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A Measure of Social Loss for Production Economies with Externalities

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Maria Gabriella Graziano* **and Vincenzo Platino**†

Abstract
In this paper we consider a production economy and adopt a cooperative approach to equilibrium analysis which allows each individual to cooperate with others and to form a coalition whose members have access to the available technologies. Our definition of the core requires a blocking coalition to take account of the consequences of its blocking for the production of the counter-coalition. Following Montesano (2002), we introduce a *measure of social loss* with respect to the core of the economy which characterizes the corresponding core allocations. Our characterization holds in the presence of consumption externalities and an optimistic attitude of coalition agents with respect to the behavior of outsiders.

JEL classification: C71, D11, D21, D62, D64.

Keywords: production economy, core, social loss, externalities.

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1 Introduction

We consider a production economy with finite numbers of commodities and households. We adopt a cooperative approach to equilibrium analysis where each individual is allowed to cooperate with others and to form a coalition. Once a coalition is formed and regardless of how the remaining economy is organized, coalition members have access to the available technologies. We employ this framework in order to introduce a *measure of social loss* associated to the core of the economy which completely characterizes core allocations. Our aim is to complete the analysis in Di Pietro et al. (2022) allowing for interdependency effects due to production as well as consumption externalities. To achieve this, we propose and study a new notion of core.

Measurement of the "welfare loss" associated with an inefficient allocation dates back to Debreu (1951), Luenberger (1992, 1994) and Montesano (1997, 2002). It was motivated by limited availability of resources and the impact on the economic environment and focuses on measuring the amounts of resources that are wasted under a given allocation compared to a Pareto optimal allocation. Montesano (2002) and Di Pietro et al. (2022) propose a measure of social loss that also takes account of core allocations. The core of the economy is the subset of Pareto optimal allocations formed by those feasible allocations which no coalition can improve upon in terms of preferences (*preferences-core*). Montesano (2002) introduces the idea of a *resources-core* where for an allocation that is not in the resources-core, there exists at least one coalition whose members can improve upon the given allocation (in terms of preferences) by saving resources. A measure of the social loss associated to the given allocation is defined by the amount of the resources that can be saved. Since the preferences-core and resources-core are equivalent notions, this measure provides a characterization of the core. This characterization holds in the context of a standard pure exchange economy with regular, continuous and monotonic (selfish) preferences, based on the key assumption that coalitions dislike resources waste. Therefore, the duality between the minimization of resources and the maximization of preferences which is used to show the equivalence between the preferences-core and the resources-core and to define the measure of social loss, builds on monotonicity arguments. However, in the presence of consumption or production externalities these arguments might not hold. Also, a blocking coalition needs to take account of the presence of the outsiders. Indeed, the coalition's resources are affected by the firms owned jointly by the coalition with the outsiders and the levels of its members' utility change with the outsiders' allocation.

In Di Pietro et al. (2022), the idea of social loss with respect to the core of an economy is extended to pure exchange economies where individual preferences are affected by the consumption of all other agents in the economy (consumption externalities). This extension is possible due to a special form of monotonicity of preference relations related to the redistribution of the surplus within a coalition (Social Group Monotonicity). Consequently, a measure of social loss can still be used to characterize the core in the presence of externalities and regardless of whether the coalitions are optimistic or pessimistic about the possible reactions of outsiders [1].

In this paper, we measure social losses caused by inefficiency for models of production economies. We describe production in very general terms. In line with Hildenbrand (1968, 1974) and Cornwall (1969), we suppose that the production capabilities of each coalition of agents that is formed to improve upon an allocation, are described by a production correspondence. This way of modeling production technologies accounts for cases where the technology is available to all agents that are described in Debreu and Scarf (1963), as well as the classical private ownership economy with a finite set of producers and firms owned by agents. In particular, it allows individuals to control technologies according to corporate shares (Xiong and Zheng (2007)).

We introduce the core of our production economy and show that core allocations can be characterized as zero points of the social loss functions. Again, our characterization is based on the idea of no waste of resources, and so our result requires an appropriate formulation of monotonicity assumptions. Moreover, the notion of blocking takes into account the additional interdependency problems created, for example, by conditions that the blocking coalition needs to respect in relation to the shareholders outside the coalition. To deal with this form of interdependency, we introduce a new notion of core, in preferences and in resources, which generalizes the *considerate* core defined by Xiong and Zheng (2007). In particular, we require that a blocking coalition S must take into account the consequence of its blocking on the counter-coalition $N \setminus S$. This requirement is relevant if the firms are controlled by corporate shareholdings and a blocking coalition S can only modify the production plans of the firms under its control. For the expectation that the counter-coalition does not react after a change in the production plans of the other firms to be plausible, S should allow the outsiders to have feasible consumption plans. Our definition of the core (in preferences) is sufficiently general to include not only the classical definitions considered in the literature (see Debreu and Scarf (1963), Aliprantis et al. (1989)), but also the core notions which involve control rights introduced by Xiong and Zheng (2007).

To analyze the core, we start by assuming selfish agents, and in the spirit of Montesano (2002) and Di Pietro et al. (2022), we introduce the notion of

[1] That is, in the case of the so-called γ-core and in the case of the α-core notions as defined in Di Pietro et al. (2022).

resources-core, which emphasizes the optimal allocation of the resources. The blocking coalition produces according to its capabilities and the interdependency problem is captured by an outsiders' feasibility condition. Under the classical monotonicity assumption, we show that the preferences-core and the resources-core coincide. This equivalence allow us to introduce a suitable measure of social loss associated to the core. In the second part of the paper, we show that our results continue to hold in the presence of consumption externalities. In this framework, which can be described also as the model with *other-regarding preferences*, the core can be defined in several different ways according to the attitude of the blocking coalition S with respect to the reaction of the outsiders [2] . The blocking procedure we adopt in this paper leads to the core notion which is described in the literature as the γ-core. To keep the framework simple, we consider only the case where coalitions of agents have an *optimistic* attitude with respect the behavior of outsiders. In this case, the deviating coalition S assumes that the counter-coalition passively accepts the deviation of S and that outsiders stick to their status quo allocation. Similarly, firms that are not controlled by S, stick to their status quo production. Under a suitable form of monotonicity referred to as *Social Group Monotonicity*, we restore the equivalence between the preferences-core and the resources-core, and characterize core allocations as zero points of the measure of social loss. The Social Group Monotonicity assumption ensures that at a given allocation, each coalition finds a way to distribute additional resources while making all of its members better off. However, to handle external effects related to production and consumption, we also assume *Boundary Aversion*. Since the object of our analysis is the production side, we adopt standard individual consumption sets, i.e. the nonnegative cone. As in Di Pietro et al. (2022), the analysis can be generalized by considering consumption sets which are affected by the coalition to which the agents belong.

The paper is organized as follows. Section 2 is the main section of the paper: Subsection 2.1 presents the model and the model assumptions; Subsection 2.2 introduces the notions of preferences-core and resources-core, and demonstrate their equivalence; Subsection 2.3 characterizes the core in terms of loss mapping. Section 3 discusses the other-regarding preferences model and Section 4 proposes characterizations of core notions in terms of prices. Section 5 presents some additional results, the technical proofs and a table of models with production covered by our model.

[2] For more detail, see e.g. Di Pietro et al. (2022), Graziano et al. (2017) and Hervés-Beloso and Moreno-García (2021).

2 A production economy

Our model is of a production economy with finitely many consumers. The aim is to consider a model that is sufficiently general to include both production economies with publicly accessible technologies and production economies where the technologies are controlled by individuals according to their corporate shares. In this latter case, we incorporate externalities due to the presence of outsiders[3]. This is the motivation for proposing the set correspondences Y, σ and Λ to describe the production possibilities of a coalition S, and the production and resource constraints on its outsiders, respectively.

2.1 The model and the basic assumptions

There is a finite number l of commodities and the commodity space is $\mathbb{R}^l$. There is a finite number of individuals (agents or traders) denoted by the subscript $i \in N := \{1, \dots, n\}$. The consumption set of agent i is the positive cone $\mathbb{R}^l_+$ and the consumption bundle of individual i is $x_i := (x_i^1, \dots, x_i^l)$. We denote by $x := (x_i)_{i \in N}$ a vector of consumption bundles. If the individual preferences of each agent depend only on his own consumption, we describe the agents in the economy as *selfish*. If this is the case, the preferences of individual i are represented formally by a binary relation $\succsim_i$ over $\mathbb{R}^l_+$[4].

The initial endowment of individual i is $\omega_i := (\omega_i^1, \dots, \omega_i^l)$, and let $\omega := (\omega_i)_{i \in N} \in \mathbb{R}^{l \cdot n}_+$ be the vector of all initial endowments.

A production plan for the economy is a point $y \in \mathbb{R}^l$, with the convention that the outputs of production are represented by the positive components of y and the inputs of production are represented by its negative components. There is a finite number of firms denoted by the subscript $j \in J := \{1, \dots, f\}$ and the production possibilities of a firm $j \in J$ are represented by the production set $Y_j \subseteq \mathbb{R}^l$.

A state of the economy $\xi \in \Xi \subseteq \mathbb{R}^{l \cdot n}_+ \times \mathbb{R}^{l \cdot f}$ is a specification of the consumption bundle $x_i \in \mathbb{R}^l_+$ for each consumer and of the production plan $y_j \in Y_j$ for each producer, i.e. $\xi := (x_1, \dots, x_n, y_1, \dots, y_f)$. A state of the economy $\xi =$

[3] The *interdependency problem* described by Xiong and Zheng (2007).

[4] In a more general situation, the individual preferences of each agent may depend on the consumption of all the agents i.e. consumption externalities are present. For simplicity, we present the results for the case of selfish traders and the results for the extension to the case of consumption externalities in two separate sections. This allows us to distinguish external effects due to production of the outsiders from external effects due to their consumption.

$(x_1, \ldots, x_n, y_1, \ldots, y_f)$ is said to be an *allocation* if it satisfies the physical feasibility condition

$$\sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i + \sum_{j \in J} y_j.$$

$\mathcal{F}$ denotes the set of all allocations.

A coalition is any nonempty subset S of the set of agents N. We use $\mathcal{P}(N)$ to denote the set of all coalitions and for each coalition S strictly contained in N, S^c denotes the complementary coalition (the members of S^c are also called *outsiders*).

