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Interim Rationalizable (and Bayes-Nash) Implementation of Functions: A full Characterization

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Ritesh Jain* and Michele Lombardi†

Abstract
Interim Rationalizable Monotonicity, due to Oury and Tercieux (2012), fully characterizes the class of social choice functions that are implementable in interim correlated rationalizable (and Bayes-Nash equilibrium) strategies.

JEL classification: C79, D82.

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I. Introduction

A social choice function (SCF) f is (fully) interim rationalizably (and Bayes-Nash) implementable on a type space (T, κ) if there exists a mechanism such that (a) every interim rationalizable strategy profile leads to the realization of f and (b) it has a pure strategy Bayes-Nash equilibrium.[1] Oury and Tercieux (2012) introduce Interim Rationalizable Monotonicity (IRM, henceforth), a generalization of Bayesian Monotonicity, and they show that IRM is sufficient for interim rationalizable implementation when combined with Assumption 1. We show that IRM fully characterizes the class of interim rationalizably implementable functions.[2]

Beyond its relevance for implementation theory, our characterization result strengthens the connection between strict continuous implementation and interim rationalizable implementation. By adopting the notion of robustness of Weinstein and Yildiz (2007) in a mechanism design setting, Oury and Tercieux (2012) introduce the notion of strict continuous implementation and show that, when combined with Assumption 1, strict continuous implementation implies (full) implementation in interim rationalizable strategies. An SCF is strictly continuously implementable if there exists a strict Bayes-Nash equilibrium that continuously implements f.[3]

Specifically, Oury and Tercieux (2012) show that only functions satisfying IRM on (T, κ) are strictly continuously implementable on (T, κ). Moreover, they show that if f satisfies IRM on (T, κ) and it satisfies Assumption 1, then f is interim rationalizably implementable on (T, κ). Our characterization result strengthens Oury and Tercieux (2012)'s connection between partial implementation and full implementation as follows: Only interim rationalizable implementable functions on (T, κ) are strictly continuously implementable on (T, κ).

Roughly speaking, Assumption 1 is a condition that allows the planner to find a

[1] The requirement of the existence of a pure strategy Bayes-Nash equilibrium is shown to be equivalent to the requirement that the implementing mechanism guarantees the non-emptiness of player's best-responses to "certain" beliefs. For further details, see Lemma 2 below.

[2] The necessity of IRM is briefly discussed by Oury and Tercieux (2012) in footnote 4 (p. 1606), though no formal arguments have been provided.

[3] Specifically, Oury and Tercieux (2012) require that, in any type space that embeds (T, κ), there exists an equilibrium that (i) is a strict equilibrium on (T, κ), and (ii) it yields the desired outcome, not only at all types of (T, κ) but also at all types "close" to (T, κ).

punishment outcome for each player, whatever the player's beliefs are. The assumption is satisfied in environments with transfers or bad outcomes that the planner does not desire. However, it may be violated in many environments, such as voting, matching, and allocation problems. For instance, Assumption 1 is violated when there is a state of the world at which a player deems all pure outcomes equally good. On this observation, in Section II, we present an interim rationalizably implementable voting rule violating Assumption 1. Moreover, it is violated in house allocation problems in which a player receives his worst house. This is the case in situations in which players have the same ranking of the houses.

As we discuss in Section II, Assumption 1 ensures that for every player, the elimination of a never-best reply starts in the first round of the iterative process of deletion of never-best replies. Indeed, the sufficiency result of Oury and Tercieux (2012) relies on this fact. However, Assumption 1 is not related to the assumption of common knowledge of rationality. Indeed, the iterative process that builds on the assumption of common knowledge of rationality neither requires deleting strategies simultaneously for all players nor requires deleting them in the first round for all players.

When Assumption 1 is relaxed, we show that IRM fully characterizes interim rationalizable implementation. This result is obtained by characterizing IRM in terms of an iterative condition, which embeds an argument of iterated deletion of never-best replies. This iterative condition is termed interim iterative monotonicity (IIM, henceforth).

Recently, Xiong (2021) and Jain et al. (2022) obtain full characterization results for rationalizable implementation of functions under complete information. The seminal paper on this class of implementation problems is Bergemann et al. (2011), which critically hinges on a condition similar to Assumption 1, named No Worst Alternatives (NWA, henceforth) and on the assumption that there are three or more players.

The idea to use iterative arguments has been shown to be fruitful by Xiong (2021) and Jain et al. (2022) in dispensing with the NWA condition and relaxing the assumption of three or more players, respectively. Indeed, the latter authors offer a novel iterative condition, named Iterative Monotonicity, and use it to provide an iterative characterization of the class of rationalizably implementable functions under complete information when there are two or more players.[4] IIM is the counterpart of iterative monotonicity in an incomplete information setup.

Following Jain et al. (2022), IIM is defined on the space of deception profiles, over which we define a decreasing sequence of deception profiles $\left(\beta^k\right)_{k\geq 0}$ (in the sense of set inclusion). The limit of the sequence, which we refer to as β^*, can be viewed as the profile of largest deceptions that the planner cannot rule out in *any* implementing mechanism. An SCF f satisfies IIM on a type space (T, κ) if for any type profiles t and t' such that $t' \in \beta^*(t)$, it holds that $f(t) = f(t')$. It is worth mentioning that IIM is a measurability condition, which is reminiscent of the classical Abreu–Matsushima measurability condition (Abreu and Matsushima (1992)).[5]

As is typical in the implementation literature, the sufficiency result of Oury and Tercieux (2012) is based on designing an "augmented" direct mechanism. However, the devised mechanism does not work when Assumption 1 is relaxed. Indeed, thanks to the assumption, the augmentation of the direct mechanism used by Oury and Tercieux (2012) relies on β^0, which is the first element of the sequence $\left(\beta^k\right)_{k\geq 0}$. However, our characterization result is obtained by devising an augmentation of the direct mechanism that may crucially hinge on the entire sequence. Therefore, we provide an iterative characterization of the class of interim rationalizably implementable functions.

Section II present our motivating example. Section III presents the implementation model. Section IV discusses IIM and relates it to IRM. Section V presents our characterization result. Appendices include proofs not in the main body.

[4] Xiong (2021) provides a complete characterization of rationalizably implementable functions when there are three or more players.

[5] Abreu and Matsushima (1992) proposed a measurability condition, now referred to as Abreu–Matsushima measurability, to characterize virtual rationalizable implementation when there is incomplete information.

II. Motivating example

Suppose that there are two players, player 1 and player 2. Assume that the sets of types are $\Theta_1 = \{\theta_1, \theta_1'\}$ for player 1 and $\Theta_2 = \{\theta_2, \theta_2'\}$ for player 2. The possible type profiles in $\Theta_1 \times \Theta_2$ are (θ_1, θ_2), (θ_1', θ_2), (θ_1, θ_2') and (θ_1', θ_2'). Let $\phi \in \Delta(\Theta_1 \times \Theta_2)$ be the common prior and assume that the type profiles (or states) are equally likely, that is, $\phi(\theta) = \frac{1}{4}$ for all $\theta \in \Theta_1 \times \Theta_2$. The type $\hat{\theta}_i \in \Theta_i$ is only observed by player i, who uses this information both to make decisions and to update his beliefs about the likelihood of his opponent's types (using the conditional probability $\phi\left(\hat{\theta}_j | \hat{\theta}_i\right)$). The set of pure outcomes is given by $A = \{a, b, c, d\}$. For player $i = 1, 2$, let $\upsilon_i : \Delta(A) \times \Theta_1 \times \Theta_2 \to \mathbb{R}$ be the state-dependent payoff function of player i. For each $\theta \in \Theta_1 \times \Theta_2$, $\upsilon_i(\cdot, \theta)$ satisfies the expected utility hypothesis for $i = 1, 2$. Players' state-dependent payoff functions over A are represented in the table below.

(θ_1, θ_2)		(θ_1', θ_2)		(θ_1, θ_2')		(θ_1', θ_2')	
υ_1	υ_2	υ_1	υ_2	υ_1	υ_2	υ_1	υ_2
a, b, c, d	a	a, b, c, d	a	c	c	d	c
	c		c	d	a	c	a
	b		b	a, b	d	a, b	d
	d		d		b		b

where, as usual, $\genfrac{}{}{0pt}{}{\alpha}{\beta}$ for player i in state θ means that he strictly prefers α to β in state θ, while α, β in state θ means that this i is indifferent between α and β in state θ.

Suppose that we want to implement f in interim correlated rationalizable strategies, where $f(\theta_1, \theta_2) = a$, $f(\theta_1', \theta_2) = b$, $f(\theta_1, \theta_2') = c$ and $f(\theta_1', \theta_2') = d$. To this end, let us consider the following direct mechanism, where player 1 is the row player and player 2 is column player.

	θ_2	θ_2'
θ_1	a	c
θ_1'	b	d

To show that the direct mechanism implements f, let us note that truth-telling is always the unique dominant strategy for player 2. Consequently, truth-telling is the only interim correlated rationalizable strategy for player 1. Observe that truth-telling is also the Bayes-Nash equilibrium of game. Thus, the above mechanism implements f in interim correlated rationalizable strategies and Bayes-Nash equilibrium strategies.

However, in this example, Assumption 1 of Oury and Tercieux (2012) is violated. This assumption is formally stated in Definition 6. The easiest way to see it is to recall that this assumption implies the condition of no total indifference. In our example, this condition requires that no player is indifferent over the entire set A at any state: for all $i = 1, 2$ and all $\theta \in \Theta_1 \times \Theta_2$, there exist $x, y \in A$ such that such that $\upsilon_i(x, \theta) \neq \upsilon_i(x, \theta)$. As it can be checked from the above table, player 1 is indifferent over the entire set A at states (θ_1, θ_2) and (θ_1', θ_2).

III. The Implementation Model

Preliminaries

Throughout the paper, if X is a topological space, we treat it as a measurable space with its Borel sigma field, and the space of Borel probability measures on X is denoted by $\Delta(X)$. Spaces $\Delta(X)$ are endowed with the topology of weak convergence of measures. Throughout the paper, we treat each countable set as a topological space endowed with the discrete topology. A subset Y of a topological space X is a dense subset of X if the closure of Y in X is equal to X. With abuse of notation, given a space X, let δ_x denote a degenerate distribution in $\Delta(X)$ assigning probability 1 to $\{x\}$.

We consider a finite set of players $\mathcal{I} = \{1, ..., I\}$. Each player i has a bounded utility function $u_i : \Delta(A) \times \Theta \to \mathbb{R}$ where A is the set of (pure) outcomes and Θ is the set of states (of nature). For each $\theta \in \Theta$, $u_i(\cdot, \theta)$ satisfies the expected utility hypothesis. We assume that Θ and A are countable and hence are separable metric spaces.

Throughout the paper, if, for each $i \in \mathcal{I}$, there is a space X_i, we write X as an abbreviation for $\Pi_{i\in\mathcal{I}} X_i$ and, for each $i \in \mathcal{I}$, X_{-i} for $\Pi_{j\in\mathcal{I}\setminus\{i\}} X_j$.

A *model* (of incomplete information) is a pair $\mathcal{T} \equiv (T, \kappa)$, where $T = \Pi_{i \in \mathcal{I}} T_i$ is a countable type space and, for each $i \in \mathcal{I}$, $\kappa(t_i) \in \Delta(\Theta \times T_{-i})$ denotes the associated beliefs for each type $t_i \in T_i$ of player i satisfying the following condition: For all $t_i \in T_i$, $\text{Supp}(\kappa(t_i)) = \Delta(\Theta \times T_{-i})$.

A typical type profile of T (*resp.*, T_{-i}) is denoted by t (*resp.*, t_{-i}). Throughout the paper, we rule out the case that $\mathcal{T}$ is a model of complete information, for the sake of simplicity.

A (stochastic) *mechanism* is a pair $\mathcal{M} \equiv (M, g)$, where $M \equiv_{i \in \mathcal{I}} M_i$ is a message space and the outcome function $g : M \to \Delta(A)$ assigns to every $m \in M$ an element of $\Delta(A)$. For each $i \in \mathcal{I}$, M_i is player i's message space, which is assumed to be a (nonempty) countable set. A message profile $m \in M$ is often written as (m_i, m_{-i}), where $m_{-i} \in M_{-i}$.

Solution concepts

Given a mechanism $\mathcal{M}$ and a model $\mathcal{T}$, $U(\mathcal{M}, \mathcal{T})$ denotes the induced game of incomplete information. In this game, a (behavioral) strategy of player i is any function $\sigma_i : T_i \to \Delta(M_i)$. We write $\sigma_i(t_i)[m_i]$ for the probability that σ_i assigns to m_i when player i is of type t_i. Player i's strategy σ_i is a pure strategy if $\sigma_i : T_i \to M_i$. Given a mechanism $\mathcal{M}$, for each $i \in \mathcal{I}$, player i's best response correspondence BR_i from $\Delta(\Theta \times M_{-i})$ to M_i be defined, for all $\pi_i \in \Delta(\Theta \times M_{-i})$, by

$$BR_i(\pi_i|\mathcal{M}) = \arg\max_{m_i \in M_i} \left(\sum_{(\theta, m_{-i}) \in \Theta \times M_{-i}} \pi_i[\theta, m_{-i}][u_i(g(m_i, m_{-i}), \theta)] \right).$$

Since we allow for infinite mechanisms, the correspondence may be empty. For all $i \in I$, all $t_i \in T_i$ and all $\sigma_{-i} \equiv (\sigma_j)_{j \in \mathcal{I} \setminus \{i\}}$, we write $\pi_i(t_i, \sigma_{-i}) \in \Delta(\Theta \times M_{-i})$ for the joint distribution on the underlying uncertainty and the messages of other players induced by t_i and σ_{-i}.[6]

[6]Formally, $\pi_i(t_i, \sigma_{-i}) \in \Delta(\Theta \times M_{-i})$ is defined by $\pi_i(t_i, \sigma_{-i}) = \sum_{t_{-i} \in T_{-i}} \kappa(t_i)[\theta, t_{-i}] \sigma_{-i}(t_{-i})[m_{-i}]$, where $\kappa(t_i)[\theta, t_{-i}]$ is the probability attached to $[\theta, t_{-i}]$ under $\kappa(t_i)$, and $\sigma_{-i}(t_{-i})[m_{-i}]$ is the probability attached to m_{-i} under $\sigma_{-i}(t_{-i})$.

