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An Iterative Approach to Rationalizable Implementation

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An Iterative Approach to Rationalizable Implementation

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Abstract
We study rationalizable implementation of social choice functions. Iterative Monotonicity is both necessary *and* sufficient for implementation when there are two or more players.

JEL classification: C79, D82.

Keywords: Implementation, iterative monotonicity, rationalizability, complete information.

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I. Introduction

Most early studies on implementation theory focused on (Bayes–)Nash equilibrium and its refinements (see, for instance, Maskin and Sjöström, 2002). These solution concepts rely on two implicit common-knowledge assumptions. (1) No strategic uncertainty: each player correctly predicts the strategic play of her opponents. (2) No higher-order uncertainty: the underlying type space, based on the set of states Θ, is assumed to be common knowledge among players. By adopting the notion of robustness of Weinstein and Yildiz (2007) in a mechanism design setting, Oury and Tercieux (2012) show that this notion is tightly connected to (full) implementation in rationalizable strategies. A social choice function (SCF) f is rationalizably implementable if there exists a mechanism such that every rationalizable strategy profile leads to the realization of f.

In a fundamental paper, Bergemann et al. (2011) (BMT) study the implementation of SCFs under complete information in rationalizable strategies. They show that a strict version of the monotonicity condition introduced by Maskin (1999) is sufficient for implementation under a no worst alternative (NWA) condition.[1] Recently, Xiong (2022) presents necessary and sufficient conditions for implementation in rationalizable strategies.

However, these studies have two significant limitations. The first is related to the fact that their conditions are not sufficient to implement two-player SCFs.[2] Indeed, in Section III, we construct an SCF that is Nash implementable and that satisfies the sufficient conditions of BMT, but that fails to be implementable in rationalizable strategies when there are two players. Therefore, two-player rationalizable implementation problems require a fundamentally different solution from that provided by the

[1] This condition, called strict Maskin monotonicity*, is also necessary under a mild restriction on the class of implementing mechanisms. NWA requires that a player never receives his worst outcome under the SCF.

[2] The two-person problem is an important one in the theory of incentives. Indeed, the two-player model is the leading case for contracting or bargaining applications (see, for instance, Moore and Repullo, 1990 and Dutta and Sen, 1991).

existing characterization results.

The second is related to the fact that their implementing conditions are stated in terms of the existence of a partition of the set Θ. However, no rules for how to prove or disprove its existence are given. This absence makes it challenging to conceptualize its existence from a game-theoretical standpoint. From a practical standpoint, the condition becomes difficult to check as the number of partitions of Θ grows exponentially with the size of Θ.[3]

Motivated by these limitations, we develop an approach to fully characterize the class of rationalizably implementable SCFs with two or more players. Our approach is constructive and is based on the idea of deceptions (Jackson (1991), Oury and Tercieux (2012)).[4] More importantly, it allows us to solve two-player implementation problems and construct the partition specified by the existing characterization results from the primitives of the implementation model. More precisely, our necessary and sufficient condition is based on an algorithm that identifies the required partition by using the limit point of an *increasing* sequence of deceptions. The sequence has a game-theoretical interpretation. Without the guidance provided by our algorithm, one is forced to search over a larger number of partitions.[5]

Our necessary and sufficient condition for rationalizable implementation is termed *Iterative Monotonicity* (IM). Player i's deception is a correspondence $\beta_i : \Theta \longrightarrow 2^{\Theta} \setminus \{\emptyset\}$ such that $\theta \in \beta_i(\theta)$. We denote a profile of deceptions by β and the set of all profiles by $\mathcal{B}^t$. The main novelty of our approach is that, for a given SCF f and a player i, we define a function $\mathcal{R}_i$ from $\mathcal{B}^t$ to $\mathcal{B}^t_i$. The self-map $\mathcal{R} \equiv \mathcal{R}_1 \times .... \times \mathcal{R}_I$ allows

[3]In combinatorial mathematics, the number of partitions of a set of size n is referred to as bell number. Bell numbers can be recursively defined as follows: for every $n+1$,

$$B(n) = \sum_{k=0}^{n} \binom{n}{k} B(k)$$

where $B(1) = 1$.

[4]Traditionally, the idea of deceptions is used to analyze implementation problems with incomplete information. However, there are notable excepts such as Mezzetti and Renou (2017).

[5]For concreteness, when $|\Theta| = 6$, the total number of partition is 216 but our algorithm checks at most 6 partitions.

us to recursively define an increasing sequence $\{\beta_k\}_{k\geq 0}$, where $\beta_k = \mathcal{R}(\beta_{k-1})$ and the deception β^0—where $\theta \in \beta_i^0(\theta)$ for all $\theta \in \Theta$ and every player i. This sequence has a clear game-theoretical interpretation, which is discussed in Section IV. The limit of this sequence which we name as β^* is the least (fixed-point) deception.[6]

An SCF f satisfies IM provided that for all $\theta, \theta' \in \Theta$, $\beta^*(\theta) \cap \beta^*(\theta')$ is empty whenever $f(\theta) \neq f(\theta')$. When f satisfies IM, β^* pins down the partition of Θ required by the existing characterization results. It is worth mentioning that IM is a measurability type condition, which is reminiscent of the classical Abreu–Matsushima measurability (Abreu and Matsushima, 1992).[7]

The rest of the paper is organized as follows. Section II presents the implementation model. Section III presents our motivating example for the two-player case. Section IV discusses our implementing condition and present our characterization result. Section V briefly concludes by connecting our analysis with the existing characterization results. Appendices include proofs not in the main body.

II. Setup

The environment consists of $I \geq 2$ players (we write $\mathcal{I} = \{1, ..., I\}$ for the set of players), a finite set of states Θ and a countable set of pure outcomes X. Let $Y \equiv \Delta(X)$ denote the set of lotteries over X. Player i's preferences over lotteries are described by a utility function $u_i : Y \times \Theta \mapsto \mathbb{R}$, with

$$u_i(y, \theta) = \sum_{x \in X} y_x u_i(x, \theta),$$

where y_x is the probability of pure outcome x. For all $\theta \in \Theta$, $u_i(\cdot, \theta)$ satisfies the expected utility hypothesis. To save writing, for all $i \in \mathcal{I}$, we write $-i$ for player i's

[6] Since $\mathcal{R}$ is monotone (i.e., increasing) on $\mathcal{B}^t$ and $\mathcal{B}^t$ is a complete lattice, Tarski's fixed-point theorem implies that there is a least fixed-point of $\mathcal{R}$.

[7] Abreu and Matsushima (1992) proposed a measurability condition, now referred to as Abreu–Matsushima measurability, to characterize virtual rationalizable implementation when there is incomplete information.

opponents. For all $i \in \mathcal{I}$, let $\Theta_{-i} \equiv \underbrace{\Theta \times .... \times \Theta}_{I-1\text{-times}}$, with $\theta_{-i} = (\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_I)$ as a typical element of Θ_{-i}.

Given a state $\theta \in \Theta$, a player $i \in \mathcal{I}$, and a lottery $x \in Y$, the lower contour set of $u_i(\cdot, \theta)$ at x is $L_i(x, \theta) = \{y \in Y | u_i(x, \theta) \geq u_i(y, \theta)\}$; the strict lower contour set of $u_i(\cdot, \theta)$ at x is $SL_i(x, \theta) = \{y \in Y | u_i(x, \theta) > u_i(y, \theta)\}$; and the strict upper contour set of $u_i(\cdot, \theta)$ at x is $SU_i(x, \theta) = \{y \in Y | u_i(y, \theta) > u_i(x, \theta)\}$.

A mechanism $\mathcal{M}$ is a pair $\mathcal{M} \equiv (M, g)$, where $M \equiv \prod_{i \in \mathcal{I}} M_i$, with each M_i being a nonempty countable set, and $g : M \longrightarrow Y$. As usual, we refer to M_i as the (pure) strategy space of $i \in \mathcal{I}$, to a member of M, denoted by m, as a (pure) strategy profile, and to g as an outcome function. As usual, $M_{-i} \equiv \prod_{j \in \mathcal{I} \setminus \{i\}} M_j$, with m_{-i} as a typical element. The same notational convention will be followed for any profile of objects. For all $M' \subseteq M$, let $g[M'] = \{g(m) \in Y | m \in M'\}$.

The environment, when combined with the mechanism, describes a game (of complete information) for all state $\theta \in \Theta$, which is denoted by $(\mathcal{M}, \theta)$. We will use (correlated) rationalizability as a solution concept. Bernheim (1984) and Pearce (1984) provide a definition of rationalizability in which players' conjectures over their opponents' play are independent. In this paper, we follow the convention of some of the recent literature (e.g., Osborne and Rubinstein (1994) in using "rationalizability" for the correlated version of rationalizability (we refer the reader to Brandenburger and Dekel (1987)). Our definition of rationalizability coincides with the standard definition when strategy spaces are compact. However, our definition allows for infinite, non-compact strategy spaces. In this case, our definition is equivalent to one introduced by Lipman (1994).

Formally, let $\mathcal{S}$ be the set of all strategy-set profiles, defined by $\mathcal{S} \equiv \prod_{i \in \mathcal{I}} \mathcal{S}_i$, where $\mathcal{S}_i \equiv 2^{M_i}$ for all $i \in \mathcal{I}$, with $S = (S_i)_{i \in \mathcal{I}}$ as a typical profile of $\mathcal{S}$. The family $\mathcal{S}$ is a lattice with the natural ordering of the set inclusion: $S \leq S'$ if $S_i \subseteq S'_i$ for all $i \in \mathcal{I}$. The smallest element of S is denoted by $\underline{S} \equiv (\emptyset, ..., \emptyset)$, whereas the largest element is denoted by $\bar{S} \equiv M$.

Fix any game $(\mathcal{M},\theta)$. The strategy $m_i \in M_i$ is player i's best-response to his belief $\lambda_i \in \Delta(M_{-i})$ at θ if

$$m_i \in \arg\max_{m_i' \in M_i} \sum_{m_{-i} \in M_{-i}} \lambda_i(m_{-i})\, u_i(g(m_i', m_{-i}), \theta).$$

Let us define an operator $b^{\mathcal{M},\theta} : \mathcal{S} \longrightarrow \mathcal{S}$ to iteratively eliminate never best responses with $b^{\mathcal{M},\theta} \equiv \left(b_i^{\mathcal{M},\theta}\right)_{i\in\mathcal{I}}$, where $b_i^{\mathcal{M},\theta} : \mathcal{S} \longrightarrow \mathcal{S}_i$ is defined, for all $S \in \mathcal{S}$, by

$$b_i^{\mathcal{M},\theta}(S) = \left\{ m_i \in M_i \;\middle|\; \begin{array}{l} \text{there exists } \lambda_i^{m_i,\theta} \in \Delta(M_{-i}) \text{ such that} \\ (1)\ \lambda_i^{m_i,\theta}(m_{-i}) > 0 \implies m_{-i} \in S_{-i}, \\ (2)\ m_i \text{ is a best response to } \lambda_i^{m_i,\theta} \text{ at } \theta \end{array} \right\}.$$

Note that $b^{\mathcal{M},\theta}$ is increasing (that is, $S \le S' \implies b^{\mathcal{M},\theta}(S) \le b^{\mathcal{M},\theta}(S')$).

By Tarski's fixed point theorem, there exists a largest fixed point of $b^{\mathcal{M},\theta}$, which is denoted by $S^{\mathcal{M},\theta}$. That is, (1) $b^{\mathcal{M},\theta}\left(S^{\mathcal{M},\theta}\right) = S^{\mathcal{M},\theta}$ and (2) $b^{\mathcal{M},\theta}(S) = S \implies S \le S^{\mathcal{M},\theta}$. Alternatively, the fixed point $S^{\mathcal{M},\theta}$ can be constructed by starting with the largest element of the lattice, $\bar{S}$, and by iteratively applying the operator $b^{\mathcal{M},\theta}$. If the strategy sets are finite, we have that

$$S_i^{\mathcal{M},\theta} \equiv \bigcap_{k\ge 0} b_i^{\mathcal{M},\theta}\left(\left[b_i^{\mathcal{M},\theta}\right]^k (\bar{S})\right).$$

In this case, the solution concept is equivalent to deletion of strictly dominated strategies (Brandenburger and Dekel (1987)). When the strategy sets are infinite sets, transfinite induction may be necessary to reach the fixed point (Lipman (1994)). Sometimes, we will use the following notation

$$S_{i,k}^{\mathcal{M},\theta} \equiv b_i^{\mathcal{M},\theta}\left(\left[b_i^{\mathcal{M},\theta}\right]^{k-1} (\bar{S})\right)$$

to denote the kth step of the iterative process. The set $S_i^{\mathcal{M},\theta}$ is the set of strategies surviving (transfinite) iterated deletion of never best responses.

We refer to $m_i \in S_i^{\mathcal{M},\theta}$ as a player i's rationalizable strategy in $\mathcal{M}$ at state θ, and to a member of $S^{\mathcal{M},\theta}$ as a rationalizable strategy profile in $\mathcal{M}$ at state θ.

We say that a profile $S \in \mathcal{S}$ has the best-response property in state θ if $S \leq b^{\mathcal{M},\theta}(S)$, or equivalently, if for all $i \in \mathcal{I}$ and all $m_i \in S_i$, there exists $\lambda_i \in \Delta(M_{-i})$ such that $\lambda_i(m_{-i}) > 0 \implies m_j \in S_j$ for all $j \in \mathcal{I} \backslash \{i\}$, and m_i is a best-response to λ_i at θ. It can be checked that $S \leq S^{\mathcal{M},\theta}$ when S has the best-response property in state θ.

Player i's mixed-strategy σ_i is a probability distribution over M_i. The space of player i's mixed-strategy is denoted by Σ_i, where $\sigma_i(m_i)$ is the probability that σ_i assigns to m_i. The space of mixed-strategy profiles is denoted by $\Sigma = \prod_{i \in \mathcal{I}} \Sigma_i$, with element σ as a typical strategy profile. A mixed-strategy may assign probability one to a single strategy m_i, that is, $\sigma_i(m_i) = 1$. We refer to such a mixed-strategy as a (pure) strategy and denote it by m_i. The support of a mixed-strategy σ_i is the set of pure strategies that are played with positive probability, that is, $supp(\sigma_i) = \{m_i \in M_i | \sigma_i(m_i) > 0\}$. A mixed-strategy profile σ is a Nash equilibrium of $(\mathcal{M}, \theta)$ if for all $i \in \mathcal{I}$,

$$u_i(g(\sigma_i, \sigma_{-i}), \theta) \geq u_i(g(\sigma_i', \sigma_{-i}), \theta),$$

for all $\sigma_i' \in \Sigma_i$. Write $NE(\mathcal{M}, \theta)$ for the set of Nash equilibrium profiles of $(\mathcal{M}, \theta)$, and write $g(NE(\mathcal{M}, \theta))$ for the set of Nash equilibrium outcomes of $(\mathcal{M}, \theta)$.

An SCF f is a function $f : \Theta \longrightarrow Y$.

Definition 1. A mechanism $\mathcal{M}$ implements $f : \Theta \longrightarrow Y$ in rationalizable strategies if for all $\theta \in \Theta$, $S^{\mathcal{M},\theta} \neq \emptyset$ and $m \in S^{\mathcal{M},\theta} \implies g(m) = f(\theta)$. If such a mechanism exists, f is said to be rationalizably implementable.

A partition of Θ is a correspondence $P : \Theta \rightrightarrows \Theta$ satisfying the following requirements: (i) $\theta \in P(\theta)$ for all $\theta \in \Theta$, (ii) $\cup_{\theta \in \Theta} P(\theta) = \Theta$, and (iii) $P(\theta) \cap P(\theta') = \emptyset$ if $P(\theta) \neq P(\theta')$. Given an SCF f, P_f is the partition of Θ induced by f, that is, $P_f = \{\Theta_y\}_{y \in f(\Theta)}$ where $\Theta_y = \{\theta \in \Theta | f(\theta) = y\}$. A partition P of Θ is at least as fine as P_f, or equivalently, P_f is coarser than P if $P(\theta) \subseteq P_f(\theta)$ for all $\theta \in \Theta$. Let $\mathcal{P}_f$ denote the set of partitions that are at least as fine as P_f, that is,

$$\mathcal{P}_f = \{P | P \text{ is a partition of } \Theta \text{ such that } P(\theta) \subseteq P_f(\theta) \text{ for all } \theta \in \Theta\}.$$

III. Contextualizing the problem

When there are at least three players, BMT show that Maskin monotonicity* and a strengthening of the NWA condition, referred to as NWA*, is sufficient for rationalizable implementation. The requirements can be stated as follows.[8]

Definition 2. $f : \Theta \to Y$ is Maskin monotonic* provided that there exists a partition $P \in \mathcal{P}_f$ such that for all $\theta, \theta' \in \Theta$,

$$\left[\begin{array}{l} \text{for all } i \in \mathcal{I} \text{ and all } \hat{\theta} \in P(\theta'), \\ L_i\left(f(\theta'), \hat{\theta}\right) \subseteq L_i(f(\theta'), \theta) \end{array}\right] \implies \theta' \in P(\theta).$$

Definition 3. Suppose that $f : \Theta \to Y$ is Maskin monotonic* with respect to the partition $P \in \mathcal{P}_f$. $f : \Theta \to Y$ is NWA* provided that for all $(i, \theta) \in \mathcal{I} \times \Theta$, there exists $\tilde{y}_i(P(\theta)) \in Y$ such that

$$u_i(f(\theta), \theta) > u_i(\tilde{y}_i(P(\theta)), \theta).$$

BMT show that Nash implementation of f is equivalent to its rationalizable implementation when there are at least three players and f satisfies both NWA* and MM*. In what follows, we show below that when there are two players, this equivalence breaks down, though the constructed f satisfies both NWA* and MM*, moreover, it is Nash implementable. Indeed, the constructed f satisfies MM* with respect to the finest partition.

Suppose that $\mathcal{I} = \{1, 2\}$, $X = \{a, b, c, d, e, f, g\}$ and $\Theta = \{\theta, \theta', \theta''\}$. Players'

[8] Under NWA*, Maskin monotonic* is equivalent to strict Maskin monotonicity*.

utilities from pure outcomes are summarized in the table below, where $\varepsilon \in \left(\frac{1}{2}, 1\right)$.

	$u_1(\cdot, \theta)$	$u_2(\cdot, \theta)$	$u_1(\cdot, \theta')$	$u_2(\cdot, \theta')$	$u_1(\cdot, \theta'')$	$u_2(\cdot, \theta'')$
a	1	$-(1-\varepsilon)$	1	-1	1	-1
b	0	0	0	0	0	0
c	-1	1	$-(1+\varepsilon)$	1	-1	1
d	1	-2	-2	-1	1	-1
e	2	$-(2-\epsilon)$	2	-2	-2	-2
f	3	-3	-3	-3	3	-3
g	0	0	0	0	-3	-3

The planner wants to implement f, which is defined by

$$f(\theta) = f(\theta') = \{b\} \text{ and } f(\theta'') = \{a\}.$$

It can easily be checked that f satisfies NWA*. Moreover, it can be checked that f satisfies MM* with respect the finest partition vacuously.[9] Finally, it can be checked that f is Nash implementable.[10] However, f is not rationalizable implementable. The easiest way to make this point without being distracted by boring details is to suppose that $\mathcal{M}$ implements f in rationalizable strategies and in (pure strategy) Nash equilibria.

For each state $\bar{\theta} \in \Theta$, let $m(\bar{\theta}) = \left(m_1(\bar{\theta}), m_2(\bar{\theta})\right)$ be a Nash equilibrium strategy profile for the game $\left(\mathcal{M}, \bar{\theta}\right)$. Since $SL_1(f(\theta), \theta) \cap L_2(f(\theta'), \theta')$ is nonempty, by construction, and since, moreover, $\mathcal{M}$ rationalizable implements f, it follows that

[9] Indeed, it can be checked that $\frac{1}{2}a + \frac{1}{2}c \in L_2(f(\theta'), \theta') \cap SU_2(f(\theta'), \theta)$, $\frac{1}{2}a + \frac{1}{2}c \in L_1(f(\theta), \theta) \cap SU_1(f(\theta), \theta')$, $\frac{1}{2}a + \frac{1}{2}d \in L_1(f(\theta'), \theta') \cap SU_1(f(\theta'), \theta'')$, $e \in L_1(f(\theta''), \theta'') \cap SU_1(f(\theta''), \theta')$, $e \in L_1(f(\theta''), \theta'') \cap SU_1(f(\theta''), \theta)$ and $\frac{2}{3}c + \frac{1}{3}d \in L_2(f(\theta), \theta) \cap SU_2(f(\theta), \theta'')$.

[10] Indeed, f satisfies the necessary and sufficient condition of Moore and Repullo (1990) under the specifications that the set $B = \Delta(X)$, $C_i\left(f(\bar{\theta}), \bar{\theta}\right) = L_i\left(f(\bar{\theta}), \bar{\theta}\right)$ for all $i \in \mathcal{I}$ and all $\bar{\theta} \in \Theta$, and the punishment outcomes are $e\left(f(\bar{\theta}), \bar{\theta}, f(\bar{\theta}), \bar{\theta}\right) = f(\bar{\theta})$ for all $\bar{\theta} \in \Theta$, $e(f(\theta), \theta, f(\theta'), \theta') = e(f(\theta''), \theta'', f(\theta'), \theta') = e(f(\theta''), \theta'', f(\theta), \theta) = g$, $e(f(\theta), \theta, f(\theta''), \theta'') = 0.1c + 0.9g$ and $e(f(\theta'), \theta', f(\theta), \theta) = e(f(\theta'), \theta', f(\theta''), \theta'') = 0.1d + 0.9g$.