A coalition S may form to improve (or block) a status quo state $\xi \in \Xi$. In this case, the production possibilities of S will depend on the coalition, and on the status quo ξ. Formally, a correspondence $Y\colon \mathcal{P}(N) \times \Xi \rightrightarrows \mathbb{R}^l$ is defined which associates to each coalition S and status quo ξ, the set $Y(S, \xi) \subseteq \mathbb{R}^l$ of production plans available for the coalition S. Given a coalition S and a status quo ξ, the correspondence $\sigma_{S,\xi} : Y(S, \xi) \rightrightarrows Y(S^c, \xi)$ defines the production plans available to the outsiders, for any production plan y' chosen by S in its production set $Y(S, \xi)$.

Finally, the correspondence $\Lambda : \mathcal{P}(N) \times \Xi \rightrightarrows \mathbb{R}^l$ describes the possible resources constraints for the outsiders if coalition S is formed to improve the status quo state ξ.

We make the following survival assumption in relation to the aggregate endowments of each coalition:

Assumption 1 *For any coalition $S \subseteq N$, the aggregate endowment $\omega(S)$ belongs to $\mathbb{R}^l_{++}$.*

The basic assumptions on preference relations are listed below.

Assumption 2 *For every individual $i \in N$, $\succsim_i$ is complete, transitive, continuous and monotone over $\mathbb{R}^l_+$.*

Notice that we do not require any convexity assumptions on preferences. Moreover, although in this paper we do not make use of utilities, the assumptions stated for preferences ensure that each agents' preference relation $\succsim_i$ can be represented by a continuous utility function u_i defined over the commodity space.

In the rest of the paper, the production set correspondence satisfies the following set of assumptions.

Assumption 3 *For any coalition $S \subseteq N$ and any status quo $\xi \in \Xi$,*

1. *$Y(S, \xi)$ is closed;*

2. *$Y(S,\xi)$ is convex;*

3. *$0 \in Y(S,\xi)$ (possibility of inaction);*

4. *$Y(S,\xi) \cap \mathbb{R}^l_+ \subseteq \{0\}$ (no free lunch);*

5. *$Y(S,\xi) - \mathbb{R}^l_+ \subseteq Y(S,\xi)$ (free disposal);*

6. *$Y(N,\xi) = Y(N) = \sum_{j\in J} Y_j$, for any $\xi \in \Xi$.*

Assumptions 3.1 − 3.3 are standard, and the no free lunch assumption 3.4 means that production of outputs requires inputs. Assumption 3.6 requires that the production possibilities of the grand coalition do not depend on the status quo allocation. It implies, in particular, that assignments for the grand coalition do not depend on a particular status quo state.

We make the following assumptions about the correspondence describing the production possibilities of the outsiders given a production plan chosen by coalition S.

Assumption 4 *For any coalition $S \subseteq N$ and any status quo $\xi \in \Xi$,*

1. *$\sigma_{S,\xi}$ is a nonempty and compact valued correspondence;*

2. *$\sigma_{S,\xi}$ is upper hemicontinuous;*

3. *$0 \in \sigma_{S,\xi}(0)$.*

Finally, we make the following assumption on the resources constraints of the outsiders.

Assumption 5 *For any coalition $S \subseteq N$ and any status quo $\xi \in \Xi$,*

1. *$\Lambda(S,\xi)$ is closed;*

2. *$\omega(S^c) \in \Lambda(S,\xi)$.*

where condition 2. means simply that for any coalition S, the complementary coalition satisfies the resources constraints at least from its initial resources. The production economy considered under Assumptions 1, 2, 3, 4 and 5 is thus formalized in the following list of elements:

$$E \coloneqq \left\langle N, \left(\mathbb{R}^l_+ \succsim_i, \omega_i\right)_{i\in N}, J, \left(Y_j\right)_{j\in J}, \left(\sigma_{S,\xi}, Y(S,\xi), \Lambda(S,\xi)\right)_{\substack{S\in\mathcal{P}(N)\\ \xi\in\Xi}} \right\rangle.$$

Subsection 2.2 shows that this way of modeling the economy is sufficiently general for the treatment of the core of several production economies studied in the literature.

Given the economy E defined above, for every coalition $S \in \mathcal{P}(N)$, and for any vector $x \in \mathbb{R}_+^{l \cdot n}$, we use $x_S := (x_i)_{i \in S}$ to denote the commodity bundles of the members of S and $x_{S^c} := (x_i)_{i \in S^c}$ to denote the commodity bundles of the members of the complementary coalition S^c. Given x_S and x_{S^c}, without loss of generality, we denote x also by (x_S, x_{S^c}), and let $x(S) := \sum_{i \in S} x_i$ be the aggregate resources of S.

Given an allocation $\xi \in \mathcal{F}$ and a coalition $S \subseteq N$, we say that $x_S = (x_i)_{i \in S}$ is an *assignment for* S (given ξ) if there exists a production plan $y \in Y(S, \xi)$ such that $x(S) \leq \omega(S) + y$. Clearly, point 6. in Assumption 3 ensures that an assignment for the grand coalition N is an allocation.

2.2 Preferences-core and resources-core

In this Subsection, we introduce the notion of core with respect to preferences and the notion of core with respect to resources for the production economy, and prove their equivalence. We also discuss some relevant particular cases of our general model. Appendix Table 1 presents a complete taxonomy.

Definition 6 (Core) *Given an allocation* $\xi = (x_1, \dots, x_n, y_1, \dots, y_f) \in \mathcal{F}$ *and a coalition* S*, we say that* S *improves upon* ξ *whenever there exists* $x'_S = (x'_i)_{i \in S}$ *and* $y' \in Y(S, \xi)$ *such that*

- *i)* $x'(S) \leq \omega(S) + y'$ *(*x'_S *is an assignment for* S *given* ξ*);*
- *ii)* $x'_i \succ_i x_i$*, for every* $i \in S$*;*
- *iii)* $\omega(S^c) + y'' \in \Lambda(S, \xi)$*, for some* $y'' \in \sigma_{S,\xi}(y')$*.*

The set of allocations which cannot be improved upon by any coalition is called the preferences-core and is denoted by $\mathcal{C}_p(E)$*. If we replace condition i) with* $x'(S) < \omega(S) + y'$ *and in condition ii)* $\succ_i$ *is replaced by* $\succsim_i$*, then we say that* S *improves upon* ξ *in resources and the corresponding core, denoted* $\mathcal{C}_r(E)$*, is the resources-core.*

Conditions $i) - iii)$ of Definition 6 define the blocking mechanism in our economy. If coalition S improves upon ξ, we say also that S is a *blocking coalition* or that S *blocks* ξ. The blocking mechanism jointly considers some relevant issues. The production plan chosen by a blocking coalition to improve upon a status quo allocation ξ may depend on the status quo allocation itself. This for example occurs if the blocking process with corporate governance allows only firms under the control of S to change their production plans while the other firms maintain their production activity at the status quo allocation ξ [5]. An

[5] As consequence of this assumption, the total production possibilities of S, which

additional feasibility requirement on outsiders' resources is taken into account through condition iii) of Definition 6. This makes the blocking mechanism *considerate* in the sense that the notion of blocking allows the coalition to consider whether the consequence of its blocking is feasible for the outsiders. Clearly, the smaller the set $\Lambda(S,\xi)$, the larger the corresponding core since for a coalition it becomes harder to improve upon a feasible allocation.

Our notion of core considers preferences and resources. The notion of resources-core directly emphasizes the optimal use of resources in the treatment of efficiency, in the sense of no waste of resources. It requires that the utility levels achieved by the members of each coalition under the allocation cannot be achieved through an alternative allocation which also allows resources saving (compare Allais (1943))[6]. These two definitions, in terms of preferences and in terms of resources, are equivalent in (selfish) pure exchange economies under standard regularity conditions on preferences (see Montesano (2002)). Theorem 7 below extends the equivalence between the preferences-core and the resources-core to production economies. Notice that Theorem 7 does not require any other assumption on the production sets than the assumption of nonemptiness, which in its turn, is implied by the possibility of inaction. On the other hand, the idea of resources-core is based on the assumption that coalitions dislike resources waste and therefore the proof builds on the monotonicity requirement on preferences.

Theorem 7 *For the production economy E, the equivalence $\mathcal{C}_r(E) = \mathcal{C}_p(E)$ holds true.*

Proof. Let $\xi = (x_1,\ldots,x_n,y_1,\ldots,y_f) \in \mathcal{C}_r(E)$ and suppose by contradiction that $\xi \notin \mathcal{C}_p(E)$. Then, there exist a coalition $S \subseteq N$ and an assignment for S given ξ, namely $((x'_i)_{i\in S}, y')$ such that $x'_i \succ_i x_i$ for all $i \in S$ and $\omega(S^c) + y'' \in \Lambda(S,\xi)$ for some $y'' \in \sigma_{S,\xi}(y')$. If $x'(S) < \omega(S) + y'$, a contradiction follows, so, we can assume that $x'(S) = \omega(S) + y'$. By continuity of preferences, there exists a positive δ such that, for all $i \in S$, if $z_i \in \mathbb{R}^l_+$ and $\|z_i - x'_i\| < \delta$ then $z_i \succ_i x_i$. By monotonicity, $x'(S) \neq 0$ and in particular, there exists an agent $h \in S$ such that $x'_h > 0$[7]. Choose $\varepsilon > 0$ such that $0 < (1-\varepsilon)\|x'_h\| < \delta$. Define x'' by choosing $x''_i = x'_i$, for $i \in S \setminus \{h\}$ and $x''_h = \varepsilon x'_h$. For every agent $i \in S$, $\|x''_i - x'_i\| \le (1-\varepsilon)\|x'_h\| < \delta$ and consequently $x''_i \succ_i x_i$. By construction, $x''(S) < x'(S) = \omega(S) + y'$, which contradicts the fact that $\xi \in \mathcal{C}_r(E)$.

Let $\xi = (x_1,\ldots,x_n,y_1,\ldots,y_f) \in \mathcal{C}_p(E)$ and suppose that there exist a coali-

depends on the joint activities of all firms, depends on the production at the status quo state.

[6] Resources saved by coalition S after exchange and production are represented by the quantity $\omega(S) + y' - x'(S) > 0$.

[7] Based on the irreflexivity property of the binary relation $\succ_h$, $x'_h \neq x_h$. Since x_h and x'_h belong to $\mathbb{R}^l_+$, by monotonicity assumption it follows that $x'_h > 0$.

tion $S \subseteq N$ and $((x'_i)_{i\in S}, y')$ such that $y' \in Y(S,\xi)$, $\upsilon := -x'(S)+\omega(S)+y' > 0$, $x'_i \succsim_i x_i$, for all $i \in S$, and $\omega(S^c)+y'' \in \Lambda(S,\xi)$ for some $y'' \in \sigma_{S,\xi}(\xi')$. Consider the vector x''_S defined by $x''_i := x'_i + \frac{\upsilon}{|S|}$, for each $i \in S$ Notice that, (x''_S, y') is an assignment for S given ξ, and $x''_i > x'_i$ for each $i \in S$. By monotonicity, $x''_i \succ_i x'_i$ for any $i \in S$ and, by transitivity, we obtain that $x''_i \succ_i x_i$, for all $i \in S$, which is a contradiction. ■

Under the assumptions of Theorem 7, we can denote the core of the economy E simply by $\mathcal{C}(E)$ making no distinction between preferences and resources.