Definition 1. Let $U(\mathcal{M},\mathcal{T})$ be any game of incomplete information. A profile of strategies $\sigma = (\sigma_i)_{i\in\mathcal{I}}$ is a *Bayes-Nash equilibrium* of $U(\mathcal{M},\mathcal{T})$ if, for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$m_i \in \text{Supp}(\sigma_i(t_i)) \implies m_i \in BR_i(\pi_i(t_i,\sigma_{-i})|\mathcal{M}).$$

We denote by $BNE(U(\mathcal{M},\mathcal{T}))$ the set of all pure strategy Bayes-Nash equilibria of $U(\mathcal{M},\mathcal{T})$. To distinguish between pure strategy and mixed strategy equilibrium, let us denote by $\overline{BNE}(U(\mathcal{M},\mathcal{T}))$ as the set of pure strategy Bayes-Nash equilibria of $U(\mathcal{M},\mathcal{T})$.

Next, let us define the solution concept of interim correlated rationalizability (ICR, henceforth), which was introduced by Dekel et al. (2007). Before introducing it, we need additional notation. Fix any pair $(\mathcal{M},\mathcal{T})$. For all $i \in \mathcal{I}$, let Σ_i be a nonempty correspondence from T_i to $2^{M_i}\backslash\{\emptyset\}$, and let $\mathfrak{S}_i^{\mathcal{M},\mathcal{T}}$ denote the set of all nonempty correspondences from T_i to $2^{M_i}\backslash\{\emptyset\}$. Let $\mathfrak{S}^{\mathcal{M},\mathcal{T}} = \Pi_{i\in\mathcal{I}}\mathfrak{S}_i^{\mathcal{M},\mathcal{T}}$, with Σ as a typical profile of $\mathfrak{S}^{\mathcal{M},\mathcal{T}}$. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $\Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i})$ be defined by

$$\Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) = \left\{\pi_i \in \Delta(\Theta \times T_{-i} \times M_{-i})\,|\text{marg}_{\Theta\times T_{-i}}\pi_i = \kappa(t_i)\right\},$$

and, for all $\Sigma_{-i} \in \mathfrak{S}_{-i}^{\mathcal{M},\mathcal{T}}$, let $\Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$ be defined by

$$\Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i}) = \left\{ \pi_i \;\middle|\; \begin{array}{l} \pi_i \in \Delta(\Theta \times T_{-i} \times M_{-i}) \text{ and} \\ \pi_i[\theta, t_{-i}, m_{-i}] > 0 \implies m_{-i} \in \Sigma_{-i}(t_{-i}) \end{array} \right\}.$$

For all $(\mathcal{M},\mathcal{T})$ and all $\Sigma \in \mathfrak{S}^{\mathcal{M},\mathcal{T}}$, Σ is a *best-reply set* in $U(\mathcal{M},\mathcal{T})$ if, for all $i \in \mathcal{I}$, all $t_i \in T_i$ and all $m_i \in \Sigma_i(t_i)$, there exists

$$\pi_i \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \cap \Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$$

such that

$$m_i \in BR_i\left(\text{marg}_{\Theta\times M_{-i}}\pi_i|\mathcal{M}\right).$$

Definition 2. For all $(\mathcal{M},\mathcal{T})$, all $i \in \mathcal{I}$ and all $t_i \in T_i$, the *set of interim correlated* *rationalizable messages at type* t_i, denoted by $S_i^{\mathcal{M},\mathcal{T}}(t_i)$, is defined by

$$S_i^{\mathcal{M},\mathcal{T}}(t_i) = \{m_i \in \Sigma_i(t_i) \,|\text{for some best-reply set } \Sigma \text{ in } U(\mathcal{M},\mathcal{T})\}.$$

For all $t \in T$, we write $S^{\mathcal{M},\mathcal{T}}(t)$ for $\Pi_{i\in\mathcal{I}} S_i^{\mathcal{M},\mathcal{T}}(t_i)$.

Alternatively, the set of interim correlated rationalizable messages can be computed iteratively, where transfinite induction may be necessary to reach the fixed point. Following Aliprantis and Border (2006), we denote by Ω the set whose elements are called ordinals, which are ordered by $\leq$. The set Ω is such that (1) it is uncountable and (2) it has a greatest element ω_1.[7]

Definition 3. For all $(\mathcal{M},\mathcal{T})$, all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $S_i^{0,\mathcal{M},\mathcal{T}}(t_i) = M_i$ and, for all ordinal numbers $\alpha \in \Omega$, define $S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i)$ as follows:

- If α is a successor ordinal, then

$$S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i) = \left\{ m_i \in S_i^{\alpha-1,\mathcal{M},\mathcal{T}}(t_i) \;\middle|\; \begin{array}{l} \text{There exists } \pi_i \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \\ \text{such that } \pi_i \in \Delta^{S_{-i}^{\alpha-1,\mathcal{M},\mathcal{T}}}(\Theta \times T_{-i} \times M_{-i}) \\ \text{and that } m_i \in BR_i\left(\text{marg}_{\Theta\times M_{-i}}\pi_i|\mathcal{M}\right). \end{array} \right\}$$

- If α is a limit ordinal, then

$$S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i) = \bigcap_{\gamma<\alpha} S_i^{\gamma,\mathcal{M},\mathcal{T}}(t_i),$$

  Let $S_i^{\infty,\mathcal{M},\mathcal{T}}(t_i) = \bigcap_{\alpha\in\Omega} S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i)$ be the set of interim correlated rationalizable messages at type t_i.

Arieli (2010) shows that the correspondence $S^{\infty,\mathcal{M},\mathcal{T}} = \Pi_{i\in\mathcal{I}} S_i^{\infty,\mathcal{M},\mathcal{T}}$ is a best-reply set of $U(\mathcal{M},\mathcal{T})$, that is, for all $i \in \mathcal{I}$, $S_i^{\infty,\mathcal{M},\mathcal{T}} \subseteq S_i^{\mathcal{M},\mathcal{T}}$. Indeed, Arieli (2010) shows the following result.

[7] The existence of this set Ω is proved in Theorem 1.14 of Aliprantis and Border (2006) p. 19.

Lemma 1 (Arieli (2010), Theorem 1, p. 914)**.** For all $(\mathcal{M}, \mathcal{T})$, all $i \in \mathcal{I}$ and all $t_i \in T_i$, there exists a least ordinal number α such that

$$S_i^{\alpha,\mathcal{M},\mathcal{T}}(t_i) = S_i^{\alpha+1,\mathcal{M},\mathcal{T}}(t_i) = S_i^{\mathcal{M},\mathcal{T}}(t_i). \quad (1)$$

Implementation

Let $\mathcal{T}$ be given. A (stochastic) *social choice function* (SCF, henceforth) is a function $f : T \to \Delta(A)$. Following Oury and Tercieux (2012), we assume that the planner cares about all profiles of types in T.

Definition 4. A mechanism $\mathcal{M}$ *implements* $f : T \to \Delta(A)$ *in interim correlated rationalizable strategies* (ICR-implements, henceforth) on $\mathcal{T}$ if the following two conditions are satisfied.

(i) For all $i \in \mathcal{I}$ and all $t_i \in T_i$, $S_i^{\mathcal{M},\mathcal{T}}(t_i) \neq \emptyset$.

(ii) For all $t \in T$, $m \in S^{\mathcal{M},\mathcal{T}}(t) \implies g(m) = f(t)$.

If such a mechanism exists, f is *interim correlated rationalizably* (ICR, henceforth) *implementable*, or simply, *ICR-implementable* on $\mathcal{T}$.

In a complete information environment, Xiong (2021) and Jain et al. (2022) fully characterize the class of implementable functions in rationalizable strategies. Their results show that every implementable function in rationalizable strategies is also Nash implementable. The reason is that the implementing mechanism in rationalizable strategies never fails to have a Nash equilibrium. This is not the case in incomplete information environments, in which implementing mechanisms may fail to have Bayes-Nash equilibria.[8] Following Oury and Tercieux (2012), we assume that the planner is interested in implementing in interim correlated rationalizable strategies and Bayes-Nash equilibria.

[8]By assuming a variant of Assumption 1 of Oury and Tercieux (2012), Kunimoto et al. (2020) study implementation problems in interim rationalizable strategies without requiring the existence of Bayes-Nash equilibria.

Definition 5. A mechanism $\mathcal{M}$ *implements* $f : T \to \Delta(A)$ on $\mathcal{T}$ *in Bayes-Nash equilibria* if (i) $\overline{BNE}(U(\mathcal{M}, \mathcal{T})) \neq \emptyset$ and (ii) for all $\sigma \in BNE(U(\mathcal{M}, \mathcal{T}))$ and for all $t \in T$, $\bigcup_{m \in supp(\sigma(t))} g(m) = f(t)$.

Remark 1. It is clear from the definition of Bayes-Nash equilibrium that for any $\sigma \in BNE(U(\mathcal{M}, \mathcal{T}))$ and for any $t \in T$, $Supp(\sigma(t)) \subseteq S^{\mathcal{M},\mathcal{T}}(t)$.

Thus, Definition 4 implies part (ii) of the definition above. Thus, a mechanism $\mathcal{M}$ that implements an f in interim rationalizable strategies also implements f in Bayes-Nash equilibrium if and only if $\overline{BNE}(U(\mathcal{M}, \mathcal{T})) \neq \emptyset$.

The lemma below shows that a mechanism $\mathcal{M}$ that ICR-implements f also implements f in Bayes-Nash equilibrium if and only if $\mathcal{M}$ satisfies the *Equilibrium Best-Response Property* (EBRP). A mechanism $\mathcal{M}$ satisfies the EBRP on $\mathcal{T}$ if there exists a pure strategy profile σ such that for all $t \in T$,

$$\sigma(t) \in S^{\mathcal{M},\mathcal{T}}(t),$$

and for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$BR_i(\pi_i(t_i, \sigma_{-i}) | \mathcal{M}) \neq \emptyset.$$

Lemma 2. Assume that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$. $\mathcal{M}$ implements f on $\mathcal{T}$ in Bayes-Nash equilibria if and only if $\mathcal{M}$ satisfies the EBRP.

Proof. Assume that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$. Assume that $\mathcal{M}$ satisfies the EBRP on $\mathcal{T}$. Let us show that $\mathcal{M}$ implements f on $\mathcal{T}$ in Bayes-Nash equilibria. To this end, we need only to show that $\overline{BNE}(U(\mathcal{M}, \mathcal{T})) \neq \emptyset$. Since $\mathcal{M}$ ICR-implements f and $\mathcal{M}$ satisfies the EBRP, it follows that there exists a pure strategy profile σ such that for all $t \in T$, $\sigma(t) \in S^{\mathcal{M},\mathcal{T}}(t)$, and for all $i \in \mathcal{I}$ and all $t_i \in T_i$, $BR_i(\pi_i(t_i, \sigma_{-i}) | \mathcal{M}) \neq \emptyset$. Let us show that $\sigma \in \overline{BNE}(U(\mathcal{M}, \mathcal{T}))$.

For all $i \in \mathcal{I}$ and all $t_i \in T_i$, since $BR_i(\pi_i(t_i, \sigma_{-i}) | \mathcal{M}) \neq \emptyset$, let $\hat{\sigma}_i(t_i) \in BR_i(\pi_i(t_i, \sigma_{-i}) | \mathcal{M})$ for all $t_i \in T_i$ and all $i \in \mathcal{I}$. Fix any $i \in \mathcal{I}$. By construction, we see that for all $t \in T$, $(\hat{\sigma}_i(t_i), \sigma_{-i}(t_{-i})) \in S^{\mathcal{M},\mathcal{T}}(t)$. Moreover, since $\mathcal{M}$ ICR-implements f on $\mathcal{T}$, we also have that for all $t \in T$, $f(t) = g(\hat{\sigma}_i(t_i), \sigma_{-i}(t_{-i}))$. Thus, we can replace $\hat{\sigma}_i$ with σ_i and see that $\sigma_i(t_i) \in BR_i(\pi_i(t_i, \sigma_{-i})|\mathcal{M})$ for all $t_i \in T_i$. Since the choice of i was arbitrary, we have that $\sigma \in \overline{BNE}(U(\mathcal{M}, \mathcal{T}))$.

For the converse, assume that $\mathcal{M}$ implements f on $\mathcal{T}$ in Bayes-Nash equilibria. This implies that $\overline{BNE}(U(\mathcal{M}, \mathcal{T})) \neq \emptyset$. Thus, $\mathcal{M}$ satisfies the EBRP on $\mathcal{T}$. □

IV. Interim Iterative Monotonicity

In the following section, we present our necessary condition. Let $\mathcal{T}$ be any model. For every player $i \in \mathcal{I}$, let us call any map $\beta_i : T_i \to 2^{T_i} \backslash \{\emptyset\}$ as player i's deception. A special deception for player i is the truth-telling deception, β_i^t, defined by $\beta_i^t(t_i) = \{t_i\}$ for all $t_i \in T_i$. Another special deception for player i is denoted by $\bar{\beta}_i$ and defined by $\bar{\beta}_i(t_i) = T_i$. For any β_i and β_i' we write $\beta_i \subseteq \beta_i'$ if $\beta_i(t_i) \subseteq \beta_i'(t_i)$ for all $t_i \in T_i$. Let $\mathcal{B}_i$ be the set of all player i's deceptions containing the truth-telling deception; that is,

$$\mathcal{B}_i = \left\{\beta_i : T_i \to 2^{T_i} \backslash \{\emptyset\} \,|\beta_i^t \subseteq \beta_i\right\}.$$

Let $\mathcal{B} = \Pi_{i \in \mathcal{I}} \mathcal{B}_i$, with $\beta = (\beta_i)_{i \in \mathcal{I}}$ as a typical deception profile of $\mathcal{B}$.