$g\left(m_1\left(\theta'\right), m_2\left(\theta\right)\right) = f\left(\theta\right)$. Since $L_1\left(f\left(\theta\right), \theta\right) \subseteq L_1\left(f\left(\theta\right), \theta''\right)$ and $L_2\left(f\left(\theta\right), \theta'\right) \subseteq L_2\left(f\left(\theta\right), \theta''\right)$, it follows that $\left(m_1\left(\theta'\right), m_2\left(\theta\right)\right)$ is a Nash equilibrium strategy profile for $\left(\mathcal{M}, \theta''\right)$. However, since $\mathcal{M}$ Nash implements f, it follows that $f\left(\theta\right) = f\left(\theta''\right)$, which is a contradiction.

f is Nash implementable because we can assign to the strategy profile $\left(m_1\left(\theta'\right), m_2\left(\theta\right)\right)$ the outcome g and this allows us to satisfy the necessary and sufficient conditions for Nash implementation (Moore and Repullo, 1991). Rationalizable implementation does not allow us to make this assignment. The reason is that any mechanism that rationalizable implements f has to assign the outcome $f\left(\theta\right)$ to the rationalizable strategy profile $\left(m_1\left(\theta'\right), m_2\left(\theta\right)\right)$. Therefore, when there are two players, rationalizable implementation imposes restrictions in the mechanism design exercise that are not available in the case of Nash implementation.

IV. Iterative Monotonicity

Fix any $i \in \mathcal{I}$ and any SCF f. Let

$$\Theta_i^f = \left\{\theta \in \Theta | SL_i\left(f\left(\theta\right), \theta\right) = \emptyset\right\}. \tag{1}$$

In words, $\theta \in \Theta_i^f$ if $f\left(\theta\right)$ is the worst outcome for i at θ. Note that if $\theta \in \Theta_i^f$ and f is rationalizably implementable, every strategy of player i is a rationalizable strategy at θ.

When $I = 2$, for every $\theta \in \Theta$, we define $A_i(\theta)$ by

$$A_i\left(\theta\right) = \left\{\theta' \middle| SL_i\left(f\left(\theta\right), \theta\right) \bigcap L_{-i}\left(f\left(\theta'\right), \theta'\right) = \emptyset\right\} \bigcup \left\{\theta\right\}. \tag{2}$$

Let us call any map $\beta_i : \Theta \longrightarrow 2^{\Theta} \backslash \left\{\emptyset\right\}$ as player i's deception. A special deception for player i is the truth-telling deception, β_i^t, defined by $\beta_i^t\left(\theta\right) = \left\{\theta\right\}$ for all $\theta \in \Theta$.

Another special deception is denoted by β_i^0 and it is defined, for all $\theta \in \Theta$, by

$$\beta_i^0(\theta) = \begin{cases} A_i(\theta) & \text{if } I = 2 \\ \\ \beta_i^t(\theta) & \text{if } i \in \mathcal{I}^\theta \text{ and } I \geq 3 \\ \\ \Theta & \text{if } i \notin \mathcal{I}^\theta \text{ and } I \geq 3. \end{cases} \tag{3}$$

Note that $\theta \in \beta_i^0(\theta)$ for all $\theta \in \Theta$.

For any β_i and β_i', we write $\beta_i \subseteq \beta_i'$ if $\beta_i(\theta) \subseteq \beta_i'(\theta)$ for all $\theta \in \Theta$. Let $\mathcal{B}_i^t$ denote the set of player i's deceptions containing the truth-telling deception, that is,

$$\mathcal{B}_i^t \equiv \left\{ \beta_i \in \mathcal{B}_i \middle| \beta_i^0 \subseteq \beta_i \right\}. \tag{4}$$

Let $\mathcal{B}^t \equiv \prod_{i \in \mathcal{I}} \mathcal{B}_i^t$, with $\beta = (\beta_i)_{i \in \mathcal{I}}$ as a typical deception profile of $\mathcal{B}^t$. For all $\beta, \beta' \in \mathcal{B}^t$, we write $\beta \subseteq \beta'$ if $\beta_i \subseteq \beta_i'$ for all $i \in \mathcal{I}$. The collection $\mathcal{B}^t$ is a complete lattice with the natural ordering set inclusion: $\beta \leq \beta'$ if $\beta \subseteq \beta'$. The largest element is $\bar{\beta} = (\Theta, ..., \Theta)$. The smallest element is β^t.

Let us define the function $\mathcal{R}_i : \mathcal{B}^t \mapsto \mathcal{B}_i^t$ as follows, where for all $\beta \in \mathcal{B}^t$, we use $\mathcal{R}_i(\beta)$ and $\mathcal{R}_i^\beta$ interchangeably. For all $\beta \in \mathcal{B}^t$, all $\theta \in \Theta$ and all $i \in \mathcal{I}$, let $\mathcal{R}_i^\beta$ be defined by

$$\mathcal{R}_i^\beta(\theta) = \bigcup_{E_i \in 2^\Theta \setminus \{\emptyset\}} \left\{ E_i \,\middle|\, \begin{array}{l} \text{there exists } E_{-i} \in \left(2^\Theta \setminus \{\emptyset\}\right)^{I-1} \text{ such that} \\ \text{for all } (\ell, \theta') \in \mathcal{I} \times E_\ell \text{, there exists} \\ \left(\tilde{\theta}, \hat{\theta}\right) \in E_\ell \times \bigcap_{j \in \mathcal{I} \setminus \{\ell\}} E_j \text{ such that} \\ \left(\theta', \hat{\theta}\right) \in \beta_\ell\left(\tilde{\theta}\right) \times \bigcap_{j \in \mathcal{I} \setminus \{\ell\}} \beta_j\left(\tilde{\theta}\right) \text{ and} \\ \textit{either } \tilde{\theta} \in \Theta_\ell^f \textit{ or } L_\ell\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \subseteq L_\ell\left(f\left(\tilde{\theta}\right), \theta\right) \end{array} \right\} \tag{5}$$

For all $\beta \in \mathcal{B}^t$, let $\mathcal{R}(\beta) = \prod_{i \in \mathcal{I}} \mathcal{R}_i(\beta)$.[11]

To proceed further, we present some useful properties of $\mathcal{R}$. To this end, we need additional notation. Let $\mathcal{E}(\mathcal{R})$ denote the set of fixed points of $\mathcal{R}$, which is defined by

$$\mathcal{E}(\mathcal{R}) = \{\beta \in \mathcal{B}^t | \mathcal{R}(\beta) = \beta\}.$$

The function $\mathcal{R} : \mathcal{B}^t \mapsto \mathcal{B}^t$ is monotone (or, increasing) on $\mathcal{B}^t$ if for all $\beta, \beta' \in \mathcal{B}^t$, $\beta \subseteq \beta'$ implies $\mathcal{R}(\beta) \subseteq \mathcal{R}(\beta')$. For each $\theta \in \Theta$, let us define $\mathcal{I}^\theta$ by $\mathcal{I}^\theta = \{i \in \mathcal{I} | SL_i(f(\theta), \theta) \neq \emptyset\}$. Finally, let $\{\beta^k\}_{k \geq 0}$ be an increasing sequence (in the sense of set inclusion) of elements of $\mathcal{B}^t$, which is defined recursively as follows. The initial point is the special deception profile β^0, where $\beta_i^0(\theta)$ is defined in (10) for all $i \in \mathcal{I}$ and all $\theta \in \Theta$, whereas for all $k \geq 1$, β^k is the computation of the mapping $\mathcal{R}$ at the profile β^{k-1}; that is,

$$\beta^k = \mathcal{R}\left(\beta^{k-1}\right). \tag{6}$$

The first property is that $\beta_i(\theta) \subseteq \mathcal{R}_i^\beta(\theta)$ for all $\beta \in \mathcal{B}^t$, all $\theta \in \Theta$ and all $i \in \mathcal{I}$. This property follows directly from (5) by setting $E_i = E_j = \{\theta\}$ for all $i, j \in \mathcal{I}$ and $\hat{\theta} = \theta' = \theta$. The second property is that $\mathcal{R}$ is monotone on $\mathcal{B}^t$.[12] The third property is that the set $\mathcal{E}(\mathcal{R})$ is a complete lattice and that the limit point of the sequence $\{\beta^k\}_{k \geq 0}$ is the smallest element of $\mathcal{E}(\mathcal{R})$. This property comes from the fact that since $\mathcal{B}^t$ is a complete lattice and since $\mathcal{R}$ is monotone, Tarski fixed point-theorem implies that $\mathcal{R}$ has a greatest and a least fixed point and that $\mathcal{E}(\mathcal{R})$ is a complete lattice. Moreover, since $\mathcal{B}^t$ is finite, the mapping $\mathcal{R}$ is continuous. Therefore, by applying Echenique (2005), we can conclude that the limit point of the sequence $\{\beta^k\}_{k \geq 0}$, denoted by β^*, is the smallest fixed point of $\mathcal{E}(\mathcal{R})$. The last property says that when if $\bigcap_{j \in \mathcal{I} \setminus \{i\}} \beta_j^*(\theta) \bigcap \Theta_i^f \neq \emptyset$, then $\beta_i^*(\theta) = \Theta$.[13]

[11] Alternatively, $\mathcal{R}^\beta$ can be defined via an iterative elimination notion that reflects the notion of iterative elimination of never best-responses. For such an approach, we refer the reader to Jain et al. (2021).

[12] It directly follows from (5) and the assumption that $\beta \subseteq \beta'$.

[13] To see it, let $\theta^* \in \bigcap_{j \in \mathcal{I} \setminus \{i\}} \beta_j^*(\theta) \cap \Theta_i^f$. Then, it follows from (10) that $\beta_i^0(\theta^*) = \Theta$. Since

Since the arguments from the previous paragraph clarify why the mapping $\mathcal{R}$ has the above-discussed properties, we summarize them in the following lemma, whose proof is omitted.

Lemma 1. $\mathcal{R} : \mathcal{B}^t \mapsto \mathcal{B}^t$ has the following properties.

(i) For all $\beta \in \mathcal{B}^t$, $\beta \subseteq \mathcal{R}(\beta)$.

(ii) $\mathcal{R}$ is monotone on $\mathcal{B}^t$.

(iii) $\mathcal{E}(\mathcal{R})$ is a complete lattice and $\{\beta^*\} = \min \mathcal{E}(\mathcal{R})$ where $\beta^* \equiv \sup_{k \geq 0} \beta^k$.

(iv) For all $i \in \mathcal{I}$ and all $\theta \in \Theta$, if $\bigcap_{j \in \mathcal{I} \setminus \{i\}} \beta_j^*(\theta) \bigcap \Theta_i^f \neq \emptyset$, then $\beta_i^*(\theta) = \Theta$.

The following condition is at the heart of our characterization result.

Definition 4. $f : \Theta \mapsto Y$ satisfies *Iterative Monotonicity* (henceforth, IM) if for all $\theta, \theta' \in \Theta$,

$$f(\theta) \neq f(\theta') \implies \beta^*(\theta) \bigcap \beta^*(\theta') = \emptyset. \tag{7}$$

Let us here briefly explain IM and its relationship with rationalizable implementation. Recall that Maskin monotonicity ensures the elimination of undesirable Nash equilibrium. Similarly, IM ensures the elimination of undesirable 'best-response sets.' This is done by computing players' largest best-response sets via the fixed point deception profile β^*.

To see it, suppose that $\mathcal{M}$ rationalizable implements f, so that $S^{\mathcal{M},\theta}$ is the set of rationalizable strategy profiles for θ. Recall that β^* is the limit point of the sequence $\{\beta^k\}_{k \geq 0}$ and that this sequence is an increasing sequence (in the sense of set inclusion) of elements of $\mathcal{B}^t$, which is defined recursively. Its starting deception is β^0, where β_i^0 is defined in (10) for all $i \in \mathcal{I}$ and all $\theta \in \Theta$, whereas for all $k \geq 1$, β^k is computed by $\mathcal{R}(\beta^{k-1})$ as in (6).

$\beta^* \in \mathcal{B}^t$, it follows that $\beta_i^*(\theta^*) = \Theta$. To show that $\beta_i^*(\theta) = \Theta$, it suffices to show that $\Theta \subseteq \mathcal{R}_i^{\beta^*}(\theta)$. This follows from (5) by setting $E_i = \Theta$ and $E_j = \beta_j^*(\theta)$ for all $j \in \mathcal{I} \setminus \{i\}$ and by setting $\tilde{\theta} = \hat{\theta} = \theta^*$ for all $\theta' \in E_i$ and setting $\tilde{\theta} = \hat{\theta} = \theta$ for all $\theta' \in E_j$.

For every $\theta \in \Theta$ and every player i, implementability requires that the set $\beta^0(\theta)$ generates a best-response set at θ that cannot be eliminated; that is, for each $i \in \mathcal{I}$, $\bigcup_{\bar{\theta} \in \beta_i^0(\theta)} S_i^{\mathcal{M},\bar{\theta}} \subseteq S_i^{\mathcal{M},\theta}$.

By applying (5), it is possible to compute $\mathcal{R}^{\beta^0}(\theta) = \beta^1(\theta)$. Since, by construction, $\beta^1(\theta)$ is computed in a way that it generates a best-response set at θ that cannot be eliminated, it holds, for each $i \in \mathcal{I}$, that $\bigcup_{\bar{\theta} \in \beta_i^1(\theta)} S_i^{\mathcal{M},\bar{\theta}} \subseteq S_i^{\mathcal{M},\theta}$.

This reasoning can be repeated to derived a best-response set at θ generated by $\mathcal{R}^{\beta^1}(\theta) = \beta^2(\theta)$. And so on. After a finite number of iterations, the largest best-response set at θ that cannot be eliminated is obtained by the limit point deception β^*; that is, for each $i \in \mathcal{I}$, it holds that

$$\bigcup_{\bar{\theta} \in \beta_i^*(\theta)} S_i^{\mathcal{M},\bar{\theta}} \subseteq S_i^{\mathcal{M},\theta}. \tag{8}$$

From this perspective, β^* can be viewed as the largest deception that is not refutable given the mechanism $\mathcal{M}$. To see that f satisfies IM if it is rationalizable implementable by the mechanism $\mathcal{M}$, fix any $\theta, \theta' \in \Theta$ such that $f(\theta) \neq f(\theta')$. Let us show that the intersection $\beta^*(\theta) \cap \beta^*(\theta')$ is empty. To see it, suppose that it is not empty. This implies that for each player i, it holds that $\theta(i) \in \beta_i^*(\theta) \cap \beta_i^*(\theta')$. Since $\mathcal{M}$ rationalizable implements f, it follows from (8) that the intersection $S_i^{\mathcal{M},\theta} \cap S_i^{\mathcal{M},\theta'}$ is nonempty for every player i, which leads to the contradiction that $f(\theta) \neq f(\theta')$.

The following theorem shows that IM is necessary and sufficient for rationalizable implementation.

Theorem 1. $f : \Theta \mapsto Y$ is rationalizably implementable if and only if f satisfies IM.

For the proof, see Appendices A and B.

V. Connection With The Partition Based Approach

In this section, we connect our approach with the partition-based approach followed by BMT and Xiong (2022). For this reason, we restrict our discussion below to the case of three or more players. Recall that the implementing conditions in the existing literature rely on the existence of a partition P of Θ. To make a connection between the two approaches, let $\mathcal{B}_f$ denote the set of deceptions that satisfy (7). Our approach can be summarized as follows: An SCF f is rationalizably implementable if, and only if, the deception $\beta^* \in \mathcal{B}_f$. When f satisfies IM, we can construct a partition $P \in \mathcal{P}_f$ by using β^* as follows:

$$P(\theta) = \left\{ \tilde{\theta} \in \Theta \middle| \beta^*(\theta) = \beta^*(\tilde{\theta}) \right\} \tag{9}$$

for all $\theta \in \Theta$. If f satisfies IM, it can be checked that P is finer than P_f.[14]

Let us first connect our approach with that used by BMT. These authors discuss the role of partition in their characterization result. In particular, they show that the required partition must be as fine as P_f and as coarse as the partition obtained by their Lemma 1, which BMT call "pairwise inclusion property" (See BMT, p. 1266 for a discussion). [15] BMT show that f is rationalizably implementable if f satisfies the NWA condition and Maskin monotonicity and, moreover, it is responsive.[16] When f satisfies these conditions, it can be shown that the truth-telling deception, $\beta^* = \beta^0 = \beta^t$.

Xiong (2022) shows that rationalizable implementation of an SCF is equivalent to the Strict Event Monotonicity** (SEM**). We show below that SEM** is equivalent to IM.

[14]Formal arguments can be found in Appendix C, where we prove that IM is equivalent to the necessary and sufficient conditions of Xiong (2022).

[15]Indeed, BMT argue that the pairwise inclusion property is insufficient to pin down the partition by stating"We finally observe that the partition P may yet have to be coarser than is indicated by the pairwise inclusion property", BMT, p. 1266.

[16]f is responsive provided that for all $\theta, \theta' \in \Theta$ such that $\theta \neq \theta'$, it holds that $f(\theta) \neq f(\theta')$.

Theorem 2. Assume that $I \geq 3$ and that that $\mathcal{I}^{\Theta} \neq \emptyset$. f satisfies IM if and only if it is SEM**.

The proof of the above theorem can be found in Appendix C. Let us briefly discuss it below. When f satisfies IM, the partition used to show that f is SEM** is defined in (9). For the converse result, we show that the partition P appearing in SEM** specifies a deception β^P such that $\mathcal{R}(\beta^P) = \beta^P$ and $\beta^P \in \mathcal{B}_f$. More precisely, β^P is defined by:

$$\beta_i^P(\theta) = \begin{cases} P(\theta) & \text{if } i \in \mathcal{I}^{P(\theta)} \\ \Theta & \text{if } i \notin \mathcal{I}^{P(\theta)}. \end{cases} \tag{10}$$

where $i \in \mathcal{I}^{P(\theta)}$ provided that $P(\theta) \bigcap \Theta_i^f = \emptyset$; otherwise, $i \notin \mathcal{I}^{P(\theta)}$. When f is SEM**, the limit deception β^* of our iterative procedure is explicitly defined by (10).

In summary, the fundamental novelty of our approach is to uncover the mapping $\mathcal{R}$, which is implicitly used in the existing characterizations. This mapping plays a critical role in the analysis of two-player implementation problems. Indeed, when we focus on partitions in $\mathcal{P}_f$, we implicitly restrict the set of deceptions in $\mathcal{B}_f$ to the set $\mathcal{B}_f(\mathcal{P}_f) = \left\{\beta \in \mathcal{B}_f \middle| \beta = \beta^P \text{ for some } P \in \mathcal{P}_f\right\}$. When there are three or more players, this restriction is without loss of generality. The reason is that for any partition $P \in \mathcal{P}_f$ such that $\beta^P \in \mathcal{E}(\mathcal{R})$, implementation requires that $\beta^0 \subseteq \beta^P$. It is clear from the definition of β^0 that this constraint is vacuously satisfied, when there are three or more players.[17] However, this is not the case for studying two-player implementation problems. Indeed, in what follows, we present an example in which there does not exist any partition $P \in \mathcal{P}_f$ such that $\beta^0 \subseteq \beta^P \in \mathcal{E}(\mathcal{R})$, though f is rationalizably implementable (that is, f satisfies IM with respect to $\beta^* = \beta^0$).

The example has two players, denoted by 1 and 2, three states, denoted by θ, θ' and θ'', and six outcomes, denoted by a, b, c, d, e and f. Players' utilities from pure

[17] Indeed, $\beta^0 = \beta^{\underline{P}}$.

outcomes are summarized in the table below, where $\varepsilon \in \left(\frac{1}{2}, 1\right)$.

	$u_1(\cdot,\theta)$	$u_2(\cdot,\theta)$	$u_1(\cdot,\theta')$	$u_2(\cdot,\theta')$	$u_1(\cdot,\theta'')$	$u_2(\cdot,\theta'')$
a	1	$-(1-\varepsilon)$	1	-1	1	$-(1-\varepsilon)$
b	0	0	0	0	0	0
c	-1	1	$-(1-\varepsilon)$	1	$-(1-\varepsilon)$	1
d	1	-2	-2	-1	-2	-2
e	2	-3	-2	-2	-2	-3
f	-3	-3	-3	3	-3	-3

The planner wants to implement f, which is defined by

$$f(\theta) = f(\theta') = \{b\} \text{ and } f(\theta'') = \{a\}.$$

It can be easily checked that f is NWA. Let us note that P_f is such that $P_f(\theta) = P_f(\theta') = \{\theta, \theta'\}$ and $P_f(\theta'') = \{\theta''\}$. Also, the finest partition $\underline{P}$ is $\underline{P}(\theta) = \{\theta\}$, $\underline{P}(\theta') = \{\theta'\}$ and $\underline{P}(\theta'') = \{\theta''\}$. It can be checked that $\mathcal{P}_f = \{P_f, \underline{P}\}$. In Appendix **??**, we show that f is rationalizably implementable.

Finally, let us show that there does not exist any partition $P \in \mathcal{P}_f$ such that $\beta^0 \subseteq \beta^P \in \mathcal{E}(\mathcal{R})$. To this end, we need to consider only the partition β^{P_f}.[18] Observe that, by construction, $\beta^0 \subseteq \beta^P_f$. We show that $\beta^{P_f} \notin \mathcal{E}(\mathcal{R})$. Assume, to the contrary, that $\beta^{P_f} \in \mathcal{E}(\mathcal{R})$. This implies that for each $i \in \mathcal{I}$, it holds that $P_f(\theta) \bigcap \mathcal{R}_i^{\beta_f^P}(\theta'') = \emptyset$.

Note that, by construction, $u_1(\cdot,\theta') = u_1(\cdot,\theta'')$ and $u_2(\cdot,\theta) = u_2(\cdot,\theta'')$. This implies that $SL_1(f(\theta),\theta') \subseteq L_1(f(\theta),\theta'')$ and $SL_2(f(\theta),\theta) \subseteq L_2(f(\theta),\theta'')$. Thus, in our environment, it is equivalent to $L_1(f(\theta),\theta') \subseteq L_1(f(\theta),\theta'')$ and $\theta' \in P_f(\theta)$ and to $L_2(f(\theta),\theta) \subseteq L_2(f(\theta),\theta'')$ and $\theta \in P_f(\theta)$. Applying (5) under the specification that $E_1 = E_2 = P_f(\theta)$, we can conclude that for each $i \in \mathcal{I}$, it holds that $P_f(\theta) \subseteq \mathcal{R}_i^{\beta_f^P}(\theta'')$, yielding a contradiction.[19]

[18]The reason is that, by construction, $\beta_1^0(\theta) = \{\theta, \theta'\}$ but $\beta_1^{\underline{P}} = \{\theta\}$.

