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WORKING PAPER NO. 649

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July 2022

This version January 2024

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ISSN: 2240-9696

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A Measure of Social Loss for Production Economies with Externalities

Maria Gabriella Graziano* **and Vincenzo Platino**†

Abstract
In this paper we consider a production economy and adopt a cooperative approach to equilibrium analysis which allows each individual to cooperate with others and to form a coalition whose members have access to the available technologies. We investigate the behavior of the core defined with respect to preferences (*preferences-core*) and with respect to resources (*resources-core*). We introduce a *measure of social loss* with respect to the core of the production economy which characterizes the corresponding core allocations. Our definition of the core requires that coalitions proposing a deviation take into account the consequences that changes in production plans may have for the counter-coalitions (*considerate dominance*). Our characterization holds in the presence of consumption externalities and an optimistic or a pessimistic attitude of coalition agents with respect to the behavior of outsiders.

JEL classification: C71, D11, D21, D62, D64.

Keywords: production economy, core, social loss, selfish preferences, other-regarding preferences.

Acknowledgments: This work was financial supported by the Project 000004 Star Linea 1 - 2020. The authors thank seminar participants at the SI&GE XI (Paris, January 2023) and 22nd SAET (Paris, July 2023) for helpful comments, questions and suggestions.

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1 Introduction

We consider a production economy with finite numbers of commodities and households. We adopt a cooperative approach to equilibrium analysis where each individual is allowed to cooperate with others and to form a coalition. Once a coalition is formed and regardless of how the remaining economy is organized, coalition members have access to the available technologies. In this framework we introduce a *measure of social loss* associated to the *resources-core* of the production economy which completely characterizes core allocations. The core of a pure exchange economy defined in terms of resources and its relationship with measures of social loss have been studied in Montesano (2002) and in Di Pietro et al. (2022). In this paper we consider the corresponding issues in the presence of production presenting a core concept under which coalitions take into account interdependency effects due to production as well as consumption externalities. The proposed formulation appears to be general enough for modeling various core notions in economies without externalities (models with selfish preferences) and new core notions in production economies with consumption externalities (models with other-regarding preferences) as well as corporate governance.

Measures of social loss and the core. The problem of measuring the "welfare loss" associated with an inefficient allocation dates back to Debreu (1951), Luenberger (1992, 1994) and Montesano (1997, 2002). It was motivated by limited availability of resources and the impact on the economic environment and focuses on measuring the amounts of resources that are wasted under a given allocation compared to a Pareto optimal allocation. Montesano (2002) and Di Pietro et al. (2022) extend the analysis to the core of an economy and propose a measure of social loss associated to core allocations. The core of the economy is the subset of Pareto optimal allocations formed by those feasible allocations which no coalition can improve upon in terms of preferences of its members (the *preferences-core*, based on maximization of preferences). Montesano (2002) introduces the idea of a *resources-core* based on minimization of resources, where for an allocation that is not in the resources-core there exists at least one coalition whose members can improve upon (or block) the given allocation by saving resources. A *measure of the social loss* associated to the given allocation is defined by the amount of the resources that can be saved so that the social loss vanishes if and only if the allocation belongs to the resources-core. Since the preferences-core and resources-core are equivalent notions, this measure also provides a characterization of the preferences-core. The characterization holds in the context of a standard pure exchange economy [2], and it is based on the key assumption that coalitions dislike resources

[2] A pure exchange economy with regular, continuous and monotonic (selfish) preferences.

waste. Therefore, the duality between the maximization of preferences and the minimization of resources which is used to show the equivalence between the preferences-core and the resources-core and to define the measure of social loss, depends critically on monotonicity arguments.

Production economies with interdependency problems. Recently, the core of a production economy has been studied assuming that technologies are controlled by individuals according to corporate shares. In such models, it is not clear whether a coalition deviating from a status quo allocation can change the production plans of firms not entirely owned by its members, and the blocking mechanism is defined addressing special forms of externalities due to production (see Xiong and Zheng (2007)). On the other hand, the literature on other-regarding preferences has widely documented that agents often fail to maximize their pure self interest (see Levine (1998), Fehr and Schmidt (1999), Sobel (2005)), leading to a growing interest in core notions defined in the presence of consumption externalities. In the present paper we measure the social losses caused by inefficiency with respect to core allocations for economic models that consider both these aspects, i.e. for models of economies with production and consumption externalities. In this context, the analysis of the core is complicated by the fact that a blocking coalition needs to take account of the presence of the outsiders. Precisely, the blocking coalition must take two aspects into account: 1. the coalition's resources might be affected by the firms owned jointly by the coalition with the counter-coalition by the assumption of **interdependency due to production** [3]; 2. the levels of its members' utility might change with the outsiders' allocation due to the assumption of **other-regarding preferences**. Moreover, since preferences depend on the total allocation of consumption bundles, the usual monotonicity arguments might not hold (Dufwenberg et al. (2011)). In order to take interdependency problems related to production into account, we describe production in very general terms. In line with Hildenbrand (1968, 1974) and Cornwall (1969), we suppose that the production capabilities of each coalition of agents that is formed to improve upon an allocation, are described by a production correspondence. This way of modeling production technologies accounts for cases where the technology is available to all agents that are described in Debreu and Scarf (1963), as well as the classical private ownership economy with a finite set of producers and firms owned by agents. In particular, it allows individuals to control technologies according to corporate shares (Xiong and Zheng (2007)). To deal with other-regarding preferences, we follow the approach of Di Pietro et al. (2022), where measures of social loss are studied assuming that individual preferences are affected by the consumption of all other agents in the economy. We adopt a special form of monotonicity of preference relations related to the redistribution of the sur-

[3] The *interdependency problem* due to production described by Xiong and Zheng (2007).

plus within a coalition (*Social Group Monotonicity*) in order to show that a measure of social loss can still be used to characterize the core in the presence of other-regarding preferences. The characterization of core allocations holds regardless of whether the notion of blocking is formulated under an optimistic or a pessimistic attitude of coalitions towards the possible reactions of outsiders. That is, in the case of the so-called γ-*core* (Dufwenberg et al. (2011)) and in the case of the α-*core* notion (Yannelis (1991)).

The result for production economies with selfish preferences. To simplify our analysis, we first study the core taking into account only the interdependency effects due to production. In a production economy whose agents have selfish preferences, we introduce a notion of resources-core, which emphasizes the optimal allocation of the resources. The blocking coalition produces according to its capabilities and the interdependency problem is captured by an outsiders' feasibility constraint. Under a classical monotonicity assumption comprising the *boundary aversion* of agents, we show that the preferences-core and the resources-core coincide. This equivalence allows us to introduce a suitable measure of social loss associated to the core and show that core allocations can be characterized as zero points of the social loss functions. Our notion of core is new and generalizes the *considerate* core defined by Xiong and Zheng (2007). In particular, we require that: i) each production plan chosen by coalition S to block a status quo allocation, affects the production possibilities of the counter-coalition S^c (the *outsiders*); the blocking coalition S must take into account the consequence of its blocking on the feasibility of outsiders' resources. These requirements are relevant if the firms are controlled by corporate share-holdings and a blocking coalition S can only modify the production plans of the firms under its control. For the expectation that the counter-coalition does not react after a change in the production plans of the other firms to be plausible, S should allow the outsiders to have feasible consumption plans. Our definition of the core (in preferences) is sufficiently general to include not only the classical definitions considered in the literature (see Debreu and Scarf (1963), Aliprantis et al. (1989)), but also the core notions which involve control rights introduced by Xiong and Zheng (2007) where a blocking coalition needs to respect additional conditions in relation to the shareholders outside the coalition.

The result for production economies with other-regarding preferences. In the second part of the paper, we jointly study the interdependency due to production and the extenal effects due to consumption and show that our results continue to hold in a production economy with consumption externalities. In a general equilibrium model with consumption externalities, the core can be defined in several different ways depending on the attitude of the blocking coalition S with respect to the reaction of the outsiders[4]. The

[4] For more details, see e.g. Yannelis (1991), Graziano et al. (2017), Hervés-Beloso

blocking procedures we adopt in the paper lead to the core notions which are described in the literature as the γ-core (Graziano et al. (2022), Dufwenberg et al. (2011)) and the α-core (Yannelis (1991)). As for the core with selfish agents, also in the case of γ-core and α-core, the blocking coalition S takes into account the consequences of its blocking on production and resources of the outsiders. Moreover, in the γ-blocking mechanism, coalitions of agents have an *optimistic* attitude with respect the behavior of outsiders. In this case, the deviating coalition S assumes that the counter-coalition passively accepts the deviation of S and that outsiders stick to their status quo allocation of consumption goods [5] . In the α-blocking mechanism, coalitions of agents have instead a *pessimistic* attitude towards the behavior of outsiders. Precisely, the deviating coalition S assumes that the counter-coalition may react to its deviation by redistributing its own resources. Moreover, S is willing to deviate only when all the redistributions ensure a better outcome to its members.
Again, our characterization is based on the idea of no waste of resources, and so our result requires an appropriate formulation of monotonicity assumptions. Under a suitable form of monotonicity referred to as *Social Group Monotonicity* and *Social Boundary Aversion*, we restore the equivalence between the preferences-core and the resources-core, and characterize core allocations as zero points of the measure of social loss in both dominance relations. However, to handle all the possible reactions of the outsiders under the α-dominance, we also assume a suitable form of separability of preferences referred to as *Social Group Separability.* The Social Group Monotonicity assumption ensures that at a given allocation, each coalition finds a way to distribute additional resources while making all of its members better off. Under Social Group Separability, a stronger form of classical separability of preferences, the preference of a trader for the consumption of a coalition S to which the trader belongs does not depend on the choice of traders outside S. Our characterization in the model with other-regarding preferences holds for core allocations that ensure a strictly positive consumption bundle to each agent. Under Social Boundary Aversion, allocations in the γ-core always satisfy this property. For the α-core allocations, strict positivity of consumption bundles must be imposed.

Outline of the paper. The paper is organized as follows. Section 2 is the section of the paper devoted to selfish models of production economies: Subsection 2.1 presents the model and the model assumptions; Subsection 2.2 introduces the notions of preferences-core and resources-core, and demonstrate their equivalence; Subsection 2.3 characterizes the core in terms of loss map-

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and Moreno-García (2021), Di Pietro et al. (2022) and Graziano et al. (2022). See also Hervés-Beloso and Martínez-Concha (2023) and Hervés-Beloso et al. (2023) for recent general equilibrium models with externalities.

[5] This behavior is in line with a model of production economies with corporate shares where the blocking coalition expects that the production plans of the firms controlled by the outsiders are fixed at the status quo (compare Example 35).

ping. Section 3 discusses the other-regarding preferences model: Subsection 3.1 extends the selfish model and presents the basic assumptions; Subsection 3.2 and Subsection 3.3 study the γ-core and its characterization in terms of social loss; Subsections 3.4 and 3.5 analyze the α-core. Section 4 and Section 5 propose applications, and possible extensions of our results. Section 6 presents some additional results, the technical proofs and a table of models with production covered by our paper.

2 A production economy with selfish preferences

Our model is of a production economy with finitely many consumers. The aim is to consider a framework that is sufficiently general to include both production economies with publicly accessible technologies and production economies where the technologies are controlled by individuals according to their corporate shares. In this latter case, we incorporate externalities due to the presence of outsiders, that is the members of a coalition that blocks by proposing alternative production plans, take into consideration the shareholders outside the coalition. This is the motivation for proposing a usual production set correspondences Y to describe the production possibilities of a coalition S, but also the set correspondences σ and Λ to describe production constraints and positive resource constraints on its outsiders when coalition S is formed. For simplicity, the model and the results will first be presented in Section 2 assuming that agents are selfish. They will be extended to also include consumption externalities in Section 3. This separation allows us to distinguish external effects due to production of the outsiders from external effects due to their consumption.

2.1 The model and the basic assumptions

There is a finite number l of commodities and $\mathbb{R}^l$ is the commodity space[6]. There is a finite number of individuals (agents or traders) denoted by the subscript $i \in N := \{1, \ldots, n\}$. The consumption set of agent i is $\mathbb{R}^l_+$ and the consumption bundle of individual i is $x_i := (x_i^1, \ldots, x_i^l)$. We denote by $x := (x_i)_{i \in N}$ a vector of consumption bundles. If the individual preferences of each agent depend only on his own consumption, we describe the agents in the economy as *selfish*. If this is the case, the preferences of individual i are

[6] With standard notations, the positive cone of $\mathbb{R}^l$ is $\mathbb{R}^l_+$, the interior and the boundary of $\mathbb{R}^l_+$ are denoted by $\operatorname{Int}\mathbb{R}^l_+$ and $\partial\mathbb{R}^l_+$, respectively.

represented formally by a binary relation $\succsim_i$ over $\mathbb{R}^l_+$ [7].

The initial endowment of individual i is $\omega_i := (\omega_i^1, \ldots, \omega_i^l)$, and let $\omega := (\omega_i)_{i \in N} \in \mathbb{R}^{l \cdot n}_+$ be the vector of all initial endowments.

A production plan for the economy is a point $y \in \mathbb{R}^l$, with the convention that the outputs of production are represented by the positive components of y and the inputs of production are represented by its negative components. There is a finite number of firms denoted by the subscript $j \in J := \{1, \ldots, f\}$ and the production possibilities of a firm $j \in J$ are represented by the production set $Y_j \subseteq \mathbb{R}^l$.

A state of the economy $\xi \in \Xi \subseteq \mathbb{R}^{l \cdot n}_+ \times \mathbb{R}^{l \cdot f}$ is a specification of the consumption bundle $x_i \in \mathbb{R}^l_+$ for each consumer and of the production plan $y_j \in Y_j$ for each producer, i.e. $\xi := (x_1, \ldots, x_n, y_1, \ldots, y_f)$. A state of the economy $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ is said to be an *allocation* if it satisfies the physical feasibility condition

$$\sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i + \sum_{j \in J} y_j.$$

$\mathcal{F}$ denotes the set of all allocations.

A coalition is any nonempty subset S of the set of agents N. We use $\mathcal{P}(N)$ to denote the set of all coalitions and for each coalition S strictly contained in N, S^c denotes the complementary coalition (the members of S^c are also called *outsiders*).

A coalition S may form to improve (or block) a status quo state $\xi \in \Xi$. In this case, the production possibilities of S will depend on the coalition, and on the status quo ξ. Formally, a correspondence $Y \colon \mathcal{P}(N) \times \Xi \rightrightarrows \mathbb{R}^l$ is defined which associates to each coalition S and status quo ξ, the set $Y(S, \xi) \subseteq \mathbb{R}^l$ of production plans available for the coalition S. Given a coalition S and a status quo ξ, the correspondence $\sigma_{S,\xi} : Y(S, \xi) \rightrightarrows Y(S^c, \xi)$ defines the production plans available to the outsiders, for any production plan y' chosen by S in its production set $Y(S, \xi)$. Finally, the correspondence $\Lambda : \mathcal{P}(N) \times \Xi \rightrightarrows \mathbb{R}^l_+$ describes the possible resources constraints for the outsiders if coalition S is formed to improve the status quo state ξ [8].

We make the following survival assumption in relation to the aggregate en-

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[7] In the more general situation analyzed in Section 3, the individual preferences of each agent may depend on the consumption of all the agents i.e. consumption externalities are present.

[8] We remark that the dependence of the correspondences Y, σ and Λ on the status quo state ξ, allow us to take into account not only the classical production models considered in the literature, but also notions involving control rights (see Examples of Section 4.

dowments of each coalition:

Assumption 1 *For any coalition $S \subseteq N$, the aggregate endowment $\omega(S) = \sum_{i \in S} \omega_i$ belongs to* $\operatorname{Int} \mathbb{R}^l_+$.