We show below that our production economy is sufficiently general to cover study of the core of many relevant cases of production economies. For each of the particular cases presented in the succeeding remarks, Theorem 7 ensures the equivalence between the preferences-core and the resources-core.

Remark 8 (Production economy with free available technology) Consider the productive economy model in Debreu and Scarf (1963) for instance. In this case, there is a unique production set Y and all coalitions have access to the same production possibilities described by $Y(S,\xi) := Y$, for every coalition S and status quo state ξ. For each production plan y' chosen by coalition S in Y, define $\sigma_{S,\xi}(y') := \{0\}$ and $\Lambda(S,\xi) := \mathbb{R}^l$, meaning that the coalition S is able to produce by itself if this improves upon ξ and ensuring condition iii) of Definition 6 always satisfied. For this production economy, in line with Debreu and Scarf (1963), Definition 6 ensures that the preferences-core is defined as the set of feasible allocations ξ for which it is not possible to find a coalition S, $x'_S = (x'_i)_{i\in S}$ and $y' \in Y$ such that

i) $x'(S) \leq \omega(S) + y'$;

ii) $x'_i \succ_i x_i$, for every $i \in S$.

The resources-core is defined analogously. Under assumptions which are standard for the production set Y, the preferences-core and the resources-core coincide.

Remark 9 (Private ownership economy) In this case, we assume that firms Y_j, $j \in J$, are owned by agents i, $i \in N$, and $\theta_{ij} \in [0,1]$ is the share of firm $j \in J$ owned by agent $i \in N$. Moreover, the condition $\sum_{i\in N} \theta_{ij} = 1$ is statisfied. Assume that for each status quo allocation ξ, the technology available to coalition S is given by

$$Y(S,\xi) := \sum_{j\in J} \sum_{i\in S} \theta_{ij} Y_j,$$

and that the production plans of the complementary coalition corresponding to any choice y' of S, are defined by

$$\sigma_{S,\xi}(y') \coloneqq \left\{ \sum_{j\in J}\sum_{i\notin S}\theta_{ij}y'_j \,|\, y'_j \in Y_j,\ y' = \sum_{j\in J}\sum_{i\in S}\theta_{ij}y'_j \right\} \subseteq Y(S^c,\xi),$$

and finally we assume that $\Lambda(S,\xi) \coloneqq \mathbb{R}^l$. Then, the preferences-core is defined as the set of feasible allocations ξ for which it is not possible to find a coalition S, $x'_S = (x'_i)_{i\in S}$ and $y'_j \in Y_j$, for each $j \in J$, such that

i) $x'(S) \leq \omega(S) + \sum_{j\in J}\sum_{i\in S}\theta_{ij}y'_j$;

ii) $x'_i \succ_i x_i$, for every $i \in S$,

with condition iii) of Definition 6 always satisfied. This notion of preferences-core is studied in Aliprantis et al. (1989) for example. The resources-core is defined analogously. Under standard assumptions for production sets Y_j, the preferences-core and the resources-core of the production economy are the same.

Remark 10 (Private ownership economy with corporate governance) Our production economy model also covers the case that considers a form of corporate governance. Following Xiong and Zheng (2007), in the private ownership production economy defined in Remark 9, we can assume that the set $\widetilde{J}(S)$ of firms controlled by a coalition S is described by a correspondence $\widetilde{J} : \mathcal{P}(N) \rightrightarrows J$ which satisfies the conditions:

(1) if $\sum_{i\in S}\theta_{ij} = 1$ then $j \in \widetilde{J}(S)$;

(2) if $\sum_{i\in S}\theta_{ij} = 0$ then $j \notin \widetilde{J}(S)$;

(3) $\widetilde{J}(S^c) = J \setminus \widetilde{J}(S)$, for each coalition S[8].

Given a status quo $\xi = (x_1,\ldots,x_n,y_1,\ldots,y_f)$, define

$$Y(S,\xi) \coloneqq \sum_{j\in\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}Y_j + \sum_{j\notin\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y_j,$$

$$\sigma_{S,\xi}(y') \coloneqq$$

$$= \left\{ \sum_{j\in\widetilde{J}(S)}\sum_{i\notin S}\theta_{ij}y'_j + \sum_{j\notin\widetilde{J}(S)}\sum_{i\notin S}\theta_{ij}y_j \,|\, y'_j \in Y_j,\ y' = \sum_{j\in\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y'_j + \sum_{j\notin\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y_j \right\},$$

[8] This condition is not a requirement for Xiong and Zheng (2007) model of an economy with production and corporate governance. We introduce it to ensure that the correspondence $\sigma_{S,\xi}$ takes values in the production set of S^c, $Y(S^c,\xi)$.

and $\Lambda(S,\xi) \coloneqq \mathbb{R}^l$. Then, the preferences-core is defined as the set of feasible allocations ξ for which it is not possible to find a coalition S, $x'_S = (x'_i)_{i\in S}$ and $y'_j \in Y_j$, for each $j \in \widetilde{J}(S)$, such that

i) $x'(S) \leq \omega(S) + \sum_{j\in\widetilde{J}(S)} \sum_{i\in S} \theta_{ij} y'_j + \sum_{j\notin\widetilde{J}(S)} \sum_{i\in S} \theta_{ij} y_j$;

ii) $x'_i \succ_i x_i$, for every $i \in S$.

The resources-core is defined similarly. The notion of preferences-core in the presence of corporate governance, is studied in Xiong and Zheng (2007) who also refer to the above Definition of blocking as *inconsiderate blocking*. In the case of inconsiderate blocking, a coalition is able to change the actions of the firms it controls, with the actions of other firms fixed at the status quo. Note that the blocking coalition does not consider whether the consequences of improving an allocation are feasible for the outsiders. The *considerate blocking mechanism* follows from the previous blocking mechanism if we define $\Lambda(S,\xi) \coloneqq \mathbb{R}^l_+$, for each S and ξ. Therefore, defining the preferences-core requires the following additional condition

iii) $\omega(S^c) + \sum_{j\in\widetilde{J}(S)} \sum_{i\notin S} \theta_{ij} y'_j + \sum_{j\notin\widetilde{J}(S)} \sum_{i\notin S} \theta_{ij} y_j \geq 0$,

to obtain the notions of considerate blocking and considerate core introduced in (Xiong and Zheng, 2007, Definition 3). We observe that in this case also and under standard assumptions about the production sets, the preferences-core and the resources-core of the production economy coincide.

2.3 Characterization of the core

Following Montesano (2002) and Di Pietro et al. (2022), we define a measure of social loss for every coalition S. Given an allocation $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f) \in \mathcal{F}$, we start by considering the set of resources which give to coalition S the possibility to reach a redistribution that is weakly preferred to x by all the members of S and allows available production. This set is denoted $\mathcal{R}_S(\xi)$. Formally,

$$\mathcal{R}_S(\xi) \coloneqq \left\{x'(S) \in \mathbb{R}^l_+ : x'_i \succsim_i x_i, i \in S\right\} - \Gamma_S(\xi),$$

where $\Gamma_S(\xi) \coloneqq \{y' \in Y(S,\xi) |\ \exists\ y'' \in \sigma_{S,\xi}(y') : \omega(S^c) + y'' \in \Lambda(S,\xi)\}$. Notice that $\mathcal{R}_S(\xi)$ is nonempty since $x(S)$ belongs to $\mathcal{R}_S(\xi)$, by the reflexivity property of the preference relation, possibility of inaction, Point 3 of Assumption 4, and Point 2 of Assumption 5. The next lemmas show important properties of the set $\mathcal{R}_S(\xi)$.

Lemma 11 *If $\omega(S) \notin \mathcal{R}_S(\xi)$, then S is not able to improve upon ξ.*

Proof. By contradiction, suppose that there exists $((x'_i)_{i\in S}, y')$ such that $y' \in$

$Y(S,\xi)$, $x'(S)-y' < \omega(S)$, $x'_i \succsim_i x_i$ for any $i \in S$, and $\omega(S^c)+y'' \in \Lambda(S,\xi)$ for some $y'' \in \sigma_{S,\xi}(y')$. Therefore, by monotonicity of preferences, we find vectors $\zeta_i > 0$, $i \in S$ such that $(x'+\zeta)(S)-y' = \omega(S)$ with $(x'_i+\zeta_i) \succ_i x_i$ for any $i \in S$, and $y' \in \Gamma_S(\xi)$, contradicting the fact that $\omega(S) \notin \mathcal{R}_S(\xi)$. ■

Lemma 12 *If* $\gamma' > \gamma$ *and* $\gamma \in \mathcal{R}_S(\xi)$, *then* $\gamma' \in \mathcal{R}_S(\xi)$.

Proof. If $\gamma' > \gamma = x'(S)-y'$, then there exist $\upsilon > 0$ and vectors $x''_i \coloneqq x_i + \frac{\upsilon}{|S|}$, $i \in S$, such that $\gamma' = \gamma + \upsilon = x''(S) - y'$. By monotonicity $x''_i \succ_i x_i$ for any $i \in S$, and $y' \in \Gamma_S(\xi)$. Therefore, $\gamma' \in \mathcal{R}_S(\xi)$. ■

Lemma 13 *The set* $\mathcal{R}_S(\xi)$ *is closed in* $\mathbb{R}^l$.

Proof. See the Appendix. ■

If we now consider the differences between endowments and elements in the set $\mathcal{R}_S(\xi)$, we can define the set $\Psi_S(\xi)$ of resources that can be saved by coalition S while still allowing S to achieve for its members a resources allocation that is at least as good as x and to produce. Formally, $\Psi_S\colon \mathcal{F} \rightrightarrows \mathbb{R}^l$,

$$\Psi_S(\xi) \coloneqq \Big\{ z \in \Big(\omega(S) + Y(S,\xi)\Big) \cap \mathbb{R}^l_+ \colon \omega(S) - z \in \mathcal{R}_S(\xi) \Big\}.$$

The next result gives the necessary and sufficient condition for the nonemptiness of $\Psi_S(\xi)$.

Lemma 14 $\omega(S) \in \mathcal{R}_S(\xi)$ *if and only if* $\Psi_S(\xi) \neq \emptyset$.