For every $i \in \mathcal{I}$, let Y_i^f be the set of mappings from T_{-i} to $\Delta(A)$ satisfying the following requirement. Whatever is player i's actual type, he would never prefer the outcome to be selected by a mapping Y_i^f to the outcome he could obtain under f if all his opponents were reporting truthfully. Formally,

$$Y_i^f = \left\{ y : T_{-i} \to \Delta(A) \;\middle|\; \begin{array}{l} \text{For all } \tilde{t}_i \in T_i, \\ \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} \kappa(\tilde{t}_i)[\theta, t_{-i}]\, u_i\left(f\left(\tilde{t}_i, t_{-i}\right), \theta\right) \geq \\ \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} \kappa(\tilde{t}_i)[\theta, t_{-i}]\, u_i(y(t_{-i}), \theta). \end{array} \right\} \quad (2)$$

Note that Y_i^f is a metrizable separable space.[9] We write Y^f for $\Pi_{i \in \mathcal{I}} Y_i^f$. For all $i \in \mathcal{I}$,

[9]To see it, observe that $\Delta(A)$ is a separable metric space under the Prohokorov metric given that A is a separable metric space Aliprantis and Border (2006); Theorem 14.15). Moreover, a countable product of the space $\Delta(A)$ is separable metric space under the standard metric (see, e.g., Ok (2011), p. 196). Thus, since Y_i^f is a subset of a separable metric space, it follows that it is a separable metric space.

let $Y_{i,s}^f$ be the set of all mappings in Y_i^f satisfying the inequality in (2) strictly for all $\tilde{t}_i \in T_i$.[10] Similarly, we write Y_s^f for $\Pi_{i \in \mathcal{I}} Y_{i,s}^f$.

Assumption 1 used by Oury and Tercieux (2012) to characterize a class of implementable SCFs can be stated as follows.

Definition 6 (Assumption 1 of Oury and Tercieux (2012))**.** Let $\mathcal{T}$ be any model and let $f : T \to \Delta(A)$ be any SCF. For all $i \in \mathcal{I}$, there exists $\bar{y}_i : T_{-i} \to \Delta(A)$ such that for all $\psi_i \in \Delta(\Theta \times T_{-i})$, there exists $y_i \in Y_i^f$ satisfying

$$\begin{array}{ll} \sum_{(\theta,t'_{-i})\in\Theta\times T_{-i}} \psi_i\left[\theta, t'_{-i}\right] u_i\left(y_i\left(t'_{-i}\right), \theta\right) & > \\ \sum_{(\theta,t'_{-i})\in\Theta\times T_{-i}} \psi_i\left[\theta, t'_{-i}\right] u_i\left(\bar{y}_i\left(t'_{-i}\right), \theta\right). & \end{array}$$

The assumption requires that player i's preferences over the mappings from T_{-i} to $\Delta(A)$ are such that there exists a mapping $\bar{y}_i : T_{-i} \to \Delta(A)$ such that, whatever his beliefs over $\Theta \times T_{-i}$ are, the mapping $\bar{y}_i$ is never his top-ranked mapping.

For the sake of clarity, in what follows, for every $i \in \mathcal{I}$, we write $T_{-i} \times \hat{T}_{-i}$ for $T_{-i} \times T_{-i}$. In the context of a mechanism, our interpretation of $\left(t_{-i}, \hat{t}_{-i}\right) \in T_{-i} \times \hat{T}_{-i}$ is that player i's opponents are of types t_{-i} but they are playing as if they were of types $\hat{t}_{-i}$.

A deception profile $\beta \in \mathcal{B}$ is *acceptable* on $\mathcal{T}$ for f if for all $t, t' \in T$, $t' \in \beta(t) \Longrightarrow f(t) = f(t')$. The following condition is due to Oury and Tercieux (2012).

Definition 7. $f : T \to \Delta(A)$ satisfies *interim (correlated) rationalizable monotonicity* (IRM, henceforth) on $\mathcal{T}$ if for every unacceptable deception profile $\beta \in \mathcal{B}$ on $\mathcal{T}$ for f, there exists $(i, t_i, t'_i) \in \mathcal{I} \times T_i \times \beta_i(t_i)$ such that for all $\psi_i(t_i) \in$

[10]Formally, for all $i \in \mathcal{I}$,

$$Y_{i,s}^f = \left\{ y : T_{-i} \to \Delta(A) \;\middle|\; \begin{array}{l} \text{For all } \tilde{t}_i \in T_i, \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa\left(\tilde{t}_i\right)\left[\theta, t_{-i}\right] u_i\left(f\left(\tilde{t}_i, t_{-i}\right), \theta\right) > \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa\left(\tilde{t}_i\right)\left[\theta, t_{-i}\right] u_i\left(y\left(t_{-i}\right), \theta\right). \end{array} \right\}$$

$\Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, there exists $y_i^* \in Y_i^f$ such that

$$\begin{array}{ll} \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\psi_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(y_i^*\left(\hat{t}_{-i}\right),\theta\right) & > \\ \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\psi_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(f\left(t_i',\hat{t}_{-i}\right),\theta\right). & \end{array} \tag{3}$$

Remark 2. Oury and Tercieux (2012) introduce a strict variant of IRM. f satisfies strict IRM on $\mathcal{T}$ if $y^* \in Y_i^f$ satisfying (3) is such that it satisfies the inequality in (2) strictly for $t_i' = \tilde{t}_i$. However, it can be shown that the two conditions are equivalent.[11]

A condition, which is equivalent to IRM, can be expressed in terms of the limit point of an iterative net of deception profiles. To define the net, we need additional notation. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $\Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ be defined by

$$\Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) = \left\{\nu_i \in \Delta\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) | \text{marg}_{\Theta\times T_{-i}}\nu_i = \kappa\left(t_i\right)\right\},$$

and, moreover, for all $\beta \in \mathcal{B}$, let $\Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ be defined by

$$\Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) = \left\{ \nu_i \;\middle|\; \begin{array}{l} \nu_i \in \Delta\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \text{ and} \\ \nu_i\left[\theta, t_{-i}, \hat{t}_{-i}\right] > 0 \implies \hat{t}_{-i} \in \beta_{-i}\left(t_{-i}\right) \end{array} \right\}.$$

The iterative net, denoted by $\left(\beta^\alpha\right)_{\alpha\in\Omega}$, is defined as follows. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let

$$\beta_i^0\left(t_i\right) = \bar{\beta}_i\left(t_i\right) = T_i,$$

and, for all ordinal numbers $\alpha \in \Omega$, define $\beta_i^\alpha\left(t_i\right)$ as follows:

[11] The formal arguments are available from authors on request.

- If α is a successor ordinal, then

$$\beta_i^{\alpha}(t_i) = \left\{ \hat{t}_i \;\middle|\; \begin{array}{l} \hat{t}_i \in \beta_i^{\alpha-1}(t_i) \text{ and there exists} \\ \nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \text{ such} \\ \text{that } \nu_i(t_i) \in \Delta^{\beta_{-i}^{\alpha-1}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \text{ and} \\ \sum_{(\theta,\hat{t}_{-i}) \in \Theta \times \hat{T}_{-i}} \left(\operatorname{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i)\left[\theta, \hat{t}_{-i}\right]\right) u_i\left(f\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) \geq \\ \sum_{(\theta,\hat{t}_{-i}) \in \Theta \times \hat{T}_{-i}} \left(\operatorname{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i)\left[\theta, \hat{t}_{-i}\right]\right) u_i\left(y_i\left(\hat{t}_{-i}\right), \theta\right), \\ \text{for all } y_i \in Y_i^f. \end{array} \right\} \tag{4}$$

- If α is a limit ordinal, then

$$\beta_i^{\alpha}(t_i) = \bigcap_{\gamma<\alpha} \beta_i^{\gamma}(t_i). \tag{5}$$

Observe that $t_i \in \beta_i^{\alpha}(t_i)$ for all $i \in \mathcal{I}$, all $t_i \in T_i$ and all $\alpha \in \Omega$. A net $(\beta^{\alpha})_{\alpha\in\Omega}$ is monotonic decreasing if $\beta^{\alpha+1} \subseteq \beta^{\alpha}$ for all $\alpha \in \Omega$. If the limit of $(\beta^{\alpha})_{\alpha\in\Omega}$ exists, we denote it by β^*; that is, $\lim_{\alpha\in\Omega} \beta^{\alpha} \to \beta^*$.

Lemma 3. Let $\mathcal{T}$ be any model. $(\beta^{\alpha})_{\alpha\in\Omega}$ is a monotonic decreasing net such that $\lim_{\alpha\in\Omega} \beta^{\alpha} \to \beta^* \in \mathcal{B}$. Moreover, there exists an ordinal $\alpha \in \Omega$ such that $\beta^{\alpha} = \beta^{\alpha+1} = \beta^*$.

Proof. Let $\mathcal{T}$ be any model. Let $(\beta^{\alpha})_{\alpha\in\Omega}$ be given. By definition $(\beta^{\alpha})_{\alpha\in\Omega}$, it holds that $\beta^t \subseteq \beta^{\alpha}$ for all $\alpha \in\Omega$. Thus, $\beta^{\alpha} \in \mathcal{B}$ for all $\alpha \in\Omega$ and it is bounded below. Moreover, since $(\beta^{\alpha})_{\alpha\in\Omega}$ is bounded below, it holds that $\lim_{\alpha\in\Omega} \beta^{\alpha} \to \beta^* \in \mathcal{B}$ if it is a monotonic decreasing net. Thus, we show that $(\beta^{\alpha})_{\alpha\in\Omega}$ is a monotonic decreasing net. Fix any ordinal $\alpha \in\Omega$. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. We show that $\beta_i^{\alpha+1}(t_i) \subseteq \beta_i^{\alpha}(t_i)$. Let us proceed according to whether α is a successor ordinal or not.

- Suppose that α is a successor ordinal. Fix any $\hat{t}_i \in \beta_i^{\alpha+1}(t_i)$. We show that $\hat{t}_i \in \beta_i^{\alpha}(t_i)$. (4) implies that $\hat{t}_i \in \beta_i^{\alpha}(t_i)$, as we sought.

- Suppose that α is a limit ordinal. Since α is a limit ordinal, it follows that $\alpha+1$

is a successor ordinal. Suppose that $\hat{t}_i \in \beta_i^{\alpha+1}(t_i)$. We show that $\hat{t}_i \in \beta_i^{\alpha}(t_i)$. Again, (4) implies that $\hat{t}_i \in \beta_i^{\alpha}(t_i)$, as we sought.

Since the choice of α was arbitrary, it follows that $\lim_{\alpha\in\Omega}\beta^{\alpha} \to \beta^* \in \mathcal{B}$. Finally, the fact that there exists an ordinal $\alpha \in \Omega$ such that $\beta^{\alpha} = \beta^{\alpha+1} = \beta^*$ follows from the assumption that T_i is a countable set for each player i and the fact that Ω is an uncountable set. To see this, fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. Assume that, for all $\alpha \in \Omega$, it holds that $\beta_i^{\alpha+1}(t_i) \subset \beta_i^{\alpha}(t_i)$.[12] Define the mapping $f : \Omega \to T_i$ by $f(\alpha) \in \beta_i^{\alpha}(t_i) \setminus \beta_i^{\alpha+1}(t_i)$, for all $\alpha \in \Omega$. Let us show that f is an injective mapping. Fix any $\alpha, \alpha' \in \Omega$ such that $\alpha \neq \alpha'$. Let us show that $f(\alpha) \neq f(\alpha')$. Since Ω is a well ordered set, it is Without loss of generality, let $\alpha' > \alpha$. Since $\beta_i^{\alpha'}(t_i) \subseteq \beta_i^{\alpha+1}(t_i) \subset \beta_i^{\alpha}(t_i)$, it follows from definition of f that $f(\alpha) \neq f(\alpha')$. Since f is an injective mapping from Ω to T_i, it follows that Ω is a countable set, which is a contradiction. Thus, for all $i \in \mathcal{I}$, all $t_i \in T_i$, there exists $\alpha \in \Omega$ such that $\beta_i^{\alpha+1}(t_i) = \beta_i^{\alpha}(t_i)$. Since Ω is an uncountable set whose elements are ordered by $\geq$, it follows that there exists $\alpha \in \Omega$ such that $\beta^{\alpha+1} = \beta^{\alpha}$. □

Our condition can be stated as follows.

Definition 8. $f : T \to \Delta(A)$ satisfies *Interim Iterative Monotonicity* (IIM, henceforth) on $\mathcal{T}$ if β^* is an acceptable deception on $\mathcal{T}$ for f.

The following result shows that IIM is equivalent to IRM.

Lemma 4. $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$ if and only if f satisfies IRM on $\mathcal{T}$.

Proof. Assume that $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$. Take any unacceptable deception profile $\beta \in \mathcal{B}$ on $\mathcal{T}$ for f. Assume, to the contrary, that for all $(i, t_i, t_i') \in \mathcal{I} \times T_i \times \beta_i(t_i)$, there exists $\psi_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ such that for all $y_i^* \in Y_i^f$, (3) is violated.[13] To derive a contradiction, let us first show that $\beta \subseteq \beta^{\alpha}$ for all $\alpha \in \Omega$. Let us proceed by transfinite induction.

By definition, $\beta \subseteq \bar{\beta} = \beta^0$. Fix an arbitrary $\alpha \in \Omega$ and suppose that for all $\gamma < \alpha$, it holds that $\beta \subseteq \beta^{\gamma}$. To complete the proof we need to show that $\beta \subseteq \beta^{\alpha}$.

[12] The symbol $\subset$ denotes strict set inclusion.

[13] Recall that Y^f is a nonempty metrizable subspace.

We proceed according to whether α is a limit ordinal or a successor ordinal. When α is a limit ordinal, the induction hypothesis and the definition of β^{α} implies that $\beta \subseteq \bigcap_{\gamma<\alpha} \beta^{\gamma} = \beta^{\alpha}$.

Suppose that α is a successor ordinal. Let us show that $\beta \subseteq \beta^{\alpha}$. By the inductive hypothesis, it holds that $\psi_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta_{-i}^{\alpha-1}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$. Fix any $i \in \mathcal{I}$ and any $t_i \in T_i$. Take any $\hat{t}_i \in \beta_i(t_i)$. It follows from the inductive hypothesis that $\hat{t}_i \in \beta_i^{\alpha-1}(t_i)$. Since (3) is violated for $y_i^* \in Y_i^f$, (4) implies that $\hat{t}_i \in \beta_i^{\alpha}(t_i)$. Since the triplet $\left(i, t_i, \hat{t}_i\right) \in \mathcal{I} \times T_i \times \beta_i(t_i)$ was chosen arbitrarily, we conclude that $\beta \subseteq \beta^{\alpha}$. By the principle of transfinite induction, it holds that $\beta \subseteq \beta^{\alpha}$ for all $\alpha \in \Omega$. Since Lemma 3 implies that the $(\beta^{\alpha})_{\alpha\in\Omega}$ is a monotonically decreasing net which converges to $\beta^* \in \mathcal{B}$, we have that $\beta \subseteq \beta^*$. Since f satisfies IIM on $\mathcal{T}$, it follows that β^* is an acceptable deception profile on $\mathcal{T}$ for f, and so β is also an acceptable deception profile on T for f, which is a contradiction.