[19]To see it, we need to set $\tilde{\theta} = \hat{\theta} = \theta'$ for player 1 and $\tilde{\theta} = \hat{\theta} = \theta$ for player 2, and observe that

Appendices

A. Proof of "only if" part of Theorem 1

Suppose that $\mathcal{M}$ implements f in rationalizable strategies. To save writing, we show below that f satisfies IM when $I = 2$. The proof for the case $I \geq 3$ is available upon request. Thus, let us assume that $I = 2$. To show that f satisfies IM, we need the following useful results and notation.

For all $i \in \mathcal{I}$, let λ_i be any player i's belief. The support of λ_i is defined by $supp(\lambda_i) = \{m_{-i} \in M_{-i} | \lambda_i(m_{-i}) > 0\}$.

Lemma 2. For all $i \in \mathcal{I}$ and all $\theta \in \Theta$, there exists $\lambda_i^\theta \in \Delta\left(S_{-i}^{\mathcal{M},\theta}\right)$ such that for all $m_i \in S_i^{\mathcal{M},\theta}$, m_i is a best-response to λ_i^θ at θ.

Proof. Take any $\theta \in \Theta$ and any $i \in \mathcal{I}$. Since f is rationalizably implementable by $\mathcal{M}$, it follows that $S^{\mathcal{M},\theta} \neq \emptyset$ and $f(\theta) = g(m)$ for all $m \in S^{\mathcal{M},\theta}$.

Fix any $m_i \in S_i^{\mathcal{M},\theta}$. Then, m_i is a best-response to some $\lambda_i^{m_i,\theta} \in \Delta\left(S_{-i}^{\mathcal{M},\theta}\right)$ at θ. Let $\lambda_i^{m_i,\theta} = \lambda_i^\theta$. Fix any $m_i^* \in S_i^{\mathcal{M},\theta}$. Since f is rationalizably implementable by $\mathcal{M}$, we have that

$$\begin{aligned} u_i(f(\theta),\theta) &= \sum_{m_{-i}\in M_{-i}} \lambda_i^\theta(m_{-i})\, u_i(g(m_i^*, m_{-i}),\theta) \\ &\geq \sum_{m_{-i}\in M_{-i}} \lambda_i^\theta(m_{-i})\, u_i(g(m_i', m_{-i}),\theta) \end{aligned}$$

for all $m_i' \in M_i$. Thus, m_i^* is a best-response to λ_i^θ at θ. Since the choice of $m_i^* \in S_i^{\mathcal{M},\theta}$ is arbitrary, the statement follows. ■

To show that f satisfies IM, we show that (7) is satisfied with respect to $\hat{\beta}$, which is define below, and then we show that $\beta^* \subseteq \hat{\beta}$.

For all $i \in \mathcal{I}$, let $\hat{\beta}_i$ be defined, for all $\theta \in \Theta$, by

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$\beta_i^{P_f}(\theta) = \beta_i^{P_f}(\theta') = \{\theta, \theta'\}$ for all $i \in \mathcal{I}$.

$$\hat{\beta}_i(\theta) = \left\{\theta' \in \Theta \middle| supp(\lambda_{-i}^{\theta'}) \subseteq S_i^{\mathcal{M},\theta}\right\}. \tag{11}$$

Let us first show that $\hat{\beta} \in \mathcal{B}^t$. Specifically, by (10), we need to show that $\beta_i^0(\theta) \subseteq \hat{\beta}_i(\theta)$ for all $i \in \mathcal{I}$ and all $\theta \in \Theta$. To this end, fix any $i \in \mathcal{I}$, any $\theta \in \Theta$ and any $\theta' \in \beta_i^0(\theta) = A_i(\theta)$. Let us show that $\theta' \in \hat{\beta}_i(\theta)$. Since $\theta \in \hat{\beta}_i(\theta)$, by (11), let us suppose that $\theta' \neq \theta$. Since $\mathcal{M}$ implements f in rationalizable strategies, it follows that

$$g\left[supp\left(\lambda_{-i}^{\theta'}\right) \times supp\left(\lambda_i^{\theta}\right)\right] \subseteq L_i\left(f\left(\theta\right), \theta\right) \bigcap L_{-i}\left(f\left(\theta'\right), \theta'\right). \tag{12}$$

Fix any $m_i^* \in supp\left(\lambda_{-i}^{\theta'}\right)$. Assume, to the contrary, that m_i^* is not a best-response to λ_i^{θ} at θ. Since $m_i^* \in supp\left(\lambda_{-i}^{\theta'}\right)$ but it is not a best-response to λ_i^{θ} at θ, it follows from (12) that

$$g\left[m_i^* \times supp\left(\lambda_i^{\theta}\right)\right] \subseteq SL_i\left(f\left(\theta\right), \theta\right) \bigcap L_{-i}\left(f\left(\theta'\right), \theta'\right).$$

(2) implies that $\theta' \notin A_i\left(\theta\right)$, which is a contradiction. Thus, we have established that every element of $supp\left(\lambda_{-i}^{\theta'}\right)$ is a best-response to λ_i^{θ} at θ if $\theta' \in \beta_i^0(\theta) = A_i\left(\theta\right)$. Thus, $supp\left(\lambda_{-i}^{\theta'}\right) \subseteq S_i^{\mathcal{M},\theta}$, and so $\theta' \in \hat{\beta}_i(\theta)$, by (11).

Let us now show that (7) is satisfied with respect to $\hat{\beta}$. To this end, take any $\theta, \theta' \in \Theta$ such that $f\left(\theta\right) \neq f\left(\theta'\right)$. We show that $\hat{\beta}\left(\theta\right) \bigcap \hat{\beta}\left(\theta'\right) = \emptyset$. Assume, to the contrary, that $\hat{\beta}\left(\theta\right) \bigcap \hat{\beta}\left(\theta'\right) \neq \emptyset$. This implies that for all $i \in \mathcal{I}$, there exists $\theta_i \in \Theta$ such that $\theta_i \in \hat{\beta}_i\left(\theta\right) \bigcap \hat{\beta}_i\left(\theta'\right)$. Since $\theta_i \in \hat{\beta}_i\left(\theta\right) \bigcap \hat{\beta}_i\left(\theta'\right)$ for all $i \in \mathcal{I}$, it follows from (11) that $S_i^{\mathcal{M},\theta} \bigcap S_i^{\mathcal{M},\theta'} \neq \emptyset$ for all $i \in \mathcal{I}$, so that $S^{\mathcal{M},\theta} \bigcap S^{\mathcal{M},\theta'} \neq \emptyset$. Since $\mathcal{M}$ implements f in rationalizable strategies, we have that $f\left(\theta\right) = f\left(\theta'\right)$, which is a contradiction.

Next, let us show that $\beta^* \subseteq \hat{\beta}$. To this end, we first show that $\mathcal{R}\left(\hat{\beta}\right) = \hat{\beta}$. The following lemmata establishes that $\mathcal{R}\left(\hat{\beta}\right) \subseteq \hat{\beta}$.

Lemma 3. For all $i \in \mathcal{I}$, all $\theta, \theta', \tilde{\theta} \in \Theta$, all $\theta_{-i} \in \Theta_{-i}$ and all $\beta \in \mathcal{B}^t$, if $\beta \subseteq \hat{\beta}$,

$\theta_{-i} \in \beta_{-i}\left(\tilde{\theta}\right)$, and either

$$\left[\begin{array}{c} \theta' \in \beta_i\left(\tilde{\theta}\right),\ L_i\left(f\left(\theta_{-i}\right),\theta_{-i}\right) \subseteq L_i\left(f\left(\tilde{\theta}\right),\theta\right) \\ \text{and}\ \tilde{\theta} \notin \Theta_i^f \end{array}\right] \tag{13}$$

or

$$\tilde{\theta} \in \Theta_i^f,$$

then every element of $supp\left(\lambda_{-i}^{\theta'}\right)$ is a best-response to $\lambda_i^{\theta_{-i}}$ at θ.

Proof. Fix any $i \in \mathcal{I}$, any $\theta, \theta', \tilde{\theta} \in \Theta$, any $\theta_{-i} \in \Theta_{-i}$ and any $\beta \in \mathcal{B}^t$ such that $\beta \subseteq \hat{\beta}$ and $\theta_{-i} \in \beta_{-i}\left(\tilde{\theta}\right)$. Since $\theta_{-i} \in \beta_{-i}\left(\tilde{\theta}\right)$ and since $\beta\left(\tilde{\theta}\right) \subseteq \hat{\beta}\left(\tilde{\theta}\right)$, it follows from (11) that $supp\left(\lambda_i^{\theta_{-i}}\right) \subseteq S_{-i}^{\mathcal{M},\tilde{\theta}}$. We proceed according to whether (13) holds or $\tilde{\theta} \in \Theta_i^f$.

Case 1: (13) holds

Since $\theta' \in \beta_i\left(\tilde{\theta}\right)$ and since $\beta_i\left(\tilde{\theta}\right) \subseteq \hat{\beta}_i\left(\tilde{\theta}\right)$, it follows from (11) that $supp\left(\lambda_{-i}^{\theta'}\right) \subseteq S_i^{\mathcal{M},\tilde{\theta}}$. Therefore, we have that $supp\left(\lambda_{-i}^{\theta'}\right) \times supp\left(\lambda_i^{\theta_{-i}}\right) \subseteq S^{\mathcal{M},\tilde{\theta}}$. Since $\mathcal{M}$ implements f, it follows that

$$g\left[supp\left(\lambda_{-i}^{\theta'}\right) \times supp\left(\lambda_i^{\theta_{-i}}\right)\right] = f\left(\tilde{\theta}\right). \tag{14}$$

Finally, fix any $m_i^* \in supp\left(\lambda_{-i}^{\theta'}\right)$. Assume, to the contrary, that m_i^* is not a best-response to $\lambda_i^{\theta_{-i}}$ at θ. It follows that there exists $\tilde{m}_i \in M_i$ such that

$$\sum_{m_{-i} \in M_{-i}} \lambda_i^{\theta_{-i}}\left(m_{-i}\right) u_i\left(g\left(\tilde{m}_i, m_{-i}\right), \theta\right) > \sum_{m_{-i} \in M_{-i}} \lambda_i^{\theta_{-i}}\left(m_{-i}\right) u_i\left(g\left(m_i^*, m_{-i}\right), \theta\right).$$

Since $g\left(m_i^*, m_{-i}\right) = f\left(\tilde{\theta}\right)$ for all $m_{-i} \in supp\left(\lambda_i^{\theta_{-i}}\right)$, by (14), it follows that

$$\sum_{m_{-i} \in M_{-i}} \lambda_i^{\theta_{-i}}\left(m_{-i}\right) u_i\left(g\left(\tilde{m}_i, m_{-i}\right), \theta\right) > u_i\left(f\left(\tilde{\theta}\right), \theta\right). \tag{15}$$

The lottery $\left(\sum_{m_{-i} \in M_{-i}} \lambda_i^{\theta_{-i}}\left(m_{-i}\right) g\left(\tilde{m}_i, m_{-i}\right)\right) \notin L_i\left(f\left(\theta_{-i}\right), \theta_{-i}\right)$ because $L_i\left(f\left(\theta_{-i}\right), \theta_{-i}\right) \subseteq$

$L_i\left(f\left(\tilde{\theta}\right),\theta\right)$, by our initial supposition that (13) holds. However, since $\mathcal{M}$ implements f, Lemma 2 implies that it must be the case that $\left(\sum_{m_{-i}\in M_{-i}}\lambda_i^{\theta_{-i}}\left(m_{-i}\right)g\left(\tilde{m}_i,m_{-i}\right)\right)\in$ $L_i\left(f\left(\theta_{-i}\right),\theta_{-i}\right)$, which is a contradiction.

Case 2: $\tilde{\theta}\in\Theta_i^f$

Then, $S_i^{\mathcal{M},\tilde{\theta}}=M_i$. Recall that $supp\left(\lambda_i^{\theta_{-i}}\right)\subseteq S_{-i}^{\mathcal{M},\tilde{\theta}}$. Since $\mathcal{M}$ implements f, it follows that

$$g\left[M_i\times supp\left(\lambda_i^{\theta_{-i}}\right)\right]=f\left(\tilde{\theta}\right).$$

Therefore, every $m_i\in supp\left(\lambda_{-i}^{\theta'}\right)$ is a best-response to $\lambda_i^{\theta_{-i}}$ at θ. ■

To proceed further, we need additional notation. For all $i\in\mathcal{I}$, all $\theta\in\Theta$ and all $\beta\in\mathcal{B}^t$, let us define the set $\Psi_i^\beta(\mathcal{M},\theta)$ by

$$\Psi_i^\beta(\mathcal{M},\theta)=\bigcup_{\theta'\in\mathcal{R}_i^\beta(\theta)}supp(\lambda_{-i}^{\theta'}).\tag{16}$$

Lemma 4. For all $\beta\in\mathcal{B}^t$ and all $\theta\in\Theta$, if $\beta\subseteq\hat{\beta}$, then $\Psi^\beta(\mathcal{M},\theta)\subseteq S^{\mathcal{M},\theta}$.

Proof. Take any $\theta\in\Theta$ and any $\beta\in\mathcal{B}^t$ such that $\beta\subseteq\hat{\beta}$. To show that $\Psi^\beta(\mathcal{M},\theta)\subseteq S^{\mathcal{M},\theta}$, it suffices to show that the set $\Psi^\beta(\mathcal{M},\theta)$ satisfies the best-response property at θ; that is, it suffices to show that for all $i\in\mathcal{I}$ and all $m_i^*\in\Psi_i^\beta(\mathcal{M},\theta)$, there exists $\lambda_i^{m_i^*}\in\Delta\left(M_{-i}\right)$ such that $\lambda_i^{m_i^*}\left(m_{-i}\right)>0\implies m_{-i}\in\Psi_{-i}^\beta(\mathcal{M},\theta)$, and m_i^* is a best-response to $\lambda_i^{m_i^*}$ at θ.

Fix any $i\in\mathcal{I}$ and any $m_i^*\in\Psi_i^\beta(\mathcal{M},\theta)$. Since $m_i^*\in\Psi_i^\beta(\mathcal{M},\theta)$, (16) implies that there exists $E_i\in 2^\Theta\backslash\{\emptyset\}$ such that $\theta'\in E_i\subseteq\mathcal{R}_i^\beta(\theta)$. (5) implies that there exists $E_{-i}\in 2^\Theta\backslash\{\emptyset\}$ such that for each $\ell\in\mathcal{I}$ and each $\theta_\ell\in E_\ell$, there exist $\tilde{\theta}\in E_\ell$ and $\theta_{-\ell}\in\beta_{-\ell}\left(\tilde{\theta}\right)\cap E_{-\ell}$ such that $\theta_\ell\in\beta_\ell\left(\tilde{\theta}\right)$ and either $\tilde{\theta}\in\Theta_\ell^f$ or $L_\ell\left(f\left(\theta_{-\ell}\right),\theta_{-\ell}\right)\subseteq L_\ell\left(f\left(\tilde{\theta}\right),\theta\right)$. By (5), it also follows that $E_{-i}\subseteq\mathcal{R}_{-i}^\beta(\theta)$, and so $\theta_{-i}\in\mathcal{R}_{-i}^\beta(\theta)$. Since $\theta_{-i}\in\mathcal{R}_{-i}^\beta(\theta)$, (16) implies that $supp\left(\lambda_i^{\theta_{-i}}\right)\subseteq\Psi_{-i}^\beta(\mathcal{M},\theta)$. Moreover, since $\beta\subseteq\hat{\beta}$

and $\theta_{-i} \in \beta_{-i}\left(\tilde{\theta}\right)$, and since, moreover, either $\tilde{\theta} \in \Theta_i^f$ or

$$\theta' \in \beta_i\left(\tilde{\theta}\right),\ L_i\left(f\left(\theta_{-i}\right), \theta_{-i}\right) \subseteq L_i\left(f\left(\tilde{\theta}\right), \theta\right) \text{ and } \tilde{\theta} \notin \Theta_i^f,$$

it follows from Lemma 3 that every element of $supp\left(\lambda_{-i}^{\theta'}\right)$ is a best-response to $\lambda_i^{\theta_{-i}}$ at θ. Thus, m_i^* is a best-response to $\lambda_i^{\theta_{-i}}$ at θ and $supp\left(\lambda_i^{\theta_{-i}}\right) \subseteq \Psi_{-i}^{\beta}(\mathcal{M}, \theta)$, as we sought. ■

Lemma 5. For all $\beta \in \mathcal{B}^t$, if $\beta \subseteq \hat{\beta}$, then $\mathcal{R}(\beta) \subseteq \hat{\beta}$.

Proof. Fix any $\beta \in \mathcal{B}^t$ such that $\beta \subseteq \hat{\beta}$. Fix any $i \in \mathcal{I}$ and any $\theta, \theta' \in \Theta$ such that $\theta' \in \mathcal{R}_i^{\beta}(\theta)$. We show that $\theta' \in \hat{\beta}_i(\theta)$. (16) implies that $supp(\lambda_{-i}^{\theta'}) \subseteq \Psi_i^{\beta}(\mathcal{M}, \theta)$. Since $\beta \subseteq \hat{\beta}$, Lemma 4 implies that $\Psi_i^{\beta}(\mathcal{M}, \theta) \subseteq S_i^{\mathcal{M},\theta}$. Thus, $supp(\lambda_{-i}^{\theta'}) \subseteq S_i^{\mathcal{M},\theta}$. By definition of $\hat{\beta}$, in (11), it follows that $\theta' \in \hat{\beta}_i(\theta)$, as we sought. ■

Since $\hat{\beta} \subseteq \hat{\beta}$, Lemma 5 implies that $\mathcal{R}\left(\hat{\beta}\right) \subseteq \hat{\beta}$. Since part (i) of Lemma 1 implies that $\mathcal{R}\left(\hat{\beta}\right) \supseteq \hat{\beta}$, it follows that $\mathcal{R}\left(\hat{\beta}\right) = \hat{\beta}$, and so $\hat{\beta} \in \mathcal{E}(\mathcal{R})$. Since $\{\beta^*\} = \min \mathcal{E}(\mathcal{R})$, by part (iii) of Lemma 1, we have that $\beta^* \subseteq \hat{\beta}$, as we sought. Thus, f satisfies IM.

B. Proof of "if" part of Theorem 1

Suppose that $f : \Theta \to Y$ satisfies IM. Below we provide the proof for $I = 2$. The reason is that the case $I \geq 3$ can be proved similarly. The details are available from authors. However, indirect arguments can be found in Section **??**. Therefore, in what follows, we suppose that $I = 2$.

For each $i \in \mathcal{I}$, recall the definition of Θ_i^f in (1). The complement of Θ_i^f is denoted by $\bar{\Theta}_i^f$. For each $i \in \mathcal{I}$, $y_i : \bar{\Theta}_i^f \to Y$ is a function such that for each $\theta \in \bar{\Theta}_i^f$,

$$y_i(\theta) \in SL_i(f(\theta), \theta). \tag{17}$$

Given the set $\{y_i(\theta)\}_{\theta \in \bar{\Theta}_i^f}$, we define the average lotteries by setting

$$\underline{y}_i = \frac{1}{\left|\bar{\Theta}_i^f\right|} \sum_{\theta \in \bar{\Theta}_i^f} y_i(\theta) \text{ and } \underline{y} = \frac{1}{I} \sum_{i \in \mathcal{I}} \underline{y}_i. \tag{18}$$

For each $i \in \mathcal{I}$, $y_i^* : \bar{\Theta}_i^f \to Y$ is a function defined, for all $\theta \in \Theta$ by

$$y_i^*(\theta) \in \arg\max_{y \in Y} u_i(y, \theta). \tag{19}$$

For each $\theta \in \Theta$, let us define $\mathcal{I}^\theta$ by $\mathcal{I}^\theta = \{i \in \mathcal{I} | SL_i(f(\theta), \theta) \neq \emptyset\}$ and $\mathcal{I}^\Theta$ by $\mathcal{I}^\Theta = \bigcap_{\theta \in \Theta} \mathcal{I}^\theta$. To save notation, we omit below the proof for the trivial case $\mathcal{I}^\Theta = \emptyset$.[20]

The following results will be used in constructing $\mathcal{M}$.

Lemma 6. For all $i \in \mathcal{I}$ and all $\theta \in \bar{\Theta}_i^f$, $u_i(y_i^*(\theta), \theta) > u_i(\underline{y}, \theta)$.

Proof. Formal arguments can be found in Bergemann et al (2011; Lemma 2, p. 1260). ■

Lemma 7. For each $i \in \mathcal{I}$, there exists a function $z_i : \Theta \times \Theta \to Y$ such that for each $(\theta, \theta') \in \bar{\Theta}_i^f \times \bar{\Theta}_i^f$,

$$u_i(f(\theta'), \theta') > u_i(z_i(\theta, \theta'), \theta'), \tag{20}$$

and for $\theta \neq \theta'$,

$$u_i(z_i(\theta, \theta'), \theta) > u_i(z_i(\theta', \theta'), \theta). \tag{21}$$

Proof. Formal arguments can be found in Bergemann et al (2011; Lemma 2, p. 1260). ■

Whereas the above results will be used in the construction of both Rule 3 and Rule 4, the following result will be used in the construction of Rule 2.

[20] The reason is that f is a constant function when $\mathcal{I}^\Theta = \emptyset$. See Appendix A of Jain et al. (2021).