Proof. Let $\omega(S) \in \mathcal{R}_S(\xi)$. Then, there exists $((x'_i)_{i\in S}, y')$ such that $\omega(S) = x'(S) - y'$, $x'_i \succsim_i x_i$ for any $i \in S$, and $y' \in \Gamma_\xi(S)$. Therefore, $\omega(S) - 0 = x'(S) - y' \in \mathcal{R}_S(\xi)$ and $0 \in (\omega(S) + Y(S,\xi)) \cap \mathbb{R}^l_+$ since $0 = \omega(S) - \omega(S)$ and $-\omega(S) \in Y(S,\xi)$ by Points 3 and 5 of Assumption 3. Thus, $0 \in \Psi_S(\xi)$ and consequently $\Psi_S(\xi) \neq \emptyset$. Coversely, suppose that $\Psi_S(\xi) \neq \emptyset$. Then, there exists z such that $z \geq 0$ and $\omega(S) - z = x'(S) - y' \in \mathcal{R}_S(\xi)$. By Lemma 12, $\omega(S) \geq \omega(S) - z$ and $\omega(S) - z \in \mathcal{R}_S(\xi)$ imply $\omega(S) \in \mathcal{R}_S(\xi)$. ■

Lemma 15 *The set* $\Psi_S(\xi)$ *is compact.*

Proof. Claim 1: $\Psi_S(\xi)$ is closed in $\mathbb{R}^l_+$. Indeed, take z in its closure and a sequence $(z^\nu)_{\nu\in\mathbb{N}} \subseteq \Psi_S(\xi)$ such that z^ν converges to z. Notice that, $(z^\nu)_{\nu\in\mathbb{N}}$ is contained in the set $\omega(S)+Y(S,\xi)$ and $(z^\nu)_{\nu\in\mathbb{N}} \subseteq \mathbb{R}^l_+$. So, $z \in \omega(S)+Y(S,\xi)$ since by Point 1 of Assumption 3, the set $Y(S,\xi)$ is closed and $z \in \mathbb{R}^l_+$. Therefore, $z \in (\omega(S) + Y(S,\xi)) \cap \mathbb{R}^l_+$. Furthermore, $\{\omega(S) - z^\nu \colon \nu \in \mathbb{N}\} \subseteq \mathcal{R}_S(\xi)$ implies that $\omega(S) - z \in \mathcal{R}_S(\xi)$ since by Lemma 13 the set $\mathcal{R}_S(\xi)$ is closed. Thus we conclude that $z \in \Psi_S(\xi)$ and the claim is proved.

Claim 2: $\Psi_S(\xi)$ is bounded. In order prove the claim, it is enough to show the boundedness of $(\omega(S) + Y(S,\xi)) \cap \mathbb{R}^l_+$. Since a translation of a set does not affect its asymptotic cone, then $\mathcal{A}(\omega(S)+Y(S,\xi)) = \mathcal{A}(Y(S,\xi))$. Furthermore, by Points 1, 2 and 3 of Assumption 3, $\mathcal{A}(Y(S,\xi)) \subseteq Y(S,\xi)$, and in particular, by definition of the asymptotic cone, $0 \in \mathcal{A}(Y(S,\xi))$. Since $\mathcal{A}(\mathbb{R}^l_+) = \mathbb{R}^l_+$, then by Points 3 and 4 of Assumption 3 we get $\mathcal{A}(\omega(S)+Y(S,\xi)) \cap \mathcal{A}(\mathbb{R}^l_+) = \{0\}$, which concludes the proof by point 5. of Proposition 33. ■

Let us fix a vector $g \in \mathbb{R}^l_+$ with $g \neq 0$. We will call g the *reference bundle*. Below, we introduce the *loss mapping* as a function measuring the maximum amount of resources that can be saved by a coalition S with respect to an allocation x in the direction of the reference bundle g. Equivalently, the loss mapping measures the loss, in terms of g, procured to coalition S by an allocation ξ.

Formally, the loss mapping $\mathcal{L}^g_S : \mathcal{F}(\omega) \to \mathbb{R}$ is defined as follows

$$\mathcal{L}^g_S(\xi) := \begin{cases} \max\{\lambda \in \mathbb{R} \colon \lambda \cdot g \in \Psi_S(\xi)\} & \text{if } \Psi_S(\xi) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, if $\Psi_S(\xi) \neq \emptyset$ then the maximum is well-defined, since according to Lemma 15, $\Psi_S(\xi)$ is compact. Furthermore, $\mathcal{L}^g_S(\xi) \geq 0$ since $g \in \mathbb{R}^l_+$ with $g \neq 0$ [9] . Note that the loss mapping may vary according to the reference bundle g. However, if there exists g such that $\mathcal{L}^g_S(x)$ is strictly positive, then for all reference bundles, the corresponding loss mappings are strictly positive.

Proposition 16 *For a given allocation ξ, if $\mathcal{L}^g_S(\xi) > 0$ for a vector $g > 0$, then $\mathcal{L}^{g'}_S(\xi) > 0$ for every $g' > 0$.*

Proof. See the Appendix. ■

Next moving from the loss (in terms of g) procured to each coalition S by an allocation ξ, we introduce a measure of social loss with respect to ξ defined as the *social loss mapping* $\mathcal{L}^g \colon \mathcal{F}(\omega) \to \mathbb{R}$ given by

$$\mathcal{L}^g(\xi) := \max_{S \subseteq N} \mathcal{L}^g_S(\xi).$$

The social loss mapping $\mathcal{L}^g(\xi)$ is well-defined because for every coalition S, the loss mapping $\mathcal{L}^g_S$ is well-defined. $\mathcal{L}^g(\xi)$ is the maximal loss procured to a coalition by the allocation ξ. Theorem 17 shows that the maximal loss vanishes if and only if the allocation belongs to the core. Consequently, we obtain a full characterization of the core in terms of loss mappings.

[9] In the literature studying Pareto optimal allocations in terms of resources, the reference bundle g is chosen arbitrarily. In a classical setting, Debreu (1951) chose $g = \omega(N)$ and Allais (1943) and Groves (1979) use $g = (1, 0 \ldots, 0)$.

Theorem 17 *For any non null reference bundle* $g \in \mathbb{R}^l_+$, $\mathcal{L}^g(\xi) = 0$ *if and only if* $\xi \in \mathcal{C}(E)$.

Proof. We start by proving that if an allocation ξ belongs to the core, then $\mathcal{L}^g(\xi) = 0$. Suppose by contradiction that $\mathcal{L}^g(\xi) > 0$. Then there exists a coalition S such that $\mathcal{L}^g_S(\xi) > 0$ and $\Psi_S(\xi)\setminus\{0\} \neq \emptyset$. Consequently, there exists $z > 0$ such that $z \in \Psi_S(\xi)$. Therefore, $0 < z = \omega(S) + \widehat{y}$ with $\widehat{y} \in Y(S,\xi)$ and $\omega(S) - z = -\widehat{y} \in \mathcal{R}_S(\xi)$. Thus $-\widehat{y} = x'(S) - y'$, for some $((x'_i)_{i\in S}, y') \in \mathbb{R}^{l\cdot|S|}_+ \times \Gamma_S(\xi)$ with $x'_i \succsim_i x_i$ for any $i \in S$. Finally, notice that $0 < z = \omega(S) + \widehat{y} = \omega(S) - x'(S) + y'$ and consequently a contradiction is obtained since S improves upon ξ. Let us show now that $\mathcal{L}^g(\xi) = 0$ implies $\xi \in \mathcal{C}(E)$. By contradiction, suppose that $\xi \notin \mathcal{C}(E)$. So, there exists a coalition $S \subseteq N$, $(x'_i)_{i\in S}$ and y' such that $x'_i \succsim_i x_i$ for every $i \in S$, $y' \in Y(S,\xi)$, $x'(S) < \omega(S)+y'$ and $\omega(S^c)+y'' \in \Lambda(S,\xi)$ for some $y'' \in \sigma_{S,\xi}(y')$. So, $g' := \omega(S)+y'-x'(S) > 0$ belongs to $(\omega(S)+Y(S,\xi))\cap\mathbb{R}^l_+$ since $y'-x'(S) \in Y(S,\xi)$ by $y' \in Y(S,\xi)$ and Point 5 of Assumption 3. Furthermore, $\omega(S) - g' = x'(S) - y' \in \mathcal{R}_S(\xi)$ since $y' \in \Gamma_S(\xi)$, and consequently, $g' \in \Psi_S(\xi)$ and $g' > 0$. Thus $\mathcal{L}^{g'}_S(\xi) > 0$ which implies $\mathcal{L}^g_S(\xi) > 0$ (by Proposition 16), contradicting the fact that $\mathcal{L}^g(\xi) = 0$. ∎

Theorem 17 proves that the core of a production economy is formed by zero points of suitable loss mappings. The result holds for the case of inconsiderate blocking and for models that take account of the interdependency effects due to production. In the next section, we allow also for interdependency effects due to consumption.

3 A production economy with consumption externalities

In Di Pietro et al. (2022), we study pure exchange economies in which individual preferences are affected by the consumption of all other agents in the economy. We introduce a measure of social loss with respect to the γ-core and the α-core of the economy which completely characterizes the corresponding core allocations. In this Section, we show that the characterization of the core of production economies proved by Theorem 17 still holds in the presence of consumption externalities. For simplicity, we consider only an optimistic attitude of a blocking coalition with respect to the behavior of the outsiders and define a suitable extension of the γ-core notion. This new notion of core will account for both, consumption externalities and interdependency due to production and includes the notions studied in Dufwenberg et al. (2011) and Xiong and Zheng (2007).

In this extension of the model presented in Section 2.1, individuals are assumed to be not selfish and their preferences may depend on the consumption of all the agents. Formally, the preferences of individual i are described by a binary relation $\succsim_i$ over $\mathbb{R}_+^{l\cdot n}$. $\widetilde{E}$ denotes the production economy being considered. In this extension Assumptions 1, 3, 4 and 5 in Section 2.1 are retained unchanged but Assumption 2 is replaced by the following new assumptions.