Assume f satisfies IRM on $\mathcal{T}$. Assume, to the contrary, that $\beta^* \in \mathcal{B}$ is not acceptable on $\mathcal{T}$ for f. Since f satisfies IRM, it follows that there exists $(i, t_i, t_i') \in \mathcal{I} \times T_i \times \beta_i^*(t_i)$ such that for all $\psi_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta^*_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, there exists $y_i^* \in Y_i^f$ such that (3) is satisfied. Lemma 3 implies that there exists an $\alpha \in \Omega$ such that $\beta^{\alpha} = \beta^{\alpha+1} = \beta^*$. Since $t_i' \in \beta_i^*(t_i)$, (4) implies that there exists $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta^*_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ such that

$$\begin{array}{l}\sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) \quad \geq \\ \sum_{(\theta,\hat{t}_{-i})\in\Theta\times\hat{T}_{-i}}\left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i(t_i)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(y_i^*\left(\hat{t}_{-i}\right),\theta\right)\end{array}$$

for all $y_i^* \in Y_i^f$, yielding a contradiction. □

Any SCF satisfying our condition on $\mathcal{T}$ is *incentive compatible* on $\mathcal{T}$. The condition can be stated as follows.

Definition 9. $f : T \to \Delta(A)$ *incentive compatible* on $\mathcal{T}$ if for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$\sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta,t_{-i}]\, u_i(f(t_i,t_{-i}),\theta) \geq \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta,t_{-i}]\, u_i(f(t_i',t_{-i}),\theta)$$

for all $t_i \in T_i$.

Lemma 5. $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$ implies that f is incentive compatible on $\mathcal{T}$.

Proof. It follows from Lemma 4 above and Lemma 3 of Oury and Tercieux (2012). □

V. A full characterization

Our main result can be stated as follows.

Theorem 1. The following statements are equivalent.

(i) $f : T \to \Delta(A)$ is ICR-implementable on $\mathcal{T}$ by a mechanism satisfying the EBRP.

(ii) $f : T \to \Delta(A)$ satisfies IRM on $\mathcal{T}$.

(iii) $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$.

(iv) $f : T \to \Delta(A)$ is ICR-implementable and Bayes-Nash implementable on $\mathcal{T}$.

Proof of Theorem 1

The proof that part (i) implies part (ii) can be found in Appendix. Lemma 4 implies that part (ii) implies (iii). Lemma 2 implies that part (iv) implies (i). Finally, we show that part (iii) implies part (iv). Thus, assume that $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$. We show that $f : T \to \Delta(A)$ is ICR-implementable on $\mathcal{T}$ by a mechanism satisfying the EBRP. Before proving this result, we need additional notation. Fix any $\beta \in \mathcal{B}$, and any $i \in \mathcal{I}$. Let $\Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$ be defined by

$$\Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right) = \left\{ \psi_i \;\middle|\; \begin{array}{l} \text{There exists } \nu_i(t_i) \in \Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \\ \text{such that } \operatorname{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i) = \psi_i. \end{array} \right\} \tag{6}$$

Since for all $t_{-i} \in T_{-i}$, $\bar{\beta}_{-i}(t_{-i}) = T_{-i}$, it follows that $\Delta^{\bar{\beta}_{-i}}\left(\Theta \times \hat{T}_{-i}\right) = \Delta\left(\Theta \times \hat{T}_{-i}\right)$.

The following definition is critical in the construction of our implementing mechanism.

Definition 10. Let $\mathcal{T}$ be any model. For all $\beta \in \mathcal{B}$ and all $i \in \mathcal{I}$, $i \in \mathcal{I}(\beta)$ if and only if for all $\psi_i \in \Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exist $y_i, \bar{y}_i \in Y_i^f$ such that

$$\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \psi_i\left[\theta, \hat{t}_{-i}\right] u_i\left(y_i\left(\hat{t}_{-i}\right), \theta\right) > \sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \psi_i\left[\theta, \hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right), \theta\right). \quad (7)$$

The above definition says that $i \in \mathcal{I}(\beta)$ provided that for every belief ψ_i of player i over $\Theta \times \hat{T}_{-i}$, there are mappings $y_i, \bar{y}_i \in Y_i^f$ that may depend on his belief ψ_i such that y_i is strictly preferred to $\bar{y}_i$, given his belief ψ_i. A stronger, though more desirable, definition would be to require that the mapping $\bar{y}_i$ does not depend on player i's belief. The definition can be stated as follows.

Definition 11. Let $\mathcal{T}$ be any model. For all $\beta \in \mathcal{B}$ and all $i \in \mathcal{I}$, $i \in \mathcal{I}^*(\beta)$ if and only if there exists $\bar{y}_i \in Y_i^f$ such that for all $\psi_i \in \Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exists $y_i \in Y_i^f$ such that

$$\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \psi_i\left[\theta, \hat{t}_{-i}\right] u_i\left(y_i\left(\hat{t}_{-i}\right), \theta\right) > \sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \psi_i\left[\theta, \hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right), \theta\right). \quad (8)$$

Observe that Definition 11 is equivalent to Assumption 1 of Oury and Tercieux (2012) when $\beta = \bar{\beta}$. We show below that Definition 10 and Definition 11 are equivalent.

Lemma 6. Let $\mathcal{T}$ be any model. For all $\beta \in \mathcal{B}$, $\mathcal{I}^*(\beta) = \mathcal{I}(\beta)$.

Proof. Let $\mathcal{T}$ be any model. Fix any $\beta \in \mathcal{B}$. Since it is clear that $\mathcal{I}^*(\beta) \subseteq \mathcal{I}(\beta)$, let us show that $\mathcal{I}(\beta) \subseteq \mathcal{I}^*(\beta)$. Assume that $i \in \mathcal{I}(\beta)$. Definition 10 implies that for all $\psi_i \in \Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exist $y_i^{\psi_i}, \bar{y}_i^{\psi_i} \in Y_i^f$ such that (8) is satisfied. Since $\Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$ is a separable metric space, let $\hat{\Delta}\left(\Theta \times \hat{T}_{-i}\right) = \cup_{k \in \mathbb{N}}\{\psi_{i,k}\}$ be a countable, dense subset of $\Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$. Let $\tilde{y}_i \in Y_i^f$ be a mapping defined by

$$\tilde{y}_i = \sum_{k=1}^{\infty} \frac{1}{2^k} \bar{y}_i^{\psi_{i,k}}.$$

For all $\bar{k} \in \mathbb{N}$, let $y_i^{\psi_{i,\bar{k}}} \in Y_i^f$ be a mapping defined by

$$y_i^{\bar{k}} = \sum_{k \neq \bar{k}} \frac{1}{2^k} \bar{y}_i^{\psi_{i,k}} + \frac{1}{2^{\bar{k}}} y_i^{\psi_{i,\bar{k}}}.$$

Thus, for all $k \in \mathbb{N}$, we have that

$$\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \psi_{i,k} \left[\theta, \hat{t}_{-i}\right] u_i \left(y_i^k \left(\hat{t}_{-i}\right), \theta\right) > \sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \psi_i \left[\theta, \hat{t}_{-i}\right] u_i \left(\tilde{y}_i \left(\hat{t}_{-i}\right), \theta\right),$$

where the strict inequality is guaranteed by (7). Since player i's preference over lotteries are continuous and since, moreover, $\hat{\Delta}\left(\Theta \times \hat{T}_{-i}\right)$ is a countable, dense subset of $\Delta^{\beta_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, it follows that $i \in \mathcal{I}^*(\beta)$. Since the choice of $i \in \mathcal{I}(\beta)$ was arbitrary, it follows that $\mathcal{I}(\beta) \subseteq \mathcal{I}^*(\beta)$. □

In what follows, to avoid trivialities, we assume that $\mathcal{I}\left(\bar{\beta}\right) \neq \emptyset$. Moreover, we will also assume that $\mathcal{I}(\beta^*) = \mathcal{I}$. The reason is that if $\mathcal{I}(\beta^*) \neq \mathcal{I}$, part (ii) of the above lemma implies that the planner's objective is constant on $\Pi_{i \in \mathcal{I}^c(\beta^*)} T_i \equiv T_{\mathcal{I}^c(\beta^*)}$, where $\mathcal{I}^c(\beta^*)$ is the complement of $\mathcal{I}^c(\beta^*)$. Therefore, the planner can, equivalently, focus on the implementation of an SCF $\hat{f} : \Pi_{i \in \mathcal{I}(\beta^*)} T_i \to \Delta(A)$ defined, for all $t \in \Pi_{i \in \mathcal{I}(\beta^*)} T_i$, by $\hat{f}(t) = f(t, t')$ for all $t' \in T_{\mathcal{I}^c(\beta^*)}$. This is justified by the following lemmata.

Lemma 7. Assume that $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$. For all $\alpha \in \Omega$ and all $i \in \mathcal{I}$, $i \in \mathcal{I}^c(\beta^\alpha) \implies \beta_i^\alpha = \beta_i^{\alpha+1} = \bar{\beta}_i$.

Proof. Assume that $f : T \to \Delta(A)$ satisfies IIM on $\mathcal{T}$. Fix any $\alpha \in \Omega$. Assume that $i \in \mathcal{I}^c(\beta^\alpha)$. Assume, to the contrary, $\beta_i^{\alpha+1} \neq \beta_i^\alpha$. Since Lemma 3 implies that $(\beta^\alpha)_{\alpha \in \Omega}$ is a monotonic decreasing net, it follows that there exists $\left(t_i, \hat{t}_i\right)$ such that $\hat{t}_i \in \beta_i^\alpha(t_i)$ and $\hat{t}_i \notin \beta_i^{\alpha+1}(t_i)$. It follows from (4) that for all $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta_{-i}^\alpha}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$,

$$\begin{aligned} &\textstyle\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \left(\text{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i) \left[\theta, \hat{t}_{-i}\right]\right) u_i \left(f\left(\hat{t}_i, \hat{t}_{-i}\right), \theta\right) \quad < \\ &\textstyle\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \left(\text{marg}_{\Theta \times \hat{T}_{-i}} \nu_i(t_i) \left[\theta, \hat{t}_{-i}\right]\right) u_i \left(\bar{y}_i \left(\hat{t}_{-i}\right), \theta\right) \end{aligned}$$

for some $\bar{y}_i \in Y_i^f$. Therefore, for all $\psi_i \in \Delta^{\beta^\alpha_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$,

$$\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)$$

for some $\bar{y}_i \in Y_i^f$. Let $y_i\left(\hat{t}_i,\cdot\right) = f\left(\hat{t}_i,\cdot\right)$. Since f satisfies IIM on $\mathcal{T}$, Lemma 4 and Lemma 5 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i\left(\hat{t}_i,\cdot\right) \in Y_i^f$. Definition 10 implies that $i \in \mathcal{I}\left(\beta^\alpha\right)$, yielding a contradiction.

Finally, let us show that $\beta_i^{\alpha+1} = \beta_i^\alpha = \bar{\beta}_i$. Assume, to the contrary, that $\beta_i^{\alpha+1} = \beta_i^\alpha \neq \bar{\beta}_i$. Since Lemma 3 implies that $\left(\beta_i^\alpha\right)_{\alpha\in\Omega}$ is a decreasing monotonic net, it follows that there exists a successor ordinal $\hat{\alpha}$ such that $0 < \hat{\alpha} \leq \alpha$ such that $\beta_i^{\hat{\alpha}} \subseteq \beta_i^{\hat{\alpha}-1}$ and $\beta_i^{\hat{\alpha}} \neq \beta_i^{\hat{\alpha}-1}$.[14] It follows that $\beta_i^{\hat{\alpha}}\left(t_i\right) \subseteq \beta_i^{\hat{\alpha}-1}\left(t_i\right)$ and $\beta_i^{\hat{\alpha}}\left(t_i\right) \neq \beta_i^{\hat{\alpha}-1}\left(t_i\right)$ for some $t_i \in T_i$, and so $\hat{t}_i \in \beta_i^{\hat{\alpha}-1}\left(t_i\right)$ and $t_i \notin \beta_i^{\hat{\alpha}}\left(t_i\right)$ for some $\hat{t}_i, t_i \in T_i$. (4) implies that there exists $\bar{y}_i \in Y_i^f$ such that

$$\begin{array}{l}\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}}\left(\operatorname{marg}_{\Theta\times\hat{T}_{-i}}\nu_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) \quad < \\ \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}}\left(\operatorname{marg}_{\Theta\times\hat{T}_{-i}}\nu_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)\end{array}$$

for all $\nu_i\left(t_i\right) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta^{\hat{\alpha}-1}_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$. By definition of $\Delta^{\beta^{\hat{\alpha}-1}_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$ in (6), it follows that there exists $\bar{y}_i \in Y_i^f$ such that

$$\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)$$

for all $\psi_i \in \Delta^{\beta^{\hat{\alpha}-1}_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$. Let $y_i\left(\hat{t}_i,\cdot\right) = f\left(\hat{t}_i,\cdot\right)$. Since f satisfies IIM on $\mathcal{T}$, Lemma 4 and Lemma 5 imply that f is incentive compatible on $\mathcal{T}$. This implies