The basic assumptions on preference relations are listed below.

Assumption 2 *For every individual $i \in N$,*

1. $\succsim_i$ *is complete, reflexive, transitive and continuous;*
2. Strict Monotonicity on the interior. $\succsim_i$ *is strictly monotone over* $\operatorname{Int} \mathbb{R}^l_+$*;*
3. Boundary Aversion. $x \succ_i z$*, for each* $x \in \operatorname{Int} \mathbb{R}^l_+$ *and* $z \in \partial \mathbb{R}^l_+$.

The condition 3. that everything in the interior of $\mathbb{R}^l_+$ is preferred to anything on the boundary of $\mathbb{R}^l_+$ is called *Boundary Aversion* in Xiong and Zheng (2007). Notice that no form of convexity is required on preferences [9] . Moreover, although in this paper we do not make use of utilities, the assumptions stated for preferences ensure that each agents' preference relation $\succsim_i$ can be represented by a continuous utility function u_i defined over the commodity space.

In the rest of the paper, the production set correspondence satisfies the following set of assumptions.

Assumption 3 *For any status quo $\xi \in \Xi$, the correspondence $Y(\cdot, \xi)$ is superadditive in the sense that $Y(S, \xi) + Y(T, \xi) \subseteq Y(S \cup T, \xi)$ for each pair of disjoint coalitions $S, T \subseteq N$. Moreover, for any coalition $S \subseteq N$ and any status quo $\xi \in \Xi$,*

1. $Y(S, \xi)$ *is closed;*
2. $Y(S, \xi)$ *is convex;*
3. $0 \in Y(S, \xi)$ *(possibility of inaction);*
4. $Y(S, \xi) \cap \mathbb{R}^l_+ \subseteq \{0\}$ *(no free lunch);*
5. $Y(S, \xi) - \mathbb{R}^l_+ \subseteq Y(S, \xi)$ *(free disposal);*
6. $Y(N, \xi) = Y(N) = \sum_{j \in J} Y_j$*, for any* $\xi \in \Xi$.

Assumptions 3.1 − 3.3 are standard, and the no free lunch assumption 3.4 means that production of outputs requires inputs. Assumption 3.6 requires

[9] Preference relations introduced with Assumption 2 which also satisfy strict convexity on the interior of $\mathbb{R}^l_+$, are *neoclassical* according to Aliprantis et al. (1989).

that the production possibilities of the grand coalition do not depend on the status quo allocation. It implies, in particular, that assignments for the grand coalition do not depend on a particular status quo state.

We make the following assumptions about the correspondence the correspondence $\sigma_{S,\xi} : Y(S,\xi) \rightrightarrows Y(S^c,\xi)$ describing the production possibilities of the outsiders given a production plan chosen by coalition S.

Assumption 4 *For any coalition $S \subseteq N$ and any status quo $\xi \in \Xi$,*

1. *$\sigma_{S,\xi}$ is a nonempty and compact valued correspondence;*

2. *$\sigma_{S,\xi}$ is upper hemicontinuous;*

3. *$0 \in \sigma_{S,\xi}(0)$.*

The first two requirements in Assumption 4 are technical and allow us to look at the limit behavior of the producers controlled by the outsiders (see the proof of point 3. of Lemma 9 in the Appendix). The last condition is related to the possibility of inaction, i.e., Point 3 of Assumption 3.

Finally, we make the following assumption on the correspondence $\Lambda : \mathcal{P}(N) \times \Xi \rightrightarrows \mathbb{R}^l_+$ describing resources constraints of the outsiders.

Assumption 5 *For any coalition $S \subseteq N$ and any status quo $\xi \in \Xi$,*

1. *$\Lambda(S,\xi)$ is a closed subset of $\mathbb{R}^l_+$;*

2. *$\omega(S^c) \in \Lambda(S,\xi)$.*

In particular, condition 2. in Assumption 5 means simply that for any coalition S, the complementary coalition satisfies the resources constraints at least from its initial resources.

The production economy considered under Assumptions 1, 2, 3, 4 and 5 is thus formalized in the following list of elements:

$$E := \left\langle N, \left(\mathbb{R}^l_+ \succsim_i, \omega_i\right)_{i\in N}, J, \left(Y_j\right)_{j\in J}, \left(\sigma_{S,\xi}, Y(S,\xi), \Lambda(S,\xi)\right)_{\substack{S\in\mathcal{P}(N)\\ \xi\in\Xi}} \right\rangle.$$

Section 4 shows that this way of modeling the economy is sufficiently general for the treatment of the core of several production economies studied in the literature. Furthermore, Assumptions 4 and 5 are implicitly satisfied by many standard production models (see Examples in in Section 4).

Given the economy E defined above, for every coalition $S \in \mathcal{P}(N)$, and for any vector $x \in \mathbb{R}^{l\cdot n}_+$, we use $x_S := (x_i)_{i\in S}$ to denote the commodity bundles of the members of S and $x_{S^c} := (x_i)_{i \in S^c}$ to denote the commodity bundles of the members of the complementary coalition S^c. Given x_S and x_{S^c}, without loss of generality, we denote x also by (x_S, x_{S^c}), and let $x(S) := \sum_{i \in S} x_i$ be the aggregate resources of S.

Given an allocation $\xi \in \mathcal{F}$ and a coalition $S \subseteq N$, we say that $x_S = (x_i)_{i \in S}$ is an *assignment for* S (given ξ) if there exists a production plan $y \in Y(S, \xi)$ such that $x(S) \leq \omega(S) + y$. Clearly, point 6. in Assumption 3 ensures that an assignment for the grand coalition N is an allocation.

2.2 Preferences-core and resources-core

Below we introduce the notion of core with respect to preferences and the notion of core with respect to resources for the production economy, and prove their equivalence.

Definition 6 (Core) *Given an allocation* $\xi = (x_1, \dots, x_n, y_1, \dots, y_f) \in \mathcal{F}$ *and a coalition* S*, we say that* S *improves upon* ξ *whenever there exists* $x'_S = (x'_i)_{i \in S}$ *and* $y' \in Y(S, \xi)$ *such that*

i) $x'(S) \leq \omega(S) + y'$ *(*x'_S *is an assignment for* S *given* ξ*);*

ii) $\omega(S^c) + y'' \in \Lambda(S, \xi)$*, for some* $y'' \in \sigma_{S,\xi}(y')$*;*

iii) $x'_i \succ_i x_i$*, for every* $i \in S$*.*

The set of allocations which cannot be improved upon by any coalition is called the preferences-core and is denoted by $\mathcal{C}_p(E)$*. If we replace condition i) with* $x'(S) < \omega(S) + y'$ *and in condition iii)* $\succ_i$ *is replaced by* $\succsim_i$*, then we say that* S *improves upon* ξ *in resources and the corresponding core, denoted* $\mathcal{C}_r(E)$*, is the resources-core.*

Conditions $i) - iii)$ of Definition 6 define the blocking mechanism in our economy. A coalition S improves upon an allocation ξ if using its own resources and a feasible production plan is able to ensure a better outcome to each of its members (conditions i) and iii)) making sure that for at least one production plan available for the outsiders, positive resources constraints are satisfied (condition ii)). If coalition S improves upon ξ, we say also that S is a *blocking coalition* or that S *blocks* ξ. The blocking mechanism introduced with Definition 6 jointly considers some relevant issues. The production plan chosen by a blocking coalition to improve upon a status quo allocation ξ may depend on the status quo allocation itself[10]. Moreover, a feasibility requirement

[10] This case for example occurs in models with corporate governance analyzed in Section 4 for which the blocking procedure allows only firms under the control of

on outsiders' resources is taken into account through condition ii). This requirement makes the blocking mechanism *considerate* in the sense that the notion of blocking allows the coalition to consider whether the consequence of its blocking is feasible for the outsiders. Clearly, the smaller the set $\Lambda(S,\xi)$, the larger the corresponding core since for a coalition it becomes harder to improve upon a feasible allocation.

Our notion of core considers preferences and resources. The notion of resources-core directly emphasizes the optimal use of resources in the treatment of efficiency, in the sense of no waste of resources. It requires that the utility levels achieved by the members of each coalition under the allocation cannot be achieved through an alternative allocation which also allows resources saving (compare Allais (1943))[11]. These two definitions, in terms of preferences and in terms of resources, are equivalent in (selfish) pure exchange economies under standard regularity conditions on preferences (see Montesano (2002)). Theorem 7 below extends the equivalence between the preferences-core and the resources-core to production economies. Notice that Theorem 7 does not require any other assumption on the production sets than the assumption of nonemptiness, which in its turn, is implied by the possibility of inaction. On the other hand, the idea of resources-core is based on the assumption that coalitions dislike resources waste and therefore the proof builds on the monotonicity requirement on preferences.

Theorem 7 *Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ be an allocation of the production economy E. Then $\xi \in \mathcal{C}_r(E)$ if and only if $\xi \in \mathcal{C}_p(E)$.*

Proof. Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f) \in \mathcal{C}_r(E)$ and suppose by contradiction that $\xi \notin \mathcal{C}_p(E)$. Then, there exist a coalition $S \subseteq N$ and an assignment for S given ξ, namely $((x'_i)_{i\in S}, y')$ such that $x'_i \succ_i x_i$ for all $i \in S$ and $\omega(S^c) + y'' \in \Lambda(S,\xi)$ for some $y'' \in \sigma_{S,\xi}(y')$. If $x'(S) < \omega(S) + y'$, a contradiction follows, so, we can assume that $x'(S) = \omega(S) + y'$. By continuity of preferences, there exists a positive δ such that, for all $i \in S$, if $z_i \in \mathbb{R}^l_+$ and $\|z_i - x'_i\| < \delta$ then $z_i \succ_i x_i$. By monotonicity and continuity assumptions, for each agent $i \in S$ it is true that $x'_i \gg 0$ [12] and consequently $x'(S) \gg 0$. Choose $\varepsilon > 0$ such that $0 < (1-\varepsilon)\|x'_h\| < \delta$. Define x'' by choosing $x''_i = x'_i$, for $i \in S \setminus \{h\}$ and

S to change their production plans while the other firms maintain their production activity at the status quo allocation ξ. As consequence of this assumption, in models with corporate governance also the total production possibilities of S, which depends on the joint activities of all firms, depends on the production at the status quo state.

[11] Resources saved by coalition S after exchange and production are represented by the quantity $\omega(S) + y' - x'(S) > 0$.

[12] The monotonicity and continuity assumptions ensure that each agent is indifferent between zero and a vector $x \in \partial R^l_+$ and then boundary consumption bundles are equivalent with respect to $\succ_i$. Hence $x'_i \succ_i x_i$ implies that x'_i is an interior consumption bundle.

$x''_h = \varepsilon x'_h$. For every agent $i \in S$, $\|x''_i - x'_i\| \le (1-\varepsilon)\|x'_h\| < \delta$ and consequently $x''_i \succ_i x_i$. By construction, $x''(S) < x'(S) = \omega(S) + y'$, which contradicts the fact that $\xi \in \mathcal{C}_r(E)$.

Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f) \in \mathcal{C}_p(E)$ and suppose by contradiction that $\xi \notin \mathcal{C}_r(E)$. There exist a coalition $S \subseteq N$ and $((x'_i)_{i\in S}, y')$ such that $y' \in Y(S,\xi)$, $\upsilon := -x'(S) + \omega(S) + y' > 0$, $x'_i \succsim_i x_i$, for all $i \in S$, and $\omega(S^c) + y'' \in \Lambda(S,\xi)$ for some $y'' \in \sigma_{S,\xi}(y')$. From $\xi \in \mathcal{C}_p(E)$ it follows that $x_i \gg 0$ for each agent $i \in N$. If not, an agent $i \in N$ with a boundary consumption bundle x_i would be able to improve upon ξ in preferences using the strictly positive initial endowment and with no production, by monotonicity assumption, point 3. of Assumption 3, point 3. of Assumption 4 and point 2. of Assumption 5. Consequently, $x'_i \succsim_i x_i$ implies that $x'_i \gg 0$, for each $i \in N$. Consider the vector x''_S defined by $x''_i := x'_i + \frac{\upsilon}{|S|}$, for each $i \in S$. Notice that, (x''_S, y') is an assignment for S given ξ, and $x''_i > x'_i$ for each $i \in S$. Then, by strict monotonicity over $\operatorname{Int}\mathbb{R}^l_+$, we have that $x''_i \succ_i x'_i$ for any $i \in S$ and, by transitivity, we obtain that $x''_i \succ_i x_i$, for all $i \in S$, which is a contradiction. ∎

As consequence of Theorem 7, we can denote the core of the economy E simply by $\mathcal{C}(E)$ making no distinction between preferences and resources. Moreover, Theorem 7 shows that, under the assumptions of the model, when the allocation $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ belongs to $\mathcal{C}(E)$, then ξ ensures a strictly positive consumption bundle to each consumer, i.e. $x = (x_1, \ldots, x_n) \gg 0$. The notion of core given in terms of resources is central to obtain in the next section the characterization of core allocations as zero points of social loss mappings.

Remark 8 Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f) \in \mathcal{F}$ be an allocation with $x = (x_1, \ldots, x_n) \gg 0$. The proof of Theorem 7 shows that when a coalition S is able to improve upon ξ in preferences, then the same coalition is able to improve upon ξ by saving resources, and vice-versa. Under the same assumptions and with similar arguments, we can also show that whenever the coalition S is able to improve upon ξ by saving resources, then S improves upon ξ by saving a strictly positive amount of each commodity. Precisely, the preferences-core and the resources-core coincide with the core defined by the following dominance:

i) $x'(S) \ll \omega(S) + y'$;

ii) $\omega(S^c) + y'' \in \Lambda(S,\xi)$, for some $y'' \in \sigma_{S,\xi}(y')$;

iii) $x'_i \succsim_i x_i$, for every $i \in S$.

Indeed, if a coalition $S \subseteq N$ is able to improve upon ξ in resources, there exists $((x'_i)_{i\in S}, y')$ such that $y' \in Y(S,\xi)$, and $x'(S) < \omega(S) + y'$, $x'_i \succsim_i x_i$, for all $i \in S$, and $\omega(S^c) + y'' \in \Lambda(S,\xi)$ for some $y'' \in \sigma_{S,\xi}(y')$. From $x'_i \succsim_i x_i$ and $x_i \gg 0$, it follows that $x'_i \gg 0$, for every $i \in S$. Consequently, as in the second part of the proof of Theorem 7, we can find the strictly positive consumption bundles

x''_i such that $x''_i \succ_i x_i$ and $x''(S) = \omega(S) + y'$. By continuity assumption, $\varepsilon x''_i \succ_i x_i$ for a positive ε and for each $i \in S$. Then the conclusion follows from $\varepsilon x''(S) \ll x''(S) = \omega(S) + y'$.