Lemma 8. For all $\theta, \theta', \theta'' \in \Theta$, if $(\theta', \theta'') \notin \beta_i^*\left(\bar{\theta}\right) \times \beta_{-i}^*\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \Theta$, then:

(i) There exists $e(\theta', \theta'') \in Y$ such that

$$e(\theta', \theta'') \in SL_i(f(\theta''), \theta'') \cap SL_{-i}(f(\theta'), \theta'). \tag{22}$$

(ii) For all $i \in \mathcal{I}$, if $\theta \in \bar{\Theta}_i^f$, then there exists $y_i^\theta(\theta', \theta'') \in Y$ such that

$$u_i\left(y_i^\theta(\theta', \theta''), \theta\right) > u_i(e(\theta', \theta''), \theta) \tag{23}$$

$$y_i^\theta(\theta', \theta'') \in SL_i(f(\theta''), \theta''). \tag{24}$$

Proof. Fix any $\theta, \theta', \theta'' \in \Theta$. Suppose that $(\theta', \theta'') \notin \beta_i^*\left(\bar{\theta}\right) \times \beta_{-i}^*\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \Theta$. Since $\beta^* \in \mathcal{E}(\mathcal{R})$, we have that $(\theta', \theta'') \notin \mathcal{R}_i^{\beta^*}\left(\bar{\theta}\right) \times \mathcal{R}_{-i}^{\beta^*}\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \Theta$. Since $\theta'' \in \mathcal{R}_{-i}^{\beta^*}(\theta'')$, it holds that $\theta' \notin \mathcal{R}_i^{\beta^*}(\theta'')$. Part (i) of Lemma 1 implies that $\beta_i^*(\theta'') \subseteq \mathcal{R}_i^{\beta^*}(\theta'')$. Since $\theta' \notin \mathcal{R}_i^{\beta^*}(\theta'')$, it follows that $\theta' \notin \beta_i^*(\theta'')$. Since $\beta^* \equiv \sup_{k \geq 0} \beta^k$ and since $\beta_i^0 \subseteq \beta_i^*$, by (6), it follows that $\theta' \notin \beta_i^0(\theta'')$. Since $\beta_i^0(\theta'') = A_i(\theta'')$, by (10), it follows from (2) that $x \in SL_i(f(\theta''), \theta'') \cap L_{-i}(f(\theta'), \theta')$ for some $x \in Y$. Similarly, we have that $\theta'' \notin A_{-i}(\theta')$, and so $y \in L_i(f(\theta''), \theta'') \cap SL_{-i}(f(\theta'), \theta')$ for some $y \in Y$. Hence, there exists a small, but positive, number $p \in (0, 1)$ such that $z = (px + (1-p)y) \in SL_i(f(\theta''), \theta'') \cap SL_{-i}(f(\theta'), \theta')$. Define

$$e(\theta', \theta'') = (1-\varepsilon) z + \varepsilon \underline{y} \tag{25}$$

with $\underline{y}$ as defined in (18). Since Θ is finite, we can find a sufficiently small, but positive, $\varepsilon > 0$ such that for all $\theta', \theta'' \in \Theta$, $e(\theta', \theta'') \in SL_i(f(\theta''), \theta'') \cap SL_{-i}(f(\theta'), \theta')$, which proves (22).

Fix any $i \in \mathcal{I}$. Suppose $\theta \in \bar{\Theta}_i^f$. Define

$$y_i^\theta(\theta', \theta'') = (1-\varepsilon) z + \varepsilon y_i^*(\theta), \tag{26}$$

with $y_i^*(\theta)$ as defined in (19).

Since Θ and $\bar{\Theta}_i^f$ are finite, we can find a sufficiently small, but positive, $\varepsilon > 0$ such that for all $\theta', \theta'' \in \Theta$ and all $\theta \in \bar{\Theta}_i^f$, $e(\theta', \theta'') \in SL_i(f(\theta''), \theta'') \cap SL_{-i}(f(\theta'), \theta')$, which establishes (22), and $y_i^\theta(\theta', \theta'') \in SL_i(f(\theta''), \theta'')$, which establishes (24). Since the only difference between $e(\theta', \theta'')$ and $y_i^\theta(\theta', \theta'')$ is that $\underline{y}$ in (25) is replaced by $y_i^*(\theta)$ in (26), and since Lemma 6 implies that $u_i(y_i^*(\theta), \theta) > u_i(\underline{y}, \theta)$, it follows that (23) is proved for all $\theta, \theta', \theta''$ such that $\theta \in \bar{\Theta}_i^f$. ■

For all $i \in \mathcal{I}$ and all $\theta, \theta' \in \Theta$, let $D_i(\theta', \theta)$ be defined by

$$D_i(\theta', \theta) = \{y \in Y | y \in L_i(f(\theta'), \theta') \cap SU_i(f(\theta'), \theta)\}, \tag{27}$$

where for all $x \in Y$, $SU_i(x, \theta) = \{y \in Y | u_i(x, \theta) < u_i(y, \theta)\}$. For every $i \in \mathcal{I}$ and every $\theta \in \bar{\Theta}_i^f$, let $\alpha_i^\theta : \Theta \times \Theta \to Y$ be a function such that for every $\theta', \theta'' \in \Theta$,

$$\alpha_i^\theta(\theta', \theta'') \in L_i(f(\theta'), \theta') \cap SU_i(f(\theta''), \theta) \tag{28}$$

if $L_i(f(\theta'), \theta') \cap SU_i(f(\theta''), \theta) \neq \emptyset$.

For every $i \in \mathcal{I}$ and every $\theta, \theta' \in \Theta$, let $\bar{D}_i(\theta', \theta)$ be any finite subset of $D_i(\theta', \theta)$ satisfying the following requirements:

(i) If $\theta, \theta' \in \bar{\Theta}_i^f$ and $z_i(\theta, \theta') \in D_i(\theta', \theta)$, then $z_i(\theta, \theta') \in \bar{D}_i(\theta', \theta)$.

(ii) For all $\theta'' \in \Theta$, if $\theta \in \bar{\Theta}_i^f$, $(\theta', \theta'') \notin \beta_i^*(\bar{\theta}) \times \beta_{-i}^*(\bar{\theta})$ for all $\bar{\theta} \in \Theta$ and $y_i^\theta(\theta', \theta'') \in D_i(\theta'', \theta)$, then $y_i^\theta(\theta', \theta'') \in \bar{D}_i(\theta'', \theta)$.

(iii) For all $\theta'' \in \Theta$, if $\theta \in \bar{\Theta}_i^f$ and $L_i(f(\theta'), \theta') \cap SU_i(f(\theta''), \theta) \neq \emptyset$, then $\alpha_i^\theta(\theta', \theta'') \in \bar{D}_i(\theta', \theta)$.

For every $i \in \mathcal{I}$ and every $\theta, \theta' \in \Theta$, fix any $\bar{D}_i(\theta', \theta)$ satisfying the above requirements. Let $B_i(\theta', \theta)$ be defined by

$$B_i(\theta', \theta) = \begin{cases} \arg\max_{y \in \bar{D}_i(\theta', \theta)} u_i(y, \theta) & \text{if } D_i(\theta', \theta) \neq \emptyset \\ f(\theta') & \text{otherwise.} \end{cases} \tag{29}$$

Note that $B_i(\theta', \theta) \neq \emptyset$ for all $\theta, \theta' \in \Theta$ and all $i \in \mathcal{I}$. Furthermore, for every $i \in \mathcal{I}$, let $x_i : \Theta \times \Theta \to Y$ be a function such that

$$x_i(\theta', \theta) \in B_i(\theta', \theta) \tag{30}$$

for all $\theta', \theta \in \Theta$.

Lemma 9. For all $i \in \mathcal{I}$ and all $\theta, \theta' \in \Theta$, if $\theta, \theta' \in \bar{\Theta}_i^f$, then $u_i(x_i(\theta', \theta), \theta) > u_i(z_i(\theta', \theta'), \theta)$.

Proof. Fix any $i \in \mathcal{I}$ and any $\theta, \theta' \in \bar{\Theta}_i^f$. Then, $x_i(\theta', \theta) \in B_i(\theta', \theta)$ by (30). Let us proceed according to whether $D_i(\theta', \theta) = \emptyset$ or not.

Suppose that $D_i(\theta', \theta) = \emptyset$, so that $x_i(\theta', \theta) = f(\theta')$ by (29). By (20), $u_i(f(\theta'), \theta') > u_i(z_i(\theta, \theta'), \theta')$. This proves the statement if $\theta = \theta'$. Suppose that $\theta \neq \theta'$. By (21), $u_i(z_i(\theta, \theta'), \theta) > u_i(z_i(\theta', \theta'), \theta)$. Since $D_i(\theta', \theta) = \emptyset$ and since $u_i(f(\theta'), \theta') > u_i(z_i(\theta, \theta'), \theta')$, it holds that $u_i(x_i(\theta', \theta), \theta) \geq u_i(z_i(\theta, \theta'), \theta)$, and so $u_i(x_i(\theta', \theta), \theta) > u_i(z_i(\theta', \theta'), \theta)$.

Suppose that $D_i(\theta', \theta) \neq \emptyset$. Then, $\theta \neq \theta'$. Suppose that $u_i(f(\theta'), \theta) \geq u_i(z_i(\theta, \theta'), \theta)$. By (21), $u_i(f(\theta'), \theta) > u_i(z_i(\theta', \theta'), \theta)$. Since $x_i(\theta', \theta) \in B_i(\theta', \theta)$, it follows that $u_i(x_i(\theta', \theta), \theta) \geq u_i(f(\theta'), \theta)$. Therefore, $u_i(x_i(\theta', \theta), \theta) > u_i(z_i(\theta', \theta'), \theta)$. Finally, suppose that $u_i(z_i(\theta, \theta'), \theta) > u_i(f(\theta'), \theta)$. By (20), $u_i(f(\theta'), \theta') > u_i(z_i(\theta, \theta'), \theta')$. Thus, $z_i(\theta, \theta') \in D_i(\theta', \theta)$, by (27), and so $z_i(\theta, \theta') \in \bar{D}_i(\theta', \theta)$, by requirement (i) of the set $\bar{D}_i(\theta', \theta)$. Since $x_i(\theta', \theta) \in B_i(\theta', \theta)$, it follows that $u_i(x_i(\theta', \theta), \theta) \geq u_i(z_i(\theta, \theta'), \theta)$. By (21), $u_i(x_i(\theta', \theta), \theta) > u_i(z_i(\theta', \theta'), \theta)$. ■

Lemma 10. For all $i \in \mathcal{I}$ and all $\theta, \theta', \theta'' \in \Theta$, if $\theta, \theta'' \in \bar{\Theta}_i^f$ and $(\theta', \theta'') \notin \beta_i^*(\bar{\theta}) \times \beta_{-i}^*(\bar{\theta})$ for all $\bar{\theta} \in \Theta$, then $u_i(x_i(\theta'', \theta), \theta) > u_i(e(\theta', \theta''), \theta)$.

Proof. Fix any $i \in \mathcal{I}$ and any $\theta, \theta', \theta'' \in \Theta$ such that $\theta, \theta'' \in \bar{\Theta}_i^f$. Then, $x_i(\theta'', \theta) \in B_i(\theta'', \theta)$ by (30). Suppose that $(\theta', \theta'') \notin \bar{\Psi}_i(\bar{\theta}) \times \bar{\Psi}_{-i}(\bar{\theta})$ for all $\bar{\theta} \in \Theta$. We proceed according to whether $D_i(\theta'', \theta) = \emptyset$ or not.

Suppose that $D_i(\theta'', \theta) = \emptyset$. Thus, $x_i(\theta'', \theta) = f(\theta'')$ by (29). Since $D_i(\theta'', \theta) = \emptyset$ and since (24) holds, it follows that $u_i(x_i(\theta'', \theta), \theta) \geq u_i\left(y_i^\theta(\theta', \theta''), \theta\right)$. Since (23) holds, we conclude that $u_i(x_i(\theta'', \theta), \theta) > u_i(e(\theta', \theta''), \theta)$.

Suppose that $D_i(\theta'', \theta) \neq \emptyset$. Then, $\theta \neq \theta''$. Suppose that $u_i(f(\theta''), \theta) \geq u_i\left(y_i^\theta(\theta', \theta''), \theta\right)$. Since $x_i(\theta'', \theta) \in B_i(\theta'', \theta)$, then $u_i(x_i(\theta'', \theta), \theta) \geq u_i(f(\theta''), \theta)$, and so $u_i(x_i(\theta'', \theta), \theta) \geq u_i\left(y_i^\theta(\theta', \theta''), \theta\right)$. By (23), we have that $u_i(x_i(\theta'', \theta), \theta) > u_i(e(\theta', \theta''), \theta)$. Otherwise, let $u_i\left(y_i^\theta(\theta', \theta''), \theta\right) > u_i(f(\theta''), \theta)$. By (24), it holds that $u_i(f(\theta''), \theta'') > u_i\left(y_i^\theta(\theta', \theta''), \theta''\right)$. Thus, $y_i^\theta(\theta', \theta'') \in D_i(\theta', \theta)$, and so $y_i^\theta(\theta', \theta'') \in \bar{D}_i(\theta', \theta)$, by requirement (ii) of the set $\bar{D}_i(\theta', \theta)$. Since $x_i(\theta'', \theta) \in B_i(\theta'', \theta)$, it follows that $u_i(x_i(\theta', \theta), \theta) \geq u_i\left(y_i^\theta(\theta', \theta''), \theta\right)$ and so, by (23), $u_i(x_i(\theta'', \theta), \theta) > u_i(e(\theta', \theta''), \theta)$. ■

Based on the above results, let us construct $\mathcal{M} = (M, g)$. We define the following countable set of lotteries:

$$\begin{aligned}\mathcal{Y} = & \{z_i(\theta', \theta)\}_{i \in \mathcal{I}, \theta, \theta' \in \bar{\Theta}_i^f} \bigcup \{y_i^*(\theta)\}_{i \in \mathcal{I}, \theta \in \bar{\Theta}_i^f} \bigcup \{B_i(\theta', \theta)\}_{i \in \mathcal{I}, \theta, \theta' \in \Theta} \bigcup \\ & \left\{e(\theta', \theta''), y_i^\theta(\theta', \theta'')\right\}_{i \in \mathcal{I}, \left\{(\theta', \theta'') | (\theta', \theta'') \notin \cup_{\bar{\theta} \in \Theta}\left(\beta_i^*(\bar{\theta}) \times \beta_{-i}^*(\bar{\theta})\right)\right\}},\end{aligned}$$

where the collection $\{z_i(\theta', \theta)\}_{i \in \mathcal{I}, \theta, \theta' \in \bar{\Theta}_i^f}$ has been defined in Lemma 7, the collection $\{y_i^*(\theta)\}_{i \in \mathcal{I}, \theta \in \bar{\Theta}_i^f}$ follows from (23), the collection

$$\left\{e(\theta', \theta''), y_i^\theta(\theta', \theta'')\right\}_{i \in \mathcal{I}, \left\{(\theta', \theta'') | (\theta', \theta'') \notin \cup_{\bar{\theta} \in \Theta}\left(\beta_i^*(\bar{\theta}) \times \beta_{-i}^*(\bar{\theta})\right)\right\}}$$

is established by Lemma 8, and the collection $\{B_i(\theta', \theta)\}_{i \in \mathcal{I}, \theta, \theta' \in \Theta}$ follows from (29).

Each $i \in \mathcal{I}$ plays a strategy $m_i = (m_i^1, m_i^2, m_i^3, m_i^4)$, where $m_i^1 \in \Theta$, $m_i^2 \in \mathbb{Z}_+$, $m_i^3 : \Theta \to \mathcal{Y}$ and $m_i^4 \in \mathcal{Y}$. By construction, M_i is a nonempty countable set for player i. Moreover, the third component of the strategy allows player i to announce a lottery in $\mathcal{Y}$ contingent on other player playing $m^1_{-i} = \theta$. The outcome function also uses $\underline{y}$, as defined in (18). Recall that for all $\theta \in \Theta$, $\mathcal{I}^\theta = \{i \in \mathcal{I} | SL_i(f(\theta), \theta) \neq \emptyset\}$ and let $\mathcal{I}^\Theta = \bigcap_{\theta \in \Theta} \mathcal{I}^\theta$. Thus, $i \in \mathcal{I}^\theta$ if and only if $\theta \in \bar{\Theta}^f_i$, if and only if $\theta \notin \Theta^f_i$. For all $\theta \in \Theta$, let

$$\mathcal{I}^{\beta^*(\theta)} = \left\{ i \in \mathcal{I} | \beta^*_{-i}(\theta) \subseteq \bar{\Theta}^f_i \right\}. \tag{31}$$

Note that if $\mathcal{I}^{\beta^*(\theta)} \neq \emptyset$, then $\theta \in \bar{\Theta}^f_i$ for all $i \in \mathcal{I}^{\beta^*(\theta)}$.

Before defining the outcome function g, let us derive the following useful results.

Lemma 11. If $f : \Theta \longmapsto Y$ satisfies IM, then for all $\theta, \theta', \bar{\theta} \in \Theta$ and all $i \in \mathcal{I}$, if $\theta' \in \beta^*_i(\bar{\theta})$ and $\theta \in \beta^*_{-i}(\bar{\theta})$, then $f(\bar{\theta}) \in L_i(f(\theta), \theta)$.

Proof. Let the premises hold. Take any $\theta, \theta', \bar{\theta} \in \Theta$ and any $i \in \mathcal{I}$. Suppose that $\theta' \in \beta^*_i(\bar{\theta})$ and $\theta \in \beta^*_{-i}(\bar{\theta})$. We show that $u_i(f(\theta), \theta) \geq u_i(f(\bar{\theta}), \theta)$.

Since $\theta \in \beta^*_{-i}(\bar{\theta})$ and since $\theta \in \beta^*_{-i}(\theta)$, it follows that

$$\theta \in \beta^*_{-i}(\bar{\theta}) \bigcap \beta^*_{-i}(\theta). \tag{32}$$

We proceed according to whether $\theta \notin \bar{\Theta}^f_i$ or not.

Suppose that $\theta \in \Theta^f_i$. Lemma 1 implies that $\beta^*_i(\theta) = \Theta$. Since $\theta' \in \beta^*_i(\bar{\theta})$, it follows that

$$\theta' \in \beta^*_i(\bar{\theta}) \bigcap \beta^*_i(\theta). \tag{33}$$

Since (32) and (33) hold, IM implies that $f(\theta) = f(\bar{\theta})$.

Suppose that $\theta \in \bar{\Theta}^f_i$. Since $u_i(f(\bar{\theta}), \theta) > u_i(f(\theta), \theta)$, it follows that $L_i(f(\theta), \theta) \subseteq L_i(f(\bar{\theta}), \theta)$. Let us show that $\beta^*_i(\bar{\theta}) \subseteq \beta^*_i(\theta)$. Since $\beta^* = \mathcal{R}(\beta^*)$, it suffices to show that $\beta^*_i(\bar{\theta}) \subseteq \mathcal{R}^{\beta^*}_i(\theta)$. To this end, let $E_i = \{\theta, \bar{\theta}\}$ and $E_{-i} = \{\theta\}$. It suffices to show that $\beta^*_i(E_i) \subseteq \mathcal{R}^{\beta^*}_i(\theta)$. Let us first consider player $-i$. Then, for all $\theta'' \in \beta^*_{-i}(E_{-i})$, it holds that $\theta \in \beta^*_{-i}(E_{-i})$, $\theta \in \beta^*_i(\theta) \cap \beta^*_i(E_i)$, $\theta'' \in \beta^*_{-i}(\theta)$ and $L_{-i}(f(\theta), \theta) \subseteq L_{-i}(f(\theta), \theta)$. Next, let us consider player i. Fix any $\theta'' \in \beta^*_i(E_i)$.

Let us proceed according to whether $\theta'' \in \beta_i^*\left(\bar{\theta}\right)$ or not.

- Suppose that $\theta'' \in \beta_i^*\left(\bar{\theta}\right)$. By our initial supposition, it holds that $\theta \in \beta_{-i}^*\left(\bar{\theta}\right)$. Moreover, we are also under the assumption that $L_i\left(f\left(\theta\right),\theta\right) \subseteq L_i\left(f\left(\bar{\theta}\right),\theta\right)$. Then, we have that $\bar{\theta} \in \beta_i^*(E_i)$, $\theta \in \beta_{-i}^*\left(\bar{\theta}\right) \cap \beta_{-i}^*(E_{-i})$, $\theta'' \in \beta_i^*\left(\bar{\theta}\right)$ and $L_i\left(f\left(\theta\right),\theta\right) \subseteq L_i\left(f\left(\bar{\theta}\right),\theta\right)$.

- Suppose that $\theta'' \notin \beta_i^*\left(\bar{\theta}\right)$. Then, $\theta'' \in \beta_i^*\left(\theta\right)$. Then, it holds that $\theta \in \beta_i^*(E_i)$, $\theta'' \in \beta_i^*\left(\theta\right)$, $\theta \in \beta_{-i}^*\left(\theta\right) \cap \beta_{-i}^*(E_{-i})$ and $L_i\left(f\left(\theta\right),\theta\right) \subseteq L_i\left(f\left(\theta\right),\theta\right)$.

(5) implies that $\beta_i^*\left(E_i\right) \subseteq \mathcal{R}_i^{\beta^*}\left(\theta\right)$. Thus, $\beta_i^*\left(\bar{\theta}\right) \subseteq \mathcal{R}_i^{\beta^*}\left(\theta\right)$.

Since $\theta' \in \beta_i^*\left(\bar{\theta}\right)$ and $\beta_i^*\left(\bar{\theta}\right) \subseteq \mathcal{R}_i^{\beta^*}\left(\theta\right) = \beta_i^*\left(\theta\right)$, we have that (33) holds. Again, since (32) and (33) hold, IM implies $f\left(\theta\right) = f\left(\bar{\theta}\right)$, which is a contradiction. ■

For all $m \in M$, the outcome $g\left(m\right)$ is defined by the following rules.[21]

Rule 1: If there exists $\bar{\theta} \in \Theta$ such that $(m_i^1)_{i\in\mathcal{I}} \in \beta^*\left(\bar{\theta}\right)$ and $m_i^2 = 0$ for all $i \in \mathcal{I}^{\beta^*\left(\bar{\theta}\right)}$, then

$$g\left(m\right) = f\left(\bar{\theta}\right).$$

Rule 2: If $m_1^2 = m_2^2 = 0$ and $(m_i^1)_{i\in\mathcal{I}} \notin \beta^*\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \Theta$, then

$$g\left(m\right) = e\left(m_i^1, m_{-i}^1\right),$$

[21] To include the case $\mathcal{I}^{\Theta} = \emptyset$, Rule 1 needs to be formulated as follows: If there exists $\bar{\theta} \in \Theta$ such that $\left(m_i^1\right)_{i\in\mathcal{I}} \in \beta^*\left(\bar{\theta}\right)$ and $[m_i^2 = 0$ for all $i \in \mathcal{I}^{\beta^*\left(\bar{\theta}\right)}$ or $\mathcal{I}^{\Theta} = \emptyset]$, then $g\left(m\right) = f\left(\bar{\theta}\right)$. It is easy to see that any $m \in M$ falls into Rule 1 when $\mathcal{I}^{\Theta} = \emptyset$.

where the existence of $e\left(m_i^1, m_{-i}^1\right)$ is established in Lemma 8.