[14] Suppose not. Then, for all successor ordinals $\hat{\alpha}$ such that $\hat{\alpha} \leq \alpha$, it holds that $\beta_i^{\hat{\alpha}} = \beta_i^{\hat{\alpha}-1}$. Suppose that $\beta_i^{\hat{\alpha}} = \bar{\beta}_i$ for all successor ordinals $\hat{\alpha}$ such that $\hat{\alpha} \leq \alpha$. It follows that for every limit ordinal $\delta \leq \alpha$, it holds that $\beta_i^\delta\left(t_i\right) = \bigcap_{\gamma<\delta}\beta_i^\gamma\left(t_i\right) = \bar{\beta}_i\left(t_i\right)$ for all $t_i \in T_i$. An immediate contradiction is obtain if α is a limit ordinal. Thus, let α be a successor ordinal, and so $\beta_i^\alpha = \bar{\beta}_i$, which is a contradiction. Thus, there exists a successor ordinal α', with $\alpha' \leq \alpha$, such that $\beta_i^{\alpha'} \neq \bar{\beta}_i$. Since for all $\tilde{\alpha} \in \Omega$, $\beta_i^{\tilde{\alpha}} = \beta_i^{\tilde{\alpha}+1}$, it follows that for all successor ordinals $\hat{\alpha}$ such that $\hat{\alpha} \leq \alpha$, it holds that $\beta_i^{\hat{\alpha}} \neq \bar{\beta}_i$. Since $1 \in \Omega$ is a successor ordinal, it follows that there exists a successor ordinal such that $\beta_i^1 \subseteq \beta_i^0 = \bar{\beta}_i$, yielding a contradiction.

that $y_i\left(\hat{t}_i,\cdot\right)\in Y_i^f$. Definition 10 implies that $i\in\mathcal{I}\left(\beta^{\hat{\alpha}-1}\right)$. Since Lemma 3 implies that $\left(\beta_i^\alpha\right)_{\alpha\in\Omega}$ is a decreasing monotonic sequence and since, moreover, $\hat{\alpha}$ is such that $0\neq\hat{\alpha}\leq\alpha$, it follows that there exist $\bar{y}_i, y_i\left(\hat{t}_i,\cdot\right)\in Y_i^f$ such that

$$\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}}\psi_i\left[\theta,\hat{t}_{-i}\right]u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right)<\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}}\psi_i\left[\theta,\hat{t}_{-i}\right]u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)$$

for all $\psi_i\in\Delta^{\beta_{-i}^{\alpha}}\left(\Theta\times\hat{T}_{-i}\right)\subseteq\Delta^{\beta_{-i}^{\hat{\alpha}-1}}\left(\Theta\times\hat{T}_{-i}\right)$. Definition 10 implies that $i\in\mathcal{I}\left(\beta^\alpha\right)$, which is a contradiction. Thus, $\beta_i^{\alpha+1}=\beta_i^\alpha=\bar{\beta}_i$. □

Lemma 8. Assume that $f:T\to\Delta\left(A\right)$ satisfies IIM on $\mathcal{T}$.

(i) If $\mathcal{I}\left(\bar{\beta}\right)=\emptyset$, then f is constant.[15]

(ii) If $\mathcal{I}\left(\beta^*\right)\neq\mathcal{I}$, then for all $i\in\mathcal{I}^c\left(\beta^*\right)$, all $t_{-i}\in T_{-i}$ and all $t_i,t_i'\in T_i$, $f\left(t_i,t_{-i}\right)=f\left(t_i',t_{-i}\right)$.

Proof. Assume that $f:T\to\Delta\left(A\right)$ satisfies IIM on $\mathcal{T}$. To show part (i), assume that $\mathcal{I}\left(\bar{\beta}\right)=\emptyset$. If $\beta^*=\bar{\beta}$, then $\bar{\beta}$ is an acceptable deception profile on $\mathcal{T}$ for f. This implies that f is constant. Thus, the complete the proof, let us show that $\beta^*=\bar{\beta}$. Assume, to the contrary, that $\beta^*\neq\bar{\beta}$. Then, there exists $\left(i,t_i\right)\in\mathcal{I}\times T_i$ such that $\beta_i^*\left(t_i\right)\neq T_i=\bar{\beta}_i\left(t_i\right)$. Since $\beta_i^*\left(t_i\right)\subseteq\bar{\beta}_i\left(t_i\right)=T_i$, it follows that there exists $\hat{t}_i\in\bar{\beta}_i\left(t_i\right)=T_i$ such that $\hat{t}_i\notin\beta_i^*\left(t_i\right)$. Since β^* is the limit point of $\left(\beta^k\right)_{k\geq0}$ and since, by Lemma 3, $\beta^*\subseteq\beta^k$ for all $k\geq0$, it follows from (4) and the fact that $\beta_i^0\left(t_i\right)=\bar{\beta}_i\left(t_i\right)$ that there exists $k+1$ such that $\hat{t}_i\notin\beta_i^{k+1}\left(t_i\right)$, $\hat{t}_i\in\beta_i^k\left(t_i\right)$ and for all $\nu_i\left(t_i\right)\in\Delta^{\kappa\left(t_i\right)}\left(\Theta\times T_{-i}\times\hat{T}_{-i}\right)\cap\Delta^{\beta_{-i}^k}\left(\Theta\times T_{-i}\times\hat{T}_{-i}\right)$,

$$\begin{array}{l}\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}}\left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right)\quad<\\ \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}}\left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right)u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)\end{array}$$

for some $\bar{y}_i\in Y_i^f$. Since $\Delta^{\beta_{-i}^k}\left(\Theta\times T_{-i}\times\hat{T}_{-i}\right)\subseteq\Delta^{\bar{\beta}_{-i}}\left(\Theta\times T_{-i}\times\hat{T}_{-i}\right)=\Delta\left(\Theta\times T_{-i}\times\hat{T}_{-i}\right)$,

[15] f is constant if for all $t,t'\in T$, $f\left(t\right)=f\left(t'\right)$.

we can write that for all $\psi_i \in \Delta\left(\Theta \times \hat{T}_{-i}\right)$,

$$\sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(f\left(\hat{t}_i,\hat{t}_{-i}\right),\theta\right) < \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \psi_i\left[\theta,\hat{t}_{-i}\right] u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right),\theta\right)$$

for some $\bar{y}_i \in Y_i^f$. Let $y_i\left(\hat{t}_i,\cdot\right) = f\left(\hat{t}_i,\cdot\right)$. Since f satisfies IIM on $\mathcal{T}$, Lemma 4 and Lemma 5 imply that f is incentive compatible on $\mathcal{T}$. This implies that $y_i\left(\hat{t}_i,\cdot\right) \in Y_i^f$. Definition 10 implies that $i \in \mathcal{I}\left(\bar{\beta}\right)$, yielding a contradiction. This completes the proof of part (i).

Let us show part (ii). Assume that $\mathcal{I}(\beta^*) \neq \mathcal{I}$. Suppose that $\beta_i^* = \bar{\beta}_i$ for all $i \in \mathcal{I}^c(\beta^*)$. Since f satisfies IIM on $\mathcal{T}$, it follows that β^* is an acceptable deception profile on $\mathcal{T}$ for f. Fix any $i \in \mathcal{I}^c(\beta^*)$ and any $t_i \in T_i$. Since $\beta_i^* = \bar{\beta}_i$, we have that $\beta_i^*(t_i) = \bar{\beta}_i(t_i) = T_i$. Since f satisfies IIM on $\mathcal{T}$, we have that for all $t_{-i} \in T_{-i}$, $f(t_i', t_{-i}) = f(t_i'', t_{-i})$ for all $t_i', t_i'' \in \beta_i^*(t_i) = \bar{\beta}_i(t_i) = T_i$. Since the choice of $i \in \mathcal{I}^c(\beta^*)$ was arbitrary, the statement of part (ii) follows if we show that $\beta_i^* = \bar{\beta}_i$ for all $i \in \mathcal{I}^c(\beta^*)$. To this end, fix any $i \in \mathcal{I}^c(\beta^*)$. Assume that $\beta_i^* \neq \bar{\beta}_i$. Then, there exists $t_i \in T_i$ such that $\beta_i^*(t_i) \neq T_i = \bar{\beta}_i(t_i)$. A contradiction can be derived by using the same reasoning used in part (i). This completes the proof of part (ii). □

Lemma 9. For all $(\alpha, i) \in \Omega \times \mathcal{I}$, if $i \in \mathcal{I}(\beta^\alpha) \backslash \mathcal{I}(\beta^0)$, then there exists $\hat{\alpha} \leq \alpha$ such that $i \in \mathcal{I}\left(\beta^{\hat{\alpha}}\right)$ and $i \in \mathcal{I}^c(\beta^\gamma)$ for all $\gamma < \hat{\alpha}$.

Proof. Fix any pair $(\alpha, i) \in \Omega \times \mathcal{I}$ such that $i \in \mathcal{I}(\beta^\alpha) \backslash \mathcal{I}(\beta^0)$. Assume, to the contrary, that there does not exist any $\hat{\alpha} \in \Omega$ with $\hat{\alpha} \leq \alpha$ such that $i \in \mathcal{I}\left(\beta^{\hat{\alpha}}\right)$ and $i \in \mathcal{I}^c(\beta^\gamma)$ for all $\gamma < \hat{\alpha}$. Thus, for all $\hat{\alpha} \in \Omega$ with $\hat{\alpha} \leq \alpha$, it holds that $i \in \mathcal{I}^c\left(\beta^{\hat{\alpha}}\right)$ or $i \in \mathcal{I}(\beta^\gamma)$ for some $\gamma < \hat{\alpha}$. Suppose that there exists $\hat{\alpha} \in \Omega$ with $\hat{\alpha} \leq \alpha$ such that $i \in \mathcal{I}(\beta^\gamma)$ for some $\gamma < \hat{\alpha}$. Let us consider the set $\bar{\Omega} = \left\{\delta \in \Omega \backslash \{0\} \,|\, \delta \leq \gamma < \hat{\alpha} \text{ and } i \in \mathcal{I}\left(\beta^\delta\right)\right\}$. Let $\gamma^* \in \bar{\Omega}$ be such that $\gamma^* \leq \delta$ for all $\delta \in \bar{\Omega}$. We have that $i \in \mathcal{I}\left(\beta^{\gamma^*}\right)$ and $i \in \mathcal{I}^c(\beta^\gamma)$ for all $\gamma < \gamma^*$, which is a contradiction. Therefore, suppose that for all $\hat{\alpha} \in \Omega$ with $\hat{\alpha} \leq \alpha$, it holds that $i \in \mathcal{I}^c(\beta^\gamma)$ for all $\gamma < \hat{\alpha}$. Since $i \in \mathcal{I}(\beta^\alpha)$ and since $i \in \mathcal{I}^c(\beta^\gamma)$ for all $\gamma < \alpha$, we have a contradiction.

□

The following result is useful in defining *Rule 3* of the mechanism.

Lemma 10. Let $\mathcal{T}$ be any model. For all $i \in \mathcal{I}(\beta^*)$, there exists $\hat{y}_i \in \Delta(A)$ such that for all $\phi_i \in \Delta(\Theta)$, there exists $y_i \in \Delta(A)$ such that

$$\sum_{\theta \in \Theta} \phi_i(\theta) u_i(y_i, \theta) > \sum_{\theta \in \Theta} \phi_i(\theta) u_i(\hat{y}_i, \theta) . \tag{9}$$

Proof. Fix any $i \in \mathcal{I}(\beta^*)$. Lemma 6 implies that $i \in \mathcal{I}^*(\beta^*)$. Definition 11 implies that there exists $\bar{y}_i \in Y_i^f$ such that for all $\psi_i \in \Delta^{\beta^*_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exists $y_i \in Y_i^f$ such that (8) holds. Since $\beta^t \subseteq \beta^*$, it follows that there exists $\bar{y}_i \in Y_i^f$ such that for all $\psi_i \in \Delta^{\beta^t_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$, there exists $y_i \in Y_i^f$ such that (8) holds. Fix any $t_i \in T_i$. Observe that $\phi_i \circ \left(\text{marg}_{T_{-i}}\kappa(t_i)\right) \in \Delta^{\beta^t_{-i}}\left(\Theta \times \hat{T}_{-i}\right)$ for all $\phi_i \in \Delta(\Theta)$. Therefore, it holds that

$$\sum_{\left(\theta, \hat{t}_{-i}\right) \in \Theta \times \hat{T}_{-i}} \left(\phi_i[\theta]\left(\text{marg}_{T_{-i}}\kappa(t_i)\left[\hat{t}_{-i}\right]\right)\right)\left[u_i\left(y_i\left(\hat{t}_{-i}\right), \theta\right) - u_i\left(\bar{y}_i\left(\hat{t}_{-i}\right), \theta\right)\right] > 0.$$

By setting

$$y_i = \sum_{\hat{t}_{-i} \in \hat{T}_{-i}} \left(\text{marg}_{T_{-i}}\kappa(t_i)\left[\hat{t}_{-i}\right]\right) y_i\left(\hat{t}_{-i}\right)$$

and

$$\hat{y}_i = \sum_{\hat{t}_{-i} \in \hat{T}_{-i}} \left(\text{marg}_{T_{-i}}\kappa(t_i)\left[\hat{t}_{-i}\right]\right) \bar{y}_i\left(\hat{t}_{-i}\right),$$

and by noting that $y_i, \hat{y}_i \in \Delta(A)$, the inequality in (9) follows for i. Since the choice of $i \in \mathcal{I}(\beta^*)$ was arbitrary, the statement follows. □

Let $\mathcal{T}$ be any model. Since $\mathcal{I}(\beta^*) = \mathcal{I}$ and since Lemma 9 guarantees the existence of the lottery $\hat{y}_i \in \Delta(A)$ for all $i \in \mathcal{I}$, let us define the lottery $\hat{y}$ by

$$\hat{y} = \frac{1}{I} \sum_{i \in \mathcal{I}} \hat{y}_i.$$

Given the net $(\beta^\alpha)_{\alpha \in \Omega}$ and our assumption that $\mathcal{I}(\beta^*) = \mathcal{I}$, Lemma 3 implies that for some $\alpha \in \Omega$, it holds that $\mathcal{I}(\beta^\alpha) = \mathcal{I}$. For every $i \in \mathcal{I} \setminus \mathcal{I}(\beta^0)$, Lemma 9 implies that there exists a least ordinal $\alpha(i)$ such that $i \in \mathcal{I}^*\left(\beta^{\alpha(i)}\right) \backslash \mathcal{I}^*\left(\beta^{\gamma}\right)$ for every $\gamma < \alpha(i)$. For player $i \in \mathcal{I}^*\left(\beta^{\alpha(i)}\right)$, Definition 8 implies that there exists $\bar{y}_i \in Y_i^f$ satisfying (8). Let us denote $\bar{y}_i$ by $\bar{y}_i^{\beta^{\alpha(i)}}$. Since $\bar{y}_i^{\beta^{\alpha(i)}} \in Y_{i,s}^f$, we can choose an $\varepsilon > 0$ sufficiently small such that the mapping $\mathfrak{y}_i^{\beta^{\alpha(i)}} : T_{-i} \to \Delta(A)$ defined by

$$\mathfrak{y}_i^{\beta^{\alpha(i)}}(t_{-i}) = (1-\varepsilon)\,\bar{y}_i^{\beta^{\alpha(i)}}(t_{-i}) + \varepsilon\hat{y} \tag{10}$$

is such that $\mathfrak{y}_i^{\beta^{\alpha(i)}} \in Y_{i,s}^f$. For $i \in \mathcal{I}(\beta^0)$, let $\alpha(i) = 0$.

