

We note that, for any $n \in \mathbb{N}$ and $\boldsymbol{x} \in H$, $\sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y) \leq \sup_{y \in \mathcal{M}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$, so that $\inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y) \leq \lim_n \mathcal{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n)$.

If a is a real number such that

$$\inf_{\boldsymbol{x}\in H} \sup_{y\in\mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x},y) < a < \lim_n \mathcal{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n),$$

there exists a point $\widetilde{\boldsymbol{x}} \in H$ such that

$$\Pi(\widetilde{\boldsymbol{x}},y) < a \quad \text{for any } y \in \mathcal{M}(\widetilde{\boldsymbol{x}}). \tag{7}$$

Due to condition vi), there exists a sequence $(\widetilde{\boldsymbol{x}}_n)_n$ converging to $\widetilde{\boldsymbol{x}}$ in H such that for every $y \in Y$ and every sequence $(y_n)_n$ converging to y in Y one has $\limsup_n \Pi(\widetilde{\boldsymbol{x}}_n, y_n) \le \Pi(\widetilde{\boldsymbol{x}}, y)$. Condition V_2) implies that $\mathcal{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \le \mathcal{P}^{\varepsilon_n}(\widetilde{\boldsymbol{x}}_n)$ for any n, so that, being

$$a < \lim_n \mathcal{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \le \lim_n \mathcal{P}^{\varepsilon_n}(\widetilde{\boldsymbol{x}}_n),$$

there exists a sequence $(y_n)_n$ in Y such that $y_n \in \mathcal{M}^{\varepsilon_n}(\widetilde{\boldsymbol{x}}_n)$ and $\Pi(\widetilde{\boldsymbol{x}}_n, y_n) > a$ for any $n \in \mathbb{N}$. A subsequence of $(y_n)_n$ has to converge to $\widetilde{y} \in \mathcal{M}(\widetilde{\boldsymbol{x}})$ (see, for example, [20, Prop 2.2]) so, by assumption vi), we get $\Pi(\widetilde{\boldsymbol{x}}, \widetilde{y}) \ge a$, which contradicts (7). □

It is worth noting that condition vi) is satisfied whenever the function $\Pi(\boldsymbol{x}, \cdot)$ is upper semicontinuous over Y for every $\boldsymbol{x} \in H$.

Finally, in the case of a general CF game we emphasize that one could regularize the set-valued map $\mathcal{M}$ in the same way as above and look for approximate and viscosity solutions of the CF game, but nor the concavity of the approximate set-valued map $\mathcal{M}^\varepsilon$ nor the existence of approximate solutions could be guaranteed, even under nice conditions on the data, differently to what happened for weighted potential CF games.
Moreover, concerning a weighted value-potential CF game, note that to regularize the set-valued map $\mathcal{M}$ in the same way as above does not guarantee that the normal form game $(\mathcal{P}_1^\varepsilon, .., \mathcal{P}_k^\varepsilon, H_1, .., H_k)$, where $\mathcal{P}_i^\varepsilon(\boldsymbol{x}) = \sup_{y\in\mathcal{M}^\varepsilon(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$ for any $i = 1, .., k$, is a weighted potential normal form game with lower semicontinuous weighted potential.

References

[1] J.P. Aubin and A. Frankowska: *Set-valued Analysis*, Birkhauser Boston, Boston, (1990).

[2] D. Aussel, M. Cervinka and M. Marechal: Deregulated electricity markets with thermal losses and production bounds: models and optimality conditions, RAIRO-Oper. Res. 50(1), 19-38, (2016).

[3] D. Aussel, P. Bendotti and M. Pištěk: Nash equilibrium in a pay-as-bid electricity market: part 1 existence and characterization. Optimization 66(6), 1013-1025, (2017).

[4] D. Aussel, P. Bendotti and M. Pištěk: Nash equilibrium in a pay-as-bid electricity market part 2 best response of a producer. Optimization 66 (6), 1027-1053, (2017).

[5] D. Aussel and A. Svensson: A Short State of the Art on Multi-Leader-Follower Games. In: Dempe S., Zemkoho A. (eds) *Bilevel Optimization*. Springer Optimization and its Applications 161. Springer, Cham. 53-76, (2020).

[6] B. Bank, J. Guddat, D. Klatte, B. Kummer and K. Tammer: *Nonlinear Parametric Optimization*. Birkhäuser, Basel, Switzerland, (1983).

[7] C. Berge: *Espaces Topologiques: Fonctions Multivoques*, Collection Universitaire de Mathématiques, Vol. III. Dunod, Paris, (1959).

[8] F. Caruso, M.B. Lignola and J. Morgan: Regularization and Approximation Methods in Stackelberg Games and Bilevel Optimization., In: Dempe S., Zemkoho A. (eds) *Bilevel Optimization*. Springer Optimization and its Applications 161. Springer, Cham. 77-138, (2020).

[9] F. Caruso, M.C. Ceparano and J. Morgan: Uniqueness of Nash equilibrium in continuous two-player weighted potential games, J. Math. Anal. Appl. 459(2), 1208-1221, (2018).

[10] G. Facchini, F. van Megen, P. Borm and S. Tijs: Congestion models and weighted bayesian potential games, Theory Decis. 42, 193-206, (1997).

[11] A.V. Fiacco and J. Kyparisis: Convexity and concavity properties of the optimal value function in parametric nonlinear programming, J. Opt. Theory Appl., 48, 95-126,(1986).

[12] M. Hu and M. Fukushima: Existence, uniqueness, and computation of robust Nash equilibria in a class of multi-leader-follower games. SIAM Journal on Optimization 23(2), 894-916, (2013).

[13] M. Hu and M. Fukushima: Multi-leader-follower games: models, methods and applications. J. Oper. Res. Soc. Jpn 58(1), 1-23, (2015).

[14] A. A. Kulkarni and U. V. Shanbhag: A shared-constraint approach to multi-leader multi-follower games, Set-Valued and Variational Anal. 22(4), 691-720, (2014).

[15] A.A. Kulkarni and U.V. Shanbhag: An Existence Result for Hierarchical Stackelberg v/s Stackelberg Games. IEEE Trans. Autom. Control 60(12), 3379-3384, (2015).

[16] G. Leitmann: On generalized Stackelberg strategies. J. Optim. Theory Appl. 26(4), 637-643, (1978).

[17] S. Leyffer and T.S. Munson: Solving multi-leader-common-follower games. Optim. Methods Softw. 25(4), 601-623, (2010).

[18] M.B. Lignola and J. Morgan: Viscosity solutions for bilevel problems with Nash equilibrium constraints. Far East J. Appl. Math. 88 (1), 15-34, (2014).

[19] M.B. Lignola and J. Morgan: Topological Existence and Stability for Stackelberg Problems, J. Optim. Theory Appl. 84(1), 145-169, (1995).

[20] M.B. Lignola and J. Morgan: Inner Regularizations and viscosity solutions for pessimistic bilevel optimization problems, J. Optim. Theory Appl. 173 (1), 183-202, (2017).

[21] D. Monderer and L. S. Shapley: Potential games, Games Econom. Behav. 14(1), 124-143, (1996).

[22] Nash, J.F. Jr.: Equilibrium points in n-person games, Proc. Nat. Acad. Sci. Usa 36(1), 48-49, (1950).

[23] J-S. Pang and M. Fukushima: Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. Comput Manag Sci 2 (1) 21-56, (2005).