Rule 3: For all $i \in \mathcal{I}$, if for some $\bar{\theta} \in \bar{\Theta}_i^f$, $\left(m_{-i}^1, m_{-i}^2\right) = \left(\bar{\theta}, 0\right)$ and $m_i^2 > 0$, then

$$g(m) = \begin{cases} \left(\frac{m_i^2}{m_i^2+1}\right) m_i^3\left(\bar{\theta}\right) + \left(\frac{1}{m_i^2+1}\right) z_i\left(\bar{\theta}, \bar{\theta}\right) & \text{if } u_i\left(f\left(\bar{\theta}\right), \bar{\theta}\right) \geq u_i\left(m_i^3\left(\bar{\theta}\right), \bar{\theta}\right) \\ z_i\left(\bar{\theta}, \bar{\theta}\right) & \text{otherwise,} \end{cases}$$

where the existence of $z_i\left(\bar{\theta}, \bar{\theta}\right)$ is established in Lemma 7.

Rule 4: In all other cases, an integer game is played: we identify a pivotal player i by requiring that $m_i^2 \geq m_{-i}^2$, and that if $m_i^2 = m_{-i}^2$, then $i < -i$. Then,

$$g(m) = \left(\frac{m_i^2}{m_i^2+1}\right) m_i^4 + \left(\frac{1}{m_i^2+1}\right) \underline{y},$$

where $\underline{y}$ is defined in (18).

To check that g is well-defined, we need only to check that Rule 1 is well-defined. Assume, to the contrary, that there exists $m \in M$ falling into Rule 1 such that for some $\bar{\theta}, \bar{\theta}' \in \Theta$, $(m_i^1)_{i\in\mathcal{I}} \in \beta^*\left(\bar{\theta}\right)$ and $m_i^2 = 0$ for all $i \in \mathcal{I}^{\beta^*\left(\bar{\theta}\right)}$, $(m_i^1)_{i\in\mathcal{I}} \in \beta^*\left(\bar{\theta}'\right)$ and $m_i^2 = 0$ for all $i \in \mathcal{I}^{\beta^*\left(\bar{\theta}'\right)}$, and $f\left(\bar{\theta}\right) \neq f\left(\bar{\theta}'\right)$. Then, $m_i^1 \in \beta_i^*\left(\bar{\theta}\right) \cap \beta_i^*\left(\bar{\theta}'\right)$ for all $i \in \mathcal{I}$, and so $\beta^*\left(\bar{\theta}\right) \cap \beta^*\left(\bar{\theta}'\right) \neq \emptyset$. Since f satisfies IM, we have that $f\left(\bar{\theta}\right) = f\left(\bar{\theta}'\right)$, which is a contradiction.

Let $\bar{m}_i = (\theta, 0, m_i^3, m_i^4) \in M_i$, for all $i \in \mathcal{I}$. Let us first show that $\bar{m} \in NE(\mathcal{M}, \theta)$, that is, for all $i \in \mathcal{I}$,

$$u_i\left(g\left(\bar{m}_i, \bar{m}_{-i}\right), \theta\right) \geq u_i\left(g\left(m_i, \bar{m}_{-i}\right), \theta\right), \tag{34}$$

for all $m_i \in M_i$.

By construction, $\bar{m}$ falls into Rule 1, and so $g(\bar{m}) = f(\theta)$. Fix any $i \in \mathcal{I}$ and any

$m_i \in M_i$. Note that no deviation of i can induce Rule 4. Thus, Rules 1-3 apply.

(A) Suppose that $(m_i, \bar{m}_{-i})$ falls into Rule 2. Then, $g(m) = e\left(m_i^1, \bar{m}_{-i}^1\right)$, and so $e(m_i^1, \theta) \in SL_i(f(\theta), \theta)$, by (22) of Lemma 8.

(B) Suppose that $(m_i, \bar{m}_{-i})$ falls into Rule 3. Suppose that $u_i(f(\theta), \theta) \geq u_i(m_i^3(\theta), \theta)$. Then, $g(m_i, \bar{m}_{-i}) = \left(\frac{m_i^2}{m_i^2+1}\right) m_i^3(\theta) + \left(\frac{1}{m_i^2+1}\right) z_i(\theta, \theta)$. Since $\theta \in \bar{\Theta}_i^f$, the inequality in (20) of Lemma 7 implies that $z_i(\theta, \theta) \in SL_i(f(\theta), \theta)$. Since $u_i(f(\theta), \theta) \geq u_i(m_i^3(\theta), \theta)$, it follows that $g(m_i, \bar{m}_{-i}) \in SL_i(f(\theta), \theta)$. Suppose that $u_i(m_i^3(\theta), \theta) > u_i(f(\theta), \theta)$. Then, $g(m_i, \bar{m}_{-i}) = z_i(\theta, \theta) \in SL_i(f(\theta), \theta)$ given that $\theta \in \bar{\Theta}_i^f$. Therefore, in either case, $g(m_i, \bar{m}_{-i}) \in SL_i(f(\theta), \theta)$

(C) Suppose that $(m_i, \bar{m}_{-i})$ falls into Rule 1. Then, there exists $\bar{\theta}' \in \Theta$ such that $\bar{m}_{-i}^1 \in \beta_{-i}^*\left(\bar{\theta}'\right)$, $m_i^1 \in \beta_i^*\left(\bar{\theta}'\right)$, $m_i^2 = 0$ if $i \in \mathcal{I}^{\beta^*\left(\bar{\theta}'\right)}$, and $g(m_i, \bar{m}_{-i}) = f\left(\bar{\theta}'\right)$. Lemma 11 implies that $u_i(f(\theta), \theta) \geq u_i\left(f\left(\bar{\theta}'\right), \theta\right)$.

Since $i \in \mathcal{I}$ and $m_i \in M_i$ are arbitrary, we conclude that the inequality in (34) is satisfied for all $i \in \mathcal{I}$ and all $m_i \in M_i$. Thus, $\bar{m} \in NE(\mathcal{M}, \theta)$. Since $\bar{m}$ is also a rationalizable strategy profile at θ, it follows that $S^{\mathcal{M},\theta}$ is a nonempty set. According to Definition 1, to complete the proof, we need to show that $m \in S^{\mathcal{M},\theta} \implies g(m) = f(\theta)$. To this end, we need additional notation.

For all $i \in \mathcal{I}$, let M_{-i}^0 be defined by

$$M_{-i}^0 = \left\{ m_{-i} \in M_{-i} | m_{-i}^1 \in \bigcup_{\bar{\theta} \in \Theta} \left(\bigcup_{\beta_{-i}^*\left(\bar{\theta}\right) \cap \Theta_i^f \neq \emptyset} \beta_{-i}^*\left(\bar{\theta}\right) \right) \text{ and } m_{-i}^2 = 0 \right\}. \tag{35}$$

Observe that M_{-i}^0 may be empty. Let $M_{-i}^0 \neq \emptyset$ and let us take any $m_{-i} \in M_{-i}^0$. Then, there exists $\bar{\theta} \in \Theta$ such that $m_{-i}^1 \in \beta_{-i}^*\left(\bar{\theta}\right)$ and $\beta_{-i}^*\left(\bar{\theta}\right) \bigcap \Theta_i^f \neq \emptyset$. Since $\beta_{-i}^*\left(\bar{\theta}\right) \bigcap \Theta_i^f \neq \emptyset$, it follows from (31) that $i \notin \mathcal{I}^{\beta^*\left(\bar{\theta}\right)}$. Part (iv) of Lemma 1 implies that $\beta_i^*\left(\bar{\theta}\right) = \Theta$. Thus, any strategy $(m_i, m_{-i}) \in M_i \times M_{-i}^0$ falls into Rule 1.

For each $i \in \mathcal{I}$, let $\hat{m}_i = (\theta, 0, x_i(\cdot,\theta), y_i^*(\theta)) \in M_i$, where $x_i(\cdot,\theta)$ is defined in (30) and $y_i^*(\theta)$ is defined in (19). The following lemmata will help us to complete the proof.

Lemma 12. For all $i \in \mathcal{I}$, all $\theta \in \bar{\Theta}_i^f$, all $m_i \in M_i$ and all $\lambda_i^\theta \in \Delta\left(M_{-i} \backslash M_{-i}^0\right)$, if $i \in \mathcal{I}^{\beta^*(\theta)}$ and m_i is a best-response to λ_i^θ at θ, then $m_i^2 = 0$.[22]

Proof. Fix any $i \in \mathcal{I}$, any $\theta \in \bar{\Theta}_i^f$, and any $\lambda_i^\theta \in \Delta\left(M_{-i} \backslash M_{-i}^0\right)$ and any $m_i \in M_i$ so that m_i is a best-response to λ_i^θ at θ. Recall that $\mathcal{I}^\Theta \neq \emptyset$. Assume, to the contrary, $m_i^2 > 0$. Since $m_i^2 > 0$ and $i \in \mathcal{I}^{\beta^*(\theta)}$, it follows that for any $m_{-i} \in supp\left(\lambda_i^\theta\right)$, (m_i, m_{-i}) falls either into Rule 3 or into Rule 4. A contradiction of the initial assumption that m_i is a best response to λ_i^θ is derived if we show that $\hat{m}_i = (\theta, \hat{m}_i^2, x_i(\cdot,\theta), y_i^*(\theta))$ strictly dominates m_i.[23]

Case 1: (m_i, m_{-i}) falls into Rule 3

Then, for some $\bar{\theta} \in \bar{\Theta}_i^f$, $\left(m_j^1, m_j^2\right) = \left(\bar{\theta}, 0\right)$ for all $j \in \mathcal{I} \backslash \{i\}$. Observe that if $u_i\left(f\left(\bar{\theta}\right), \bar{\theta}\right) \geq u_i\left(m_i^3\left(\bar{\theta}\right), \bar{\theta}\right)$, then $u_i\left(x_i\left(\bar{\theta},\theta\right), \theta\right) \geq u_i\left(m_i^3\left(\bar{\theta}\right), \theta\right)$ because $x_i\left(\bar{\theta},\theta\right) \in B_i\left(\bar{\theta},\theta\right)$, by (30). Since $\bar{\theta}, \theta \in \bar{\Theta}_i^f$, Lemma 9 implies that $u_i\left(x_i\left(\bar{\theta},\theta\right), \theta\right) > u_i\left(z_i\left(\bar{\theta},\bar{\theta}\right), \theta\right)$. By choosing an appropriate integer $\hat{m}_i^2 > 0$ and by changing m_i into $\hat{m}_i$, player i can induce Rule 3 with $u_i\left(f\left(\bar{\theta}\right), \bar{\theta}\right) \geq u_i\left(x_i\left(\bar{\theta},\theta\right), \bar{\theta}\right)$, obtain $g(\hat{m}_i, m_{-i})$ and be strictly better off at θ since $u_i\left(x_i\left(\bar{\theta},\theta\right), \theta\right) > u_i\left(z_i\left(\bar{\theta},\bar{\theta}\right), \theta\right)$.

Case 2: (m_i, m_{-i}) falls into Rule 4

Suppose that $j \in \mathcal{I}$ is the pivotal player. By definition of $y_i^*(\theta)$ in (19), $u_i(y_i^*(\theta), \theta) \geq u_i\left(m_j^4, \theta\right)$. Since $\theta \in \bar{\Theta}_i^f$, Lemma 6 implies that $u_i(y_i^*(\theta), \theta) > u_i\left(\underline{y}, \theta\right)$. By choosing an appropriate integer $\hat{m}_i^2 > 0$ and by changing m_i into $\hat{m}_i$, player i can induce Rule

[22] This lemma holds only in the case $\mathcal{I}^\Theta \neq \emptyset$.

[23] $\hat{m}_i$ strictly dominates m_i if $u_i(g(\hat{m}_i, m_{-i}), \theta) > u_i(g(m_i, m_{-i}), \theta)$ for all $m_{-i} \in M_{-i}$.

4, obtain $g(\hat{m}_i, m_{-i})$ and be strictly better off at θ since $u_i(y_i^*(\theta), \theta) > u_i(\underline{y}, \theta)$.

Since Θ is finite and since the choice of $m_{-i} \in supp(\lambda_i^\theta)$ is arbitrary, we see that $\hat{m}_i$ strictly dominates m_i by an appropriate choice of $\hat{m}_i^2 > 0$. ■

Lemma 13. For all $i \in \mathcal{I}$, all $\theta \in \bar{\Theta}_i^f$, all $m_i \in M_i$ and all $\lambda_i^\theta \in \Delta(M_{-i})$, if $i \in \mathcal{I}^{\beta^*(\theta)}$ and m_i is a best-response to λ_i^θ at θ, then there exists $m_{-i} \in supp(\lambda_i^\theta)$ such that m_i is a best-response to m_{-i} at θ.[24]

Proof. Fix any $i \in \mathcal{I}$, any $\theta \in \bar{\Theta}_i^f$, any $\lambda_i^\theta \in \Delta(M_{-i})$ and any $m_i \in M_i$ such that m_i is a best-response to λ_i^θ at θ. Assume, to the contrary, that m_i is not a best-response to any $m_{-i} \in supp(\lambda_i^\theta)$ at θ. We proceed according to whether $supp(\lambda_i^\theta) \cap M_{-i}^0 \neq \emptyset$ or not.

Suppose that $supp(\lambda_i^\theta) \cap M_{-i}^0 \neq \emptyset$. Take any $m_{-i} \in supp(\lambda_i^\theta) \cap M_{-i}^0$. By (35), $m_{-i}^2 = 0$ and $m_{-i}^1 \in \bigcap_{j \in \mathcal{I} \setminus \{i\}} \beta_j^*(\bar{\theta})$ for some $\bar{\theta} \in \Theta$ such that $\left(\bigcap_{j \in \mathcal{I} \setminus \{i\}} \beta_j^*(\bar{\theta})\right) \bigcap \Theta_i^f \neq \emptyset$. Since $\left(\bigcap_{j \in \mathcal{I} \setminus \{i\}} \beta_j^*(\bar{\theta})\right) \bigcap \Theta_i^f \neq \emptyset$, it follows from (31) that $i \notin \mathcal{I}^{\beta^*(\bar{\theta})}$. Lemma **??** implies that $\beta_i^*(\bar{\theta}) = \Theta$. Thus, (m_i, m_{-i}) falls into Rule 1 and $g(m_i, m_{-i}) = f(\bar{\theta})$. Since i can induce only Rule 1 by changing his strategy, it follows that m_i is a best-response to $m_{-i} \in supp(\lambda_i^\theta) \cap M_{-i}^0$ at θ, which is a contradiction.

Suppose that $supp(\lambda_i^\theta) \cap M_{-i}^0 = \emptyset$. Since $\lambda_i^\theta \in \Delta(M_{-i} \setminus M_{-i}^0)$, Lemma 12 implies that $m_i^2 = 0$. A contradiction of the initial assumption that m_i is a best response to λ_i^θ is derived if we show that $\hat{m}_i = (\theta, \hat{m}_i^2, x_i(\cdot, \theta), y_i^*(\theta))$ strictly dominates m_i. To this end, fix any $m_{-i} \in supp(\lambda_i^\theta)$. Since $m_{-i} \notin M_{-i}^0$, it follows from the definition of M_{-i}^0 in (35) that $m_j^2 \neq 0$ for some $j \in \mathcal{I} \setminus \{i\}$ or $m_{-i}^1 \notin \beta_{-i}^*(\tilde{\theta})$ for all $\tilde{\theta} \in \Theta$ such that $\left(\bigcap_{j \in \mathcal{I} \setminus \{i\}} \beta_j^*(\tilde{\theta})\right) \bigcap \Theta_i^f \neq \emptyset$. We proceed according to whether (m_i, m_{-i}) falls either

[24] m_i is a best-response to m_{-i} at θ if $u_i(g(m_i, m_{-i}), \theta) \geq u_i(g(m_i', m_{-i}), \theta)$ for all $m_i' \in M_i$.

into Rule 1, or into Rule 2, or into Rule 3, or into Rule 4.

Case 1: (m_i, m_{-i}) falls into Rule 1

Then, there exists $\bar{\theta}$ such that $\left(m_j^1\right)_{j\in\mathcal{I}} \in \beta^*\left(\bar{\theta}\right)$ and $m_j^2 = 0$ for all $j \in \mathcal{I}^{\beta^*(\bar{\theta})}$ and $g(m_i, m_{-i}) = f\left(\bar{\theta}\right)$. Since $m_{-i} \notin M_{-i}^0$ and (m_i, m_{-i}) falls into Rule 1, it cannot be that $i \notin \mathcal{I}^{\beta^*(\bar{\theta})}$. Thus, let $i \in \mathcal{I}^{\beta^*(\bar{\theta})}$. We proceed according to whether $m_j^2 \neq 0$ for some $j \in \mathcal{I}\backslash\{i\}$ or not.

Sub-case 1.1: $m_j^2 = 0$ for all $j \in \mathcal{I}\backslash\{i\}$

Let us proceed according to whether there exists $\theta' \in \bar{\Theta}_i^f$ such that $m_j^1 = \theta'$ for all $j \in \mathcal{I}\backslash\{i\}$ or not.

Sub-case 1.1.1: For some $\theta' \in \bar{\Theta}_i^f$, $m_j^1 = \theta'$ for all $j \in \mathcal{I}\backslash\{i\}$

Since m_i is not a best-response to m_{-i} at θ, it holds that $L_i(f(\theta'), \theta') \cap SU_i\left(f\left(\bar{\theta}\right), \theta\right) \neq \emptyset$. To see it, assume, to the contrary, that $L_i(f(\theta'), \theta') \cap SU_i\left(f\left(\bar{\theta}\right), \theta\right) = \emptyset$, and so $L_i(f(\theta'), \theta') \subseteq L_i\left(f\left(\bar{\theta}\right), \theta\right)$. Note that player i cannot induce Rule 4 since $m_j^1 = \theta' \in \bar{\Theta}_i^f$ for all $j \in \mathcal{I}\backslash\{i\}$. Take any $\tilde{m}_i \in M_i$.

- If $(\tilde{m}_i, m_{-i})$ falls into Rule 2, then $g(\tilde{m}_i, m_{-i}) = e(\tilde{m}_i^1, \theta') \in SL_i(f(\theta'), \theta')$.

- If $(\tilde{m}_i, m_{-i})$ falls into Rule 3, then $g(\tilde{m}_i, m_{-i}) = \left(\frac{\tilde{m}_i^2}{\tilde{m}_i^2+1}\right)\tilde{m}_i^3(\theta') + \left(\frac{1}{\tilde{m}_i^2+1}\right) z_i(\theta', \theta')$ if $u_i(f(\theta'), \theta') \geq u_i(m_i^3(\theta'), \theta')$, otherwise, $g(\tilde{m}_i, m_{-i}) = z_i(\theta', \theta')$. Since $z_i(\theta', \theta') \in L_i(f(\theta'), \theta')$ by (20) and since $u_i(f(\theta'), \theta') \geq u_i(m_i^3(\theta'), \theta')$, it follows that $g(\tilde{m}_i, m_{-i}) \in L_i(f(\theta'), \theta')$.

- If $(\tilde{m}_i, m_{-i})$ falls into Rule 1, then there exists $\bar{\theta}'$ such that $\left(\tilde{m}_i^1, m_{-i}^1\right) \in \beta^*\left(\bar{\theta}'\right)$, $m_j^2 = 0$ for all $j \in \mathcal{I}^{\beta^*(\bar{\theta}')}$ and $g(\tilde{m}_i, m_{-i}) = f\left(\bar{\theta}'\right)$. By arguing as above, we can see that $(m_i', m_{-i}) \in NE(\Gamma, \theta')$, where $m_i' = (\theta', 0, m_i^3, m_i^4)$ and $m_{-i} = \left(\theta', 0, m_{-i}^3, m_{-i}^4\right)$, that (m_i', m_{-i}) falls into Rule 1 and that $g(m_i', m_{-i}) = f(\theta')$. This implies that $g(\tilde{m}_i, m_{-i}) = f\left(\bar{\theta}'\right) \in L_i(f(\theta'), \theta')$.

Since $L_i(f(\theta'),\theta') \subseteq L_i\left(f\left(\bar{\theta}\right),\theta\right)$, it follows that $g(\tilde{m}_i, m_{-i}) \in L_i\left(f\left(\bar{\theta}\right),\theta\right)$. Since the choice of $\tilde{m}_i \in M_i$ is arbitrary, we have that m_i is a best-response to m_{-i}, which is a contradiction. Thus, $L_i(f(\theta'),\theta') \cap SU_i\left(f\left(\bar{\theta}\right),\theta\right) \neq \emptyset$.

Since $L_i(f(\theta'),\theta') \cap SU_i\left(f\left(\bar{\theta}\right),\theta\right) \neq \emptyset$, it follows that $\alpha_i^{\theta}\left(\theta',\bar{\theta}\right) \in L_i(f(\theta'),\theta') \cap SU_i\left(f\left(\bar{\theta}\right),\theta\right)$ by definition of $\alpha_i^{\theta}\left(\theta',\bar{\theta}\right)$ given in (28). Since $\theta \in \bar{\Theta}_i^f$, requirement (iii) of the definition of $\bar{D}_i(\theta',\theta)$ implies that $\alpha_i^{\theta}\left(\theta',\bar{\theta}\right) \in \bar{D}_i(\theta',\theta)$.

We show that $u_i(x_i(\theta',\theta),\theta) > u_i\left(f\left(\bar{\theta}\right),\theta\right)$. Let us proceed according to whether $D_i(\theta',\theta) \neq \emptyset$ or not.