Let us now define the mechanism $\mathcal{M}$. For all $i \in \mathcal{I}$, let

$$M_i = M_i^1 \times M_i^2 \times M_i^3 \times M_i^4,$$

where

$$M_i^1 = T_i,\ M_i^2 = \mathbb{N},\ M_i^3 = Y_i^* \text{ and } M_i^4 = \Delta^*(A),$$

where $\mathbb{N}$ is the set of natural numbers, Y_i^* is a countable, dense subset of Y_i^f, and $\Delta^*(A)$ is a countable, dense subset of $\Delta(A)$. For all $m \in M$, let $g : M \to \Delta(A)$ be defined as follows.

Rule 1: If $m_i^2 = 1$ for all $i \in \mathcal{I}$, then $g(m) = f(m^1)$.

Rule 2: For all $i \in \mathcal{I}$, if $m_j^2 = 1$ for all $j \in \mathcal{I}\backslash\{i\}$ and $m_i^2 > 1$, then

$$g(m) = m_i^3\left(m_{-i}^1\right)\left(1 - \frac{1}{1+m_i^2}\right) \oplus \mathfrak{y}_i^{\beta^{\alpha(i)}}\left(m_{-i}^1\right)\left(\frac{1}{1+m_i^2}\right), \tag{11}$$

where $\mathfrak{y}_i^{\beta^{\alpha(i)}} \in Y_{i,s}^f$ is defined in (10).

Rule 3: Otherwise, for each $i \in \mathcal{I}$, m_i^4 is picked with probability $\frac{1}{I}\left(1 - \frac{1}{1+m_i^2}\right)$ and $\hat{y}_i$ is picked with probability $\frac{1}{I}\left(\frac{1}{1+m_i^2}\right)$; that is,

$$g(m) = \frac{1}{I}\left[m_i^4\left(1 - \frac{1}{1+m_i^2}\right) \oplus \hat{y}_i\left(\frac{1}{1+m_i^2}\right)\right], \tag{12}$$

where $\hat{y}_i$ is specified by Lemma 10.

Suppose that f satisfies IIM on $\mathcal{T}$. In what follows, we prove that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$ and that $\mathcal{M}$ satisfies the EBRP. The following lemmata will help us to complete the proof.

Lemma 11. $\overline{BNE}\left(U\left(\mathcal{M},\mathcal{T}\right)\right)\neq\emptyset$.

Proof. For all $i\in\mathcal{I}$, let $\sigma_i:T_i\rightarrow M_i$ be defined by $\sigma_i\left(t_i\right)=\left(t_i,1,\cdot,\cdot\right)$. For all $i\in\mathcal{I}$ and all $t_i\in T_i$, let $\pi_i\left(t_i\right)\in\Delta\left(\Theta\times T_i\times M_{-i}\right)$ be defined by

$$\pi_i\left(t_i\right)\left[\theta,t_i,m_{-i}\right]=\kappa\left(t_i\right)\left[\theta,t_{-i}\right]\delta_{\sigma_{-i}\left(t_{-i}\right)}\left[m_{-i}\right],$$

where $\delta_{\sigma_{-i}(t_{-i})}$ is the dirac measure on $\left\{\sigma_{-i}\left(t_{-i}\right)\right\}$. By construction, for all $t_i\in T_i$ and all $\left(\theta,t_{-i},m_{-i}\right)\in\Theta\times T_i\times M_{-i}$, $\pi_i\left(t_i\right)\left[\theta,t_i,m_{-i}\right]>0\implies m_{-i}=\sigma_{-i}\left(t_{-i}\right)$. Moreover, by construction and *Rule 1*, for all $i\in\mathcal{I}$ and all $t_i\in T_i$,

$$\begin{aligned}&\sum_{(\theta,m_{-i})\in\Theta\times M_{-i}}\operatorname{marg}_{\Theta\times M_{-i}}\pi_i\left(t_i\right)\left[\theta,m_{-i}\right]u_i\left(g\left(\sigma_i\left(t_i\right),m_{-i}\right),\theta\right)\\=&\sum_{(\theta,t_{-i})\in\Theta\times T_{-i}}\kappa\left(t_i\right)\left[\theta,t_{-i}\right]u_i\left(f\left(t_i,t_{-i}\right),\theta\right).\end{aligned}$$

Finally, by definition of g and the fact that f is incentive compatible on $\mathcal{T}$ (Lemma 5), it follows that for all $i\in\mathcal{I}$ and all $t_i\in T_i$, $\operatorname{Supp}(\sigma_i\left(t_i\right))\subseteq BR_i\left(\operatorname{marg}_{\Theta\times M_{-i}}\pi_i|\mathcal{M}\right)$, and so $\sigma\in BNE\left(U\left(\mathcal{M},\mathcal{T}\right)\right)$.

Before proving the following lemma, let us introduce the following definitions. For all $\beta\in\mathcal{B}$ and all $i\in\mathcal{I}$, define $\Sigma_i^{\beta_i}:T_i\rightarrow 2^{M_i}\backslash\left\{\emptyset\right\}$ by

$$\Sigma_i^{\beta_i}\left(t_i\right)=\left\{m_i\in M_i|m_i^1\in\beta_i\left(t_i\right)\right\},\tag{13}$$

and define $\tilde{\Sigma}_i^{\beta_i}:T_i\rightarrow 2^{M_i}\backslash\left\{\emptyset\right\}$ by

$$\tilde{\Sigma}_i^{\beta_i}\left(t_i\right)=\left\{m_i\in\Sigma_i^{\beta_i}\left(t_i\right)|m_i^2=1\right\}.\tag{14}$$

It can be checked that $\Sigma^\beta,\tilde{\Sigma}^\beta\in\mathfrak{S}^{\mathcal{M},\mathcal{T}}$. □

Lemma 12. For all $\alpha \in \Omega$, all $i \in \mathcal{I}(\beta^\alpha)$ and all $\pi_i \in \Delta(\Theta \times T_{-i} \times M_{-i})$, if

$$m_i \in BR_i\left(\text{marg}_{\Theta \times M_{-i}} \pi_i | \mathcal{M}\right) \tag{15}$$

and

$$\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^\alpha}}(\Theta \times T_{-i} \times M_{-i}),$$

then $m_i^2 = 1$ and

$$\pi_i \in \Delta^{\tilde{\Sigma}_{-i}^{\beta_{-i}^\alpha}}(\Theta \times T_{-i} \times M_{-i}),$$

and $m_i^1 \in \beta_i^{\alpha+1}(t_i)$ for all $t_i \in T_i$.

Proof. Fix any $\alpha \in \Omega$ and any $i \in \mathcal{I}(\beta^\alpha)$. Suppose that $\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^\alpha}}(\Theta \times T_{-i} \times M_{-i})$ and that $m_i \in BR_i\left(\text{marg}_{\Theta \times M_{-i}} \pi_i | \mathcal{M}\right)$. Let us first show that $m_i^2 = 0$. Assume, to the contrary, that $m_i^2 > 0$. Let us proceed according to whether *Rule 2* applies or *Rule 3* applies. To this end, let us first show that $\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^\alpha}}(\Theta \times T_{-i} \times M_{-i})$ implies that

$$\underbrace{\sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} \sum_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta_{-i}^\alpha}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}_{Prob[Rule2]} + \underbrace{\sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} \sum_{m_{-i} \in M_{-i} \setminus \tilde{\Sigma}_{-i}^{\beta_{-i}^\alpha}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}_{Prob[Rule3]} = 1. \tag{16}$$

For all $i \in \mathcal{I}$ and all $t_i \in T_{-i}$, define $\nu_i(t_i) \in \Delta(\Theta \times T_{-i} \times M_{-i}^1)$ by

$$\nu_i(t_i)[\theta, t_{-i}, m_{-i}^1] = \frac{\sum_{m_{-i} \in \tilde{\Sigma}_{-i}^{\beta_{-i}^\alpha}(t_{-i})[m_{-i}^1]} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}{Prob[Rule2]}. \tag{17}$$

Since $\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^k}}(\Theta \times T_{-i} \times M_{-i})$, it follows that $\nu_i(t_i) \in \Delta^{\beta_{-i}^k}(\Theta \times T_{-i}^1 \times M_{-i}^1)$. Let $\psi_i = marg_{\Theta \times M_{-i}^1} \nu_i(t_i)$. Since $\nu_i(t_i) \in \Delta^{\beta_{-i}^k}(\Theta \times T_{-i}^1 \times M_{-i}^1)$, it holds that

$$\psi_i \in \Delta^{\beta_{-i}^\alpha}(\Theta \times M_{-i}^1). \tag{18}$$

Next, let $\phi_i(\theta) \in \Delta(\Theta)$ be defined by

$$\phi_i(\theta) = \frac{\sum_{t_{-i} \in T_{-i}} \sum_{m_{-i} \in M_{-i} \setminus \tilde{\Sigma}^{\beta^\alpha_{-i}}_{-i}(t_{-i})} \pi_i(t_i)[\theta, t_{-i}, m_{-i}]}{Prob[Rule3]}. \tag{19}$$

The utility of m_i under the beliefs $marg_{\Theta \times M_{-i}} \pi_i$, which is denoted by $U_i(m_i, marg_{\Theta \times M_{-i}} \pi_i)$, is given by

$$\begin{aligned} U_i(m_i, marg_{\Theta \times M_{-i}} \pi_i) = \alpha & \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} \psi_i(\theta, t_{-i}) u_i \Big[\big(1 - \frac{1}{m_i^2 + 1}) m_i^3(t_{-i}) \oplus \frac{1}{m_i^2 + 1} \mathfrak{y}_i^{\beta^{\alpha(i)}}(t_{-i}) \Big), \theta \Big] \\ & + (1 - \alpha) \sum_{\theta \in \Theta} \phi_i(\theta) u_i \Big[\big(1 - \frac{1}{m_i^2 + 1}) m_i^4 \oplus \frac{1}{m_i^2 + 1} \hat{y}_i \Big), \theta \Big] \end{aligned} \tag{20}$$

where $\alpha = Prob[Rule2]$.

Since $\psi_i \in \Delta^{\beta^k_{-i}}(\Theta \times \hat{T}_{-i})$, Definition 11 implies that there exists $y_i' \in Y_i^f$ such that

$$\sum_{(\theta, \hat{t}_{-i}) \in \Theta \times \hat{T}_{-i}} \psi_i(\theta, \hat{t}_{-i}) u_i(y_i'(\hat{t}_{-i}), \theta) > \sum_{(\theta, \hat{t}_{-i}) \in \Theta \times \hat{T}_{-i}} \psi_i(\theta, \hat{t}_{-i}) u_i(\mathfrak{y}^{\beta^{\alpha(i)}}(\hat{t}_{-i}), \theta). \tag{21}$$

Furthermore, Lemma 10 implies that there exists $y_i \in \Delta(A)$ such that

$$\sum_{\theta \in \Theta} \phi_i(\theta) u_i(y_i, \theta) > \sum_{\theta \in \Theta} \phi_i(\theta) u_i(\hat{y}_i, \theta). \tag{22}$$

Since $m_i \in BR_i(marg_{\Theta \times M_{-i}} \pi_i | \mathcal{M})$, it follows that

$$\sum_{(\theta, \hat{t}_{-i}) \in \Theta \times \hat{T}_{-i}} \psi_i(\theta, \hat{t}_{-i}) u_i(m_i^3(\hat{t}_{-i}), \theta) \geq \sum_{(\theta, \hat{t}_{-i}) \in \Theta \times \hat{T}_{-i}} \psi_i(\theta, \hat{t}_{-i}) u_i(y_i'(\hat{t}_{-i}), \theta) \tag{23}$$

and that

$$\sum_{\theta \in \Theta} \phi_i(\theta) u_i(m_i^4, \theta) \geq \sum_{\theta \in \Theta} \phi_i(\theta) u_i(y_i, \theta). \tag{24}$$

Inequalities in (21)-(24) imply that $U_i(m_i, marg_{\Theta\times M_{-i}}\pi_i)$ is strictly increasing in m_i^2, which is a contradiction. Thus, $m_i^2 = 1$.