2.3 Core allocations and zero points of social loss mappings

The aim of this section is to prove that core allocations are zero points of suitable social loss mappings. Since allocations in the core $\mathcal{C}(E)$ ensure a strictly positive consumption bundle to each consumer, in the rest of the section we shall focus our attention on allocations $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ such that $x = (x_1, \ldots, x_n) \gg 0$. Following Montesano (2002) and Di Pietro et al. (2022), we define a measure of social loss for every coalition S. Given an allocation $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f) \in \mathcal{F}$ with $x = (x_1, \ldots, x_n) \gg 0$, we start by considering the set of resources which give to coalition S the possibility to reach a redistribution that is weakly preferred to x by all the members of S and allows available production. This set is denoted $\mathcal{R}_S(\xi)$. Formally,

$$\mathcal{R}_S(\xi) \coloneqq \Big\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\Big\} - \Gamma_S(\xi),$$

where $\Gamma_S(\xi) \coloneqq \{y' \in Y(S,\xi) |\ \exists\ y'' \in \sigma_{S,\xi}(y') \colon \omega(S^c) + y'' \in \Lambda(S,\xi)\}$. Notice that $\mathcal{R}_S(\xi)$ is nonempty since $x(S)$ belongs to $\mathcal{R}_S(\xi)$, by the reflexivity property of the preference relation, possibility of inaction, Point 3 of Assumption 4, and Point 2 of Assumption 5. The next lemmas show important properties of the set $\mathcal{R}_S(\xi)$ when $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f) \in \mathcal{F}$ is an allocation ensuring a strictly positive consumption bundle to each consumer.

Lemma 9 *Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ be an allocation such that $x = (x_1, \ldots, x_n) \gg 0$. Then the set $\mathcal{R}_S(\xi)$ satisfies the following properties:*

1. *If $\omega(S) \notin \mathcal{R}_S(\xi)$, then S is not able to improve upon ξ;*
2. *if $\gamma' > \gamma$ and $\gamma \in \mathcal{R}_S(\xi)$, then $\gamma' \in \mathcal{R}_S(\xi)$;*
3. *the set $\mathcal{R}_S(\xi)$ is closed in $\mathbb{R}^l$.*

Proof. See the Appendix. ■

If we now consider the differences between endowments and elements in the set $\mathcal{R}_S(\xi)$, we can define the set $\Psi_S(\xi)$ of resources that can be saved by coalition S while still allowing S to achieve for its members a resources allocation that is at least as good as x and to produce. Formally, $\Psi_S \colon \mathcal{F} \rightrightarrows \mathbb{R}^l$,

$$\Psi_S(\xi) \coloneqq \Big\{z \in \Big(\omega(S) + Y(S,\xi)\Big) \cap \mathbb{R}^l_+ \colon \omega(S) - z \in \mathcal{R}_S(\xi)\Big\}.$$

The next result gives the necessary and sufficient condition for the nonemptiness of $\Psi_S(\xi)$.

Lemma 10 *$\omega(S) \in \mathcal{R}_S(\xi)$ if and only if $\Psi_S(\xi) \neq \emptyset$.*

Proof. Let $\omega(S) \in \mathcal{R}_S(\xi)$. Then, there exists $((x'_i)_{i\in S}, y')$ such that $\omega(S) = x'(S) - y'$, $x'_i \succsim_i x_i$ for any $i \in S$, and $y' \in \Gamma_\xi(S)$. Therefore, $\omega(S) - 0 = x'(S) - y' \in \mathcal{R}_S(\xi)$ and $0 \in (\omega(S) + Y(S,\xi)) \cap \mathbb{R}^l_+$ since $0 = \omega(S) - \omega(S)$ and $-\omega(S) \in Y(S,\xi)$ by Points 3 and 5 of Assumption 3. Thus, $0 \in \Psi_S(\xi)$ and consequently $\Psi_S(\xi) \neq \emptyset$. Coversely, suppose that $\Psi_S(\xi) \neq \emptyset$. Then, there exists z such that $z \geq 0$ and $\omega(S) - z = x'(S) - y' \in \mathcal{R}_S(\xi)$. If $z = 0$, then $\omega(S) \in \mathcal{R}_S(\xi)$. If $z > 0$, by point 2. of Lemma 9, $\omega(S) \geq \omega(S) - z$ and $\omega(S) - z \in \mathcal{R}_S(\xi)$ imply $\omega(S) \in \mathcal{R}_S(\xi)$. ∎

Lemma 11 *The set $\Psi_S(\xi)$ is compact.*

Proof. Claim 1: $\Psi_S(\xi)$ is closed in $\mathbb{R}^l_+$. Indeed, take z in its closure and a sequence $(z^\nu)_{\nu\in\mathbb{N}} \subseteq \Psi_S(\xi)$ such that z^ν converges to z. Notice that, $(z^\nu)_{\nu\in\mathbb{N}}$ is contained in the set $\omega(S) + Y(S,\xi)$ and $(z^\nu)_{\nu\in\mathbb{N}} \subseteq \mathbb{R}^l_+$. So, $z \in \omega(S) + Y(S,\xi)$ since by Point 1 of Assumption 3, the set $Y(S,\xi)$ is closed and $z \in \mathbb{R}^l_+$. Therefore, $z \in (\omega(S) + Y(S,\xi)) \cap \mathbb{R}^l_+$. Furthermore, $\{\omega(S) - z^\nu : \nu \in \mathbb{N}\} \subseteq \mathcal{R}_S(\xi)$ implies that $\omega(S) - z \in \mathcal{R}_S(\xi)$ since by point 3. of Lemma 9 the set $\mathcal{R}_S(\xi)$ is closed. Thus we conclude that $z \in \Psi_S(\xi)$ and the claim is proved.

Claim 2: $\Psi_S(\xi)$ is bounded. In order prove the claim, it is enough to show the boundedness of $(\omega(S) + Y(S,\xi)) \cap \mathbb{R}^l_+$. Since a translation of a set does not affect its asymptotic cone, then $\mathcal{A}(\omega(S)+Y(S,\xi)) = \mathcal{A}(Y(S,\xi))$. Furthermore, by Points 1, 2 and 3 of Assumption 3, $\mathcal{A}(Y(S,\xi)) \subseteq Y(S,\xi)$, and in particular, by definition of the asymptotic cone, $0 \in \mathcal{A}(Y(S,\xi))$. Since $\mathcal{A}(\mathbb{R}^l_+) = \mathbb{R}^l_+$, then by Points 3 and 4 of Assumption 3 we get $\mathcal{A}(\omega(S) + Y(S,\xi)) \cap \mathcal{A}(\mathbb{R}^l_+) = \{0\}$, which concludes the proof by Point 5. of Proposition 40. ∎

Let us fix a vector $g \in \mathbb{R}^l_+$ with $g \neq 0$. We will call g the *reference bundle*. Below, we introduce the *loss mapping* as a function measuring the maximum amount of resources that can be saved by a coalition S with respect to an allocation x in the direction of the reference bundle g. Equivalently, the loss mapping measures the loss, in terms of g, procured to coalition S by an allocation ξ.

Formally, the loss mapping $\mathcal{L}_{g,S} : \mathcal{F}(\omega) \to \mathbb{R}$ is defined as follows

$$\mathcal{L}_{g,S}(\xi) := \begin{cases} \max\{\lambda \in \mathbb{R} : \lambda \cdot g \in \Psi_S(\xi)\} & \text{if } \Psi_S(\xi) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, if $\Psi_S(\xi) \neq \emptyset$ then the maximum is well-defined, since according to Lemma 11, $\Psi_S(\xi)$ is compact. Furthermore, $\mathcal{L}_{g,S}(\xi) \geq 0$ since $g \in \mathbb{R}^l_+$ with

$g \neq 0$[13]. Note that the loss mapping may vary according to the reference bundle g. However, if there exists g such that $\mathcal{L}_{g,S}(x)$ is strictly positive, then for all reference bundles, the corresponding loss mappings are strictly positive.

Proposition 12 *For a given allocation ξ, if $\mathcal{L}_{g,S}(\xi) > 0$ for a vector $g > 0$, then $\mathcal{L}_{g',S}(\xi) > 0$ for every $g' > 0$.*

Proof. See the Appendix. ■

Next moving from the loss (in terms of g) procured to each coalition S by an allocation ξ, we introduce a measure of social loss with respect to ξ defined as the *social loss mapping* $\mathcal{L}_g \colon \mathcal{F}(\omega) \to \mathbb{R}$ given by

$$\mathcal{L}_g(\xi) \coloneqq \max_{S \subseteq N} \mathcal{L}_{g,S}(\xi).$$

The social loss mapping $\mathcal{L}_g(\xi)$ is well-defined because for every coalition S, the loss mapping $\mathcal{L}_{g,S}$ is well-defined. $\mathcal{L}_g(\xi)$ is the maximal loss procured to a coalition by the allocation ξ. Theorem 13 shows that the maximal loss vanishes if and only if the allocation belongs to the core. Consequently, we obtain a full characterization of the core in terms of loss mappings.

Theorem 13 *Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ be an allocation such that $x = (x_1, \ldots, x_n) \gg 0$. For any non null reference bundle $g \in \mathbb{R}^l_+$, $\mathcal{L}_g(\xi) = 0$ if and only if $\xi \in \mathcal{C}(E)$.*

Proof. We start by proving that if the allocation ξ belongs to the core, then $\mathcal{L}_g(\xi) = 0$. Suppose by contradiction that $\mathcal{L}_g(\xi) > 0$. Then there exists a coalition S such that $\mathcal{L}_{g,S}(\xi) > 0$ and $\Psi_S(\xi) \setminus \{0\} \neq \emptyset$. Consequently, there exists $z > 0$ such that $z \in \Psi_S(\xi)$. Therefore, $0 < z = \omega(S) + \widehat{y}$ with $\widehat{y} \in Y(S, \xi)$ and $\omega(S) - z = -\widehat{y} \in \mathcal{R}_S(\xi)$. Thus $-\widehat{y} = x'(S) - y'$, for some $((x'_i)_{i \in S}, y') \in \mathbb{R}^{l \cdot |S|}_+ \times \Gamma_S(\xi)$ with $x'_i \succsim_i x_i$ for any $i \in S$. Finally, notice that $0 < z = \omega(S) + \widehat{y} = \omega(S) - x'(S) + y'$ and consequently a contradiction is obtained since S improves upon ξ. Let us show now that $\mathcal{L}_g(\xi) = 0$ implies $\xi \in \mathcal{C}(E)$. By contradiction, suppose that $\xi \notin \mathcal{C}(E)$. So, there exists a coalition $S \subseteq N$, $(x'_i)_{i \in S}$ and y' such that $x'_i \succsim_i x_i$ for every $i \in S$, $y' \in Y(S, \xi)$, $x'(S) < \omega(S) + y'$ and $\omega(S^c) + y'' \in \Lambda(S, \xi)$ for some $y'' \in \sigma_{S,\xi}(y')$. So, $g' \coloneqq \omega(S) + y' - x'(S) > 0$ belongs to $(\omega(S) + Y(S, \xi)) \cap \mathbb{R}^l_+$ since $y' - x'(S) \in Y(S, \xi)$ by $y' \in Y(S, \xi)$ and Point 5 of Assumption 3. Furthermore, $\omega(S) - g' = x'(S) - y' \in \mathcal{R}_S(\xi)$ since $y' \in \Gamma_S(\xi)$, and consequently, $g' \in \Psi_S(\xi)$ and $g' > 0$. Thus $\mathcal{L}_{g',S}(\xi) > 0$ which implies $\mathcal{L}_{g,S}(\xi) > 0$ (by Proposition 12), contradicting the fact that $\mathcal{L}_g(\xi) = 0$. ■

[13] In the literature studying Pareto optimal allocations in terms of resources, the reference bundle g is chosen arbitrarily. In a classical setting, Debreu (1951) chose $g = \omega(N)$ and Allais (1943) and Groves (1979) use $g = (1, 0 \ldots, 0)$.

Remark 14 We have already observed that under the assumptions of the model, when the allocation $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ belongs to $\mathcal{C}(E)$, then ξ ensures a strictly positive consumption bundle to each consumer, i.e. $x = (x_1, \ldots, x_n) \gg 0$ (see the proof of Theorem 7). Therefore, Theorem 13 proves that the core $\mathcal{C}(E)$ of a production economy is formed by zero points of suitable loss mappings. The result holds for the case of considerate blocking and for models that take account of the interdependency effects due to production. A quick look at the proofs of the results of this Section, shows that point 1. of Assumption 5 could well be replaced by the weaker requirement that $\Lambda(S, \xi)$ is a closed subset of $\mathbb{R}^l$, i.e. without imposing positive constraints on the resources of the counter-coalition. This observation will be useful in Section 4 to formally include in our model also examples of inconsiderate dominance.

In the next Section, we consider a more general model of production economies and allow also for interdependency effects due to consumption.

3 A production economy with other-regarding preferences

In this Section, we study production economies in which individual preferences are affected by the consumption of all other agents in the economy (production economies with consumption externalities) and show that the characterization of the core allocations proved by Theorem 13 still holds. The notion of dominance for production economies with consumption externalities is not obvious and can be based on several elements. In what follows, yet a blocking coalition S is able to reallocate its resources and use feasible production plans to make its members better off (compare Definition 6, conditions i) and iii)). It also takes due account of any interdependence due to production (Definition 6, condition ii)). But in addition, since a whole distribution of resources influences the preferences of its members, the coalition S also explicitly evaluates the possible reactions of the complementary coalition to its deviation. In particular, we propose a classification of dominance based on coalition S's view of the reaction of the complementary coalition. This point of view can be *optimistic* or *pessimistic*. Under an optimistic attitude, the blocking coalition S believes that the members of the complementary coalition simply stick to their status quo allocation and do not react (γ-*dominance*). In a pessimistic view, the coalition S considers possible any feasible redistribution of initial resources among outsiders and is willing to deviate from the status quo only when all potential reactions ensure a better outcome for its members (α-*dominance*). The corresponding core notions in the two dominance relations are called γ-*core* and α-*core*. We introduce a measure of social loss with respect to the γ-core and the α-core of the economy which provides a characterization of the corresponding core allocations. The new core notions introduced in this Section will take into account both, consumption externalities and interdependency due to production, and will include and generalize classical core notions for production economies with selfish agents analyzed in Section 2 as well as the notions studied in Yannelis (1991), Dufwenberg et al. (2011) and Di Pietro et al. (2022), among others. Since the sets of assumptions needed in the two cases are rather different, we will divide the analysis of the γ-core and α-core in separated subsections.

3.1 The model and the basic assumptions

In this extension of the model presented in Section 2.1, individuals are assumed to be not selfish and their preferences may depend on the consumption of all the agents. Formally, the preferences of individual i are described by a binary relation $\succsim_i$ over $\mathbb{R}_+^{l\cdot n}$. With innocuous abuse of notation, we still denote by E the production economy being considered. For a given coalition $S \subseteq N$ and for a vector $z \in \mathbb{R}_+^{l\cdot n}$, we define $z_S \coloneqq (z_i)_{i\in S}$ and $z_{S^c} = (z_i)_{i\in N\setminus S}$. Given z_S and z_{S^c}, without loss of generality, we denote z by (z_S, z_{S^c}). For a given coalition $S \subseteq N$ and a vector $z \in \mathbb{R}_+^{l\cdot n}$, define the two sets S_z° and S_z^* as follows,

$$S_z^\circ \coloneqq \{i \in S \colon z_i \in \partial\mathbb{R}_+^l\} \text{ and } S_z^* \coloneqq \{i \in S \colon z_i \in \operatorname{Int}\mathbb{R}_+^l\}.$$

Then $S = S_z^\circ \cup S_z^*$ and the vector z_S can also be denoted by $(z_{S_z^\circ}, z_{S_z^*})$. In this Section, Assumptions 1, 3, 4 and 5 in Section 2.1 are retained unchanged but Assumption 2 is replaced by the following new assumptions.