- Suppose that $D_i(\theta',\theta) \neq \emptyset$. Then, $L_i(f(\theta'),\theta') \cap SU_i(f(\theta'),\theta) \neq \emptyset$ by definition of $D_i(\theta',\theta)$ in (27). Since $\emptyset \neq \bar{D}_i(\theta',\theta) \subseteq D_i(\theta',\theta)$ by definition and, moreover, since $x_i(\theta',\theta) \in \arg\max_{y \in \bar{D}_i(\theta',\theta)} u_i(y,\theta)$ by (29)-(30), it follows that $u_i(x_i(\theta',\theta),\theta) > u_i(f(\theta'),\theta)$. Clearly, $u_i(x_i(\theta',\theta),\theta) > u_i\left(f\left(\bar{\theta}\right),\theta\right)$ if $u_i(f(\theta'),\theta) \geq u_i\left(f\left(\bar{\theta}\right),\theta\right)$. Otherwise, let $u_i\left(f\left(\bar{\theta}\right),\theta\right) > u_i(f(\theta'),\theta)$. Since $\alpha_i^{\theta}\left(\theta',\bar{\theta}\right) \in \bar{D}_i(\theta',\theta) \cap SU_i\left(f\left(\bar{\theta}\right),\theta\right)$ and since $u_i(x_i(\theta',\theta),\theta) \geq u_i\left(\alpha_i^{\theta}\left(\theta',\bar{\theta}\right),\theta\right)$, it follows that $u_i(x_i(\theta',\theta),\theta) > u_i\left(f\left(\bar{\theta}\right),\theta\right)$.

- Suppose that $D_i(\theta',\theta) = \emptyset$. Then, $L_i(f(\theta'),\theta') \subseteq L_i(f(\theta'),\theta)$. By definition of $B_i(\theta',\theta)$ in (29) and by definition of $x_i(\cdot,\theta)$ in (30), $x_i(\theta',\theta) = f(\theta')$. Assume, to the contrary, that $u_i\left(f\left(\bar{\theta}\right),\theta\right) \geq u_i(f(\theta'),\theta)$. We have already established above that $\alpha_i^{\theta}\left(\theta',\bar{\theta}\right) \in L_i(f(\theta'),\theta') \cap SU_i\left(f\left(\bar{\theta}\right),\theta\right)$. Thus, $u_i(f(\theta'),\theta') \geq u_i\left(\alpha_i^{\theta}\left(\theta',\bar{\theta}\right),\theta'\right)$ and $u_i\left(\alpha_i^{\theta}\left(\theta',\bar{\theta}\right),\theta\right) > u_i(f(\theta'),\theta)$. This implies that $D_i(\theta',\theta) \neq \emptyset$, which is a contradiction. Thus, $u_i(x_i(\theta',\theta),\theta) > u_i\left(f\left(\bar{\theta}\right),\theta\right)$ where $x_i(\theta',\theta) = f(\theta')$.

We conclude that $u_i(x_i(\theta',\theta),\theta) > u_i\left(f\left(\bar{\theta}\right),\theta\right)$. Since $m_j^2 = 0$ and $m_j^1 = \theta' \in \bar{\Theta}_i^f$ for all $j \in \mathcal{I}\backslash\{i\}$ and $i \in \mathcal{I}^{\beta^*(\bar{\theta})}$, by choosing an appropriate integer $\hat{m}_i^2 > 0$ and by changing m_i into $\hat{m}_i$, player i can induce Rule 3 with $u_i(f(\theta'),\theta') \geq u_i(x_i(\theta',\theta),\theta')$, obtain $g(\hat{m}_i, m_{-i})$ and be strictly better off at θ since $u_i(x_i(\theta', \theta), \theta) > u_i(f(\bar{\theta}), \theta)$.

Sub-case 1.1.2: There does not exist any $\theta' \in \bar{\Theta}_i^f$ such that $m_j^1 = \theta'$ for all $j \in \mathcal{I} \setminus \{i\}$

Since $m_j^2 = 0$ for all $j \in \mathcal{I} \setminus \{i\}$ and $m_{-i} \notin M_{-i}^0$, it follows from the definition of M_{-i}^0 in (35) that $m_{-i}^1 \notin \beta_{-i}^*(\tilde{\theta})$ for all $\tilde{\theta} \in \Theta$ such that $\left(\bigcap_{j \in \mathcal{I} \setminus \{i\}} \beta_j^*(\tilde{\theta}) \right) \bigcap \Theta_i^f \neq \emptyset$. Suppose that $I = 2$. Then, $m_{-i}^1 \notin \bar{\Theta}_i^f$. Since $\theta' \in \beta_{-i}^*(\theta')$ for all $\theta' \in \Theta_i^f$ and since $m_{-i}^1 \notin \beta_{-i}^*(\tilde{\theta})$ for all $\tilde{\theta} \in \Theta$ such that $\beta_{-i}^*(\tilde{\theta}) \bigcap \Theta_i^f \neq \emptyset$, it follows that $m_{-i}^1 \in \bar{\Theta}_i^f$, which is a contradiction. Thus, let us suppose that $I \geq 3$.

Suppose that there exists $j, j' \in \mathcal{I} \setminus \{i\}$, with $j \neq j'$, such that $m_j^1 \neq m_{j'}^1$. By definition of $y_i^*(\theta)$ in (19), $u_i(y_i^*(\theta), \theta) \geq u_i(f(\bar{\theta}), \theta)$. Since $\theta \in \bar{\Theta}_i^f$, Lemma 6 implies that $u_i(y_i^*(\theta), \theta) > u_i(\underline{y}, \theta)$. Suppose that $u_i(y_i^*(\theta), \theta) = u_i(f(\bar{\theta}), \theta)$. Then, by definition of g, we have that m_i is a best-response to m_{-i} at θ, which is a contradiction. Suppose that $u_i(y_i^*(\theta), \theta) > u_i(f(\bar{\theta}), \theta)$. By choosing an appropriate integer $\hat{m}_i^2 > 0$ and by changing m_i into $\hat{m}_i$, player i can induce Rule 4, obtain $g(\hat{m}_i, m_{-i})$ and be strictly better off at θ since $u_i(y_i^*(\theta), \theta) > u_i(f(\bar{\theta}), \theta)$.

Suppose that for all $j, j' \in \mathcal{I} \setminus \{i\}$, $m_j^1 = m_{j'}^1$. Since there does not exist any $\theta' \in \bar{\Theta}_i^f$ such that $m_j^1 = \theta'$ for all $j \in \mathcal{I} \setminus \{i\}$, it follows that $m_j^1 = \tilde{\theta} \in \Theta_i^f$ and $m_j^1 = m_{j'}^1 = \tilde{\theta}$ for all $j, j' \in \mathcal{I} \setminus \{i\}$. Since $m_{-i}^1 \in \beta_{-i}^*(\bar{\theta})$, we have that $m_j^1 = \tilde{\theta} \in \bigcap_{j \in \mathcal{I} \setminus \{i\}} \beta_j^*(\bar{\theta})$. However, since $i \in \mathcal{I}^{\beta^*(\bar{\theta})}$, it follows from (31) that $m_j^1 = \tilde{\theta} \notin \Theta_i^f$, which is a contradiction.

Sub-case 1.2: $m_j^2 \neq 0$ for some $j \in \mathcal{I} \setminus \{i\}$

Since $i \in \mathcal{I}^{\beta^*(\bar{\theta})}$, by choosing an appropriate integer $\hat{m}_i^2 > 0$ and by changing m_i into $\hat{m}_i$, player i can induce Rule 4. Suppose that $u_i(y_i^*(\theta), \theta) = u_i(f(\bar{\theta}), \theta)$. Then, by definition of g, we have that m_i is a best-response to m_{-i} at θ, which is a contradiction. Suppose that $u_i(y_i^*(\theta), \theta) > u_i(f(\bar{\theta}), \theta)$. Player i can obtain

$g(\hat{m}_i, m_{-i})$ and be strictly better off at θ since $u_i(y_i^*(\theta), \theta) > u_i\left(f\left(\bar{\theta}\right), \theta\right)$.

Case 2: (m_i, m_{-i}) falls into Rule 2

Then, $m_1^2 = m_2^2 = 0$, $(m_i^1)_{i\in\mathcal{I}} \notin \beta^*\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \Theta$ and $g(m) = e\left(m_i^1, m_{-i}^1\right)$. Recall that $\theta \in \bar{\Theta}_i^f$. Then, to apply Lemma 10, we need to guarantee that $m_{-i}^1 \in \bar{\Theta}_i^f$. Since $m_{-i} \notin M_{-i}^0$ and $m_{-i}^2 = 0$, $m_{-i}^1 \notin \beta_{-i}^*\left(\tilde{\theta}\right)$ for all $\tilde{\theta} \in \Theta$ such that $\beta_{-i}^*\left(\tilde{\theta}\right) \cap \Theta_i^f \neq \emptyset$, by (35). Since $\theta' \in \beta_{-i}^*(\theta')$ for all $\theta' \in \Theta_i^f$, it follows that $m_{-i}^1 \notin \Theta_i^f$, and so $m_{-i}^1 \in \bar{\Theta}_i^f$. Since $\theta, m_{-i}^1 \in \bar{\Theta}_i^f$, Lemma 10 implies that $u_i\left(x_i\left(m_{-i}^1, \theta\right), \theta\right) > u_i\left(e\left(m_i^1, m_{-i}^1\right), \theta\right)$.

Suppose that i changes m_i into $\hat{m}_i$ in which $\hat{m}_i^2 > 0$. Then, $(\hat{m}_i, m_{-i})$ cannot fall into Rule 1. To see it, assume, on the contrary, that it falls into Rule 1. Then, there exists $\bar{\theta}'$ such that $\left(\hat{m}_i^1, m_{-i}^1\right) \in \beta_i^*\left(\bar{\theta}'\right) \times \beta_{-i}^*\left(\bar{\theta}'\right)$. Since $\hat{m}_i^2 > 0$ and $(\hat{m}_i, m_{-i})$ falls into Rule 1, it follows that $\mathcal{I}^{\beta^*\left(\bar{\theta}'\right)} = \{-i\}$ given that $\mathcal{I}^{\Theta} \neq \emptyset$. Since $i \notin \mathcal{I}^{\beta^*\left(\bar{\theta}'\right)}$, it follows from (31) that $\beta_{-i}^*\left(\bar{\theta}'\right) \bigcap \Theta_i^f \neq \emptyset$. Since $m_{-i} \notin M_{-i}^0$ and $m_{-i}^2 = 0$ and since $\beta_{-i}^*\left(\bar{\theta}'\right) \bigcap \Theta_i^f \neq \emptyset$, (35) implies that $m_{-i}^1 \notin \beta_{-i}^*\left(\bar{\theta}'\right)$, which is a contradiction. Thus, $(\hat{m}_i, m_{-i})$ cannot fall into Rule 1.

Since $m_{-i}^1 \in \bar{\Theta}_i^f$, by choosing an appropriate integer $\hat{m}_i^2 > 0$ and by changing m_i into $\hat{m}_i$, player i can induce Rule 3 with $u_i\left(f\left(m_{-i}^1\right), m_{-i}^1\right) \geq u_i\left(x_i\left(m_{-i}^1, \theta\right), m_{-i}^1\right)$–by definition of $x_i\left(m_{-i}^1, \theta\right)$ in (30), obtain $g(\hat{m}_i, m_{-i})$ and be strictly better off at θ since $u_i\left(x_i\left(m_{-i}^1, \theta\right), \theta\right) > u_i\left(e\left(m_i^1, m_{-i}^1\right), \theta\right)$.

Case 3: (m_i, m_{-i}) falls into Rule 3

Then, for some $h \in \mathcal{I}$ and some $\bar{\theta} \in \bar{\Theta}_h^f$, $\left(m_j^1, m_j^2\right) = \left(\bar{\theta}, 0\right)$ for all $j \in \mathcal{I}\setminus\{h\}$, and either $m_h^2 > 0$ or [$m_h^2 = 0$, $I \neq 2$ and $(m_k^1)_{k\in\mathcal{I}} \notin \beta^*\left(\hat{\theta}\right)$ for all $\hat{\theta} \in \Theta$]. Let us proceed according to whether $m_i^1 = \bar{\theta}$ or not.

Sub-case 3.1: $m_i^1 = \bar{\theta}$.

Since $m_i^2 = 0$, it follows that $h \neq i$. Since $\theta \in \bar{\Theta}_i^f$, Lemma 6 implies that $u_i(y_i^*(\theta), \theta) > u_i\left(\underline{y}, \theta\right)$. Moreover, by definition of $y_i^*(\theta)$ in (19), it holds that

$u_i(y_i^*(\theta),\theta) \geq u_i\left(m_h^3\left(\bar{\theta}\right),\theta\right)$ and $u_i(y_i^*(\theta),\theta) \geq u_i\left(z_h\left(\bar{\theta},\bar{\theta}\right),\theta\right)$. We proceed according to whether $u_i(y_i^*(\theta),\theta) = u_i\left(m_h^3\left(\bar{\theta}\right),\theta\right)$ and $u_i(y_i^*(\theta),\theta) = u_i\left(z_h\left(\bar{\theta},\bar{\theta}\right),\theta\right)$ or not.

- Suppose that $u_i(y_i^*(\theta),\theta) = u_i\left(m_h^3\left(\bar{\theta}\right),\theta\right)$ and $u_i(y_i^*(\theta),\theta) = u_i\left(z_h\left(\bar{\theta},\bar{\theta}\right),\theta\right)$. Since i cannot be strictly better off by changing m_i, it follows that m_i is a best-response to m_{-i}, which is a contradiction.

- Suppose that $u_i(y_i^*(\theta),\theta) > u_i\left(m_h^3\left(\bar{\theta}\right),\theta\right)$ or $u_i(y_i^*(\theta),\theta) > u_i\left(z_h\left(\bar{\theta},\bar{\theta}\right),\theta\right)$. By choosing an appropriate integer $\hat{m}_i^2 > 0$ and by changing m_i into $\hat{m}_i$, player i can induce Rule 4, obtain $g(\hat{m}_i, m_{-i})$ and be strictly better off at θ since $u_i(y_i^*(\theta),\theta) > u_i\left(m_h^3\left(\bar{\theta}\right),\theta\right)$ or $u_i(y_i^*(\theta),\theta) > u_i\left(z_h\left(\bar{\theta},\bar{\theta}\right),\theta\right)$.

Sub-case 3.2: $m_i^1 \neq \bar{\theta}$.

Since $m_i^2 = 0$, we have that either $i \neq h$ if $I = 2$ or $i = h$ if $I \neq 2$. If $I = 2$, then i can find a profitable deviation by changing m_i into $\hat{m}_i$–by arguing as in sub-case 3.1. Suppose that $I \neq 2$. Then, $g(m) = z_i\left(\bar{\theta},\bar{\theta}\right)$ because $m_i^2 = 0$. Since $\bar{\theta},\theta \in \bar{\Theta}_i^f$, Lemma 9 implies that $u_i\left(x_i\left(\bar{\theta},\theta\right),\theta\right) > u_i\left(z_i\left(\bar{\theta},\bar{\theta}\right),\theta\right)$. By choosing an appropriate integer $\hat{m}_i^2 > 0$ and by changing m_i into $\hat{m}_i$, player i can induce Rule 3, obtain $g(\hat{m}_i, m_{-i})$ and be strictly better off at θ since $u_i\left(x_i\left(\bar{\theta},\theta\right),\theta\right) > u_i\left(z_i\left(\bar{\theta},\bar{\theta}\right),\theta\right)$.

Case 4: (m_i, m_{-i}) falls into Rule 4

Suppose that j is the pivotal player. By definition of $y_i^*(\theta)$ in (19), $u_i(y_i^*(\theta),\theta) \geq u_i\left(m_j^4,\theta\right)$. Recall that $m_i^2 = 0$. We proceed according to whether $m_h^2 \neq 0$ for some $h \in \mathcal{I}\backslash\{i\}$ or not.

Sub-case 4.1: $m_h^2 \neq 0$ for some $h \in \mathcal{I}\backslash\{i\}$

Since $\theta \in \bar{\Theta}_i^f$, Lemma 6 implies that $u_i(y_i^*(\theta),\theta) > u_i\left(\underline{y},\theta\right)$. By choosing an appropriate integer $\hat{m}_i^2 > 0$ and by changing m_i into $\hat{m}_i$, player i can induce Rule 4, obtain $g(\hat{m}_i, m_{-i})$ and be strictly better off at θ since $u_i(y_i^*(\theta), \theta) > u_i(\underline{y}, \theta)$.

Sub-case 4.2: $m_h^2 = 0$ for all $h \in \mathcal{I} \setminus \{i\}$

Thus, $m_i^2 = m_h^2 = 0$ for all $h \in \mathcal{I} \setminus \{i\}$ and $g(m) = \underline{y}$. Since (m_i, m_{-i}) does not fall into Rule 1 and $m_k^2 = 0$ for all $k \in \mathcal{I}$, it holds that $(m_k^1)_{k \in \mathcal{I}} \notin \beta^*(\bar{\theta})$ for all $\bar{\theta} \in \Theta$. If $I = 2$, then (m_i, m_{-i}) falls into Rule 2, which is a contradiction. Therefore, let $I \neq 2$. Since (m_i, m_{-i}) falls into Rule 4 and $m_i^2 = m_h^2 = 0$ for all $h \in \mathcal{I} \setminus \{i\}$, it cannot be that there exists $\bar{\theta} \in \Theta$ such that $(m_j^1)_{j \in \mathcal{I}} \in \beta^*(\bar{\theta})$–otherwise, m would fall into Rule 1. Therefore, $(m_j^1)_{j \in \mathcal{I}} \notin \bigcup_{\bar{\theta} \in \Theta} \beta^*(\bar{\theta})$. Since (m_i, m_{-i}) falls into Rule 4 and $m_i^2 = m_h^2 = 0$ for all $h \in \mathcal{I} \setminus \{i\}$, it follows that either $m_k^1 = m_j^1$ for all $k, j \in \mathcal{I} \setminus \{i\}$ and $m_k^1 \notin \bar{\Theta}_i^f$ or there exists $k \in \mathcal{I} \setminus \{i\}$ such that $m_k^1 \neq m_j^1$ for some $j \in \mathcal{I} \setminus \{i\}$. In either case, by changing m_i into $\hat{m}_i$ with $\hat{m}_i^2 > 0$, player i induces Rule 4. Since $\theta \in \bar{\Theta}_i^f$, Lemma 6 implies that $u_i(y_i^*(\theta), \theta) > u_i(\underline{y}, \theta)$. Then, by choosing an appropriate integer $\hat{m}_i^2 > 0$, i can obtain $g(\hat{m}_i, m_{-i})$ and be strictly better off at θ since $u_i(y_i^*(\theta), \theta) > u_i(\underline{y}, \theta)$.

Since Θ is finite and since the choice of $m_{-i} \in supp(\lambda_i^\theta)$ is arbitrary, we see that $\hat{m}_i$ strictly dominates every element of $supp(\lambda_i^\theta)$ by an appropriate choice of $\hat{m}_i^2 > 0$. ■

Lemma 14. For all $i \in \mathcal{I}$, all $\theta \in \bar{\Theta}_i^f$, all $m_i \in M_i$ and all $m_{-i} \in M_{-i}$, if $i \in \mathcal{I}^{\beta^*(\theta)}$ and m_i is a best-response to m_{-i} at θ, then (m_i, m_{-i}) falls into Rule 1.

Proof. Fix any $i \in \mathcal{I}$, any $\theta \in \bar{\Theta}_i^f$, any $m_{-i} \in M_{-i}$ and any $m_i \in M_i$ such that m_i is a best-response to m_{-i} at θ. Suppose that $i \in \mathcal{I}^{\beta^*(\theta)}$. Assume, to the contrary, that (m_i, m_{-i}) does not fall into Rule 1. This implies that $(m_i^1, m_{-i}^1) \notin \beta^*(\bar{\theta})$ for all $\bar{\theta} \in \Theta$ or [for some $\ell \in \mathcal{I}$ such that $\ell \in \mathcal{I}^{\beta^*(\bar{\theta})}$, it holds that $m_\ell^2 \neq 0$, where $\bar{\theta} \in \Theta$ is such that $(m_i^1, m_{-i}^1) \in \beta^*(\bar{\theta})$].

Let us show that $m_{-i} \notin M_{-i}^0$. Suppose that $m_{-i} \in M_{-i}^0$. By (35), $m_{-i}^2 = 0$ and $m_{-i}^1 \in \beta_{-i}^*(\tilde{\theta})$ for some $\tilde{\theta} \in \Theta$ such that $\beta_{-i}^*(\tilde{\theta}) \cap \Theta_i^f \neq \emptyset$. This implies that $i \notin \mathcal{I}^{\beta^*(\tilde{\theta})}$. Moreover, Lemma 1 implies that $\beta_i^*(\tilde{\theta}) = \Theta$. Since $m_{-i}^1 \in \beta_{-i}^*(\tilde{\theta})$ and

$m^2_{-i} = 0$, it follows that (m_i, m_{-i}) falls into Rule 1, which is a contradiction. Thus, $m_{-i} \notin M^0_{-i}$.

Since $i \in \mathcal{I}^{\beta^*(\theta)}$, Lemma 12 implies that $m_i^2 = 0$. Thus, (m_i, m_{-i}) falls either into Rule 2, or into Rule 3, or into Rule 4. By arguing as in Cases 2-4 of the proof of Lemma 13, we see that m_i is not a best-response to m_{-i} at θ, which is a contradiction. ■

For all $i \in \mathcal{I}$, let

$$E_i = \left\{ m_i^1 \in M_i^1 \middle| m_i \in S_i^{\mathcal{M},\theta} \right\}. \tag{36}$$

Since $\bar{m} \in NE(\mathcal{M}, \theta)$, it follows that $E_i \neq \emptyset$ for all $i \in \mathcal{I}$. Specifically, it follows that $\theta \in E_i$ for all $i \in \mathcal{I}$. Let us show that $E_i \subseteq \beta_i^*(\theta)$ for all $i \in \mathcal{I}$. Fix any $i \in \mathcal{I}$. Suppose that $\theta \in \Theta_i^f$. Part (iv) of Lemma 1 implies that $\beta_i^*(\theta) = \Theta$, and so $E_i \subseteq \beta_i^*(\theta)$, as we sought. Otherwise, let us suppose that $\theta \in \bar{\Theta}_i^f$. Since $\beta^* \in \mathcal{E}(\mathcal{R})$, it suffices to show that $E_i \subseteq \mathcal{R}_i^{\beta^*}(\theta)$.