Next, let us show that $\pi_i \in \Delta^{\tilde{\Sigma}_{-i}^{\beta^{\alpha}_{-i}}}(\Theta \times T_{-i} \times M_{-i})$. Assume, to the contrary, that $\pi_i \notin \Delta^{\tilde{\Sigma}_{-i}^{\beta^{\alpha}_{-i}}}(\Theta \times T_{-i} \times M_{-i})$. Then, since $m_i^2 = 1$, either *Rule 2* applies where $m_j^2 > 1$ for some $j \in \mathcal{I}\backslash\{i\}$ or *Rule 3* applies. In what follows, we focus only on the case that *Rule 2* applies.[16]

By the definition of g, for all $(\theta, m_{-i}) \in Supp(\underset{\Theta\times M_{-i}}{marg}\pi_i)$, it holds that

$$g(m_i, m_{-i}) = (1 - \frac{1}{m_j^2+1})m_j^3(m_{-j}^1) + \frac{1}{m_j^2+1}\mathfrak{y}_j^{\beta^{\alpha(j)}}(m_{-j}^1), \tag{25}$$

where, for $\varepsilon > 0$ sufficiently small,

$$\mathfrak{y}_j^{\beta^{\alpha(j)}}(t_{-j}) = (1-\varepsilon)\,\bar{y}_j^{\beta^{\alpha(j)}}(t_{-j}) + \varepsilon\hat{y}. \tag{26}$$

Define $\tilde{g}(m_i, m_{-i})$ as

$$\tilde{g}(m_i, m_{-i}) = (1 - \frac{1}{m_j^2+1})m_j^3(m_{-j}^1) + \frac{1}{m_j^2+1}\tilde{\mathfrak{y}}_j^{\beta^{\alpha(j)}}(m_{-j}^1) \tag{27}$$

where $\tilde{\mathfrak{y}}_j^{\beta^{\alpha(j)}}(t_{-j}) = (1-\varepsilon)\,\bar{y}_j^{\beta^{\alpha(j)}}(t_{-j}) + \varepsilon[\sum_{j\neq i}\frac{1}{I}\hat{y}_j + \frac{1}{I}y_i]$ and y_i is such that (9) is satisfied. Finally, let us define $\hat{m}_i^4$ by

$$\hat{m}_i^4 = \sum marg_{\Theta\times M_{-i}}\pi_i(\theta, m_{-i})\tilde{g}(\cdot, m_{-i}). \tag{28}$$

Since player i's utility is strictly higher under $\tilde{g}(m_i, m_{-i})$ than under $g(m_i, m_{-i})$, for every $(\theta, m_{-i}) \in Supp(\underset{\Theta\times M_{-i}}{marg}\pi_i)$, and since player i's utility function is continuous, we can assume without lost of generality that $\hat{m}_i^4 \in \Delta^*(A) = M_i^4$. Since player i's utility is strictly higher under $\tilde{g}(m_i, m_{-i})$ than under $g(m_i, m_{-i})$, for every $(\theta, m_{-i}) \in Supp(\underset{\Theta\times M_{-i}}{marg}\pi_i)$, player i can play any $\hat{m}_i \in M_i$ such that its fourth component is $\hat{m}_i^4$ and its second component is $\hat{m}_i^2 > 1$ and trigger *Rule 3*. In this way, player i obtains a strictly higher utility. Since the utility gain is ob-

[16] When *Rule 3* applies, we can see, by the arguments provided above, that player i can find a profitable deviation

tained point-wise in the $Supp(\underset{\Theta\times M_{-i}}{marg}\pi_i)$, we obtain the desired contradiction. Thus, $\pi_i \in \Delta^{\tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$.

Finally, let us show that $m_i^1 \in \beta_i^{\alpha+1}(t_i)$ for all $t_i \in T_i$. Fix any $t_i \in T_i$. Since $\pi_i \in \Delta^{\tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}}(\Theta \times T_{-i} \times M_{-i})$, we have that

$$\sum_{t_{-i}\in T_{-i}} \sum_{m_{-i}\in\tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}(t_{-i})} \pi_i[\theta, t_{-i}, m_{-i}] = 1.$$

Let $\nu_i(t_i) \in \Delta\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$ be defined by

$$\nu_i(t_i)\left[\theta, t_{-i}, m_{-i}^1\right] = \sum_{m_{-i}\in\tilde{\Sigma}_{-i}^{\beta_{-i}^{\alpha}}\left(m_{-i}^1\right)} \pi_i[\theta, t_{-i}, m_{-i}]. \tag{29}$$

By definition, we can see that $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times M_{-i}^1\right) \cap \Delta^{\beta_{-i}^{\alpha}}\left(\Theta \times T_{-i} \times M_{-i}^1\right)$. Since $m_1^2 = 1$, then *Rule 1* applies with probability 1, and so

$$\begin{array}{rl} \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}}\left(\text{marg}_{\Theta\times M_{-i}}\pi_i[\theta, m_{-i}]\right) u_i\left(g\left(m_i, m_{-i}\right), \theta\right) & = \\ \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}}\left(\text{marg}_{\Theta\times M_{-i}}\pi_i[\theta, m_{-i}]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right), & \end{array} \tag{30}$$

and so, by (29),

$$\begin{array}{rl} \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}}\left(\text{marg}_{\Theta\times M_{-i}}\pi_i[\theta, m_{-i}]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right) & = \\ \sum_{\left(\theta,m_{-i}^1\right)\in\Theta\times M_{-i}^1}\left(\text{marg}_{\Theta\times M_{-i}^1}\nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right). & \end{array}$$

Moreover, since $m_i \in BR_i\left(\text{marg}_{\Theta\times M_{-i}}\pi_i|\mathcal{M}\right)$ and since, moreover, player i can never induce *Rule 3*, it follows from the definition of g that

$$\begin{array}{rl} \sum_{\left(\theta,m_{-i}^1\right)\in\Theta\times M_{-i}^1}\left(\text{marg}_{\Theta\times M_{-i}^1}\nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i\left(f\left(m_i^1, m_{-i}^1\right), \theta\right) & \geq \\ \sum_{\left(\theta,m_{-i}^1\right)\in\Theta\times M_{-i}^1}\left(\text{marg}_{\Theta\times M_{-i}^1}\nu_i(t_i)\left[\theta, m_{-i}^1\right]\right) u_i\left(m_i^3\left(m_{-i}^1\right), \theta\right), & \end{array} \tag{31}$$

for all $m_i^3 \in Y_i^*$. Since Y_i^* is a countable, dense subset of Y_i^f and since u_i is continuous, we have that the inequality in (31) holds for all $m_i^3 \in Y_i^f$. Since $\nu_i(t_i) \in$

$\Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times M^1_{-i}\right) \cap \Delta^{\beta^\alpha_{-i}}\left(\Theta \times T_{-i} \times M^1_{-i}\right)$ and since, moreover, the inequality in (31) holds for all $m^3_i \in Y^f_i$, and $m^1_i \in \beta^\alpha_i(t_i)$, it follows from (4) that $m^1_i \in \beta^{\alpha+1}_i(t_i)$, as we sought.

□

Lemma 13. For all $\alpha \in \Omega$ and all $i \in \mathcal{I}$, $S^{\alpha,\mathcal{M},\mathcal{T}}_i \subseteq \Sigma^{\beta^\alpha_i}_i$

Proof. Let us proceed by transfinite induction over Ω. It is clear that $S^{\alpha,\mathcal{M},\mathcal{T}}_i \subseteq \Sigma^{\beta^\alpha_i}_i = M_i$ for all $i \in \mathcal{I}$ if $\alpha = 0$. Fix any $\alpha \in \Omega \setminus \{0\}$. Suppose that for all $\gamma < \alpha$, $S^{\gamma,\mathcal{M},\mathcal{T}}_i \subseteq \Sigma^{\beta^\gamma_i}_i$ for all $i \in \mathcal{I}$. Fix any $i \in \mathcal{I}$. We proceed according to whether α is a successor ordinal or not. Suppose that α is a limit ordinal. Since $\bigcap_{\gamma<\alpha} S^{\gamma,\mathcal{M},\mathcal{T}}_i = S^{\alpha,\mathcal{M},\mathcal{T}}_i$, by Definition 3, it follows that $S^{\alpha,\mathcal{M},\mathcal{T}}_i \subseteq \bigcap_{\gamma<\alpha} \Sigma^{\beta^\gamma_i}_i$. Fix any $t_i \in T_i$ and any $m_i \in \bigcap_{\gamma<\alpha} \Sigma^{\beta^\gamma_i}_i(t_i)$. Then, $m^1_i \in \bigcap_{\gamma<\alpha} \beta^\gamma_i(t_i)$. It follows from (5) that $m^1_i \in \beta^\alpha_i(t_i)$. Since the choice of $t_i \in T_i$ was arbitrary, we have that $\bigcap_{\gamma<\alpha} \Sigma^{\beta^\gamma_i}_i \subseteq \Sigma^{\beta^\alpha_i}_i$. Since $S^{\alpha,\mathcal{M},\mathcal{T}}_i \subseteq \bigcap_{\gamma<\alpha} \Sigma^{\beta^\gamma_i}_i$, we have that $S^{\alpha,\mathcal{M},\mathcal{T}}_i \subseteq \Sigma^{\beta^\alpha_i}_i$.

Suppose that α is a successor ordinal. We proceed according to whether $i \in \mathcal{I}(\beta^{\alpha-1})$ or $i \in \mathcal{I}^c(\beta^{\alpha-1})$. Suppose that $i \in \mathcal{I}(\beta^{\alpha-1})$, we proceed according to whether $i \in \mathcal{I}(\beta^\alpha)$ or not. Suppose that $i \in \mathcal{I}^c(\beta^\alpha)$. Lemma 7 implies that $\beta^\alpha_i = \bar{\beta}_i$. It follows from (13) that $S^{\alpha,\mathcal{M},\mathcal{T}}_i \subseteq \Sigma^{\beta^\alpha}_i$. Suppose that $i \in \mathcal{I}(\beta^\alpha)$. Fix any $t_i \in T_i$ and any $m_i \in S^{\alpha,\mathcal{M},\mathcal{T}}_i(t_i)$. The inductive hypothesis implies that $S^{\alpha-1,\mathcal{M},\mathcal{T}}_{-i} \subseteq \Sigma^{\beta^{\alpha-1}_{-i}}_{-i}$.

Since $m_i \in S^{\alpha,\mathcal{M},\mathcal{T}}_i$, Definition 3 implies that $m_i \in S^{\alpha-1,\mathcal{M},\mathcal{T}}_i$ and there exists $\pi_i \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i})$ such that $\pi_i \in \Delta^{S^{\alpha-1,\mathcal{M},\mathcal{T}}_{-i}}(\Theta \times T_{-i} \times M_{-i})$ and that $m_i \in \mathrm{BR}_i\left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i|\mathcal{M}\right)$. Since $S^{\alpha-1,\mathcal{M},\mathcal{T}}_{-i} \subseteq \Sigma^{\beta^{\alpha-1}}_{-i}$, it follows that

$$\pi_i \in \Delta^{\Sigma^{\beta^{\alpha-1}}_{-i}}(\Theta \times T_{-i} \times M_{-i}).$$

Since $i \in \mathcal{I}(\beta^{\alpha-1})$ and since, moreover, $m_i \in BR_i\left(\mathrm{marg}_{\Theta\times M_{-i}}\pi_i|\mathcal{M}\right)$ and $\pi_i \in \Delta^{\Sigma^{\beta^{\alpha-1}_{-i}}_{-i}}(\Theta \times T_{-i} \times M_{-i})$, Lemma 12 implies that $m^2_i = 1$ and that $m^1_i \in \beta^\alpha_i(t_i)$. Thus, $m_i \in \Sigma^{\beta^\alpha_i}_i$.

Suppose that $i \in \mathcal{I}^c(\beta^{\alpha-1})$. Lemma 7 implies that $\beta^\alpha_i = \bar{\beta}_i$. It follows from (13) that $S^{\alpha,\mathcal{M},\mathcal{T}}_i \subseteq \Sigma^{\beta^\alpha}_i$. □

Lemma 14. For all $\alpha \in \Omega$, all $i \in \mathcal{I}(\beta^\alpha)$ and all $t_i \in T_i$, if $m_i \in S_i^{\alpha+1,\mathcal{M},\mathcal{T}}(t_i)$, then $m_i^2 = 1$ and $m_i^1 \in \beta_i^{\alpha+1}(t_i)$.

Proof. Let us proceed by transfinite induction over α. Let $\alpha = 0$. Assume that $i \in \mathcal{I}(\beta^0)$ and fix any $t_i \in T_i$. Assume that $m_i \in S_i^{1,\mathcal{M},\mathcal{T}}(t_i)$. We show that $m_i^2 = 1$ and $m_i^1 \in \beta_i^1(t_i)$. Since $m_i \in S_i^{1,\mathcal{M},\mathcal{T}}(t_i)$, it follows from Definition 3 that there exists $\pi_i \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \cap \Delta^{\Sigma_{-i}^{\beta_{-i}^0}}(\Theta \times T_{-i} \times M_{-i})$, where $\Sigma_{-i}^{\beta_{-i}^0} = S_{-i}^{0,\mathcal{M},\mathcal{T}} = M_{-i}$, such that $m_i \in BR_i\left(\mathrm{marg}_{\Theta \times M_{-i}}\pi_i|\mathcal{M}\right)$. Since $m_i \in BR_i\left(\mathrm{marg}_{\Theta \times M_{-i}}\pi_i|\mathcal{M}\right)$ and $\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^0}}(\Theta \times T_{-i} \times M_{-i})$, Lemma 12 implies that $m_i^2 = 1$ and that $m_i^1 \in \beta_i^1(t_i)$. Since the choice of (i, t_i) was arbitrary, we have that the statement holds for all $i \in \mathcal{I}(\beta^0)$ and all $t_i \in T_i$.

Fix any $\alpha \neq 0$. Suppose that for all $\gamma < \alpha$, all $i \in \mathcal{I}(\beta^\gamma)$ and all $t_i \in T_i$, if $m_i \in S_i^{\gamma+1,\mathcal{M},\mathcal{T}}(t_i)$, then $m_i^2 = 1$ and $m_i^1 \in \beta_i^{\gamma+1}(t_i)$. Suppose that $i \in \mathcal{I}(\beta^\alpha)$ and that $m_i \in S_i^{\alpha+1,\mathcal{M},\mathcal{T}}(t_i)$. We show that $m_i^2 = 1$ and $m_i^1 \in \beta_i^{\alpha+1}(t_i)$. We proceed according to whether α is a limit ordinal or not.

Suppose that α is a limit ordinal. Since $m_i \in S_i^{\alpha+1,\mathcal{M},\mathcal{T}}(t_i)$, it follows from Definition 3 that there exists $\pi_i \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \cap \Delta^{S_{-i}^{\alpha,\mathcal{M},\mathcal{T}}}(\Theta \times T_{-i} \times M_{-i})$ such that $m_i \in BR_i\left(\mathrm{marg}_{\Theta \times M_{-i}}\pi_i|\mathcal{M}\right)$. Since α is a limit ordinal, Lemma 13 implies that

$$S_{-i}^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_{-i}^{\beta_{-i}^\alpha}, \tag{32}$$

and so $\pi_i \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \cap \Delta^{\Sigma_{-i}^{\beta_{-i}^\alpha}}(\Theta \times T_{-i} \times M_{-i})$. Lemma 12 implies that $m_i^2 = 1$ and that $m_i^1 \in \beta_i^{\alpha+1}(t_i)$.