Assumption 15 *For every individual $i \in N$,*

1. *$\succsim_i$ are complete, reflexive, transitive, and continuous over $\mathbb{R}_+^{l\cdot n}$;*
2. Social Group Monotonicity (SGM). *For any coalition $S \subseteq N$, any vector $x \in \mathbb{R}_+^{l\cdot n}$ with $x(S) \in \operatorname{Int}\mathbb{R}_+^l$ and $z > x(S)$, there exist vectors $x'_i \in \mathbb{R}_+^l$, $i \in S$, with $x'(S) = z$, and $(x'_S, x_{S^c}) \succ_i (x_S, x_{S^c})$, for all $i \in S$;*
3. Social Boundary Aversion (SBA). *For any vector $x \in \mathbb{R}_+^{l\cdot n}$, for any coalition $S \subseteq N$, $(z_{S_x^\circ}, x_{S_x^*}, x_{N\setminus S}) \succ_i (x_{S_x^\circ}, x_{S_x^*}, x_{N\setminus S})$ for any $z_{S_x^\circ} \in \operatorname{Int}\mathbb{R}_+^{l\cdot|S_x^\circ|}$ and for any $i \in S$.*

The Social Group Monotonicity and the Social Boundary Aversion extend to other-regarding preferences the assumptions of Strict Monotonicity on the interior and Boundary Aversion introduced in Section 2.1 for selfish preferences. The (SGM) condition states that any increase in the resources available to the coalition S can be redistributed to make every member of S better off. (SGM) may fail in the presence of hateful agents and generalizes the Social Monotonicity condition adopted by Dufwenberg et al. (2011) in order to prove the Second Welfare Theorem [14]. Condition (SBA) is standard in the study of cooperative solutions in selfish models. For a continuous and monotone selfish preference, it is equivalent to require that all commodity bundles on the boundary are equivalent in terms of preferences. In Condition 3. of Assumption 15, this requirement is adapted to preferences with consumption externalities. The condition states that each trader i in a coalition S is strictly better off if the boundary components of a given distribution of resources x for S are replaced by interior commodity bundles [15].

To construct an example of preference satisfying our assumptions, we refer to the so called *separable preference*, i.e. a preference relation $\succsim_i$ where $(x_i, x_{i^c}) \succsim_i (x'_i, x_{i^c})$ for some x_{i^c} implies that $(x_i, x'_{i^c}) \succsim_i (x'_i, x'_{i^c})$, for each $x'_{i^c} \in \mathbb{R}_+^{l\cdot(n-1)}$. Under separability of $\succsim_i$, it is possible to introduce a well-defined preference relation $\succsim_i^{(i)}$ over $\mathbb{R}_+^l$ i.e. over the individual consumption vectors, sometimes called *internal preference* of trader i [16].

Below we give an example of a separable preference inspired by classical Edgeworth well-being externalities (see Dufwenberg et al. (2011)). In this example, agent i cares about his own internal utility and the sum of the internal utilities of the other agents.

Example 16 Each agent $i \in N$ has an (internal) utility function u_i which depends only on his own consumption x_i, and an interdependent utility function U_i which for each agent aggregates these individual utilities according to the formula:

$$U_i(x) \coloneqq u_i(x_i) + \frac{\beta_i}{n-1} \sum_{j \neq i} u_j(x_j).$$

If β_i is positive, then agent i is altruistic or benevolent and the (SGM) assumption is satisfied. The (SBA) condition is satisfied for the preference represented by U_i, when each individual selfish utility satisfies (SBA), for example if each individual utility is a Cobb-Douglas utility function.

In this Section the notions of assignment and allocation are the same as in Section 2.1.

[14] A similar condition is assumed in Borglin (1973) to ensure that the Second Welfare Theorem holds true in the case of separable preferences. Based on standard arguments, it can be seen that (strict) increasing preferences in their domain satisfy the (SGM) condition.

[15] The assumption of social boundary aversion turns out to be indispensable for core equivalence theorems in models with production if the interdependency problem is captured by the outsiders' feasibility condition, see Xiong and Zheng (2007).

[16] By definition $x_i \succsim_i^{(i)} x'_i$, if and only if $(x_i, x_{i^c}) \succsim_i (x'_i, x_{i^c})$, for some x_{i^c}.

As in Section 2.2, we introduce the notion of preferences-core and resources-core, and prove that under our assumptions these two notions coincide. The γ-core defined below is given in the spirit of Dufwenberg et al. (2011), Di Pietro et al. (2022), Graziano et al. (2022).

Definition 17 (γ-Core) *Given an allocation $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f) \in \mathcal{F}$ and a coalition S, we say that S improves upon ξ, whenever there exist $x'_S = (x'_i)_{i\in S}$ and $y' \in Y(S, \xi)$ such that*

i) $x'(S) \leq \omega(S) + y'$ *(x'_S is an assignment for S given ξ);*

ii) $\omega(S^c) + y'' \in \Lambda(S, \xi)$, *for some* $y'' \in \sigma_{S,\xi}(y')$*;*

iii) $(x'_S, x_{S^c}) \succ_i (x_S, x_{S^c})$, *for every* $i \in S$.

The set of allocations which cannot be improved upon by any coalition is called the γ-preferences core and is denoted $\mathcal{C}^\gamma_p(E)$. If we add $x'(S) < \omega(S) + y'$ in condition i) and in condition iii) $\succ_i$ is replaced by $\succsim_i$, then we say that S improves upon ξ in resources and the corresponding core, which is denoted $\mathcal{C}^\gamma_r(E)$, is the γ-resources core.

The notion of γ-dominance is, in spirit, a generalization of the definition of competitive behavior. A coalition S deviates assuming that outsiders do not change their consumption. As in pure exchange economies analyzed in Dufwenberg et al. (2011), Di Pietro et al. (2022), Graziano et al. (2022), also in our production economy the γ-blocking mechanism may produce a final distribution of resources $(x'_S, x_{S^C}, y' + y'')$ which is not feasible for the society, despite the additional condition *ii*) in Definition 17 related to resource constraints. This is due to the fact that in the γ-dominance, agents in the counter-coalition stick to their status quo consumption.

With this notion of stability, we can generalize the results in Section 2.2. In particular, the next Theorem shows that the two notions of γ-core in preferences and in resources coincide. Also in this case, the equivalence result does not require any assumptions about production sets other than non-emptiness.

Theorem 18 *Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ be an allocation of the production economy E with other-regarding preferences. Under the previous assumptions, $\xi \in \mathcal{C}^\gamma_r(E)$ if and only if $\xi \in \mathcal{C}^\gamma_p(E)$.*

Proof. Let $\xi \in \mathcal{C}^\gamma_r(E)$ and suppose by contradiction that $\xi \notin \mathcal{C}^\gamma_p(E)$. Then, there exist a coalition $S \subseteq N$, $(x'_i)_{i\in S}$ and a vector $y' \in Y(S, \xi)$ such that $x'(S) \leq \omega(S)+y'$, $(x'_S, x_{S^c}) \succ_i (x_S, x_{S^c})$ for all $i \in S$ and $\omega(S^c)+y'' \in \Lambda(S, \xi)$, for some $y'' \in \sigma_{S,\xi}(y')$. If $x'(S) < \omega(S)+y'$, a contradiction follows, so, we can assume that $x'(S) = \omega(S) + y'$. By the continuity of preferences, there exists a positive δ such that, if $z_i \in \mathbb{R}^l_+$ for all $i \in S$ and $\|(z_S, x_{S^c}) - (x'_S, x_{S^c})\| < \delta$ then $(z_S, x_{S^c}) \succ_i (x'_S, x_{S^c})$, for every $i \in S$. Moreover, by condition (SBA) and continuity, the vector $x'(S)$ is strictly positive[17]. Consider an agent $h \in S$ such that $x'_h > 0$ and choose $\varepsilon > 0$ such that $0 < (1-\varepsilon)\|x'_h\| < \delta$. Define x'' by choosing $x''_i = x'_i$, for $i \in S \setminus \{h\}$ and $x''_h = \varepsilon x'_h$. For every agent $i \in S$, $\|(x''_S, x_{S^c}) - (x'_S, x_{S^c})\| \leq (1-\varepsilon)\|x'_h\| < \delta$ and consequently $(x''_S, x_{S^c}) \succ_i x$. By construction and strict positivity of $x'(S)$, $x''(S) \ll x'(S) = \omega(S) + y'$, which contradicts the fact that $\xi \in \mathcal{C}^\gamma_r(E)$.

Let $\xi \in \mathcal{C}^\gamma_p(E)$. First observe that from $\xi \in \mathcal{C}^\gamma_p(E)$, it follows that $x_i \gg 0$ for each agent $i \in N$. If not, an agent $i \in N$ with a boundary consumption bundle x_i would be able to γ-improve upon ξ in preferences using the strictly positive initial endowment and with no production, by condition (SBA), point 3. of Assumption 3, point 3. of Assumption 4 and point 2. of Assumption 5. If ξ is not in the γ-resources core, there exist a coalition $S \subseteq N$, $(x'_i)_{i\in S}$ and a vector $y' \in Y(S,\xi)$ such that $x'(S) < \omega(S) + y'$, $(x'_S, x_{S^c}) \succsim_i (x_S, x_{S^c})$, for all $i \in S$ and $\omega(S^c) + y'' \in \Lambda(S,\xi)$, for some $y'' \in \sigma_{S,\xi}(\xi')$. Condition (SBA) and strict positivity of x ensure that $x'(S)$ is strictly positive. Then, by condition (SGM), there exist vectors x''_i, $i \in S$, such that $x''(S) = \omega(S) + y'$ and $(x''_S, x_{S^c}) \succ_i (x'_S, x_{S^c})$ for any $i \in S$. Notice that, (x''_S, y') is an assignment for S given ξ. Finally, by transitivity, we obtain $(x''_S, x_{S^c}) \succ_i x$, for all $i \in S$, which is a contradiction. ■

Under the assumptions of Theorem 18, the γ-core of the production economy E can be denoted $\mathcal{C}^\gamma(E)$ with no distinction between preferences and resources.

Remark 19 Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f) \in \mathcal{F}$ be an allocation with $x = (x_1, \ldots, x_n) \gg 0$. The proof of Theorem 18 shows that when a coalition S is able to γ-improve upon ξ in preferences, then the same coalition is able to γ-improve upon ξ by saving resources, and vice-versa. Under the same assumptions and with similar arguments, we can also show that whenever the coalition S is able to γ-improve upon ξ by saving resources then S γ-improves upon ξ by saving a strictly positive amount of each commodity (compare Remark 8).

[17] If $x'(S)$ is on the boundary, then for each $j \in S$, x'_j is a boundary vector and by (SBA) $(x_S + \varepsilon_n, x_{S^c}) \succ_i (x'_S, x_{S^c})$, for every $i \in S$ and for any sequence ε_n of strictly positive vectors converging to zero. Then, by continuity, $(x_S, x_{S^c}) \succsim (x'_S, x_{S^c}) \succ x$ and a contradiction.

As in Section 2.3, we shall focus on allocations $\xi = (x_1, \dots, x_n, y_1, \dots, y_f)$ such that $x = (x_1, \dots, x_n) \gg 0$. In order to introduce a measure of social loss for every coalition S and for a given allocation ξ, we first introduce the sets $\mathcal{R}^\gamma_S(\xi)$ and study some of their properties. Define

$$\mathcal{R}^\gamma_S(\xi) := \left\{x'(S) \in \mathbb{R}^l_+ : (x'_S, x_{S^c}) \succsim_i x, i \in S\right\} - \Gamma_S(\xi),$$

where $\Gamma_S(\xi)$ is defined in Section 2.3. As for the model without externalities, the set $\mathcal{R}^\gamma_S(\xi)$ is nonempty and satisfies the following properties.

Lemma 20 *Let $\xi = (x_1, \dots, x_n, y_1, \dots, y_f)$ be an allocation such that $x = (x_1, \dots, x_n) \gg 0$. Then the set $\mathcal{R}^\gamma_S(\xi)$ satisfies the following properties:*

1. *if $\omega(S) \notin \mathcal{R}^\gamma_S(\xi)$, then S is not a blocking coalition;*
2. *if $\gamma' > \gamma$ and $\gamma \in \mathcal{R}^\gamma_S(\xi)$, then $\gamma' \in \mathcal{R}^\gamma_S(\xi)$;*
3. *the set $\mathcal{R}^\gamma_S(\xi)$ is closed in $\mathbb{R}^l$.*

Proof. See the Appendix. ∎

In the next step, we define the set of resources that can be saved by coalition S still allowing for its members to achieve a resources allocation that is at least as good as x. Formally, $\Psi^\gamma_S : \mathcal{F} \rightrightarrows \mathbb{R}^l$,

$$\Psi^\gamma_S(\xi) := \left\{z \in \Big(\omega(S) + Y(S, \xi)\Big) \cap \mathbb{R}^l_+ : \omega(S) - z \in \mathcal{R}^\gamma_S(\xi)\right\},$$

and using the same arguments as used to proof Lemma 10 and Lemma 11, we obtain the following results.

Lemma 21 *$\omega(S) \in \mathcal{R}^\gamma_S(\xi)$ if and only if $\Psi^\gamma_S(\xi) \neq \emptyset$.*

Lemma 22 *The set $\Psi^\gamma_S(\xi)$ is compact.*

We now fix a reference bundle $g > 0$ and introduce the *loss mapping* $\mathcal{L}^\gamma_{g,S} : \mathcal{F} \to \mathbb{R}$, for a production economy with consumption externalities as follows:

$$\mathcal{L}^\gamma_{g,S}(\xi) := \begin{cases} \max\left\{\lambda \in \mathbb{R} : \lambda \cdot g \in \Psi^\gamma_S(\xi)\right\} & \text{if } \Psi^\gamma_S(\xi) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

If $\Psi^\gamma_S(\xi)$ is nonempty, the loss mapping has a finite value, since the set $\Psi^\gamma_S(\xi)$ is compact. Notice also that $\mathcal{L}^\gamma_{g,S}(\xi) \geq 0$ since $g \in \mathbb{R}^l_+$ with $g \neq 0$. The loss mappings are different if we vary the reference bundles. However, if there exists

g such that $\mathcal{L}_{g,S}(x)$ is strictly positive, then for all reference bundles, the loss mappings are strictly positive.

Proposition 23 *For a given allocation ξ, if $\mathcal{L}^{\gamma}_{g,S}(\xi) > 0$ for a vector $g > 0$, then $\mathcal{L}^{\gamma}_{g',S}(\xi) > 0$ for every $g' > 0$.*

Proof. See the Appendix. ■

From the loss procured to each coalition S by an allocation ξ, we can introduce the measure of social loss with respect to ξ as the *social loss mapping* $\mathcal{L}^{\gamma}_{g} \colon \mathcal{F} \to \mathbb{R}$ defined as

$$\mathcal{L}^{\gamma}_{g}(\xi) \coloneqq \max_{S \subseteq N} \mathcal{L}^{\gamma}_{g,S}(\xi).$$

The social loss mapping is well-defined because for every coalition S, the loss mapping $\mathcal{L}^{\gamma}_{g,S}$ is well-defined. Theorem 24 shows that the maximal loss vanishes if and only if the allocation belongs to the γ-core. Consequently, we obtain a characterization of the core allocations in terms of loss mappings in a production economy with externalities.

Theorem 24 *Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ be an allocation such that $x = (x_1, \ldots, x_n) \gg 0$. For any non null reference bundle $g \in \mathbb{R}^l_+$, $\mathcal{L}^{\gamma}_{g}(\xi) = 0$ if and only if $\xi \in \mathcal{C}^{\gamma}(E)$.*

Proof. We start by proving that if an allocation ξ belongs to the core, then $\mathcal{L}^{\gamma}_{g}(\xi) = 0$. Suppose by contradiction that $\mathcal{L}^{\gamma}_{g}(\xi) > 0$. Then there exists a coalition S such that $\mathcal{L}^{\gamma}_{g,S}(\xi) > 0$ and $\Psi^{\gamma}_{S}(\xi) \setminus \{0\} \neq \emptyset$. Consequently, there exists $z > 0$ such that $z \in \Psi^{\gamma}_{S}(\xi)$. Therefore, $0 < z = \omega(S) + \widehat{y}$ with $\widehat{y} \in Y(S, \xi)$ and $\omega(S) - z = -\widehat{y} \in \mathcal{R}^{\gamma}_{S}(\xi)$. Thus $\omega(S) - z = x'(S) - y'$, for some $(x', y') \in \mathbb{R}^{l \cdot n}_{+} \times \Gamma_S(\xi)$ with $(x'_S, x_{S^c}) \succsim_i x$ for any $i \in S$. Finally, notice that $\omega(S) + y' = x'(S) + z > x'(S)$ implies a contradiction.