Assumption 18 *For every individual $i \in N$,*

1. $\succsim_i$ *are complete, transitive, continuous over* $\mathbb{R}_+^{l\cdot n}$*;*

2. Social Group Monotonicity (SGM). *For any coalition $S \subseteq N$, any vector $x \in \mathbb{R}_+^{l\cdot n}$ and $z > x(S)$, there exists vectors $x'_i \in \mathbb{R}_+^l$, $i \in S$, with $x'(S) = z$, and $(x'_S, x_{N\backslash S}) \succ_i (x_S, x_{N\backslash S})$, for all $i \in S$;*

3. Boundary Aversion (BA). *For any coalition $S \subseteq N$, and for any vector $x \in \mathbb{R}_+^{l\cdot n}$ with $x_S \in \partial\mathbb{R}_+^{l\cdot|S|}$, $(z_S, x_{N\backslash S}) \succ_i x$ for any $z_S = (z_i)_{i\in S} \in \operatorname{Int}\mathbb{R}_+^{l\cdot|S|}$ and for any $i \in S$.*

The SGM condition states that any increase in the resources available to the coalition S can be redistributed to make every member of S better off. SGM may fail in the presence of hateful agents and generalizes the Social Monotonicity condition adopted by Dufwenberg et al. (2011) in order to prove the second welfare theorem [10] . Condition BA is standard in the study of cooperative solutions in selfish models. For a continuous and monotone selfish preference, it is equivalent to require that all commodity bundles on the boundary are equivalent in terms of preferences. In Condition 3. of Assumption 18, this requirement is adapted to preferences under externalities.

To construct an example of preference satisfying our assumptions, we refer to the so called *separable preference*, i.e. a preference relation $\succsim_i$ where $(x_i, x_{N\backslash i}) \succsim_i (x'_i, x_{N\backslash i})$ for some $x_{N\backslash i}$ implies that $(x_i, x'_{N\backslash i}) \succsim_i (x'_i, x'_{N\backslash i})$, for each $x'_{N\backslash i} \in \mathbb{R}_+^{l\cdot(n-1)}$. Under separability of $\succsim_i$, it is possible to introduce a well-defined preference relation $\succsim_i^{(i)}$ over $\mathbb{R}_+^l$ i.e. over the individual consumption vectors, sometimes called *internal preference* of trader i [11] .

Below we give an example of a separable preference inspired by classical Edge-

[10] A similar condition is assumed in Borglin (1973) to ensure that the second welfare theorem holds true in the case of separable preferences. Based on standard arguments, it can be seen that (strict) increasing preferences in their domain satisfy the (SGM) condition.

[11] By definition $x_i \succsim_i^{(i)} x'_i$, if and only if $(x_i, x_{N\backslash i}) \succsim_i (x'_i, x_{N\backslash i})$, for some $x_{N\backslash i}$.

worth well-being externalities (see Dufwenberg et al. (2011)). In this example, agent i cares about his own internal utility and the sum of the internal utilities of the other agents.

Example 19 *Each agent $i \in N$ has an (internal) utility function u_i which depends only on his own consumption x_i, and an interdependent utility function U_i which for each agent aggregates these individual utilities according to the formula:*

$$U_i(x) \coloneqq u_i(x_i) + \frac{\beta_i}{n-1} \sum_{j \neq i} u_j(x_j).$$

If β_i is positive, then agent i is altruistic or benevolent and the SGM assumption is satisfied. The BA condition is satisfied for the preference represented by U_i, when each individual selfish utility satisfies BA, for example if each individual utility is a Cobb-Douglas utility function.

In this Section the definitions of assignment and allocation are the same as in Section 2.1.

3.2 Preferences-core and resources-core

As in Section 2.2, we introduce the notion of preferences-core and resources-core, and prove that under our assumptions these two notions coincide [12].

Definition 20 (γ-Core) *Given an allocation $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f) \in \mathcal{F}$ and a coalition S, we say that S improves upon ξ, whenever there exist $x'_S = (x'_i)_{i \in S}$ and $y' \in Y(S, \xi)$ such that*

i) $x'(S) \leq \omega(S) + y'$ (x'_S is an assignment for S given ξ);

ii) $(x'_S, x_{S^c}) \succ_i (x_S, x_{S^c})$, for every $i \in S$;

iii) $\omega(S^c) + y'' \in \Lambda(S, \xi)$, for some $y'' \in \sigma_{S,\xi}(y')$.

The set of allocations which cannot be improved upon by any coalition is called the γ-preferences core and is denoted $\mathcal{C}_p(\widetilde{E})$. If we add $x'(S) < \omega(S) + y'$ in condition i) and in condition ii) $\succ_i$ is replaced by $\succsim_i$, then we say that S improves upon ξ in resources and the corresponding core, which is denoted $\mathcal{C}_r(\widetilde{E})$, is the γ-resources core.

In the next Theorem, we show that the two notions of core coincide. Also

[12] The notion of core used in this work is commonly called γ-core. In the presence of externalities, a different blocking mechanism might be defined depending on the attitude of the blocking coalition with respect to the behavior of the outsiders. See for instance Di Pietro et al. (2022).

in this case, the equivalence result does not require any assumptions about production sets other than nonemptiness.

Theorem 21 *Under the previous assumptions, $\mathcal{C}_r(\widetilde{E}) = \mathcal{C}_p(\widetilde{E})$ holds true.*

Proof. Let $\xi \in \mathcal{C}_r(\widetilde{E})$ and suppose by contradiction that $\xi \notin \mathcal{C}_p(\widetilde{E})$. Then, there exist a coalition $S \subseteq N$, $(x'_i)_{i \in S}$ and a vector $y' \in Y(S, \xi)$ such that $x'(S) \leq \omega(S) + y'$, $(x'_S, x_{N \setminus S}) \succ_i (x_S, x_{N \setminus S})$ for all $i \in S$ and $\omega(S^c) + y'' \in \Lambda(S, \xi)$, for some $y'' \in \sigma_{S,\xi}(y')$. If $x'(S) < \omega(S) + y'$, a contradiction follows, so, we can assume that $x'(S) = \omega(S) + y'$. By the continuity of preferences, there exists a positive δ such that, if $z_i \in \mathbb{R}^l_+$ for all $i \in S$ and $\|(z_S, x_{N \setminus S}) - (x'_S, x_{N \setminus S})\| < \delta$ then $(z_S, x_{N \setminus S}) \succ_i (x'_S, x_{N \setminus S})$, for every $i \in S$.

With innocuous abuse of notation, suppose there exists an agent $h \in S$ such that $x'_h > 0$[13]. Choose $\varepsilon > 0$ such that $0 < (1 - \varepsilon)\|x'_h\| < \delta$. Define x'' by choosing $x''_i = x'_i$, for $i \in S \setminus \{h\}$ and $x''_h = \varepsilon x'_h$. For every agent $i \in S$, $\|(x''_S, x_{N \setminus S}) - (x'_S, x_{N \setminus S})\| \leq (1 - \varepsilon)\|x'_h\| < \delta$ and consequently $(x''_S, x_{N \setminus S}) \succ_i x$. By construction, $x''(S) < x'(S) = \omega(S) + y'$, which contradicts the fact that $\xi \in \mathcal{C}_r(\widetilde{E})$.

Let $\xi \in \mathcal{C}_p(\widetilde{E})$ and suppose that there exist a coalition $S \subseteq N$, $(x'_i)_{i \in S}$ and a vector $y' \in Y(S, \xi)$ such that $x'(S) < \omega(S) + y'$, $(x'_S, x_{N \setminus S}) \succsim_i (x_S, x_{N \setminus S})$, for all $i \in S$ and $\omega(S^c) + y'' \in \Lambda(S, \xi)$, for some $y'' \in \sigma_{S,\xi}(\xi')$. Under SGM, there exist vectors x''_i, $i \in S$, such that $x''(S) = \omega(S) + y'$ and $(x''_S, x_{N \setminus S}) \succ_i (x'_S, x_{N \setminus S})$ for any $i \in S$. Notice that, (x''_S, y') is an assignment for S given ξ. Finally, by transitivity, we obtain $(x''_S, x_{N \setminus S}) \succ_i x$, for all $i \in S$, which is a contradiction.

■

Under the assumptions of Theorem 21, the γ-core of the production economy $\widetilde{E}$ can be denoted $\mathcal{C}(\widetilde{E})$.

3.3 Characterization of the γ-core

As in Section 2.3, in order to introduce a measure of social loss for every coalition S and for a given allocation ξ, we first introduce the sets $\widetilde{\mathcal{R}}_S(\xi)$ and

[13] By $x'_S \in \mathbb{R}^{l \cdot |S|}_+$ and $\omega(S) \gg 0$, if $y'^c \geq 0$ for some commodity c, then $x'(S) \neq 0$. Otherwise, by $x'_S \in \mathbb{R}^{l \cdot |S|}_+$, we get $-\omega(S) \leq y \ll 0$, and consequently $x'(S) \ll \omega(S)$. Thus, using (SGM) on $(x'_S, x_{N \setminus S})$ and transitivity, we find nonnegative vectors $\widetilde{x}_i$, $i \in S$, with $\widetilde{x}(S) = \omega(S)$ and a production plan $\widetilde{y} = 0$ such that $(\widetilde{x}_S, x_{N \setminus S}) \succ_i (x_S, x_{N \setminus S})$ for any $i \in S$, $\widetilde{x}(S) = \omega(S) + \widetilde{y}$, and $\omega(S^c) + 0 \in \Lambda(S, \xi)$ (by Point 3 of Assumption 3 and Point 2 of Assumption 5).

study some of their properties. Define

$$\widetilde{\mathcal{R}}_S(\xi) \coloneqq \Big\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{N\setminus S}) \succsim_i x, i \in S\Big\} - \Gamma_S(\xi),$$

where $\Gamma_S(\xi)$ is defined in Section 2.3. As for the model without externalities, the set $\widetilde{\mathcal{R}}_S(\xi)$ is nonempty and satisfies the following properties.