Suppose that α is a successor ordinal. To apply Lemma 12, we need to show that $\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^\alpha}}(\Theta \times T_{-i} \times M_{-i})$. This can be done by showing that

$$S_{-i}^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_{-i}^{\beta_{-i}^\alpha}. \tag{33}$$

Fix any $j \in \mathcal{I} \backslash \{i\}$. We proceed according to whether $j \in \mathcal{I}(\beta^{\alpha-1})$ or not.

Suppose that $j \in \mathcal{I}(\beta^{\alpha-1})$. Fix any $t_j \in T_j$ and any $m_j \in S_j^{\alpha-1,\mathcal{M},\mathcal{T}}(t_j)$. The inductive hypothesis implies that $m_j^2 = 1$ and $m_j^1 \in \beta_j^\alpha(t_j)$. It follows from (13) that

$m_j \in \Sigma_j^{\beta_j^\alpha}(t_j)$. Since the choice of $(j,t_j) \in \mathcal{I}(\beta^{\alpha-1}) \times T_j$ was arbitrary, it follows that $S_j^{\alpha,\mathcal{M},\mathcal{T}} \subseteq \Sigma_j^{\beta_j^\alpha}$ for all $j \in (\mathcal{I} \cap \mathcal{I}(\beta^{\alpha-1})) \setminus \{i\}$.

Suppose that $j \in \mathcal{I}^c(\beta^{\alpha-1})$. Since f satisfies IIM on $\mathcal{T}$, Lemma 7 implies that $\beta_j^\alpha = \beta_j^{\alpha-1} = \bar{\beta}_j$. Then, it follows from (13) that $m_j \in \Sigma_j^{\beta_j^\alpha}(t_j)$. Again, since the choice of $j \in \mathcal{I}^c(\beta^{\alpha-1})$ was arbitrary, we conclude that (33) holds.

Since $m_i \in BR_i\left(\text{marg}_{\Theta \times M_{-i}}\pi_i|\mathcal{M}\right)$ and since $\pi_i \in \Delta^{\Sigma_{-i}^{\beta_{-i}^\alpha}}(\Theta \times T_{-i} \times M_{-i})$, Lemma 12 implies that $m_i^2 = 1$ and that $m_i^1 \in \beta_i^{\alpha+1}(t_i)$, as we sought.

Since the choice of (i,t_i) was arbitrary, we have that the statement holds for all $i \in \mathcal{I}(\beta^\alpha)$ and all $t_i \in T_i$.

□

Let us show that $\mathcal{M}$ ICR-implements f on $\mathcal{T}$. Lemma 11 implies that for all $i \in \mathcal{I}$ and $t_i \in T_i$, $S_i^{\mathcal{M},\mathcal{T}}(t_i) \neq \emptyset$. Thus, part (i) of Definition 4 is satisfied. Recall that Lemma 3 implies that there exists an α such that $\beta^\alpha = \beta^{\alpha+1} = \beta^*$. Recall that by Lemma 8, we are under the assumption that $\mathcal{I}(\beta^*) = \mathcal{I}$. Thus, $\mathcal{I}(\beta^\alpha) = \mathcal{I}$. Fix any $t \in T$ and any $m \in S^{\mathcal{M},\mathcal{T}}(t)$. Since $S^{\mathcal{M},\mathcal{T}}(t) \subseteq S^{\alpha+1,\mathcal{M},\mathcal{T}}(t)$, then $m \in S^{\alpha+1,\mathcal{M},\mathcal{T}}(t)$. Lemma 14 implies that $m_i^2 = 1$ and $m_i^1 \in \beta_i^{\alpha+1}(t_i) = \beta_i^*(t_i)$ for all $i \in \mathcal{I}(\beta^\alpha) = \mathcal{I}(\beta^*)$. *Rule 1* implies that $g(m) = f(m^1)$. Since f satisfies IIM on $\mathcal{T}$, it follows that β^* is an acceptable deception on $\mathcal{T}$ for f. This implies that $f(m^1) = f(t)$. Since the choice of $(t,m) \in T \times S^{\mathcal{M},\mathcal{T}}(t)$ was arbitrary, we conclude that part (ii) of Definition 4 is satisfied. Thus, f is ICR-implementable on $\mathcal{T}$. Finally, in light of Remark 1, Lemma 11 implies that $\mathcal{M}$ also implements f in Bayes-Nash equilibria.

References

Abreu, D. and Matsushima, H. (1992). Virtual implementation in iteratively undominated strategies II: Incomplete information. Princeton University working paper.

Aliprantis, C. D. and Border, K. C. (2006). *Infinite dimensional analysis*. Springer.

Arieli, I. (2010). Rationalizability in continuous games. *Journal of Mathematical Economics*, 46:912–924.

Bergemann, D., Morris, S., and Tercieux, O. (2011). Rationalizable implementation. *Journal of Economic Theory*, 146(3):1253–1274.

Dekel, E., Fudenberg, D., and Morris, S. (2007). Interim correlated rationalizability. *Theoretical Economics*.

Jain, R., Korpela, V., and Lombardi, M. (2022). An iterative approach to rationalizable implementation. *Available at SSRN 3861192*.

Kunimoto, T., Saran, R., and Serrano, R. (2020). Interim rationalizable implementation of functions.

Ok, E. A. (2011). *Real analysis with economic applications*. Princeton University Press.

Oury, M. and Tercieux, O. (2012). Continuous implementation. *Econometrica*, 80(4):1605–1637.

Weinstein, J. and Yildiz, M. (2007). A structure theorem for rationalizability with application to robust predictions of refinements. *Econometrica*, 75(2):365–400.

Xiong, S. (2021). Rationalizable implementation of social choice functions: complete characterization. *Theoretical Economics*.

Appendices

A. Proof of Theorem 1: Part (*i*) implies part (*ii*)

Let $\mathcal{T}$ be any model. Let $f : T \to \Delta(A)$ be any SCF. Assume that $\mathcal{M}$ satisfies the EBRP and that $\mathcal{M}$ ICR-implements f. Lemma 2 implies that there exists a pure strategy $\sigma \in BNE(U(\mathcal{M}, \mathcal{T}))$. This implies that for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$\begin{array}{lr} \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta, t_{-i}]\, u_i(g(\sigma(t)), \theta) & \geq \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta, t_{-i}]\, u_i(g((m_i, \sigma_{-i}(t_{-i}))), \theta) & \end{array}$$

for all $m_i \in M_i$. Since $\mathcal{M}$ ICR-implements f, it follows that for all $i \in \mathcal{I}$ and all $t_i \in T_i$,

$$\begin{array}{lr} \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta, t_{-i}]\, u_i(f(t), \theta) & \geq \\ \sum_{(\theta,t_{-i})\in\Theta\times T_{-i}} \kappa(t_i)[\theta, t_{-i}]\, u_i(g((m_i, \sigma_{-i}(t_{-i}))), \theta) & \end{array} \tag{34}$$

for all $m_i \in M_i$.

Suppose that the deception β is unacceptable. For all $i \in \mathcal{I}$ and all $t_i \in T_i$, $\Sigma_i(t_i) = \{\sigma_i(t_i') \in M_i | t_i' \in \beta_i(t_i)\}$. Then, Σ_i is a correspondence from T_i to $2^{M_i} \setminus \{\emptyset\}$, and so $\Sigma_i \in \mathfrak{S}_i^{\mathcal{M},\mathcal{T}}$. Since $\mathcal{M}$ ICR-implements f, it follows that $\Sigma \in \mathfrak{S}^{\mathcal{M},\mathcal{T}}$ cannot be a best-reply set in $U(\mathcal{M}, \mathcal{T})$. Then, for some $\left(i, t_i, \sigma\left(\hat{t}_i\right)\right) \in \mathcal{I} \times T_i \times \Sigma_i(t_i)$ and all $\pi_i(t_i) \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \cap \Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$, it holds that

$$\sigma_i\left(\hat{t}_i\right) \notin BR_i\left(\text{marg}_{\Theta\times M_{-i}}\pi_i(t_i) | \mathcal{M}\right),$$

and so

$$\begin{array}{lr} \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\text{marg}_{\Theta\times M_{-i}}\pi_i(t_i)[\theta, m_{-i}]\right)[u_i(g(m_i, m_{-i}), \theta)] & > \\ \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\text{marg}_{\Theta\times M_{-i}}\pi_i(t_i)[\theta, m_{-i}]\right)\left[u_i\left(g\left(\sigma_i\left(\hat{t}_i\right), m_{-i}\right), \theta\right)\right] & \end{array} \tag{35}$$

for some $m_i \in M_i$.

For all $i \in \mathcal{I}$ and all $t_i \in T_i$, let $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right) \cap \Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$

be any distribution. For all $i \in \mathcal{I}$, all $t_i \in T_i$, let $\bar{\pi}_i(t_i) \in \Delta(\Theta \times T_{-i} \times M_{-i})$ be defined, for all $(\theta, t_{-i}, m_{-i}) \in \Theta \times T_{-i} \times M_{-i}$, by

$$\bar{\pi}_i(t_i)[\theta, t_{-i}, m_{-i}] = \sum_{\hat{t}_{-i} \in \sigma_{-i}^{-1}(m_{-i})} \nu_i(t_i)\left[\theta, t_{-i}, \hat{t}_{-i}\right],$$

where $\sigma_{-i}^{-1}(m_{-i}) = \Pi_{j \in \mathcal{I} \setminus \{i\}} \sigma_j^{-1}(m_j)$ and $\sigma_j^{-1}(m_j) = \{t_j \in T_j | m_j = \sigma_j(t_j)\}$. Since $\nu_i(t_i) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, we have that $\text{marg}_{\Theta \times T_{-i}} \nu_i(t_i) = \kappa(t_i)$. Moreover, by construction, $\text{marg}_{\Theta \times T_{-i}} \nu_i(t_i) = \text{marg}_{\Theta \times T_{-i}} \bar{\pi}_i(t_i)$.[17] Moreover, since $\nu_i(t_i)$ belongs to $\in \Delta^{\beta_{-i}}\left(\Theta \times T_{-i} \times \hat{T}_{-i}\right)$, it also follows that for all $(\theta, t_{-i}, m_{-i}) \in \Theta \times T_{-i} \times M_{-i}$, $\bar{\pi}_i(t_i)[\theta, t_{-i}, m_{-i}] > 0 \implies m_{-i} \in \Sigma_{-i}(t_{-i})$. Thus, we have that $\bar{\pi}_i(t_i) \in \Delta^{\kappa(t_i)}(\Theta \times T_{-i} \times M_{-i}) \cap \Delta^{\Sigma_{-i}}(\Theta \times T_{-i} \times M_{-i})$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$. Moreover,

[17] Observe that for all $(\theta, t_{-i}) \in \Theta \times T_{-i}$,

$$\begin{aligned} \text{marg}_{\Theta \times T_{-i}} \bar{\pi}_i(t_i)[\theta, t_{-i}] &= \sum_{m_{-i} \in M_{-i}} \bar{\pi}_i(t_i)[\theta, t_{-i}, m_{-i}] \\ &= \sum_{m_{-i} \in M_{-i}} \left(\sum_{\hat{t}_{-i} \in \sigma_{-i}^{-1}(m_{-i})} \nu_i(t_i)\left[\theta, t_{-i}, \hat{t}_{-i}\right] \right) \\ &= \sum_{\hat{t}_{-i} \in \hat{T}_{-i}} \nu_i(t_i)\left[\theta, t_{-i}, \hat{t}_{-i}\right] \\ &= \text{marg}_{\Theta \times T_{-i}} \nu_i(t_i)[\theta, t_{-i}]. \end{aligned}$$

by construction, we also have that for all $i \in \mathcal{I}$ and all $m_i \in M_i$,[18]

$$\begin{array}{c} \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\text{marg}_{\Theta\times M_{-i}}\bar{\pi}_i\left(t_i\right)\left[\theta,m_{-i}\right]\right) u_i\left(g\left(m_i,m_{-i}\right),\theta\right) \quad = \\ \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right). \end{array} \tag{36}$$

Since $\bar{\pi}_i\left(t_i\right) \in \Delta^{\kappa(t_i)}\left(\Theta \times T_{-i} \times M_{-i}\right) \cap \Delta^{\Sigma_{-i}}\left(\Theta \times T_{-i} \times M_{-i}\right)$ for all $i \in \mathcal{I}$ and all $t_i \in T_i$, from (35) and (36), we have that for some $\left(i, t_i, \sigma\left(\hat{t}_i\right)\right) \in \mathcal{I} \times T_i \times \Sigma_i\left(t_i\right)$,

$$\begin{array}{c} \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right) \quad > \\ \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(\sigma_i\left(\hat{t}_i\right),\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right). \end{array} \tag{37}$$

Define $y_i(\cdot) = g(m_i, \sigma_{-i}(\cdot))$. (34) implies that $y_i \in Y_i^f$. Thus, f satisfies IRM on $\mathcal{T}$.

[18] To see it, observe that

$$\begin{aligned} & \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\text{marg}_{\Theta\times M_{-i}}\bar{\pi}_i\left(t_i\right)\left[\theta,m_{-i}\right]\right) u_i\left(g\left(m_i,m_{-i}\right),\theta\right) \\ = & \sum_{(\theta,t_{-i},m_{-i})\in\Theta\times T_{-i}\times M_{-i}} \bar{\pi}_i\left(t_i\right)\left[\theta,t_{-i},m_{-i}\right] u_i\left(g\left(m_i,m_{-i}\right),\theta\right) \\ = & \sum_{(\theta,t_{-i},m_{-i})\in\Theta\times T_{-i}\times M_{-i}} \left(\sum_{\hat{t}_{-i}\in\sigma_{-i}^{-1}(m_{-i})} \nu_i\left(t_i\right)\left[\theta,t_{-i},\hat{t}_{-i}\right] u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right)\right) \\ = & \sum_{(\theta,m_{-i})\in\Theta\times M_{-i}} \left(\sum_{\hat{t}_{-i}\in\sigma_{-i}^{-1}(m_{-i})} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right)\right) \\ = & \sum_{\left(\theta,\hat{t}_{-i}\right)\in\Theta\times\hat{T}_{-i}} \left(\text{marg}_{\Theta\times\hat{T}_{-i}}\nu_i\left(t_i\right)\left[\theta,\hat{t}_{-i}\right]\right) u_i\left(g\left(m_i,\sigma_{-i}\left(\hat{t}_{-i}\right)\right),\theta\right). \end{aligned}$$