Let us show now that $\mathcal{L}^{\gamma}_{g}(\xi) = 0$ implies $\xi \in \mathcal{C}^{\gamma}(E)$. By contradiction, suppose that $\xi \notin \mathcal{C}^{\gamma}(E)$. Then there exist a coalition $S \subseteq N$, $(x'_i)_{i \in S}$ and y' such that $(x'_S, x_{S^c}) \succsim_i x$ for every $i \in S$, $y' \in Y(S, \xi)$, $x'(S) < \omega(S) + y'$ and $\omega(S^c) + y'' \in \Lambda(S, \xi)$ for some $y'' \in \sigma_{S,\xi}(y')$. Consequently, $g' \coloneqq \omega(S) + y' - x'(S) > 0$ belongs to $(\omega(S) + Y(S, \xi)) \cap \mathbb{R}^l_+$ since $y' - x'(S) \in Y(S, \xi)$ by $y' \in Y(S, \xi)$ and by Point 5 of Assumption 3. Furthermore, $\omega(S) - g' = x'(S) - y' \in \mathcal{R}^{\gamma}_{S}(\xi)$ since $y' \in \Gamma_S(\xi)$, and consequently, $g' \in \Psi_S(\xi)$ and $g' > 0$. Thus $\mathcal{L}^{\gamma}_{g',S}(\xi) > 0$ which implies $\mathcal{L}^{\gamma}_{g,S}(\xi) > 0$, contradicting the fact that $\mathcal{L}^{\gamma}_{g}(\xi) = 0$. ■

We have observed that under the assumptions of the model, when the allocation $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ belongs to $\mathcal{C}^{\gamma}(E)$, then ξ ensures a strictly positive consumption bundle to each consumer, i.e. $x = (x_1, \ldots, x_n) \gg 0$ (see the proof of Theorem 18). Therefore, Theorem 24 proves that the core $\mathcal{C}^{\gamma}(E)$ of a production economy with other-regarding preferences is formed by zero points of suitable loss mappings.

In the presence of consumption externalities, a different blocking mechanism might be defined depending on the attitude of the blocking coalition with respect to the behavior of the outsiders. This is for instance the case of the α-dominance and the corresponding α-core analyzed in Yannelis (1991) and Di Pietro et al. (2022), among others. In this Section we follow a similar idea and analyze a scenario with consumption externalities in which a blocking coalition S maintains a pessimistic attitude with respect to the behavior of the outsiders. Precisely, S considers all the redistributions of resources available for the counter-coalition S^c as possible reactions by S^c. The extreme prudence of the blocking coalitions, the presence of external effects due to both consumption and production, make the notion of α-core particularly complex. Nevertheless, results in terms of resources can be given also for α-core allocations when each agent receives a strictly positive consumption bundle.

In order to deal with the several possible reactions by the outsiders, we associate the (SGM) and (SBA) conditions to the special case of preferences which are separable with respect to coalitions. The formal definition is an extension of the separability introduced in Section 3.1 (see Borglin (1973) and Dufwenberg et al. (2011) for standard separability).

Assumption 25 (Social Group Separability (SGS)) *For any coalition S and $i \in S$, the preference relations $\succsim_i$ are S-separable: for all x_S and x'_S in $\mathbb{R}_+^{l\cdot|S|}$, if there exists $x_{S^c} \in \mathbb{R}_+^{l\cdot|S^c|}$ such that $(x'_S, x_{S^c}) \succsim_i (x_S, x_{S^c})$ (resp. $(x'_S, x_{S^c}) \succ_i (x_S, x_{S^c})$) then $(x'_S, x'_{S^c}) \succsim_i (x_S, x'_{S^c})$ (resp. $(x'_S, x'_{S^c}) \succ_i (x_S, x'_{S^c})$) for all $x'_{S^c} \in \mathbb{R}_+^{l\cdot|S^c|}$.*

Condition (SGS) states that if a member of coalition S likes the S-assignment x'_S better than the S-assignment x_S when the outsiders consume x_{S^c}, then the coalition member will also prefer x'_S to x_S if each of them is joined with any other consumption by the outsiders. Consequently, the preference of i for the consumption of a coalition S to which i belongs, does not depend on the choice of others outside S. Notice that in each comparison the consumption of the counter coalition S^c is held constant. Hence, the (SGS) condition on its own is not enough to identify the γ and α dominance. Notice also that the preference relations defined in Example 16 satisfies the (SGS) condition.

Now we introduce the notion of α-preferences-core and α-resources-core, and prove that under our assumptions these two notions coincide. The α-core defined below is given in the spirit of Yannelis (1991), Graziano et al. (2017) and Di Pietro et al. (2022).

Definition 26 (α-Core) *Given an allocation $\xi = (x_1, \dots, x_n, y_1, \dots, y_f) \in$*

$\mathcal{F}$ and a coalition S, we say that S α-improves upon ξ, whenever there exist $x'_S = (x'_i)_{i\in S}$ and $y' \in Y(S,\xi)$ such that

i) $x'(S) \leq \omega(S) + y'$ *(x'_S is an assignment for S given ξ);*

ii) $\omega(S^c) + y'' \in \Lambda(S,\xi)$, *for at least one* $y'' \in \sigma_{S,\xi}(y')$;

iii) $(x'_S, z_{S^c}) \succ_i (x_S, x_{S^c})$, *for every* $i \in S$ *and for every* $z \in \Phi_{S,\xi}(y')$

where

$$\Phi_{S,\xi}(y') \coloneqq \{z_{S^c} \in \mathbb{R}_+^{l\cdot|S^c|} \mid \exists\, y'' \in \sigma_{S,\xi}(y') : \ z(S^c) \leq \omega(S^c)+y'', \omega(S^c)+y'' \in \Lambda(S,\xi)\}.$$

The set of allocations which cannot be α-improved upon by any coalition is called the α-preferences core and is denoted $\mathcal{C}^\alpha_p(E)$. If we add $x'(S) < \omega(S)+y'$ in condition i) and in condition ii) $\succ_i$ is replaced by $\succsim_i$, then we say that S α-improves upon ξ in resources and the corresponding core, which is denoted $\mathcal{C}^\alpha_r(E)$, is the α-resources core.

We notice that by point *ii*) of Definition 26 and by Assumption 5, the set $\Phi_{S,\xi}(y')$ is non-empty since it contains the assignment which gives the zero commodity bundles to the outsiders in at least one $y'' \in \sigma_{S,\xi}(y')$. Moreover, it is easy to verify that Definition 26 gives back the usual notion of α-core when there is no production (see Yannelis (1991), Di Pietro et al. (2022)) as well as the standard notion of core for selfish models. When compared with the γ-core, the α-core ensures feasibility of the final allocation $(x'_S, z_{S^c}, y' + y'')$, for each possible reaction (z_{S^c}, y'') of the outsiders, where z_{S^c} and y'' are given as in the definition of the set $\Phi_{S,\xi}(y')$ [18] . In particular, the two notions of core formulated with respect to γ-dominance and α-dominance are not related.

We now have to prove relationship between allocations of the α-core in preferences and allocations of the α-resources core. The next Theorem shows that for an allocation ensuring a strictly positive consumption bundle to each consumer, the two notions of α-core, the one given in preferences and the one given in resources, coincide.

Theorem 27 *Let $\xi = (x_1,\dots,x_n,y_1,\dots,y_f)$ be an allocation of the production economy E with other-regarding preferences ensuring a strictly positive consumption bundle to each consumer, i.e. with $x = (x_1,\dots,x_n) \gg 0$. Under the previous assumptions, $\xi \in \mathcal{C}^\alpha_r(E)$ if and only if $\xi \in \mathcal{C}^\alpha_p(E)$.*

[18] We point out that one may restore the feasibility of the final distribution of resources also with the δ-core recently studied in Graziano et al. (2022). A notion of dominance for exchange economies defining a core smaller than the α-core, is given in Chander and Tulkens (1997) assuming that the outsiders consume their initial endowments.

Proof. Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f) \in \mathcal{C}_r^\alpha(E)$ and suppose by contradiction that there exists a coalition $S \subseteq N$ and $((x_i')_{i\in S}, y')$ such that conditions $i) - iii)$ of Definition 26 are satisfied. Notice that in condition $i)$ we should have $x'(S) = \omega(S) + y'$ otherwise we easily get a contradiction with the fact that ξ belongs to $\mathcal{C}_r^\alpha$. We may suppose that $x'(S) \gg 0$. Otherwise, from $0 \in \Phi_{S,\xi}(y')$ and condition $iii)$, we would obtain $(x_S^{'}, 0_{S^c}) \succ_i (x_S, x_{S^c})$, for every $i \in S$ with each x_i' on the boundary, contradicting Assumption (SBA).

Define the sets

$$\mathrm{K}_{S,\xi}(x_S', y') := \{(x_S', z_{S^c}) \in \mathbb{R}_+^{l\cdot n} \mid z_{S^c} \in \Phi_{S,\xi}(y')\}$$

$$\mathrm{NBT}_S(x) \coloneqq \bigcup_{i\in S}\{\zeta \in \mathbb{R}_+^{l\cdot n} | \; x \succsim_i \zeta \,\}$$

and their distance

$$\mathrm{d}_{S,\xi}(\mathrm{K}_{S,\xi}(x_S', y'), \mathrm{NBT}_{S,\xi}(x)) \coloneqq \inf\left\{\|\zeta - \eta\| \colon \eta \in \mathrm{K}_{S,\xi}(x_S', y'),\ \zeta \in \mathrm{NBT}(\xi)\right\}.$$

Notice that the sets $\mathrm{K}_{S,\xi}(x_S', y')$ and $\mathrm{NBT}_{S,\xi}(x))$ are nonempty by construction. Clearly, $\mathrm{NBT}_S(x)$ is a closed subset of $\mathbb{R}_+^{l\cdot n}$. Moreover, we claim that $\mathrm{K}_{S,\xi}(x_S', y')$ is compact and $\mathrm{K}_{S,\xi}(x_S', y') \cap \mathrm{NBT}_S(x) = \emptyset$[19]. Therefore the distance is strictly positive. Denote the distance $\mathrm{d}_{S,\xi}(\mathrm{K}_{S,\xi}(x_S', y'), \mathrm{NBT}_S(x))$ simply by δ. For every element $(x_S', z_{S^c}) \in \mathrm{K}_{S,\xi}(x_S', y')$ consider the open ball $B((x_S', z_{S^c}); \delta)$ centered in (x_S', z_{S^c}) and with ray $\delta > 0$. Then, for any $\zeta \in B((x_S', z_{S^c}); \delta) \cap \mathbb{R}_+^{l\cdot n}$, we must have $\zeta \succ_i x$ for each $i \in S$. Let $\varepsilon > 0$ be such that $0 < (1-\varepsilon)\|x_S'\| < \delta$. Then, from $(\epsilon x_S', z_{S^c}) \in B((x_S', z_{S^c}); \delta) \cap \mathbb{R}_+^{l\cdot n}$, it follows that $(\epsilon x_S', z_{S^c}) \succ_i (x_S, x_{S^c})$ for every agent $i \in S$, where $z_{S^c} \in \Phi_{S,\xi}(y')$, $\epsilon x'(S) \ll x'(S) = \omega(S) + y'$. Hence a contradiction to the fact that $(x, y) \in \mathcal{C}_r^\alpha(E)$ follows.

Vice-versa, let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f) \in \mathcal{C}_p^\alpha(E)$ and suppose by contradiction that there exist a coalition $S \subseteq N$ and an assignment for S given ξ, i.e., $((x_i')_{i\in S}, y')$ such that for every agent $i \in S$, (x_S', y') α-dominates ξ with respect to the use of resources. Then, $0 \leq x'(S) < \omega(S) + y'$ holds true, and for all $i \in S$, we must have $(x_S', z_{S^c}) \succsim_i (x_S, x_{S^c})$, for all $z_{S^c} \in \mathbb{R}_+^{l\cdot|S^c|}$ such that $z_{S^c} \in \Phi_{S,\xi}(y')$. Hence from $0 \in \Phi_{S,\xi}(y')$, we obtain $(x_S^{'}, 0_{S^c}) \succsim_i (x_S, x_{S^c})$, for every $i \in S$. Therefore, by (SBA), $x^{'}(S)$ is a strictly positive vector. Consider one of the vectors z_{S^c}. Since $\omega(S) + y' \in \mathbb{R}_+^l$, $y' + \omega(S) > x'(S)$ and $x'(S) \gg 0$, by (SGM), there exists x'' such that $x''(S) = \omega(S) + y'$ and $(x_S'', z_{S^c}) \succ_i (x_S', z_{S^c})$. By (SGS) it is also true that $(x_S'', z_{S^c}') \succ_i (x_S', z_{S^c}')$ for every z_{S^c}' and for all $i \in S$. Consequently, using the transitivity of $\succsim_i$, one obtains $(x_S'', z_{S^c}') \succ_i (x_S, x_{S^c})$ for all $i \in S$ and for all z_{S^c}' such that $z_{S^c} \in \Phi_{S,\xi}(y')$. This contradicts the fact that $\xi \in \mathcal{C}_p^\alpha(E)$. ∎

[19] For the proof of this claim, see the Appendix.

Under (SGM) and (SGS), Theorem 27 establishes an equivalence between allocations which cannot be dominated in terms of preferences and allocations which cannot be dominated in terms of resources in the case where blocking coalitions are assumed to be pessimistic and take into account each possible redistribution by the outsiders. Notice that the structure of α-dominance does not guarantee that for an allocation $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ in the preference core $\mathcal{C}_p^\alpha(E)$ it is true that $x = (x_1, \ldots, x_n) \gg 0$ [20]. Consequently, the characterization of α-core allocations in terms of social loss mappings that we are going to prove in the next section, only applies to the subset of $\mathcal{C}_p^\alpha(E)$ formed by allocations ensuring a strictly positive consumption bundle to each consumer.

3.5 α-core allocations and zero points of social loss mappings

In order to introduce a measure of social loss for every coalition S and for a given allocation $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ with $x \gg 0$, as in the case of dominance and γ-dominance, we first introduce the set $\mathcal{R}_S^\alpha(\xi)$ of resources which allows the coalition S to reach at least one redistribution x'_S which is weakly preferred with respect to x for any reaction of the counter coalition S^c, and allows available production, which restore the feasibility of the outsiders, and we study some of its properties. Formally, for each $y' \in Y(S, \xi)$ define

$$\mathcal{R}_S^\alpha(\xi, y') := \left\{ x'(S) - y' \in \mathbb{R}^l : \; (x'_S, z_{S^c}) \succsim_i x, \; \forall\, i \in S, \; \forall\, z_{S^c} \in \Phi_S(\xi, y') \right\}$$

and

$$\mathcal{R}_S^\alpha(\xi) := \bigcup_{y' \in Y(S,\xi)} \mathcal{R}_S^\alpha(\xi, y').$$

Notice that $\mathcal{R}_S^\alpha(\xi)$ might be empty. However, the set satisfies the following properties.