Lemma 22 *If $\omega(S) \notin \widetilde{\mathcal{R}}_S(\xi)$, then S is not a blocking coalition.*

Proof. By contradiction, suppose that there exists $(x'_i)_{i\in S}$ and a vector $y' \in Y(S,\xi)$ such that $x'(S) < \omega(S) + y'$, $(x'_S, x_{N\setminus S}) \succsim_i (x_S, x_{N\setminus S})$ for all $i \in S$ and $\omega(S^c) + y'' \in \Lambda(S,\xi)$, for some $y'' \in \sigma_{S,\xi}(y')$. Therefore, under SGM, we find vectors $\zeta_i \geq 0$, $i \in S$ such that $\zeta(S) = \omega(S) + y'$ with $(\zeta_S, x_{N\setminus S}) \succ_i (x'_S, x_{N\setminus S})$ for any $i \in S$, and $\omega(S^c) + y'' \in \Lambda(S,\xi)$. Finally by transitivity of preferences, we get $(\zeta_S, x_{N\setminus S}) \succ_i x$, and a contradiction to the fact that $\omega(S) = \zeta(S) - y' \notin \widetilde{\mathcal{R}}_S(\xi)$. ■

Lemma 23 *If $\gamma' > \gamma$ and $\gamma \in \widetilde{\mathcal{R}}_S(S)$, then $\gamma' \in \widetilde{\mathcal{R}}_S(\xi)$.*

Proof. If $\gamma' > \gamma = x'(S) - y'$ then according to SGM there exist vectors $\zeta_i \geq 0$, $i \in S$, such that $\zeta(S) = \gamma' + y' > x'(S)$, with $(\zeta_S, x_{N\setminus S}) \succ_i (x'_S, x_{N\setminus S})$ for each $i \in S$, and $\omega(S^c) + y'' \in \lambda(S,\xi)$ with $y'' \in \sigma_{S,\xi}(y')$. By transitivity of preferences we have $(\zeta_S, x_{N\setminus S}) \succ_i x$ for each $i \in S$ and thus $\gamma' = \zeta(S) - y'$ belongs to $\in \mathcal{R}_S(\xi)$. ■

Lemma 24 *The set $\widetilde{\mathcal{R}}_S(\xi)$ is closed in $\mathbb{R}^l$.*

Proof. See the Appendix. ■

In the next step, we define the set of resources that can be saved by coalition S still allowing for its members to achieve a resources allocation that is at least as good as x. Formally, $\widetilde{\Psi}_S \colon \mathcal{F} \rightrightarrows \mathbb{R}^l$,

$$\widetilde{\Psi}_S(\xi) \coloneqq \Big\{z \in \Big(\omega(S) + Y(S,\xi)\Big) \cap \mathbb{R}^l_+ \colon \omega(S) - z \in \widetilde{\mathcal{R}}_S(\xi)\Big\},$$

and using the same arguments as used to proof Lemma 14 and Lemma 15, we obtain the following results.

Lemma 25 *$\omega(S) \in \widetilde{\mathcal{R}}_S(\xi)$ if and only if $\widetilde{\Psi}_S(\xi) \neq \emptyset$.*

Lemma 26 *The set $\widetilde{\Psi}_S(\xi)$ is compact.*

We now fix a reference bundle $g > 0$ and introduce the *loss mapping* $\widetilde{\mathcal{L}}^g_S \colon \mathcal{F} \to \mathbb{R}$, for a production economy with consumption externalities as follows:

$$\widetilde{\mathcal{L}}^g_S(\xi) \coloneqq \begin{cases} \max\Big\{\lambda \in \mathbb{R} \colon \lambda \cdot g \in \widetilde{\Psi}_S(\xi)\Big\} & \text{if } \widetilde{\Psi}_S(\xi) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

If $\widetilde{\Psi}_S(\xi)$ is nonempty, the loss mapping has a finite value, since the set $\widetilde{\Psi}_S(\xi)$ is compact. Notice also that $\widetilde{\mathcal{L}}^g_S(\xi) \geq 0$ since $g \in \mathbb{R}^l_+$ with $g \neq 0$. The loss mappings are different if we vary the reference bundles. However, if there exists g such that $\mathcal{L}^g_S(x)$ is strictly positive, then for all reference bundles, the loss mappings are strictly positive.

Proposition 27 *For a given allocation ξ, if $\widetilde{\mathcal{L}}^g_S(\xi) > 0$ for a vector $g > 0$, then $\widetilde{\mathcal{L}}^{g'}_S(\xi) > 0$ for every $g' > 0$.*

Proof. See the Appendix. ■

From the loss procured to each coalition S by an allocation ξ, we can introduce the measure of social loss with respect to ξ as the *social loss mapping* $\widetilde{\mathcal{L}}^g : \mathcal{F} \to \mathbb{R}$ defined as

$$\widetilde{\mathcal{L}}^g(\xi) \coloneqq \max_{S \subseteq N} \widetilde{\mathcal{L}}^g_S(\xi).$$

The social loss mapping is well-defined because for every coalition S, the loss mapping $\widetilde{\mathcal{L}}^g_S$ is well-defined. Theorem 28 shows that the maximal loss vanishes if and only if the allocation belongs to the γ-core. Consequently, we obtain a full characterization of the core in terms of loss mappings in a production economy with externalities.

Theorem 28 *For any non null reference bundle $g \in \mathbb{R}^l_+$, $\widetilde{\mathcal{L}}^g(\xi) = 0$ if and only if $\xi \in \widetilde{\mathcal{C}}(\widetilde{E})$.*

Proof. We start by proving that if an allocation ξ belongs to the core, then $\widetilde{\mathcal{L}}^g(\xi) = 0$. Suppose by contradiction that $\widetilde{\mathcal{L}}^g(\xi) > 0$. Then there exists a coalition S such that $\widetilde{\mathcal{L}}^g_S(\xi) > 0$ and $\widetilde{\Psi}_S(\xi) \backslash \{0\} \neq \emptyset$. Consequently, there exists $z > 0$ such that $z \in \widetilde{\Psi}_S(\xi)$. Therefore, $0 < z = \omega(S) + \widehat{y}$ with $\widehat{y} \in Y(S, \xi)$ and $\omega(S) - z = -\widehat{y} \in \widetilde{\mathcal{R}}_S(\xi)$. Thus $-\widehat{y} = x'(S) - y'$, for some $(x', y') \in \mathbb{R}^{l \cdot n}_+ \times \Gamma_S(\xi)$ with $(x'_S, x_{N\backslash S}) \succsim_i x$ for any $i \in S$. Finally, notice that $0 < z = \omega(S) + \widehat{y} = \omega(S) - x'(S) + y'$ and a contradiction is obtained.

Let us show now that $\widetilde{\mathcal{L}}^g(\xi) = 0$ implies $\xi \in \widetilde{\mathcal{C}}(\widetilde{E})$. By contradiction, suppose that $\xi \notin \widetilde{\mathcal{C}}(\widetilde{E})$. Then there exist a coalition $S \subseteq N$, $(x'_i)_{i \in S}$ and y' such that $(x'_S, x_{N\backslash S}) \succsim_i x$ for every $i \in S$, $y' \in Y(S, \xi)$, $x'(S) < \omega(S) + y'$ and $\omega(S^c) + y'' \in \Lambda(S, \xi)$ for some $y'' \in \sigma_{S,\xi}(y')$. Consequently, $g' \coloneqq \omega(S) + y' - x'(S) > 0$ belongs to $(\omega(S) + Y(S, \xi)) \cap \mathbb{R}^l_+$ since $y' - x'(S) \in Y(S, \xi)$ by $y' \in Y(S, \xi)$ and by Point 5 of Assumption 3. Furthermore, $\omega(S) - g' = x'(S) - y' \in \widetilde{\mathcal{R}}_S(\xi)$ since $y' \in \Gamma_S(\xi)$, and consequently, $g' \in \widetilde{\Psi}_S(\xi)$ and $g' > 0$. Thus $\widetilde{\mathcal{L}}^{g'}_S(\xi) > 0$ which implies $\widetilde{\mathcal{L}}^g_S(\xi) > 0$, contradicting the fact that $\widetilde{\mathcal{L}}^g(\xi) = 0$. ■

4 Concluding Remarks

We have shown that the preferences-core and the resources-core of a production economy coincide. We have shown also that the core of a production economy can be characterized in terms of a measure of social loss. To this end, we have introduced a new core notion which is sufficiently general to cover both the core of private ownership production economies and also the considerate core in a model with corporate governance where the actions of a blocking coalition may affect the production of the firms not under its control. Consequently, our characterization holds despite the interdependence effects due to the presence of production. Moreover, it holds also for models that include consumption externalities. The main assumptions in the case of consumption externalities are the redistribution property known as Social Group Monotonicity and a Boundary Aversion condition.

For simplicity, we assume that the consumption set is the same for all traders and coincide with the positive cone of the commodity space. However, in a more general framework in which the consumption of each agent depends on which coalition the trader joins, the results would be similar.

Also, assumptions more general than SGM could be considered. We claim that the results obtained in this paper could be proved under conditions ensuring that the Second Welfare Theorem holds true [14]. Take for instance the case of the Social redistribution assumption introduced recently in del Mercato and Nguyen (2021). This condition is weaker than SGM and other relevant assumptions that have been studied in the literature (see e.g. Osana (1972)).

We conclude by commenting on the Debreu-Scarf core equivalence Theorems in the context of our core notions.

The notion of (considerate) core which we introduced in Section 2.2, includes the core of a private ownership production economy and the core of private ownership production economies with corporate governance. For these two notions, under the BA condition, Xiong and Zheng (2007) proved equivalence with Walrasian equilibrium allocations. This equivalence result can be easily adapted to our core notion.

On the other hand, in the presence of externalities, equivalence theorems for the core and competitive allocations are generally not valid. Therefore, we should not expect the core $C(\widetilde{E})$ introduced in Section 3.2 to coincide with competitive equilibria in the absence of very strong assumptions.

[14] More generally, under the conditions usually imposed to show that the set of Pareto optimal allocations is included in the internal Pareto optimal allocations in models with separable preferences.

With separable preferences and assuming SGM, a result similar to (Dufwenberg et al., 2011, Lemma 1) is valid also for the core $C(\widetilde{E})$, i.e. that the γ-core of the production economy is included in the core of the internal economy, defined by internal preferences and, consequently, the replica core is also contained in the set of competitive equilibria (see (Dufwenberg et al., 2011, Theorem 6)). Finally, it can be shown that under stronger conditions the core coincides with Walrasian equilibrium allocations (see (Dufwenberg et al., 2011, Theorem 7)).

5 Appendix

5.1 Basic properties of correspondences and asymptotic cones.

In this section we recall some basic definitions and properties of correspondences and asymptotic cones in Euclidean spaces [15] .

Definition 29 *A correspondence $\varphi\colon X \rightrightarrows Y$ between topological spaces is upper hemicontinuous at a point $x \in X$ if for all open neighborhoods $U \subseteq Y$ such that $\varphi(x) \subseteq U$, there exists a neighborhood V of x such that $\varphi(V) \subseteq U$.*

The following result is a sequential characterization of upper hemicontinuity [16] .

Theorem 30 *Let X and Y be two metric spaces. A compact valued correspondence $\varphi\colon X \rightrightarrows Y$ is upper hemicontinuous at a point $x \in X$ if and only if for every sequence $(x^\nu, y^\nu)_{\nu\in\mathbb{N}} \subseteq X \times Y$ such that x^ν converges to some $x \in X$, and $y^\nu \in \varphi(x^\nu)$ for any $\nu \in \mathbb{N}$, the sequence y^ν has a limit point in $\varphi(x)$.*

Recall that a subset $C \subseteq \mathbb{R}^d$ is a cone with vertex $x \in C$, if for any $y \in C$, it contains the set $\{z \in \mathbb{R}^d \colon \exists\, \tau \in \mathbb{R}_+, z = x + \tau(y - x)\}$.