Fix any $\ell \in \mathcal{I}$ and any $m_\ell^1 \in E_\ell$. (36) implies that there exists $m_\ell \in S_\ell^{\mathcal{M},\theta}$. Suppose that $\beta_{-\ell}^*(\theta) \cap \Theta_\ell^f \neq \emptyset$. Part (iv) of Lemma 1 implies that $\beta_\ell^*(\theta) = \Theta$. We can apply (5) to player ℓ under the specification that $\tilde{\theta} = \hat{\theta} = \theta$. Otherwise, let us suppose that $\beta_{-\ell}^*(\theta) \subseteq \bar{\Theta}_\ell^f$, so that $\ell \in \mathcal{I}^{\beta^*(\theta)}$. Since $m_\ell \in S_\ell^{\mathcal{M},\theta}$, it follows that m_ℓ is a best-response to some λ_ℓ^θ at θ. Since $\beta_{-\ell}^*(\theta) \subseteq \bar{\Theta}_\ell^f$, it follows that $M_\ell^0 = \emptyset$, and so $\lambda_\ell^\theta \in \Delta\left(M_{-\ell} \backslash M_{-\ell}^0\right)$. Lemma 12 implies that $m_\ell^2 = 0$, Lemma 13 implies that m_ℓ is a best-response to some $m_{-\ell} \in supp\left(\lambda_\ell^\theta\right)$ at θ, and Lemma 14 implies that $(m_\ell, m_{-\ell})$ falls into Rule 1. Then, there exists $\bar{\theta}$ such that $m^1 \in \beta^*\left(\bar{\theta}\right)$ and $g(m_\ell, m_{-\ell}) = f\left(\bar{\theta}\right)$. By definition of $S_\ell^{\mathcal{M},\theta}$, it also follows that $m_{-\ell} \in S_{-\ell}^{\mathcal{M},\theta}$, and so $m_{-\ell}^1 \in E_{-\ell}$. Since m_ℓ is a best-response to some $m_{-\ell}$, it holds that $L_\ell\left(f\left(m_{-\ell}^1\right), m_{-\ell}^1\right) \subseteq L_\ell\left(f\left(\bar{\theta}\right), \theta\right)$. Thus, we can apply (5) to player ℓ under the specification that $\tilde{\theta} = \bar{\theta}$ and $\hat{\theta} = m_{-\ell}^1$.

Since the choice of $\ell \in \mathcal{I}$ is arbitrary, we have that (5) applies to every player $\ell \in \mathcal{I}$. It follows that $E_i \subseteq \mathcal{R}_i^{\beta^*}(\theta)$. Since the choice of $i \in \mathcal{I}$ was arbitrary, we have that

$$E_i \subseteq \mathcal{R}_i^{\beta^*}(\theta) = \beta_i^*(\theta) \tag{37}$$

for all $i \in \mathcal{I}$.

To complete the proof, let us show that $g(m) = f(\theta)$ for all $m \in S^{\mathcal{M},\theta}$. Fix any $m \in S^{\mathcal{M},\theta}$. (36) implies that $\left(m_i^1, m_{-i}^1\right) \in E_i \times E_{-i}$. (36) implies that $\left(m_i^1, m_{-i}^1\right) \in \beta^*(\theta)$. If $\mathcal{I}^{\beta^*(\theta)} = \emptyset$, then m falls into Rule 1 and that $g(m) = f(\theta)$. Suppose that $\mathcal{I}^{\beta^*(\theta)} \neq \emptyset$. Fix any $\ell \in \mathcal{I}^{\beta^*(\theta)}$. Since $m_\ell \in S_\ell^{\mathcal{M},\theta}$, it follows that m_ℓ is a best-response to some λ_ℓ^θ at θ. Since $\beta_{-\ell}^*(\theta) \subseteq \bar{\Theta}_\ell^f$, it follows that $M_\ell^0 = \emptyset$, and so $\lambda_\ell^\theta \in \Delta\left(M_{-\ell} \backslash M_{-\ell}^0\right)$. Lemma 12 implies that $m_\ell^2 = 0$. Since the choice of $\ell \in \mathcal{I}^{\beta^*(\theta)}$ is arbitrary, it follows that $m_\ell^2 = 0$ for all $\ell \in \mathcal{I}^{\beta^*(\theta)}$. Therefore, m falls into Rule 1 and $g(m) = f(\theta)$. Since the choice of $m \in S^{\mathcal{M},\theta}$ was arbitrary, it follows that $g(m) = f(\theta)$ for all $m \in S^{\mathcal{M},\theta}$.

C. Proof of Theorem 2

In this Appendix, we study the relationship between IM and SEM**. The implementing condition of Xiong (2022) termed Strict Event Monotonicity** with respect to the partition P. To introduce SEM**, we need additional notation. For each $\theta \in \Theta$, let us define $\mathcal{I}^\theta$ by $\mathcal{I}^\theta = \{i \in \mathcal{I} | SL_i(f(\theta), \theta) \neq \emptyset\}$, and for each $E \in 2^\Theta \setminus \{\emptyset\}$, let us define $\mathcal{I}^E$ by $\mathcal{I}^E = \bigcap_{\theta \in E} \mathcal{I}^\theta$.

Definition 5. $f : \Theta \to Y$ is Strict Event Monotonic** (henceforth, SEM**) if there exists $P \in \mathcal{P}_f$ such that the following conditions are satisfied.

1. For all $(\theta', E) \in \Theta \times 2^\Theta \setminus \{\emptyset\}$,

$$\left[\begin{array}{l} \text{for all } (i,\theta) \in \mathcal{I}^{\cup_{\theta \in E} P(\theta)} \times E, \text{there exists } \hat{\theta}(i) \in P(\theta) \\ \text{such that } SL_i\left(f(\theta), \hat{\theta}(i)\right) \subseteq L_i(f(\theta), \theta') \end{array}\right] \implies P(\theta') = \bigcup_{\theta \in E} P(\theta).$$

2. For all $i \in \mathcal{I}$ and all $\theta, \theta', \hat{\theta} \in \Theta$,

$$\left[\begin{array}{c} \{i\} = \mathcal{I}^\theta \\ \text{and } P(\theta) \neq P(\theta') \end{array}\right] \implies L_i\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \cap SU_i(f(\theta), \theta') \neq \emptyset.$$

Before presenting this equivalence result, let us state and prove the following claim.

Lemma 15. Suppose f is IM and for any $\theta, \theta', \beta^*(\theta') \subseteq \beta^*(\theta)$, then $\beta^*(\theta') \subseteq \beta^*(\theta)$

Proof. Suppose f is IM and for some $\theta, \theta', \beta^*(\theta') \subseteq \beta^*(\theta)$. $\theta \in E$ and let us show that $\beta^*(\theta') \subseteq \beta^*(\theta)$. Again, since $\beta^* \in \mathcal{E}(\mathcal{R})$, it is sufficient to show that $\beta_i^*(\theta') \subseteq \mathcal{R}_i^{\beta^*}(\theta)$ for all $i \in \mathcal{I}$. Fix any $i \in \mathcal{I}$. Since we have already shown that $\beta^*(\theta) \subseteq \beta^*(\theta')$, it follows that $\theta \in \beta_j^*(\theta')$ for all $j \in \mathcal{I}$. Moreover, IM implies that $f(\theta') = f(\theta)$. Let $E_j' = \beta_j^*(\theta')$ for all $j \in \mathcal{I}$.

Fix any $\ell \in \mathcal{I}$ and any $\bar{\theta} \in E_\ell'$. Then, $\theta' \in E_\ell'$, $\theta \in \bigcap_{j \in \mathcal{I}\setminus\{\ell\}} E_j'$, $\bar{\theta} \in \beta_\ell^*(\theta')$, $\theta \in \bigcap_{j \in \mathcal{I}\setminus\{\ell\}} \beta_j^*(\theta')$ and $L_\ell(f(\theta), \theta) \subseteq L_\ell(f(\theta'), \theta)$. Since this holds for any $\ell \in \mathcal{I}$ and any $\bar{\theta} \in E_\ell'$, it follows from (5) that $\beta_i^*(\theta') = E_i' \subseteq \mathcal{R}_i^{\beta^*}(\theta)$. Since the choice of i was arbitrary, we conclude that $\beta^*(\theta') \subseteq \mathcal{R}^{\beta^*}(\theta)$. ■

PROOF OF THEOREM 2 Assume that $I \geq 3$ and that $\mathcal{I}^\Theta \neq \emptyset$. Suppose that f satisfies IM. We show that f is SEM**. Let $P : \Theta \to 2^\Theta \setminus \{\emptyset\}$ be defined by

$$P(\theta) = \{\theta' \in \Theta | \beta^*(\theta) = \beta^*(\theta')\}. \tag{38}$$

Let us show that $P \in \mathcal{P}_f$. Since $\theta \in \beta^*(\theta) \subseteq P(\theta)$, it follows that $P(\theta) \neq \emptyset$ and $\cup_{\theta \in \Theta} P(\theta) = \Theta$. Fix any $\theta, \theta' \in \Theta$ and suppose that $P(\theta) \neq P(\theta')$. We show that $P(\theta) \cap P(\theta') = \emptyset$. Assume, to the contrary, that there exists $\bar{\theta} \in P(\theta) \cap P(\theta')$. By (38), $\beta^*(\theta) = \beta^*(\bar{\theta})$ and $\beta^*(\theta') = \beta^*(\bar{\theta})$, and so $\beta^*(\theta) = \beta^*(\theta')$. Therefore, $P(\theta) = P(\theta')$, which is a contraction. Thus, P is a partition of Θ. Finally, since f satisfies IM, it can be seen that P is at least as fine as P_f. Thus, $P \in \mathcal{P}_f$.

Let us show that f satisfies part (1) of SEM**. To this end, fix any $(\theta', E) \in \Theta \times 2^\Theta \setminus \{\emptyset\}$. Suppose that for all $(i, \theta) \in \mathcal{I}^{\cup_{\theta \in E} P(\theta)} \times E$, there exists $\hat{\theta}(i) \in P(\theta)$ such that $SL_i\left(f(\theta), \hat{\theta}(i)\right) \subseteq L_i(f(\theta), \theta')$. This is equivalent to the following statement: for all $(i, \theta) \in \mathcal{I}^{\cup_{\theta \in E} P(\theta)} \times E$, there exists $\hat{\theta}(i) \in P(\theta)$ such that $L_i\left(f(\theta), \hat{\theta}(i)\right) \subseteq L_i(f(\theta), \theta')$. We show that $P(\theta') = \cup_{\theta \in E} P(\theta)$. By (38), it is sufficient to show that

$\beta^* (\theta) = \beta^* (\theta')$ for all $\theta \in E$. To this end, we first show that $\beta_i^* (E) \subseteq \beta_i^* (\theta')$ for all $i \in \mathcal{I}$.

Fix any $i \in \mathcal{I}$. Since $\beta^* \in \mathcal{E} (\mathcal{R})$, it is sufficient to show that $\beta_i^* (E) \subseteq \mathcal{R}_i^{\beta^*} (\theta')$. Let $E' = \cup_{\theta \in E} P (\theta)$. Since $\beta_i^* (E) \subseteq \beta_i^* (E')$, the proof is complete if we show that $\beta_i^* (E') \subseteq \beta_i^* (\theta')$. Fix any $\ell \in \mathcal{I}$. Let us proceed according to whether $\ell \in \mathcal{I}^{E'}$ or not.

Suppose that $\ell \in \mathcal{I}^{E'}$. Fix any $\bar{\theta} \in E'$. Then, there exists $\theta \in E$ such that $\bar{\theta} \in P (\theta)$. By (38), $\beta \left(\bar{\theta}\right) = \beta (\theta)$, and so $\bar{\theta} \in \beta_i (\theta)$. Since $\mathcal{I}^{E'} \subseteq \mathcal{I}^E$, it follows by our initial supposition that there exists $\hat{\theta} (\ell) \in P (\theta) \subseteq E'$ such that $L_\ell \left(f (\theta), \hat{\theta} (\ell)\right) \subseteq L_\ell (f (\theta), \theta')$. Since $\hat{\theta} (\ell) \in P (\theta)$, it follows from (38) that $\beta^* (\theta) = \beta^* \left(\hat{\theta} (\ell)\right)$, and so $\hat{\theta} (\ell) \in \bigcap_{j \in \mathcal{I} \setminus \{\ell\}} \beta_j^* (\theta)$. Since $\beta^* (\theta) = \beta^* \left(\hat{\theta} (\ell)\right)$, IM implies that $f (\theta) = f \left(\hat{\theta} (\ell)\right)$. Since the choice of $\bar{\theta} \in E'$ was arbitrary, we have that for all $\bar{\theta} \in E'$, there exist $\theta \in E'$ and $\hat{\theta} (\ell) \in E'$ such that $\bar{\theta} \in \beta_i^* (\theta)$, $\hat{\theta} (\ell) \in \bigcap_{j \in \mathcal{I} \setminus \{\ell\}} \beta_j^* (\theta)$ and $L_\ell \left(f \left(\hat{\theta} (\ell)\right), \hat{\theta} (\ell)\right) \subseteq L_\ell (f (\theta), \theta')$.

Suppose $\ell \notin \mathcal{I}^{E'}$. Then, there exists $\tilde{\theta} \in \Theta_\ell^f \cap E'$. By definition of E', there exists $\theta \in E$ such that $\tilde{\theta} \in P (\theta)$. It follows from (38) that $\beta^* (\theta) = \beta^* \left(\tilde{\theta}\right)$, and so $\theta \in \bigcap_{j \in \mathcal{I} \setminus \{\ell\}} \beta_j^* \left(\tilde{\theta}\right)$. Since $\tilde{\theta} \in \Theta_\ell^f$, it follows from (10) and the fact that $\beta^0 \subseteq \beta^*$ that $\beta_\ell^* \left(\tilde{\theta}\right) = \beta_\ell^* (\theta) = \Theta$, and so $E' \subseteq \beta_\ell^* \left(\tilde{\theta}\right)$. Therefore, for all $\bar{\theta} \in E'$, there exist $\tilde{\theta} \in E'$ and $\theta \in E'$ such that $\bar{\theta} \in \beta_\ell^* \left(\tilde{\theta}\right)$, $\theta \in \bigcap_{j \in \mathcal{I} \setminus \{\ell\}} \beta_j^* \left(\tilde{\theta}\right)$ and $\tilde{\theta} \in \Theta_\ell^f$.

Since the choice of ℓ was arbitrary, it follows from (5) that $\beta_i^* (E') \subseteq \mathcal{R}_i^{\beta^*} (\theta')$. Since the choice of i was arbitrary, we conclude that $\beta_i^* (E') \subseteq \mathcal{R}_i^{\beta^*} (\theta')$ for all $i \in \mathcal{I}$. Since $\beta^* \in \mathcal{E} (\mathcal{R})$ and since $E' = \cup_{\theta \in E} P (\theta)$, it follows that for all $\theta \in E$, $\beta^* (\theta) \subseteq \beta^* (\theta')$. Since f is IM, Lemma 15 implies that $\beta^* (\theta) = \beta^* (\theta')$. Thus, f satisfies part (1) of SEM**.

Let us show that f satisfies part (2) of SEM**. Fix any $\theta, \theta', \theta'' \in \Theta$ and any $i \in \mathcal{I}$. Assume that $\mathcal{I}^\theta = \{i\}$ and that $P (\theta) \neq P (\theta')$. Assume, to the contrary, that $L_i (f (\theta''), \theta'') \subseteq L_i (f (\theta), \theta')$.

Let us show that f satisfies part (2) of SEM**. Fix any $\theta, \theta', \theta'' \in \Theta$ and any

$i \in \mathcal{I}$. Assume that $\mathcal{I}^{\theta} = \{i\}$ and that $P(\theta) \neq P(\theta')$. Assume, to the contrary, that $L_i(f(\theta''), \theta'') \subseteq L_i(f(\theta), \theta')$. Since $P(\theta) \neq P(\theta')$, we have from (38) that $\beta^*(\theta) \neq \beta^*(\theta')$. We show that $\beta^*(\theta) = \beta^*(\theta')$.

Since $\mathcal{I}^{\theta} = \{i\}$, we have that $\theta \in \Theta_j^f$ for all $j \in \mathcal{I} \setminus \{i\}$. It follows from the definition of β^0 in (10) and the fact that $\beta^0 \subseteq \beta^*$ that $\beta_j^*(\theta) = \Theta$ for all $j \in \mathcal{I} \setminus \{i\}$.

Let us first show that $\beta^*(\theta) \subseteq \beta^*(\theta')$. Since $\beta^* \in \mathcal{E}(\mathcal{R})$, it is sufficient to show that $\beta^*(\theta) \subseteq \mathcal{R}^{\beta^*}(\theta')$. To this end, let $E_i = \beta_i^*(\theta)$ and $E_j = \Theta$ for all $j \in \mathcal{I} \setminus \{i\}$. Fix any $k \in \mathcal{I}$. Fix any $\ell \in \mathcal{I}$ and any $\bar{\theta} \in E_\ell$. We proceed according to whether $\ell = i$ or not.

- Suppose that $\ell = i$. Since $\bar{\theta} \in E_\ell$, it follows that $\bar{\theta} \in \beta_\ell^*(\theta)$. Then, (5) applies to player ℓ because $\theta \in E_\ell$, $\theta'' \in \bigcap_{j \in \mathcal{I} \setminus \{\ell\}} E_j$, $\bar{\theta} \in \beta_\ell^*(\theta)$, $\theta'' \in \bigcap_{j \in \mathcal{I} \setminus \{\ell\}} \beta_j^*(\theta) = \Theta$ and $L_\ell(f(\theta''), \theta'') \subseteq L_\ell(f(\theta), \theta')$.

- Suppose that $\ell \neq i$. Then, (5) applies to player ℓ because $\theta \in E_\ell$, $\theta \in \bigcap_{j \in \mathcal{I} \setminus \{\ell\}} E_j$, $\bar{\theta} \in \beta_\ell^*(\theta)$, $\theta \in \bigcap_{j \in \mathcal{I} \setminus \{\ell\}} \beta_j^*(\theta)$ and $\theta \in \Theta_\ell^f$.

Since this holds for any $\ell \in \mathcal{I}$ and any $\bar{\theta} \in E_\ell$, it follows from (5) that $\beta_k^*(\theta) \subseteq \mathcal{R}_k^{\beta^*}(\theta')$. Since the choice of $k \in \mathcal{I}$ was arbitrary, we conclude that $\beta^*(\theta) \subseteq \mathcal{R}^{\beta^*}(\theta') = \beta^*(\theta')$. Since f is IM, Lemma 15 implies that $\beta^*(\theta) = \beta^*(\theta')$, we conclude that f is SEM**.

Suppose that f is SEM**. Then, there exists $P \in \mathcal{P}_f$ satisfying parts (1)-(2) of the condition. We show that f satisfies IM. To this end, for all $i \in \mathcal{I}$, let $\beta_i : \Theta \mapsto 2^{\Theta} \setminus \{\emptyset\}$ be defined, for all $\theta \in \Theta$, by

$$\beta_i(\theta) = \begin{cases} P(\theta) & \text{if } i \in \mathcal{I}^{P(\theta)} \\ \Theta & \text{otherwise.} \end{cases} \tag{39}$$

Since $\theta \in \beta_i(\theta)$ for all $\theta \in \Theta$ and all $i \in \mathcal{I}$, it follows from (10) that $\beta^0 \subseteq \beta$, and so $\beta \in \mathcal{B}^t$. To complete the proof, we need only to show that $\beta \in \mathcal{E}(\mathcal{R})$. The reason is that f is IM with respect to β because $P \in \mathcal{P}_f$. Moreover, since $\beta^* = \min \mathcal{E}(\mathcal{R})$, it follows that f is IM. Thus, let us show that $\beta \in \mathcal{E}(\mathcal{R})$. Since Lemma 1 implies that $\beta \subseteq \mathcal{R}(\beta)$, we show below that $\mathcal{R}(\beta) \subseteq \beta$.

Fix any $\theta' \in \Theta$ and any $i \in \mathcal{I}$. We show that $\mathcal{R}_i^\beta(\theta') \subseteq \beta_i(\theta')$. Suppose that there is a $E = \prod_{i \in \mathcal{I}} E_i$ such that E satisfies (5). We show that $E_i \subseteq \beta_i(\theta')$. Suppose that $i \notin \mathcal{I}^{P(\theta')}$. It follows from (39) that that $\beta_i(\theta') = \Theta$, and so $E_i \subseteq \beta_i(\theta')$.