Lemma 28 *Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ be an allocation such that $x = (x_1, \ldots, x_n) \gg 0$. Then the set $\mathcal{R}_S^\alpha(\xi)$ satisfies the following properties:*

1. *if $\omega(S) \notin \mathcal{R}_S^\alpha(\xi)$, then S is not a blocking coalition for the allocation ξ;*
2. *if $\gamma' > \gamma$ and $\gamma \in \mathcal{R}_S^\alpha(S)$, then $\gamma' \in \mathcal{R}_S^\alpha(\xi)$;*

Proof. See the Appendix. ■

[20] Precisely, the fact that a single individual i cannot improve upon the allocation ξ in the α-dominance using the initial endowment ω_i (α-individual rationality), only entails that $(x_i, x_{i^c}) \succsim_i (\omega_i, z_{i^c})$ for a redistribution z_{i^c} of the outsiders $\{i\}^c$ with $z_{i^c} \leq \omega_{i^c}$ which is not enough to guarantee the strict positivity of x_i.

We now define the set of resources that can be saved by coalition S still allowing for its members to achieve a resources allocation that is at least as good as x. Formally, $\Psi_S^\alpha \colon \mathcal{F} \rightrightarrows \mathbb{R}^l$,

$$\Psi_S^\alpha(\xi) := \Big\{\gamma \in \Big(\omega(S) + Y(S,\xi)\Big) \cap \mathbb{R}^l_+ \colon \omega(S) - \gamma \in \mathcal{R}_S^\alpha(\xi)\Big\}.$$

Using (SGS) condition, we can prove that the set Ψ_S^α satisfies the following properties. The proofs are contained in the Appendix.

Lemma 29 $\omega(S) \in \mathcal{R}_S^\alpha(\xi)$ *if and only if* $\Psi_S^\alpha(\xi) \neq \emptyset$.

Lemma 30 *The set* $\Psi_S^\alpha(\xi)$ *is bounded.*

In particular, the set $\Psi_S^\alpha(\xi)$ is bounded. It is not compact, since the set $\mathcal{R}_S^\alpha(\xi)$ might not be a closed subset of $\mathbb{R}^l$.

We now fix a reference bundle $g > 0$ and introduce the *loss mapping* $\mathcal{L}_{g,S}^\alpha$: $\mathcal{F} \to \mathbb{R}$, for a production economy with consumption externalities as follows:

$$\mathcal{L}_{g,S}^\alpha(\xi) := \begin{cases} \sup\{\lambda \in \mathbb{R} \colon \lambda \cdot g \in \Psi_S^\alpha(\xi)\} & \text{if} \quad \Psi_S^\alpha(\xi) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

If $\Psi_S^\alpha(\xi)$ is nonempty, the loss mapping has a finite value since the set $\Psi_S^\alpha(\xi)$ is bounded, and consequently, $\mathcal{L}_{g,S}^\alpha$ is well-defined[21] . Notice also that $\mathcal{L}_{g,S}^\alpha(\xi) \geq$ since $g \in \mathbb{R}^l_+$ with $g \neq 0$ and $\Psi_S^\alpha(\xi) \subseteq \mathbb{R}^l_+$. The loss mappings are different if we vary the reference bundles. However, if there exists g such that $\mathcal{L}_S^{g;\alpha}(x) > 0$, then for all reference bundles, the loss mappings are strictly positive.

Proposition 31 *For a given allocation* ξ*, if* $\mathcal{L}_S^{g;\alpha}(\xi) >$ *for a vector* $g > 0$*, then* $\mathcal{L}_{g',S}^\alpha(\xi) >$ *for every* $g' > 0$.

Proof. See the Appendix. ■

From the loss procured to each coalition S by an allocation ξ, we can introduce the measure of social loss with respect to ξ as the *social loss mapping* $\mathcal{L}_g^\alpha \colon \mathcal{F} \to \mathbb{R}$ defined as

$$\mathcal{L}_g^\alpha(\xi) := \max_{S \subseteq N} \mathcal{L}_S^{g;\alpha}(\xi).$$

The social loss mapping is well-defined because for every coalition S, the loss mapping $\mathcal{L}_S^g$ is well-defined. As for the case of the γ-core, Theorem 32 shows that the maximal loss vanishes if and only if the allocation belongs to the α-core. It provides a characterization in terms of zero points of social loss mappings of the α-core allocations with strictly positive consumption bundles.

[21] Note that, if $\Psi_S^\alpha(\xi) \neq \emptyset$, then, as a consequence of Lemma 29, $0 \in \Psi_S^\alpha(\xi)$. Thus, for any reference bundle g, the set $\Psi_S^\alpha(\xi) \cap \mathrm{span}(g)$ is nonempty. So, for any g, there exists a scalar λ such that $\lambda \cdot g \in \Psi_S^\alpha(\xi)$.

Theorem 32 *Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_f)$ be an allocation such that $x = (x_1, \ldots, x_n) \gg 0$. Under the previous assumptions, for any non null reference bundle $g \in \mathbb{R}^l_+$, $\mathcal{L}^\alpha_g(\xi) = 0$ if and only if $\xi \in \mathcal{C}^\alpha(E)$.*

Proof. We start by proving that if an allocation ξ belongs to the α-core, then $\mathcal{L}^\alpha_g(\xi) = 0$. Suppose by contradiction that $\mathcal{L}^\alpha_g(\xi) > 0$. Then there exists a coalition S such that $\mathcal{L}^\alpha_{g,S}(\xi) > 0$ and $\Psi^\alpha_S(\xi) \setminus \{0\} \neq \emptyset$. Consequently, there exists $\zeta > 0$ such that $\zeta \in \Psi^\alpha_S(\xi)$. Therefore, $0 < \zeta = \omega(S) + \widehat{y}$ with $\widehat{y} \in Y(S, \xi)$ and $\omega(S) - \zeta = -\widehat{y} \in \mathcal{R}^\alpha_S(\xi)$. Thus, $-\widehat{y} = x'(S) - y'$ for some $(x', y') \in \mathbb{R}^{l \cdot n}_+ \times Y(S, \xi)$ with $(x'_S, z_{S^c}) \succsim_i x$ for any $z_{S^c} \in \Phi_S(\xi, y')$. Finally, notice that $0 < \zeta = \omega(S) + \widehat{y} = \omega(S) - x'(S) + y'$ and a contradiction is obtained. Let us show now that $\mathcal{L}^\alpha_g(\xi) = 0$ implies $\xi \in \mathcal{C}^\alpha(E)$. By contradiction, suppose that $\xi \notin \mathcal{C}^\alpha(E)$. Then there exist a coalition $S \subseteq N$, $(x'_i)_{i \in S}$ and y' such that, they are an assignment for S which allow the coalition to strictly save the resources available to them, and for every $i \in S$ $(x'_S, z_{S^c}) \succsim_i x$ holds true for all $z_{S^c} \in \Phi_S(\xi, y')$. Consequently, $g' \coloneqq \omega(S) + y' - x'(S) > 0$ belongs to $(\omega(S) + Y(S, \xi)) \cap \mathbb{R}^l_+$ since $y' - x'(S) \in Y(S, \xi)$ by $y' \in Y(S, \xi)$ and by Point 5 of Assumption 3. Furthermore, $\omega(S) - g' = x'(S) - y' \in \mathcal{R}^\alpha_S(\xi)$ by construction, and consequently, $g' \in \Psi^\alpha_S(\xi)$ and $g' > 0$. Thus $\mathcal{L}^\alpha_{g',S}(\xi) > 0$ which implies $\mathcal{L}^\alpha_{g,S}(\xi) > 0$, contradicting the fact that $\mathcal{L}^\alpha_g(\xi) = 0$. ■

4 Some Examples

We show below that our production economy is sufficiently general to cover study of the core of relevant cases of production economies. For each of the particular cases presented in the succeeding examples, Theorem 7 provides conditions for the equivalence between the preferences-core and the resources-core. Notice that in the Examples below, the correspondences Y, σ and Λ explicitly depend on the status quo state ξ only in the case of Production with corporate governance (Example 35).

Example 33 (Production economy with free available technology) Consider the productive economy model in Debreu and Scarf (1963) for instance. In this case, there is a unique production set Y and all coalitions have access to the same production possibilities described by $Y(S, \xi) \coloneqq Y$, for every coalition S and status quo state ξ. For each production plan y' chosen by coalition S in Y, define $\sigma_{S,\xi}(y') \coloneqq \{0\}$ and $\Lambda(S, \xi) \coloneqq \mathbb{R}^l_+$, meaning that the coalition S is able to produce by itself if this improves upon ξ and ensuring condition ii) of Definition 6 always satisfied. For this production economy, in line with Debreu and Scarf (1963), Definition 6 ensures that the preferences-core is defined as the set of feasible allocations ξ for which it is not possible to find a coalition S, $x'_S = (x'_i)_{i \in S}$ and $y' \in Y$ such that

i) $x'(S) \leq \omega(S) + y'$;

ii) $x'_i \succ_i x_i$, for every $i \in S$.

The resources-core is defined analogously. Under assumptions which are standard for the production set Y, the preferences-core and the resources-core coincide.

Example 34 (Private ownership economy) In this case, we assume that firms Y_j, $j \in J$, are owned by agents i, $i \in N$, and $\theta_{ij} \in [0, 1]$ is the share of firm $j \in J$ owned by agent i. Moreover, the condition $\sum_{i \in N} \theta_{ij} = 1$ is statisfied. Assume that for each status quo allocation ξ, the technology available to coalition S is given by

$$Y(S, \xi) \coloneqq \sum_{j \in J} \sum_{i \in S} \theta_{ij} Y_j,$$

and that the production plans of the complementary coalition corresponding to any choice y' of S, are defined by

$$\sigma_{S,\xi}(y') \coloneqq \left\{ \sum_{j \in J} \sum_{i \notin S} \theta_{ij} y'_j \,|\, y'_j \in Y_j,\ y' = \sum_{j \in J} \sum_{i \in S} \theta_{ij} y'_j \right\} \subseteq Y(S^c, \xi),$$

and finally assume that $\Lambda(S, \xi) \coloneqq \mathbb{R}^l_+$. Then, the preferences-core according to the considerate dominance introduced with Definition 6 is the set of feasible allocations ξ for which it is not possible to find a coalition S, $x'_S = (x'_i)_{i \in S}$ and $y'_j \in Y_j$, for each $j \in J$, such that

i) $x'(S) \leq \omega(S) + \sum_{j \in J} \sum_{i \in S} \theta_{ij} y'_j$;

ii) $\omega(S^c) + \sum_{j \in J} \sum_{i \notin S} \theta_{ij} y'_j \geq 0$;

iii) $x'_i \succ_i x_i$, for every $i \in S$

with condition ii) ensuring that the positive resource constraints for the outsiders is satisfied. The resources-core is defined analogously. Notice that the results proved in Section 2 formally apply also to the notion of preferences-core studied in Aliprantis et al. (1989) in which condition ii) is not imposed. It is enough in this case to define $\sigma_{S,\xi}(y') \coloneqq \{0\}$ and $\Lambda(S, \xi) \coloneqq \mathbb{R}^l_+$.

Example 35 (Private ownership economy with corporate governance) Our production economy model also covers the case that considers forms of corporate governance. In the private ownership production economy defined in Example 34, the resources of a blocking coalition S come also from the firms that the coalition shares with outsiders. Therefore, the question may arise whether the coalition can change the action of firms not completely owned by its members. Following Xiong and Zheng (2007), we can introduce the set $\widetilde{J}(S)$ of firms controlled by a coalition S as described by a correspondence $\widetilde{J} : \mathcal{P}(N) \rightrightarrows J$ which satisfies the conditions:

(1) if $\sum_{i\in S}\theta_{ij}=1$ then $j\in\widetilde{J}(S)$;

(2) if $\sum_{i\in S}\theta_{ij}=0$ then $j\notin\widetilde{J}(S)$;

(3) $\widetilde{J}(S^c)=J\setminus\widetilde{J}(S)$, for each coalition S [22].

Given a status quo $\xi=(x_1,\dots,x_n,y_1,\dots,y_f)$, define

$$Y(S,\xi) \coloneqq \sum_{j\in\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}Y_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y_j,$$

$$\sigma_{S,\xi}(y') \coloneqq$$

$$=\left\{\sum_{j\in\widetilde{J}(S)}\sum_{i\notin S}\theta_{ij}y'_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\notin S}\theta_{ij}y_j\,|\,y'_j\in Y_j,\ y'=\sum_{j\in\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y'_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y_j\right\},$$

and $\Lambda(S,\xi)\coloneqq\mathbb{R}^l_+$. Then, the preferences-core is defined as the set of feasible allocations ξ for which it is not possible to find a coalition S, $x'_S=(x'_i)_{i\in S}$ and $y'_j\in Y_j$, for each $j\in\widetilde{J}(S)$, such that

i) $x'(S)\le\omega(S)+\sum_{j\in\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y'_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y_j$;

ii) $\omega(S^c)+\sum_{j\in\widetilde{J}(S)}\sum_{i\notin S}\theta_{ij}y'_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\notin S}\theta_{ij}y_j\ge 0$;

iii) $x'_i\succ_i x_i$, for every $i\in S$.

The resources-core is defined similarly. The notion of preferences-core in the presence of corporate governance, is studied in Xiong and Zheng (2007) who also refer to the above definition of blocking as *considerate blocking*. In the case of considerate blocking, a coalition is able to change the actions of the firms it controls, with the actions of other firms fixed at the status quo. Moreover, the blocking coalition considers whether the consequences of improving an allocation are feasible for the outsiders through condition *ii)* [23]. It should be emphasized that in the present paper the *considered core* of Xiong and Zheng (2007) has been extended to the more general framework of other-regarding preferences.

[22] This condition is not a requirement for Xiong and Zheng (2007) model of an economy with production and corporate governance. We introduce it to ensure that the correspondence $\sigma_{S,\xi}$ takes values in the production set of S^c, $Y(S^c,\xi)$.

[23] The *inconsiderate blocking mechanism* introduced in (Xiong and Zheng, 2007, Definition 2) follows from the considerate blocking mechanism if we drop in the considerate dominance condition *ii)*. By Remark 14, the inconsiderate dominance and the results for the corresponding core can be formally obtained by results of Section 2 when we define $\Lambda(S,\xi)\coloneqq\mathbb{R}^l$, for each S and ξ.

5 Concluding Remarks

We have shown that the preferences-core and the resources-core of a production economy coincide. We have shown also that allocations in the core of a production economy can be characterized as zero points of measures of social loss. To this end, we have introduced a new core notion which is sufficiently general to cover both the core of private ownership production economies and also the considerate core in a model with corporate governance where the actions of a blocking coalition may affect the production of the firms not under its control. Consequently, our characterization holds despite the interdependence effects due to the presence of production. Moreover, it holds also for models that include consumption externalities. The main assumptions in the case of consumption externalities are the redistribution property known as Social Group Monotonicity and a Social Group Separability of preferences.

For simplicity, we assume that the consumption set is the same for all traders and coincides with the positive cone of the commodity space. However, in a more general framework in which the consumption of each agent depends on which coalition the trader joins, the results would be similar. Also, assumptions more general than (SGM) could be considered. We claim that the results obtained in this paper could be proved under conditions ensuring that the Second Welfare Theorem holds true [24] . Take for instance the case of the Social redistribution assumption introduced recently in del Mercato and Nguyen (2023). This condition is weaker than (SGM) and other relevant assumptions that have been studied in the literature (see e.g. Osana (1972)).

We conclude by commenting on the Debreu-Scarf core equivalence Theorems in the context of our core notions.