Definition 31 *A collection of n cones $C_k \subseteq \mathbb{R}^d$, $k = 1, \ldots, n$ with vertex 0 is positively semi-independent if $x_k \in C_k$, with $k = 1, \ldots, n$, and $\sum_{k=1}^n x_k = 0$ implies $x_k = 0$ for any k.*

From the previous definition, positively semi-independent sets do not contain elements in opposite directions.

[15] For the case of asymptotic cones and their properties we refer to Debreu (1959), Section 1.9 and Villar (2000), Chapter 12.
[16] See Aliprantis and Border (2006), Chapter 17, Corollary 17.17 for the proof of the characterization. We also refer to Appendix A in Carmona (2013) for a concise collection of mathematical results in metric spaces.

Let D be a subset of $\mathbb{R}^d$ and k a nonnegative scalar. Define by D^k the set of element of D whose norm is greater than k, i.e., $D^k \coloneqq \{x \in D \colon ||x|| \geq k\}$, and denote by $\Gamma(D^k)$ the intersection of all closed cones with vertex 0 containing D^k. The asymptotic cone of D is a closed cone with vertex zero that contains all unbounded directions of D. Formally,

Definition 32 *The asymptotic cone of $D \subseteq \mathbb{R}^d$ is $\mathcal{A}(D) \coloneqq \bigcap_{k \geq 0} \Gamma(D^k)$.*

The next proposition states some of the properties of asymptotic cones.

Proposition 33 *Let D and T be two subsets of $\mathbb{R}^d$. Then,*

1. *$\mathcal{A}(D)$ is a closed cone with vertex zero;*
2. *If D is a closed and convex set containing the null vector, then $\mathcal{A}(D) \subseteq D$;*
3. *Let x be a vector in $\mathbb{R}^d$. Then $\mathcal{A}(D + \{x\}) = \mathcal{A}(D)$;*
4. *If $D \subseteq T$, then $\mathcal{A}(D) \subseteq \mathcal{A}(T)$;*
5. *Let $\{D_i\}_{i \in I}$ be an indexed family of subsets of $\mathbb{R}^d$. If $\bigcap_{i \in I} \mathcal{A}(D_i) = \{0\}$, then $\bigcap_{i \in I} D_i$ is bounded;*
6. *Let $\{D_i\}_{k=1}^n$ be a family of closed subsets of $\mathbb{R}^d$. If the asymptotic cones $\mathcal{A}(D_k)$, with $k = 1, \ldots, n$ are positively semi-independent, then the set $\sum_{k=1}^n D_k$ is closed in $\mathbb{R}^d$.*

5.2 Proofs of technical results.

Below we presents the proofs of the technical results.

Proof of Lemma 13. In order to prove that the set $\mathcal{R}_S(\xi)$ is closed, we need first to show that the sets $\left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\}$ and $\Gamma_S(\xi)$ are closed. Then we verify that their asymptotic cones are positively semi-independent and use Point 6 of Proposition 33 in the Appendix.

<u>Claim 1:</u> The set $\left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\}$ is closed in $\mathbb{R}^l_+$. Indeed, take z in its closure. So, there exists a sequence $(z^\nu(S))_{\nu \in \mathbb{N}} \subseteq \left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\}$ such that $z^\nu(S)$ converges to z and $z^\nu_i \succsim x_i$ for any $i \in S$ and for any $\nu \in \mathbb{N}$. Let $\varepsilon > 0$. From the convergence of $z^\nu(S)$, there exists $n \in \mathbb{N}$ such that for any $\nu \geq n$ and for any $c = 1, \ldots, l$, $z^{c\nu}(S) \leq z^c(S) + \varepsilon$. Consider the vector $b \in \mathbb{R}^l$ with $b^c \coloneqq \max\{z^\nu(S) \colon \nu = 1, \ldots, n-1\} \cup \{z^c(S) + \varepsilon\}$. Since $0 \leq z^\nu_i \leq z^\nu(S)$ for any $\nu \in \mathbb{N}$, then $\{z^\nu_i \colon \nu \in \mathbb{N}\} \subseteq [0, b]$, which is a compact set. Thus, up to subsequence, z^ν_i converges to some $\overline{z}_i$, and by continuity of the preferences, $\overline{z}_i \succsim_i x_i$. Therefore, $z = \lim\limits_{\nu \to \infty} z^\nu(S) = \lim\limits_{\nu \to \infty} \sum_{i \in S} z^\nu_i = \sum_{i \in S} \lim\limits_{\nu \to \infty} z^\nu_i = \sum_{i \in S} \overline{z}_i$, which concludes the proof of the claim.

Claim 2: $\Gamma_S(\xi)$ is closed in $\mathbb{R}^l$. Indeed, take z in its closure. So, there exists a sequence $(z^\nu)_{\nu\in\mathbb{N}} \subseteq \Gamma_S(\xi)$ such that z^ν converges to z. Since $(z^\nu)_{\nu\in\mathbb{N}} \subseteq Y(S,\xi)$, and $Y(S,\xi)$ is a closed set by Point 1 of Assumption 3, then $z \in Y(S,\xi)$. By definition of $\Gamma_S(\xi)$, for any z^ν, there exists $\eta^\nu \in \sigma_{S,\xi}(z^\nu)$ such that $\omega(S^c)+\eta^\nu \in \Lambda(S,\xi)$. By Points 1 and 2 of Assumption 4, the sequence $(\eta^\nu)_{\nu\in\mathbb{N}}$ has a limit point η in $\sigma_{S,\xi}(z)$. Finally, by Point 1 of Assumption 5 the set $\Lambda(S,\xi)$ is closed, and so, $\omega(S^c)+\eta \in \Lambda(S,\xi)$. Thus $z \in \Gamma_S(\xi)$, which concludes the proof of the claim. As a consequence of claim 2, the set $-\Gamma(S,\xi)$ is closed in $\mathbb{R}^l$.

Claim 3: $\mathcal{A}\left(\left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\}\right)$ is a subset of $\mathbb{R}^l_+$. Indeed, since $\left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\} \subseteq \mathbb{R}^l_+$, then $\mathcal{A}\left(\left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\}\right) \subseteq \mathcal{A}(\mathbb{R}^l_+)$. This conclude the proof of the claim, since $\mathcal{A}(\mathbb{R}^l_+) = \mathbb{R}^l_+$.

Claim 4: $\mathcal{A}\left(-\Gamma(S,\xi)\right)$ is a subset of $-Y(S,\xi)$, and $-Y(S,\xi)$ is closed in $\mathbb{R}^l$. Since $-\Gamma(S,\xi)$ is a subset of $-Y(S,\xi)$, then $\mathcal{A}\left(-\Gamma(S,\xi)\right) \subseteq \mathcal{A}\left(-Y(S,\xi)\right)$. So, the result trivially follows by Points 1, 2 and 3 of Assumption 3.

Claim 5: $\mathcal{A}\left(\left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\}\right)$ and $\mathcal{A}\left(-\Gamma(S,\xi)\right)$ are positively semi-independent. By Claims $1-4$, take $\alpha \in \mathbb{R}^l_+$ and $-\beta \in -Y(S,\xi)$ such that $\alpha + (-\beta) = 0$. So, $\alpha = \beta$ and consequently, $\beta \in Y(S,\xi) \cap \mathbb{R}^l_+$. By Points 3 and 4 of Assumption 3, $\beta = 0$ and so, $\alpha = -\beta = 0$. ■

Proof of Lemma 24. We first prove that the sets $\left\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{N\setminus S}) \succsim_i x, i \in S\right\}$ and $\Gamma_S(\xi)$ are closed, and then we show that their asymptotic cones are positively semi-independent. An appeal to Proposition 33 in the Appendix, will conlcude the proof.

Claim 1: The set $\left\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{N\setminus S}) \succsim_i x, i \in S\right\}$ is closed in $\mathbb{R}^l_+$. Indeed, take z in its closure. So, there exists a sequence $(z^\nu(S))_{\nu\in\mathbb{N}} \subseteq \left\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{N\setminus S}) \succsim_i x, i \in S\right\}$ such that $z^\nu(S)$ converges to z and $(z^\nu_S, x_{N\setminus S}) \succsim x_i$ for any $i \in S$ and for any $\nu \in \mathbb{N}$. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be such that for any $\nu \geq n$ and for any $c = 1,\ldots,l$, $z^{c\nu}(S) \leq z^c(S) + \varepsilon$. So, we can take a vector $b \in \mathbb{R}^l$ with $b^c := \max\{z^\nu(S) \colon \nu = 1,\ldots,n-1\} \cup \{z^c(S)+\varepsilon\}$. Since $0 \leq z^\nu_i \leq z^\nu(S)$ for any $\nu \in \mathbb{N}$, then $\{z^\nu_i \colon \nu \in \mathbb{N}\} \subseteq [0,b]$, which is a compact set. Thus, up to subsequence, z^ν_i converges to some $\overline{z}_i$, and by the continuity of preferences, $(\overline{z}_S, x_{N\setminus S}) \succsim_i x$. Therefore, $z = \lim\limits_{\nu\to\infty} z^\nu(S) = \lim\limits_{\nu\to\infty} \sum_{i\in S} z^\nu_i = \sum_{i\in S} \lim\limits_{\nu\to\infty} z^\nu_i = \sum_{i\in S} \overline{z}_i = \overline{z}(S)$, which concludes the proof of the claim.

Claim 2: $\Gamma_S(\xi)$ is closed in $\mathbb{R}^l$. See Claim 2 in the proof of Lemma 13.

Claim 3: $\mathcal{A}\left(\left\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{N\setminus S}) \succsim_i x, i \in S\right\}\right)$ is a subset of $\mathbb{R}^l_+$. Indeed, since $\left\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{N\setminus S}) \succsim_i x, i \in S\right\}$ belongs to $\mathbb{R}^l_+$, we must have $\mathcal{A}\left(\left\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{N\setminus S}) \succsim_i x, i \in S\right\}\right) \subseteq \mathcal{A}(\mathbb{R}^l_+)$. This conclude the proof of the claim, since $\mathcal{A}(\mathbb{R}^l_+) = \mathbb{R}^l_+$.

Claim 4: $\mathcal{A}(-\Gamma(S,\xi))$ is a subset of $-Y(S,\xi)$, and $-Y(S,\xi)$ is closed in $\mathbb{R}^l$. See Claim 2 in the proof of Lemma 13.