Otherwise, assume that $i \in \mathcal{I}^{P(\theta')}$. Since $E_i \subseteq \mathcal{R}_i^\beta(\theta')$, it follows that there exists $E_{-i} \in \left(2^\Theta \setminus \{\emptyset\}\right)^{I-1}$ such that (5) is satisfied. To apply SEM**, we need to construct the set $\bar{E}$, which is defined by

$$\bar{E} = \bigcup_{i \in \mathcal{I}} \left\{ \theta \in E_i \;\middle|\; \begin{array}{l} \text{there exist } \theta' \in E_i \text{ and } \hat{\theta} \in \bigcap_{j \in \mathcal{I} \setminus \{i\}} E_j \\ \text{such that } \theta' \in \beta_i(\theta) \text{ and } \hat{\theta} \in \bigcap_{j \in \mathcal{I} \setminus \{i\}} \beta_j(\theta). \end{array} \right\}$$

It can be shown that for every $i \in \mathcal{I}^{\cup_{\theta \in \bar{E}} P(\theta)}$, the set E_i satisfying (5) can be replaced with $E_i' = \cup_{\theta \in \bar{E}} P(\theta)$.[25] In what follows, for all $i \in \mathcal{I}$, let us define

$$E_i' = \begin{cases} \cup_{\theta \in \bar{E}} P(\theta) & \text{if } i \in \mathcal{I}^{\cup_{\theta \in \bar{E}} P(\theta)} \\ E_i & \text{otherwise.} \end{cases}$$

[25] To see it, observe that for every $i \in \mathcal{I}^{\cup_{\theta \in \bar{E}} P(\theta)}$, it holds that $P(\theta) \cap E_i \neq \emptyset$ for all $\theta \in \bar{E}$ and that for each $\theta' \in E_i$, there exists $\theta \in \bar{E}$ such that $\theta' \in P(\theta)$. These observations follow from the fact that $\beta_i(\theta) = P(\theta)$ for all $\theta \in \bar{E}$. Next, let us show that $E_i' = \cup_{\theta \in E_i} P(\theta)$. Fix any $\theta' \in P(\theta)$ for some $\theta \in E_i$. Since $\theta \in E_i$, there exists $\tilde{\theta} \in \bar{E}$ such that $\theta \in P\left(\tilde{\theta}\right)$. Since $P \in P_f$ and since $\theta' \in P(\theta)$ and $\theta \in P\left(\tilde{\theta}\right)$, it can be seen that $\theta' \in E_i'$. For the converse, fix any $\theta' \in P(\theta)$ for some $\theta \in \bar{E}$. Since $\theta \in \bar{E}$, we have that there exists $\tilde{\theta} \in P(\theta) \cap E_i$. Since $P \in P_f$ and since $\theta' \in P(\theta)$ and $\tilde{\theta} \in P(\theta) \cap E_i$, it follows that $\theta' \in P\left(\tilde{\theta}\right)$ for some $\tilde{\theta} \in E_i$. Thus, $E_i' = \cup_{\theta \in E_i} P(\theta)$. To see that the set E_i satisfying (5) can be replaced with E_i', fix any $\bar{\theta} \in E_i'$. Then, there exists $\theta \in E_i$ such that $\bar{\theta} \in P(\theta)$. Since $E_i \subseteq \mathcal{R}_i^\beta(\theta')$, it follows that there exist $\tilde{\theta} \in E_i$ and $\hat{\theta} \in \bigcap_{j \in \mathcal{I} \setminus \{i\}} E_j$ such that $\theta \in \beta_i\left(\tilde{\theta}\right)$, $\hat{\theta} \in \bigcap_{j \in \mathcal{I} \setminus \{i\}} \beta_j\left(\hat{\theta}\right)$ and either $\tilde{\theta} \in \Theta_i^f$ or $L_i\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \subseteq L_i\left(f\left(\tilde{\theta}\right), \theta'\right)$. Since $P \in P_f$ and since, moreover, $\theta \in \beta_i\left(\tilde{\theta}\right) = P\left(\tilde{\theta}\right)$ and $\bar{\theta} \in P(\theta)$, it follows that $\bar{\theta} \in \beta_i\left(\tilde{\theta}\right)$. Since the choice of $\bar{\theta} \in E_i'$ was arbitrary, we can conclude that the set E_i can be replaced with the set E_i'.

Fix any $(\ell,\theta) \in \mathcal{I}^{\cup_{\theta\in\bar{E}}P(\theta)} \times \bar{E}$. Since $\ell \in \mathcal{I}^{\cup_{\theta\in\bar{E}}P(\theta)}$, it follows from (5) that for every $\theta \in E'_\ell$, there exist $\tilde{\theta} \in E'_\ell$ and $\hat{\theta} \in \bigcap_{j\in\mathcal{I}\setminus\{\ell\}} E'_j$ such that $\theta \in \beta_\ell(\tilde{\theta})$, $\hat{\theta} \in \bigcap_{j\in\mathcal{I}\setminus\{\ell\}} \beta_j\left(\tilde{\theta}\right)$ and

$$L_\ell\left(f\left(\hat{\theta}\right),\hat{\theta}\right) \subseteq L_\ell\left(f\left(\tilde{\theta}\right),\theta'\right) \tag{40}$$

Since, by (39), $\beta_\ell\left(\tilde{\theta}\right) = P(\tilde{\theta})$, we have that $\theta \in P(\tilde{\theta})$. Since $P \in \mathcal{P}_f$, it holds that $P(\theta) = P\left(\tilde{\theta}\right)$ and $f(\theta) = f\left(\tilde{\theta}\right)$.

We proceed according to whether $P\left(\tilde{\theta}\right) = P(\theta')$ or not. Suppose that $P\left(\tilde{\theta}\right) = P(\theta')$. Then, $\theta \in P(\theta')$, and so $P(\theta) = P(\theta')$. Since $P \in \mathcal{P}_f$, it follows that $f(\theta) = f(\theta')$. Thus, we have that $\theta' \in P(\theta)$ such that $L_\ell(f(\theta),\theta') \subseteq L_\ell(f(\theta),\theta')$.

Suppose that $P\left(\tilde{\theta}\right) \neq P(\theta')$. Part (2) of SEM** implies that $\hat{\theta} \in P\left(\tilde{\theta}\right)$.[26] Since $P \in \mathcal{P}_f$ and since, moreover, $P(\theta) = P\left(\tilde{\theta}\right)$, it follows that $\hat{\theta} \in P(\theta)$ and $f(\theta) = f\left(\tilde{\theta}\right) = f\left(\hat{\theta}\right)$. Since (40) holds, we have there exists $\hat{\theta} \in P(\theta)$ such that $L_\ell\left(f(\theta),\hat{\theta}\right) \subseteq L_\ell(f(\theta),\theta')$.

Since the choice of $(\ell,\theta) \in \mathcal{I}^{\cup_{\theta\in\bar{E}}P(\theta)} \times \bar{E}$ was arbitrary, we have fulfilled the premises of part (1) of SME** for the pair $\left(\theta',\bar{E}\right)$. SME** implies that $P(\theta') = \cup_{\theta\in\bar{E}}P(\theta)$.

Since $i \in \mathcal{I}^{P(\theta')} = \mathcal{I}^{\cup_{\theta\in\bar{E}}P(\theta)}$, it follows from (39) that $\beta_i(\theta') = P(\theta')$ and that $\beta_i(\theta) = P(\theta)$ for all $\theta \in \bar{E}$. Since $P(\theta') = \cup_{\theta\in\bar{E}}P(\theta)$, it follows that $\cup_{\theta\in\bar{E}}\beta_i(\theta) = \cup_{\theta\in\bar{E}}P(\theta) \subseteq \beta_i(\theta')$. Since $i \in \mathcal{I}^{\cup_{\theta\in\bar{E}}P(\theta)}$, it follows that $E'_i = \cup_{\theta\in\bar{E}}P(\theta)$. Since $E_i \subseteq E'_i$ and $\cup_{\theta\in\bar{E}}P(\theta) \subseteq \beta_i(\theta')$, it follows that $E_i \subseteq \beta_i(\theta')$.

Since the choice of player $i \in \mathcal{I}$ and the choice of the set E_i were arbitrary, we have that $\mathcal{R}_i^\beta(\theta') \subseteq \beta_i(\theta')$ for all $i \in \mathcal{I}$. Since the choice of θ' was arbitrary, we have that $\mathcal{R}(\beta) \subseteq \beta$. Thus, $\beta \in \mathcal{E}(\mathcal{R})$.

[26] To see it, suppose $\hat{\theta} \notin P\left(\tilde{\theta}\right)$. Since $\hat{\theta} \in \bigcap_{j\in\mathcal{I}\setminus\{\ell\}} \beta_j\left(\hat{\theta}\right)$, it follows that $\mathcal{I}^{P(\tilde{\theta})} = \{\ell\}$. Since $\mathcal{I}^{P(\tilde{\theta})} = \{\ell\}$ and $P\left(\tilde{\theta}\right) \neq P(\theta')$, Part (2) of SEM** implies that $L_\ell\left(f\left(\hat{\theta}\right),\hat{\theta}\right) \cap SU_\ell\left(f\left(\tilde{\theta}\right),\theta'\right) \neq \emptyset$, which contradicts (40).

D. Two Player Example of Section V

Let us reconsider the two-player example presented in Section V. For all $\bar{\theta} \in \{\theta, \theta', \theta''\}$, the set $A_i\left(\bar{\theta}\right)$ is defined in (2). We see that $A_1(\theta) = A_2(\theta') = \{\theta, \theta'\}$, $A_1(\theta') = \{\theta'\}$, $A_2(\theta) = \{\theta\}$, and $A_1(\theta'') = A_2(\theta'') = \{\theta''\}$. Note that $A_1(\theta) = \{\theta, \theta'\}$ because for each lottery $z \in L_1(f(\theta), \theta) \bigcap L_2(f(\theta'), \theta')$, it holds that $u_1(z, \theta) = u_2(z, \theta') = 0$.[27]

In what follows, we show that the SCF f presented in that example is rationalizably implementable. By Theorem 1, it suffices to show that that f is IM. We do it by setting $\beta_i^* = A_i$ for all $i = 1, 2$ and by showing that $\beta^* = \mathcal{R}(\beta^*)$. To this end, suppose that $\beta_i^*\left(\bar{\theta}\right) = A_i\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \{\theta, \theta', \theta''\}$ and all $i = 1, 2$. Since $\beta_i^0 \subseteq \beta_i^*$ for every player $i = 1, 2$, it follows that $\beta^* \in \mathcal{B}^t$. It is also clear that f satisfies IM. Thus, we are only left to show that for every player $i = 1, 2$, $\beta_i^*\left(\bar{\theta}\right) = \mathcal{R}_i^{\beta^*}\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \{\theta, \theta', \theta''\}$. Since Lemma 1 implies that $\beta^* \subseteq \mathcal{R}(\beta^*)$, we need only to show that for every player $i = 1, 2$, $\mathcal{R}_i^{\beta^*}\left(\bar{\theta}\right) \subseteq \beta_i^*\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \{\theta, \theta', \theta''\}$. To this end, fix any $\bar{\theta} \in \{\theta, \theta', \theta''\}$ and any $\hat{\theta} \in \mathcal{R}_i^{\beta^*}\left(\bar{\theta}\right)$. Assume, to the contrary, that $\hat{\theta} \notin \beta_i^*\left(\bar{\theta}\right)$. This implies that $\hat{\theta} \notin A_i\left(\bar{\theta}\right) = \beta_i^0(\bar{\theta})$. Since $\hat{\theta} \in \mathcal{R}_i^{\beta^*}\left(\bar{\theta}\right)$, it follows that there exists $E_i \subseteq \mathcal{R}_i^{\beta^*}\left(\bar{\theta}\right)$ such that $\hat{\theta} \in E_i$. (5) implies that there exists $E_{-i} \subseteq \mathcal{R}_i^{\beta^*}\left(\bar{\theta}\right)$ such that, given that $\hat{\theta} \in E_i$, there exists $\left(\tilde{\theta}, \theta_{-i}\right) \in E_i \times E_{-i}$ such that $\left(\hat{\theta}, \theta_{-i}\right) \in \beta_i^*\left(\tilde{\theta}\right) \times \beta_{-i}^*\left(\tilde{\theta}\right)$ and either $\tilde{\theta} \in \Theta_i^f$ or $L_i(f(\theta_{-i}), \theta_{-i}) \subseteq L_i\left(f\left(\tilde{\theta}\right), \bar{\theta}\right)$. Since f satisfies NWA and $\hat{\theta} \in E_i$, we have that there exists $\left(\tilde{\theta}, \theta_{-i}\right) \in E_i \times E_{-i}$ such that $\left(\hat{\theta}, \theta_{-i}\right) \in \beta_i^*\left(\tilde{\theta}\right) \times \beta_{-i}^*\left(\tilde{\theta}\right)$ and

$$L_i(f(\theta_{-i}), \theta_{-i}) \subseteq L_i\left(f\left(\tilde{\theta}\right), \bar{\theta}\right). \tag{41}$$

We proceed according to whether $\bar{\theta} = \theta$, $\bar{\theta} = \theta'$ or $\bar{\theta} = \theta''$. Let z be a lottery that assigns $\frac{1}{2}$ to the pure outcome a and $\frac{1}{2}$ to the pure outcome c, and let z' be a lottery that assigns $\frac{1}{2}$ to the pure outcome c and $\frac{1}{2}$ to the pure outcome d.

[27] To see it, take any $z \in L_1(f(\theta), \theta) \bigcap L_2(f(\theta'), \theta')$. Then, $u_1(z, \theta) \leq u_1(f(\theta), \theta) = 0$ and $u_2(z, \theta') \leq u_2(f(\theta'), \theta') = 0$. Since $u_1(\cdot, \theta) = -u_2(\cdot, \theta')$, it follows that $u_1(z, \theta) + u_2(z, \theta') = 0$. Since $u_1(z, \theta) \leq 0$ and $u_2(z, \theta') \leq 0$, we have that $u_1(z, \theta) = u_2(z, \theta') = 0$.

Case 1: $\bar{\theta} = \theta$

Suppose that $i = 1$. Since $\hat{\theta} \notin \beta_1^* (\theta) = A_1 (\theta)$, it follows that $\hat{\theta} = \theta''$. Since $\hat{\theta} \in \beta_1^* \left(\tilde{\theta}\right)$ and $\hat{\theta} = \theta''$ and since $\theta_{-1} \in \beta_{-1}^* \left(\tilde{\theta}\right)$, we have that $\theta'' = \hat{\theta} = \tilde{\theta} = \theta_{-1}$. However, by construction, $e \in L_1 (f (\theta''), \theta'') \bigcap SU_1 (f (\theta''), \theta)$, which is in contradiction to (41), where $\theta'' = \tilde{\theta} = \theta_{-1}$ and $\bar{\theta} = \theta$. Thus, $\mathcal{R}_1^{\beta^*} (\theta) = \beta_1^* (\theta)$.

Suppose that $i = 2$. Since $\mathcal{R}_1^{\beta^*} (\theta) = \beta_1^* (\theta)$, it follows that $\theta_{-2} \in \{\theta, \theta'\}$. Moreover, since $\theta_{-2} \in \beta_{-2}^* \left(\tilde{\theta}\right) \bigcap \{\theta, \theta'\}$, it holds that $\tilde{\theta} \neq \theta''$. Since $\hat{\theta} \neq \theta$, $\hat{\theta} \in \beta_2^* \left(\tilde{\theta}\right)$ and $\beta_2^* (\theta) = \{\theta\}$, we also have that $\tilde{\theta} \neq \theta$. Therefore, $\tilde{\theta} \neq \theta''$ and $\tilde{\theta} \neq \theta$, and so $\tilde{\theta} = \theta'$. Then, $\hat{\theta} \in \beta_2^* (\theta')$ and $\theta_{-2} \in \beta_{-2}^* (\theta')$. Since $\theta_{-2} \in \{\theta, \theta'\} \bigcap \beta_{-2}^* (\theta')$, it follows that $\theta_{-2} = \theta'$. Therefore, $\tilde{\theta} = \theta_{-2} = \theta'$. However, by construction, $z \in L_2 (f (\theta'), \theta') \bigcap SU_2 (f (\theta'), \theta)$, which is in contradiction to (41), where $\tilde{\theta} = \theta_{-2} = \theta'$ and $\bar{\theta} = \theta$. Thus, $\mathcal{R}_2^{\beta^*} (\theta) = \beta_2^* (\theta)$.

Case 2: $\bar{\theta} = \theta'$

Suppose that $i = 2$. Since $\hat{\theta} \notin \beta_2^* (\theta') = A_2 (\theta')$, it follows that $\hat{\theta} = \theta''$. Since $\hat{\theta} \in \beta_2^* \left(\tilde{\theta}\right)$ and $\theta_{-2} \in \beta_{-2}^* \left(\tilde{\theta}\right)$, we have that $\hat{\theta} = \tilde{\theta} = \theta_{-2} = \theta''$. By construction, $z' \in L_2 (f (\theta''), \theta'') \bigcap SU_2 (f (\theta''), \theta')$, which is in contradiction to (41), where $\tilde{\theta} = \theta_{-2} = \theta''$ and $\bar{\theta} = \theta'$. Thus, $\mathcal{R}_2^{\beta^*} (\theta') = \beta_2^* (\theta')$.

Suppose that $i = 1$. Since $\mathcal{R}_2^{\beta^*} (\theta') = \beta_2^* (\theta')$, it follows that $\theta_{-1} \in \{\theta, \theta'\}$. Moreover, since $\theta_{-1} \in \beta_{-1}^* \left(\tilde{\theta}\right) \bigcap \{\theta, \theta'\}$, it holds that $\tilde{\theta} \neq \theta''$. Since $\hat{\theta} \neq \theta'$, $\hat{\theta} \in \beta_1^* \left(\tilde{\theta}\right)$ and $\beta_1^* (\theta') = \{\theta'\}$, we also have that $\tilde{\theta} \neq \theta'$. Therefore, $\tilde{\theta} \notin \{\theta', \theta''\}$, and so $\tilde{\theta} = \theta$. Then, $\hat{\theta} \in \beta_1^* (\theta)$ and $\theta_{-1} \in \beta_{-1}^* (\theta)$. Since $\theta_{-1} \in \{\theta, \theta'\} \bigcap \beta_{-1}^* (\theta)$, it also follows that $\theta_{-1} = \theta$. Therefore, $\tilde{\theta} = \theta_{-1} = \theta$. By construction, $z \in L_1 (f (\theta), \theta) \bigcap SU_1 (f (\theta), \theta')$, which is in contradiction to (41), where $\tilde{\theta} = \theta_{-1} = \theta$ and $\bar{\theta} = \theta'$. Thus, $\mathcal{R}_1^{\beta^*} (\theta') = \beta_1^* (\theta')$.

Case 3: $\bar{\theta} = \theta''$

To derive a contradiction, it suffices to show that $\hat{\theta} = \theta''$. Assume, to the contrary, that $\hat{\theta} \neq \theta''$. Since $\hat{\theta} \in \beta_i^* \left(\tilde{\theta}\right)$, it holds that $\tilde{\theta} \neq \theta''$. Thus, $\hat{\theta}, \tilde{\theta} \in \{\theta, \theta'\}$.

Suppose that $\hat{\theta} = \theta$ and $i = 1$. Since $\theta \in \beta_1^* \left(\tilde{\theta}\right)$, we have that $\tilde{\theta} \neq \theta'$. Since

$\tilde{\theta} \in \{\theta, \theta'\}$, it follows that $\tilde{\theta} = \theta$. Since $\theta_{-1} \in \beta^*_{-1}(\theta)$, we have that $\hat{\theta} = \tilde{\theta} = \theta_{-1} = \theta$. However, by construction, $z \in L_1(f(\theta), \theta) \bigcap SU_1(f(\theta), \theta'')$, which is in contradiction to (41), where $\tilde{\theta} = \theta_{-1} = \theta$ and $\bar{\theta} = \theta''$. Thus, $\theta \notin \mathcal{R}_1^{\beta^*}(\theta'')$.

Suppose that $\hat{\theta} = \theta$ and $i = 2$. Then, $\theta \in \beta^*_2\left(\tilde{\theta}\right)$ and $\theta_{-2} \in \beta^*_{-2}\left(\tilde{\theta}\right)$. Since $\theta \notin \mathcal{R}_1^{\beta^*}(\theta'')$, it follows that $\theta_{-2} \in \mathcal{R}_{-2}^{\beta^*}(\theta'') \bigcap \{\theta', \theta''\}$. It cannot be that $\theta_{-2} = \theta''$ because this implies that $\tilde{\theta} = \theta''$, which is a contradiction. Therefore, it must be that $\theta_{-2} = \theta'$. Since $\theta' \in \beta^*_{-2}\left(\tilde{\theta}\right)$, we have that $\tilde{\theta} = \theta'$. Thus, $\tilde{\theta} = \theta_{-2} = \theta'$. However, by construction, $z \in L_2(f(\theta'), \theta') \bigcap SU_2(f(\theta'), \theta'')$, which is in contradiction to (41), where $\tilde{\theta} = \theta_{-2} = \theta'$ and $\bar{\theta} = \theta''$. Thus, $\theta \notin \mathcal{R}_2^{\beta^*}(\theta'')$.

Suppose that $\hat{\theta} = \theta'$ and $i = 2$. Then, $\theta' \in \beta^*_2\left(\tilde{\theta}\right)$ and $\theta_{-2} \in \beta^*_{-2}\left(\tilde{\theta}\right)$. Since $\tilde{\theta} \in \{\theta, \theta'\}$ and $\theta' \in \beta^*_2\left(\tilde{\theta}\right)$, we have that $\tilde{\theta} \neq \theta$. Thus, $\tilde{\theta} = \theta'$. Since $\theta_{-2} \in \beta^*_{-2}(\theta')$, it holds that $\theta_{-2} = \theta'$. Therefore, $\hat{\theta} = \tilde{\theta} = \theta_{-2} = \theta'$. However, by construction, $z \in L_2(f(\theta'), \theta') \bigcap SU_2(f(\theta'), \theta'')$, which is in contradiction to (41), where $\tilde{\theta} = \theta_{-2} = \theta'$ and $\bar{\theta} = \theta''$. Thus, $\theta' \notin \mathcal{R}_2^{\beta^*}(\theta'')$.

Suppose that $\hat{\theta} = \theta'$ and $i = 1$. Then, $\theta' \in \beta^*_1\left(\tilde{\theta}\right)$ and $\theta_{-1} \in \beta^*_{-1}\left(\tilde{\theta}\right)$. Since $\theta' \notin \mathcal{R}_2^{\beta^*}(\theta'')$ and $\theta \notin \mathcal{R}_2^{\beta^*}(\theta'')$, it follows that $\theta_{-1} = \theta''$. It follows from $\theta_{-1} \in \beta^*_{-1}\left(\tilde{\theta}\right)$ that $\tilde{\theta} = \theta''$, which is a contradiction. Thus, $\theta' \notin \mathcal{R}_1^{\beta^*}(\theta'')$.

Thus, we conclude that $\beta^* = \mathcal{R}(\beta^*)$ for the case that $\beta^*_i = A_i$ for all $i = 1, 2$, and so f is rationalizably implementable.

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