The notion of (considerate) core which we introduced in Section 2.2, includes the core of a private ownership production economy and the core of private ownership production economies with corporate governance. For these two notions, under the (SBA) condition, Xiong and Zheng (2007) proved equivalence with Walrasian equilibrium allocations. This equivalence result can be easily adapted to our core notion.

On the other hand, in the presence of externalities, equivalence theorems for the core and competitive allocations are generally not valid. Therefore, we should not expect the γ-core and the α-core introduced in Section 3.2 to coincide with competitive equilibria in the absence of very strong assumptions.

[24] More generally, under the conditions usually imposed to show that the set of Pareto optimal allocations is included in the internal Pareto optimal allocations in models with separable preferences.

With separable preferences and assuming (SGM), a result similar to (Dufwenberg et al., 2011, Lemma 1) is valid also for the core $C^{\gamma}(E)$, i.e. that the γ-core of the production economy is included in the core of the internal economy, defined by internal preferences and, consequently, the replica core is also contained in the set of competitive equilibria (see (Dufwenberg et al., 2011, Theorem 6)). Finally, it can be shown that under stronger conditions the core coincides with Walrasian equilibrium allocations (see (Dufwenberg et al., 2011, Theorem 7)). On the other hand, conditions under which the α-core of a pure exchange economy coincides with competitive allocations have not yet been investigated.

6 Appendix

6.1 Basic properties of correspondences and asymptotic cones.

In this section we recall some basic definitions and properties of correspondences and asymptotic cones in Euclidean spaces [25].

Definition 36 *A correspondence $\varphi\colon X \rightrightarrows Y$ between topological spaces is upper hemicontinuous at a point $x \in X$ if for all open neighborhoods $U \subseteq Y$ such that $\varphi(x) \subseteq U$, there exists a neighborhood V of x such that $\varphi(V) \subseteq U$.*

The following result is a sequential characterization of upper hemicontinuity [26].

Theorem 37 *Let X and Y be two metric spaces. A compact valued correspondence $\varphi\colon X \rightrightarrows Y$ is upper hemicontinuous at a point $x \in X$ if and only if for every sequence $(x^{\nu}, y^{\nu})_{\nu\in\mathbb{N}} \subseteq X \times Y$ such that x^{ν} converges to some $x \in X$, and $y^{\nu} \in \varphi(x^{\nu})$ for any $\nu \in \mathbb{N}$, the sequence y^{ν} has a limit point in $\varphi(x)$.*

Recall that a subset $C \subseteq \mathbb{R}^d$ is a cone with vertex $x \in C$, if for any $y \in C$, it contains the set $\{z \in \mathbb{R}^d\colon \exists\, \tau \in \mathbb{R}_+, z = x + \tau(y - x)\}$.

Definition 38 *A collection of n cones $C_k \subseteq \mathbb{R}^d$, $k = 1, \dots, n$ with vertex 0 is positively semi-independent if $x_k \in C_k$, with $k = 1, \dots, n$, and $\sum_{k=1}^{n} x_k = 0$ implies $x_k = 0$ for any k.*

[25] For the case of asymptotic cones and their properties we refer to Debreu (1959), Section 1.9 and Villar (2000), Chapter 12.
[26] See Aliprantis and Border (2006), Chapter 17, Corollary 17.17 for the proof of the characterization. We also refer to Appendix A in Carmona (2013) for a concise collection of mathematical results in metric spaces.

From the previous definition, positively semi-independent sets do not contain elements in opposite directions.

Let D be a subset of $\mathbb{R}^d$ and k a nonnegative scalar. Define by D^k the set of element of D whose norm is greater than k, i.e., $D^k \coloneqq \{x \in D \colon ||x|| \geq k\}$, and denote by $\Gamma(D^k)$ the intersection of all closed cones with vertex 0 containing D^k. The asymptotic cone of D is a closed cone with vertex zero that contains all unbounded directions of D. Formally,

Definition 39 *The asymptotic cone of $D \subseteq \mathbb{R}^d$ is $\mathcal{A}(D) \coloneqq \bigcap_{k\geq 0}\Gamma(D^k)$.*

The next proposition states some of the properties of asymptotic cones.

Proposition 40 *Let D and T be two subsets of $\mathbb{R}^d$. Then,*

1. *$\mathcal{A}(D)$ is a closed cone with vertex zero;*
2. *If D is a closed and convex set containing the null vector, then $\mathcal{A}(D) \subseteq D$;*
3. *Let x be a vector in $\mathbb{R}^d$. Then $\mathcal{A}(D + \{x\}) = \mathcal{A}(D)$;*
4. *If $D \subseteq T$, then $\mathcal{A}(D) \subseteq \mathcal{A}(T)$;*
5. *Let $\{D_i\}_{i\in I}$ be an indexed family of subsets of $\mathbb{R}^d$. If $\bigcap_{i\in I} \mathcal{A}(D_i) = \{0\}$, then $\bigcap_{i\in I} D_i$ is bounded;*
6. *Let $\{D_i\}_{k=1}^n$ be a family of closed subsets of $\mathbb{R}^d$. If the asymptotic cones $\mathcal{A}(D_k)$, with $k = 1,\ldots,n$ are positively semi-independent, then the set $\sum_{k=1}^n D_k$ is closed in $\mathbb{R}^d$.*

6.2 Proofs of technical results.

Below we presents the proofs of the technical results.

Proof of Lemma 9. We start with the proof of point 1. By contradiction, suppose that there exists $((x'_i)_{i\in S}, y')$ such that $y' \in Y(S,\xi)$, $x'(S)-y' < \omega(S)$, $x'_i \succsim_i x_i$ for any $i \in S$, and $\omega(S^c) + y'' \in \Lambda(S,\xi)$ for some $y'' \in \sigma_{S,\xi}(y')$. Let $\upsilon \coloneqq -x'(S) + \omega(S) + y' > 0$. As in proof of Theorem 7, consider the vector x''_S defined by $x''_i \coloneqq x'_i + \frac{\upsilon}{|S|}$, for each $i \in S$. Notice that, $x''_S = \omega(S) + y'$ and $x''_i > x'_i$ for each $i \in S$. Then, by strict monotonicity over $\operatorname{Int}\mathbb{R}^l_+$, we have that $x''_i \succ_i x'_i$ for any $i \in S$ and, by transitivity, we obtain that $x''_i \succ_i x_i$, for all $i \in S$, which implies $\omega(S) \in \mathcal{R}_S(\xi)$ and a contradiction.

To prove point 2., notice that if $\gamma' > \gamma = x'(S) - y'$, then there exist $\upsilon > 0$ and vectors $x''_i \coloneqq x_i + \frac{\upsilon}{|S|}$, $i \in S$, such that $\gamma' = \gamma + \upsilon = x''(S) - y'$. Since x''_i, x'_i are strictly positive and $x''_i > x'_i$, by monotonicity $x''_i \succ_i x'_i$, and then by transitivity of preferences $x''_i \succ_i x_i$ for any $i \in S$, and $y' \in \Gamma_S(\xi)$. Therefore, $\gamma' \in \mathcal{R}_S(\xi)$.

Finally, in order to prove that the set $\mathcal{R}_S(\xi)$ is closed, we need first to show that the sets $\left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\}$ and $\Gamma_S(\xi)$ are closed. Then we verify that their asymptotic cones are positively semi-independent and use Point 6 of Proposition 40 in the Appendix.

Claim 1: The set $\left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\}$ is closed in $\mathbb{R}^l_+$. Indeed, take z in its closure. So, there exists a sequence $(z^\nu(S))_{\nu\in\mathbb{N}} \subseteq \left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\}$ such that $z^\nu(S)$ converges to z and $z^\nu_i \succsim x_i$ for any $i \in S$ and for any $\nu \in \mathbb{N}$. Let $\varepsilon > 0$. From the convergence of $z^\nu(S)$, there exists $n \in \mathbb{N}$ such that for any $\nu \geq n$ and for any $c = 1, \ldots, l$, $z^{c\nu}(S) \leq z^c + \varepsilon$. Consider the vector $b \in \mathbb{R}^l$ with $b^c \coloneqq \max\{z^\nu(S) \colon \nu = 1, \ldots, n-1\} \cup \{z^c + \varepsilon\}$. Since $0 \leq z^\nu_i \leq z^\nu(S)$ for any $\nu \in \mathbb{N}$, then $\{z^\nu_i \colon \nu \in \mathbb{N}\} \subseteq [0, b]$, which is a compact set. Thus, up to subsequence, z^ν_i converges to some $\overline{z}_i$, and by continuity of the preferences, $\overline{z}_i \succsim_i x_i$. Therefore, $z = \lim\limits_{\nu\to\infty} z^\nu(S) = \lim\limits_{\nu\to\infty} \sum_{i\in S} z^\nu_i = \sum_{i\in S} \lim\limits_{\nu\to\infty} z^\nu_i = \sum_{i\in S} \overline{z}_i$, which concludes the proof of the claim.

Claim 2: $\Gamma_S(\xi)$ is closed in $\mathbb{R}^l$. Indeed, take z in its closure. So, there exists a sequence $(z^\nu)_{\nu\in\mathbb{N}} \subseteq \Gamma_S(\xi)$ such that z^ν converges to z. Since $(z^\nu)_{\nu\in\mathbb{N}} \subseteq Y(S, \xi)$, and $Y(S, \xi)$ is a closed set by Point 1 of Assumption 3, then $z \in Y(S, \xi)$. By definition of $\Gamma_S(\xi)$, for any z^ν, there exists $\eta^\nu \in \sigma_{S,\xi}(z^\nu)$ such that $\omega(S^c) + \eta^\nu \in \Lambda(S, \xi)$. By Points 1 and 2 of Assumption 4, the sequence $(\eta^\nu)_{\nu\in\mathbb{N}}$ has a limit point η in $\sigma_{S,\xi}(z)$. Finally, by Point 1 of Assumption 5 the set $\Lambda(S, \xi)$ is closed, and so, $\omega(S^c) + \eta \in \Lambda(S, \xi)$. Thus $z \in \Gamma_S(\xi)$, which concludes the proof of the claim. As a consequence of claim 2, the set $-\Gamma_S(\xi)$ is closed in $\mathbb{R}^l$.

Claim 3: $\mathcal{A}\left(\left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\}\right)$ is a subset of $\mathbb{R}^l_+$. Indeed, since $\left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\} \subseteq \mathbb{R}^l_+$, then $\mathcal{A}\left(\left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\}\right) \subseteq \mathcal{A}(\mathbb{R}^l_+)$. This conclude the proof of the claim, since $\mathcal{A}(\mathbb{R}^l_+) = \mathbb{R}^l_+$.

Claim 4: $\mathcal{A}(-\Gamma_S(\xi))$ is a subset of $-Y(S, \xi)$, and $-Y(S, \xi)$ is closed in $\mathbb{R}^l$. Since $-\Gamma_S(\xi)$ is a subset of $-Y(S, \xi)$, then $\mathcal{A}(-\Gamma_S(\xi)) \subseteq \mathcal{A}(-Y(S, \xi))$. So, the result trivially follows by Points 1, 2 and 3 of Assumption 3.

Claim 5: $\mathcal{A}\left(\left\{x'(S) \in \mathbb{R}^l_+ \colon x'_i \succsim_i x_i, i \in S\right\}\right)$ and $\mathcal{A}(-\Gamma_S(\xi))$ are positively semi-independent. By Claims $1-4$, take $\alpha \in \mathbb{R}^l_+$ and $-\beta \in -Y(S, \xi)$ such that $\alpha + (-\beta) = 0$. So, $\alpha = \beta$ and consequently, $\beta \in Y(S, \xi) \cap \mathbb{R}^l_+$. By Points 3 and 4 of Assumption 3, $\beta = 0$ and so, $\alpha = -\beta = 0$. ■

Proof of Proposition 12. Suppose that $\mathcal{L}_{g,S}(\xi) > 0$ for some $g > 0$. So, there exists $0 < \lambda \leq \mathcal{L}_{g,S}(\xi)$ such that $0 < \lambda \cdot g \in \Psi_S(\xi)$. Therefore, $\lambda \cdot g \in (\{\omega(S)\} + Y(S, \xi)) \cap \mathbb{R}^l_+$ and $\omega(S) - \lambda \cdot g = x'(S) - y'$, for some $(x'_i)_{i\in S} \in \mathbb{R}^{l\cdot|S|}_+$ with $x'_i \succsim_i x_i$ for any $i \in S$, and $y' \in \Gamma_S(\xi)$. From $x_i \gg 0$ and $x'_i \succsim_i x_i$, it follows that $x'_i \gg 0$, for each $i \in S$. Then by strict monotonicity on $\operatorname{Int}\mathbb{R}^l_+$ and transitivity, there exists $(x''_i)_{i\in S}$ such that $\omega(S) = x''(S) - y'$, $x''_i \gg 0$

and $x''_i \succ x_i$ for any $i \in S$ [27]. Take any arbitrary reference bundle $g' > 0$ with $g' \neq g$. By the continuity of $\succsim_i$, we can choose a sufficiently small scalar $\lambda' > 0$ such that $\omega(S) - \lambda' \cdot g' = \eta(S) - y'$ and $\eta_i := x''_i - \lambda' \cdot \frac{g'}{|S|} \succ_i x_i$ for each $i \in S$ [28], i.e., $\omega(S) - \lambda' \cdot g' \in \mathcal{R}_S(\xi)$. Notice that $\lambda' \cdot g'$ belongs to $(\omega(S) + Y(S, \xi)) \cap \mathbb{R}^l_+$. Indeed, (1) $\lambda' \cdot g' > 0$ by $\lambda' > 0$ and $g' \in \mathbb{R}^l_+$ with $g' \neq 0$; (2) $\lambda' \cdot g' \in (\{\omega(S)\} + Y(S, \xi))$ since $\lambda' \cdot g' = \omega(S) + (\lambda' \cdot g' - \omega(S))$ and $\lambda' \cdot g' - \omega(S) = y' - \eta(S) \in \{y'\} - \mathbb{R}^l_+ \subseteq Y(S, \xi)$ by $y' \in Y(S, \xi)$ and Point 5 of Assumption 3. Finally, since $0 < \lambda' \cdot g' \in \Psi_S(\xi)$, then $\mathcal{L}_{g',S}(\xi) > 0$. ■

Proof of Lemma 20. To prove condition 1., assume, by contradiction, that there exists $(x'_i)_{i \in S}$ and a vector $y' \in Y(S, \xi)$ such that $x'(S) < \omega(S) + y'$, $(x'_S, x_{S^c}) \succsim_i (x_S, x_{S^c})$ for all $i \in S$ and $\omega(S^c) + y'' \in \Lambda(S, \xi)$, for some $y'' \in \sigma_{S,\xi}(y')$. As in the proof of Theorem 18, we see that $x'(S)$ is a strictly positive vector. Therefore, under (SGM), we find vectors $\zeta_i \geq 0$, $i \in S$ such that $\zeta(S) = \omega(S) + y'$ with $(\zeta_S, x_{S^c}) \succ_i (x'_S, x_{S^c})$ for any $i \in S$, and $\omega(S^c) + y'' \in \Lambda(S, \xi)$. Finally by transitivity of preferences, we get $(\zeta_S, x_{S^c}) \succ_i x$, and a contradiction to the fact that $\omega(S) = \zeta(S) - y' \notin \mathcal{R}^\gamma_S(\xi)$.