Claim 5: $\mathcal{A}\left(\left\{x'(S) \in \mathbb{R}^l_+ : (x'_S, x_{N\setminus S}) \succsim_i x, i \in S\right\}\right)$ and $\mathcal{A}(-\Gamma(S,\xi))$ are positively semi-independent. By Claims 1 to 4, take $\alpha \in \mathbb{R}^l_+$ and $-\beta \in -Y(S,\xi)$ such that $\alpha + (-\beta) = 0$. So, $\alpha = \beta$ and consequently, $\beta \in Y(S,\xi) \cap \mathbb{R}^l_+$. By Points 3 and 4 of Assumption 3, $\beta = 0$ and so, $\alpha = -\beta = 0$. ■

Proof of Proposition 16 Suppose that $\mathcal{L}^g_S(\xi) > 0$ for some $g > 0$. So, there exists $0 < \lambda \leq \mathcal{L}^g_S(\xi)$ such that $0 < \lambda \cdot g \in \Psi_S(\xi)$. Therefore, $\lambda \cdot g \in (\{\omega(S)\} + Y(S,\xi)) \cap \mathbb{R}^l_+$ and $\omega(S) - \lambda \cdot g = x'(S) - y'$, for some $(x'_i)_{i\in S} \in \mathbb{R}^{l\cdot|S|}_+$ with $x'_i \succsim x_i$ for any $i \in S$, and $y' \in \Gamma(S,\xi)$. Since $\omega(S) \gg 0$, by monotonicity and transitivity, there exists $(x''_i)_{i\in S}$ such that $\omega(S) = x''(S) - y'$, with $x''_i \gg 0$ and $x''_i \succ x_i$ for any $i \in S$[17]. Take any arbitrary reference bundle $g' > 0$ with $g' \neq g$. By the continuity of $\succsim_i$, we can choose a sufficiently small scalar $\lambda' > 0$ such that $\omega(S) - \lambda' \cdot g' = \eta(S) - y'$ and $\eta_i := x''_i - \lambda' \cdot \frac{g'}{|S|} \succ_i x_i$ for each $i \in S$[18], i.e., $\omega(S) - \lambda' \cdot g' \in \mathcal{R}_S(\xi)$. Notice that $\lambda' \cdot g'$ belongs to $(\omega(S) + Y(S,\xi)) \cap \mathbb{R}^l_+$. Indeed, (1) $\lambda' \cdot g' > 0$ by $\lambda' > 0$ and $g' \in \mathbb{R}^l_+$ with $g' \neq 0$; (2) $\lambda' \cdot g' \in (\{\omega(S)\} + Y(S,\xi))$ since $\lambda' \cdot g' = \omega(S) + (\lambda' \cdot g' - \omega(S))$ and $\lambda' \cdot g' - \omega(S) = y' - \eta(S) \in \{y'\} - \mathbb{R}^l_+ \subseteq Y(S,\xi)$ by $y' \in Y(S,\xi)$ and Point 5 of Assumption 3. Finally, since $0 < \lambda' \cdot g' \in \Psi_S(\xi)$, then $\mathcal{L}^{g'}_S(\xi) > 0$. ■

Proof of Proposition 27. Suppose that $\widetilde{\mathcal{L}}^g_S(\xi) > 0$ for some $g > 0$. So, there exists $0 < \lambda \leq \widetilde{\mathcal{L}}^g_S(\xi)$ such that $0 < \lambda \cdot g \in \widetilde{\Psi}_S(\xi)$. Therefore, $\lambda \cdot g \in (\omega(S) + Y(S,\xi)) \cap \mathbb{R}^l_+$ and $\omega(S) - \lambda \cdot g = x'(S) - y'$, for some $x' \in \mathbb{R}^{l\cdot n}_+$ with $(x'_S, x_{N\setminus S}) \succsim_i x$ for any $i \in S$, and $y' \in \Gamma_S(\xi)$. Since $\omega(S) \gg 0$, then $\omega(S) + y' > \omega(S) - \lambda' \cdot g + y' = x'(S)$. By (SGM) and transitivity, there exists x'' such that $\omega(S) = x''(S) - y'$, with $x''_i \geq 0$ and $(x''_S, x_{N\setminus S}) \succ_i x$ for any $i \in S$.

Take any arbitrary reference bundle $g' > 0$ with $g' \neq g$. If $x''_i \gg 0$ for each agent $i \in S$, then by the continuity of $\succsim_i$, we can choose a scalar $\lambda' > 0$ that is small enough for $\omega(S) - \lambda' \cdot g' = \eta(S) - y'$ and $\eta_i := x''_i - \lambda' \cdot \frac{g'}{|S|}$ with $(\eta_S, x_{N\setminus S}) \succ_i x_i$ for each $i \in S$[19], i.e., $\omega(S) - \lambda' \cdot g' \in \widetilde{\mathcal{R}}_S(\xi)$. Notice that $\lambda' \cdot g'$

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[17] It is enough to take $x''_i := x'_i + \lambda \cdot \frac{\sum_{c=1}^{l} g^c}{l\cdot|S|}\mathbf{e}$ for any $i \in S$, where $\mathbf{e} := (1,\ldots,1) \in \mathbb{R}^l$.
[18] Since the preference relations are continuous and $x''_i \gg 0$ for any $i \in S$, there exists $\varepsilon > 0$ such that for any z_i in the open ball $B_\varepsilon(x''_i) \subseteq \mathbb{R}^l_{++}$, one obtains $z_i \succ_i x_i$ for any $i \in S$. So, taking $0 < \lambda' < \frac{\varepsilon|S|}{2||g||}$ we get $\eta_i \in B_\varepsilon(x''_i)$ for any $i \in S$ and $\omega(S) - \lambda' \cdot g' = \eta(S) - y'$.
[19] Since the preference relations are continuous and $x''_i \gg 0$ for any $i \in S$, there exists $\varepsilon > 0$ such that for any z which belongs to the open ball $B_\varepsilon(x''_S, x_{N\setminus S}) \subseteq \mathbb{R}^{l\cdot n}_{++}$,

belongs to $(\{\omega(S)\}+Y(S,\xi))\cap\mathbb{R}^l_+$. Indeed, (1) $\lambda'\cdot g'>0$ by $\lambda'>0$ and $g'\in\mathbb{R}^l_+$ with $g'\neq 0$; (2) $\lambda'\cdot g'\in(\omega(S)+Y(S,\xi))$ since $\lambda'\cdot g'=\omega(S)+(\lambda'\cdot g'-\omega(S))$ and $\lambda'\cdot g'-\omega(S)=y'-\eta(S)\in\{y'\}-\mathbb{R}^l_+\subseteq Y(S,\xi)$ by $y'\in Y(S,\xi)$ and Point 5 of Assumption 3. Finally, since $\lambda'\cdot g'\in\widetilde{\Psi}_S(\xi)$, then $\widetilde{\mathcal{L}}^{g'}_S(\xi)>0$.

Suppose now that there exists a commodity c for which $x''^c_h=0$ for some agent h. Then, since $\omega(S)\gg 0$, by boundary aversion, we may consider vector $(\widetilde{x},\widetilde{y}):=(\omega_S,x_{S^c}),0)\in\mathbb{R}^{l\cdot|S|}_{++}\times\{x_{S^c}\}\times Y(S,\xi)$ such that $(\omega_S,x_{N\setminus S})\succ_i(x''_S,x_{N\setminus S})$ for any $i\in S$ and $\omega(S)+\widetilde{y}=\widetilde{x}(S)\gg 0$. So, again, there exists $\lambda'>0$ such that $\omega(S)-\lambda'\cdot g'=\eta(S)-\widetilde{y}\in\widetilde{\mathcal{R}}_S(\xi)$, with $\eta_i:=\widetilde{x}_i-\lambda'\cdot\frac{g'}{|S|}$, $i\in S$. Furthermore, $\lambda'\cdot g'\in(\{\omega(S)\}+Y(S,\xi))\cap\mathbb{R}^l_+$ since $\lambda'\cdot g'>0$, and $\lambda'\cdot g'=\omega(S)+(\lambda'\cdot g'-\omega(S))$ with $\lambda'\cdot g'-\omega(S)=\widetilde{y}-\eta(S)\in\{\widetilde{y}\}-\mathbb{R}^l_+\subseteq Y(S,\xi)$ by possibility of inaction and Point 5 of Assumption 3. So, $\lambda'\cdot g'\in\widetilde{\Psi}_S(\xi)$ and thus $\widetilde{\mathcal{L}}^{g'}_S(\xi)>0$. ■

We conclude the Appendix with a Table presenting the models covered in the paper.

then $z\succ_i x$ for any $i\in S$. In particular for any $(\eta_S,x_{N\setminus S})\in B_\varepsilon(x''_S,x_{N\setminus S})\subseteq\mathbb{R}^{l\cdot|S|}_{++}\times\{x_{N\setminus S}\}$ we obtain $(\eta_S,x_{N\setminus S})\succ_i x$, for any $i\in S$. So, taking $0<\lambda'<\frac{\varepsilon|S|}{2||g||}$ we get $(\eta_S,x_{N\setminus S})\in B_\varepsilon(x''_S,x_{N\setminus S})$ for any $i\in S$.

Table 1

$\succsim$	$Y(S,\xi)$	$\sigma_{S,\xi}(y')$	$\Lambda(S)$	Economic Model
Selfish	Y	$\{0\}$	$\mathbb{R}^l$	Production economy in Debreu and Scarf (1963)
Selfish	$\sum_{j\in J}\sum_{i\in S}\theta_{ij}Y_j$	$\left\{\sum_{j\in J}\sum_{i\notin S}\theta_{ij}y'_j \,\vert\, y'_j\in Y_j\ y'=\sum_{j\in J}\sum_{i\in S}\theta_{ij}y'_j\right\}$	$\mathbb{R}^l$	Private ownership economy
Selfish	$\sum_{j\in\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}Y_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y_j$	$\left\{\sum_{j\in\widetilde{J}(S)}\sum_{i\notin S}\theta_{ij}y'_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\notin S}\theta_{ij}y_j \,\vert\, y'_j\in Y_j,\ y'=\sum_{j\in\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y'_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y_j\right\}$	$\mathbb{R}^l$	Private ownership economy in Xiong and Zheng (2007)
Selfish	$\sum_{j\in\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}Y_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y_j$	$\left\{\sum_{j\in\widetilde{J}(S)}\sum_{i\notin S}\theta_{ij}y'_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\notin S}\theta_{ij}y_j \,\vert\, y'_j\in Y_j,\ y'=\sum_{j\in\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y'_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y_j\right\}$	$\mathbb{R}^l_+$	Corporate share in Xiong and Zheng (2007)
Cons. ext.	$-\mathbb{R}^l_+$	$\{0\}$	$\mathbb{R}^l$	Exchange economy in Di Pietro et al. (2022)
Cons. ext.	General	General	General	This paper

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