For the proof of condition 2., we notice that if $\gamma' > \gamma = x'(S) - y'$, then $\gamma' + y' > \gamma = x'(S)$ where $x'(S)$ is strictly positive. Then according to (SGM) there exist vectors $\zeta_i \geq 0$, $i \in S$, such that $\zeta(S) = \gamma' + y' > x'(S)$, with $(\zeta_S, x_{S^c}) \succ_i (x'_S, x_{S^c})$ for each $i \in S$, and $\omega(S^c) + y'' \in \Lambda(S, \xi)$ with $y'' \in \sigma_{S,\xi}(y')$. By transitivity of preferences we have $(\zeta_S, x_{S^c}) \succ_i x$ for each $i \in S$ and thus $\gamma' = \zeta(S) - y'$ belongs to $\in \mathcal{R}_S(\xi)$.

To show condition 3., we first prove that the sets $\Big\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{S^c}) \succsim_i x, i \in S\Big\}$ and $\Gamma_S(\xi)$ are closed, and then we show that their asymptotic cones are positively semi-independent. An appeal to Proposition 40 in the Appendix, will conclude the proof.

<u>Claim 1:</u> The set $\Big\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{S^c}) \succsim_i x, i \in S\Big\}$ is closed in $\mathbb{R}^l_+$. Indeed, take z in its closure. So, there exists a sequence $(z^\nu(S))_{\nu \in \mathbb{N}} \subseteq \Big\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{S^c}) \succsim_i x, i \in S\Big\}$ such that $z^\nu(S)$ converges to z and $(z^\nu_S, x_{S^c}) \succsim x_i$ for any $i \in S$ and for any $\nu \in \mathbb{N}$. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be such that for any $\nu \geq n$ and for any $c = 1, \dots, l$, $z^{c\nu}(S) \leq z^c + \varepsilon$. So, we can take a vector $b \in \mathbb{R}^l$ with $b^c := \max\{z^\nu(S) \colon \nu = 1, \dots, n-1\} \cup \{z^c + \varepsilon\}$. Since $0 \leq z^\nu_i \leq z^\nu(S)$ for any $\nu \in \mathbb{N}$, then $\{z^\nu_i \colon \nu \in \mathbb{N}\} \subseteq [0, b]$, which is a compact set. Thus, up to subsequence, z^ν_i converges to some $\overline{z}_i$, and by the continuity of preferences, $(\overline{z}_S, x_{S^c}) \succsim_i x$.

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[27] It is enough to take $x''_i := x'_i + \frac{\lambda \cdot g}{|S|}$ for any $i \in S$.

[28] Since the preference relations are continuous and $x''_i \gg 0$ for any $i \in S$, there exists $\varepsilon > 0$ such that for any z_i in the open ball $B_\varepsilon(x''_i) \subseteq \mathbb{R}^l_{++}$, one obtains $z_i \succ_i x_i$ for any $i \in S$. So, taking $0 < \lambda' < \frac{\varepsilon |S|}{2||g||}$ we get $\eta_i \in B_\varepsilon(x''_i)$ for any $i \in S$ and $\omega(S) - \lambda' \cdot g' = \eta(S) - y'$.

Therefore, $z = \lim_{\nu\to\infty} z^\nu(S) = \lim_{\nu\to\infty} \sum_{i\in S} z_i^\nu = \sum_{i\in S} \lim_{\nu\to\infty} z_i^\nu = \sum_{i\in S} \overline{z}_i = \overline{z}(S)$, which concludes the proof of the claim.

<u>Claim 2:</u> $\Gamma_S(\xi)$ is closed in $\mathbb{R}^l$. See Claim 2 in the proof of Lemma 9.

<u>Claim 3:</u> $\mathcal{A}\left(\left\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{S^c}) \succsim_i x, i \in S\right\}\right)$ is a subset of $\mathbb{R}^l_+$. Indeed, since $\left\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{S^c}) \succsim_i x, i \in S\right\}$ belongs to $\mathbb{R}^l_+$, we must have $\mathcal{A}\left(\left\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{S^c}) \succsim_i x, i \in S\right\}\right) \subseteq \mathcal{A}(\mathbb{R}^l_+)$. This conclude the proof of the claim, since $\mathcal{A}(\mathbb{R}^l_+) = \mathbb{R}^l_+$.

<u>Claim 4:</u> $\mathcal{A}\left(-\Gamma_S(\xi)\right)$ is a subset of $-Y(S,\xi)$, and $-Y(S,\xi)$ is closed in $\mathbb{R}^l$. See Claim 4 in the proof of Lemma 9.

<u>Claim 5:</u> $\mathcal{A}\left(\left\{x'(S) \in \mathbb{R}^l_+ \colon (x'_S, x_{S^c}) \succsim_i x, i \in S\right\}\right)$ and $\mathcal{A}\left(-\Gamma_S(\xi)\right)$ are positively semi-independent. By Claims 1 to 4, take $\alpha \in \mathbb{R}^l_+$ and $-\beta \in -Y(S,\xi)$ such that $\alpha + (-\beta) = 0$. So, $\alpha = \beta$ and consequently, $\beta \in Y(S,\xi) \cap \mathbb{R}^l_+$. By Points 3 and 4 of Assumption 3, $\beta = 0$ and so, $\alpha = -\beta = 0$. ∎

Proof of Proposition 23. Suppose that $\mathcal{L}^\gamma_{g,S}(\xi) > 0$ for some $g > 0$. So, there exists $0 < \lambda \le \mathcal{L}^\gamma_{g,S}(\xi)$ such that $0 < \lambda \cdot g \in \Psi_S(\xi)$. Therefore, $\lambda \cdot g \in (\omega(S) + Y(S,\xi)) \cap \mathbb{R}^l_+$ and $\omega(S) - \lambda \cdot g = x'(S) - y'$, for some $x' \in \mathbb{R}^{l\cdot n}_+$ with $(x'_S, x_{S^c}) \succsim_i x$ for any $i \in S$, and $y' \in \Gamma_S(\xi)$. Since $\omega(S) + y' = x'(S) + \lambda' \cdot g > x'(S)$, by (SGM) and transitivity, there exists x'' such that $\omega(S) = x''(S) - y'$, with $x''_i \ge 0$ and $(x''_S, x_{S^c}) \succ_i x$ for any $i \in S$.

Take any arbitrary reference bundle $g' > 0$ with $g' \ne g$. Consider the two sets $S^\circ_{x''} = \{i \in S \colon x''_i \in \partial\mathbb{R}^l_+\}$ and $S^*_{x''} = \{i \in S \colon x''_i \in \operatorname{Int}\mathbb{R}^l_+\}$. By (SBA), transitivity and $(x''_S, x_{S^c}) \succ_i x$ for any $i \in S$, the case in which $S^*_{x''} = \emptyset$ does not occur. So, we may have the following two cases: (1) $S^\circ_{x''} = \emptyset$; (2) $S^\circ_{x''} \ne \emptyset$ and $S^*_{x''} \ne \emptyset$.

If $S^\circ_{x''} = \emptyset$, that is, $x''_i \gg 0$ for each agent $i \in S$, then by the continuity, we can choose a scalar $\lambda' > 0$ that is small enough for $\omega(S) - \lambda' \cdot g' = \eta(S) - y'$ and $\eta_i \coloneqq x''_i - \lambda' \cdot \frac{g'}{|S|}$ with $(\eta_S, x_{S^c}) \succ_i x_i$ for each $i \in S$ [29], i.e., $\omega(S) - \lambda' \cdot g' \in \mathcal{R}^\gamma_S(\xi)$. Notice that $\lambda' \cdot g'$ belongs to $(\{\omega(S)\} + Y(S,\xi)) \cap \mathbb{R}^l_+$. Indeed, (1) $\lambda' \cdot g' > 0$ by $\lambda' > 0$ and $g' \in \mathbb{R}^l_+$ with $g' \ne 0$; (2) $\lambda' \cdot g' \in (\omega(S) + Y(S,\xi))$ since $\lambda' \cdot g' = \omega(S) + (\lambda' \cdot g' - \omega(S))$ and $\lambda' \cdot g' - \omega(S) = y' - \eta(S) \in \{y'\} - \mathbb{R}^l_+ \subseteq Y(S,\xi)$ by $y' \in Y(S,\xi)$ and Point 5 of Assumption 3. Finally, since $\lambda' \cdot g' \in \Psi^\gamma_S(\xi)$,

[29] Since the preference relations are continuous and $x''_i \gg 0$ for any $i \in S$, there exists $\varepsilon > 0$ such that for any z which belongs to the open ball $B_\varepsilon(x''_S, x_{S^c}) \subseteq \operatorname{Int}\mathbb{R}^{l\cdot n}_+$, then $z \succ_i x$ for any $i \in S$. In particular for any $(\eta_S, x_{S^c}) \in B_\varepsilon(x''_S, x_{S^c}) \cap (\operatorname{Int}\mathbb{R}^{l\cdot|S|}_+ \times \{x_{S^c}\})$ we obtain $(\eta_S, x_{S^c}) \succ_i x$, for any $i \in S$. So, taking $0 < \lambda' < \frac{\varepsilon|S|}{2||g'||}$ we get $(\eta_S, x_{S^c}) \in B_\varepsilon(x''_S, x_{S^c})$ for any $i \in S$.

then $\mathcal{L}^{\gamma}_{g',S}(\xi) > 0$.

If $S^{\circ}_{x''} \neq \emptyset$ and $S^{*}_{x''} \neq \emptyset$, since $(x''_S, x_{S^c}) \succ_i x$ for any $i \in S$, by continuity of $\succsim$, there exists $\delta > 0$ such that if $z \in B_{\delta}(x''_S, x_{S^c}) \cap \mathbb{R}^l_+$ then $z \succ_i x$, for any $i \in S$. Let $\zeta = (\zeta_S, x_{S^c})$, with $\zeta_S := (x''_{S^{\circ}}, (1-\varepsilon)x''_{S^*})$, and notice that $\|(\zeta_S, x_{S^c}) - (x''_S, x_{S^c})\| = \|(0, \varepsilon x''_{S^*}, 0)\| \leq \varepsilon \sum_{i \in S^*} \|x''_i\| < \delta$ if $\varepsilon < \frac{\delta}{\sum_{i \in S^*} \|x''_i\|}$. Thus, $(\zeta_S, x_{S^c}) \geq 0$ and $(\zeta_S, x_{S^c}) \succ_i x$ for any $i \in S$. By (SBA), one may consider the vector $\widetilde{\zeta} = (\widetilde{\zeta}_S, x_{S^c})$, with $\widetilde{\zeta}_S := (\widetilde{x}_{S^{\circ}_{x''}}, (1-\varepsilon)x''_{S^*_{x''}}) \gg 0$, with $\widetilde{x}_i := x''_i + \frac{\varepsilon x''(S^*_{x''})}{|S^{\circ}_{x''}|} \gg 0$ for any $i \in S^{\circ}_{x''}$, in order to have $(\widetilde{\zeta}_S, x_{S^c}) \succsim (\zeta_S, x_{S^c}) \succ_i x$ for any $i \in S$. Notice that $\widetilde{\zeta}(S) - y' = \widetilde{x}(S^{\circ}_{x''}) + (1-\varepsilon)x''(S^*_{x''}) - y' = x''(S^{\circ}_{x''}) + \varepsilon x''(S^*_{x''}) + (1-\varepsilon)x''(S^*_{x''}) - y' = x''(S) - y' = \omega(S)$. Therefore, we come back to the previous case, since $S^{\circ}_{\widetilde{\zeta}} = \emptyset$. ■

Proof of the claim: $\mathrm{K}_{S,\xi}(x'_S, y')$ **is compact and** $\mathrm{K}_{S,\xi}(x'_S, y') \cap \mathrm{NBT}_S(x) = \emptyset$**.** The set $\mathrm{K}_{S,\xi}(x'_S, y')$ is bounded since x'_S is fixed, $z_{S^c} \geq 0$ and by Point 1 of Assumption 4, there exists $b \in \mathbb{R}^l_+$ such that $\|y''\| \leq b$ for any $y'' \in \sigma_{S,\xi}(y')$. Therefore, $0 \leq z_i \leq z(S^c) \leq \omega(S^c) + y' \leq \omega(S^c) + b$ for any $i \in S^c$. The set $\mathrm{K}_{S,\xi}(x'_S, y')$ is closed. Indeed, take $\overline{\eta} = (x'_S, \overline{z}_{S^c}) \in \mathrm{cl}_{\mathbb{R}^{l \cdot n}_+} \mathrm{K}_{S,\xi}(x'_S, y')$, so there exists a sequence $(x'_S, z^{\nu}_{S^c})_{\nu \in \mathbb{N}} \subseteq \mathrm{K}_{S,\xi}(x'_S, y')$ converging to $\overline{\eta}$. For any $\nu \in \mathbb{N}$, by $(x'_S, z^{\nu}_{S^c}) \in \mathrm{K}_{S,\xi}(x'_S, y')$, there exists $y''_{\nu} \in \sigma_{S,\xi}(y')$ such that $z^{\nu}(S^c) \leq \omega(S^c) + y''_{\nu}$ and $\omega(S^c) + y''_{\nu} \in \Lambda(S, \xi)$. By Point 1 of Assumption 4, up to a subsequence, $(y''_{\nu})_{\nu \in \mathbb{N}}$ converges to some $y^{\overline{z}} \in \sigma_{S,\xi}(y')$. Therefore, taking the limit and using Point 1 of Assumption 5, one gets $\overline{z}(S^c) \leq \omega(S^c) + y''$ and $\omega(S^c) + y'' \in \Lambda(S, \xi)$. This completes the proof since it shows that $\overline{\eta} \in \mathrm{K}_{S,\xi}(x'_S, y')$.

To show that $\mathrm{K}_{S,\xi}(x'_S, y') \cap \mathrm{NBT}_S(x) = \emptyset$, notice that if $\mathrm{K}_{S,\xi}(x'_S, y') \cap \mathrm{NBT}_S(x) \neq \emptyset$, then there exist $z'_{S^c} \in \mathbb{R}^{l \cdot |S^c|}_+$ and an element $y'' \in \sigma_{S,\xi}(y')$ with $z'(S^c) \leq \omega(S^c) \leq y^{z'}$ and $\omega(S^c) + y'' \in \Lambda(S, \xi)$ such that the vector (x'_S, z'_{S^c}) belongs to $\mathrm{NBT}_S(x)$. This contradicts the fact that (x'_S, y') α-dominates ξ in preferences.

■

Proof of Lemma 28. To show condition 1., by contradiction, suppose that there exists $(x'_i)_{i \in S}$ and a vector $y' \in Y(S, \xi)$ such that: $x'(S) \leq \omega(S) + y'$, $\omega(S^c) + y'' \in \Lambda(S, \xi)$, for at least one $y'' \in \sigma_{S,\xi}(y')$ and $(x'_S, z_{S^c}) \succ_i (x_S, x_{S^c})$, for every $i \in S$ and for every $z \in \Phi_{S,\xi}(y')$, where $\Phi_{S,\xi}(y')$ is defined according to Definition 26. Consider one of the vectors $z_{S^c} \in \Phi_{S,\xi}(y')$. Since $x^{'}(S)$ is strictly positive, by (SGM), one might finds vectors $\zeta_i \geq 0$, $i \in S$, such that $\zeta(S) = \omega(S) + y' > x'(S)$ with $(\zeta_S, z_{S^c}) \succ_i (x'_S, z_{S^c})$ for any $i \in S$. Then, by (SGS), for any agent $i \in S$, $(\zeta_S, z'_{S^c}) \succ_i (x'_S, z'_{S^c})$ holds true for any vector $z'_{S^c} \in \Phi_{S,\xi}(y')$. Finally using the transitivity of preferences, one easily gets a contradiction with the the fact that $\omega(S) = \zeta(S) - y' \notin \mathcal{R}^{\alpha}_S(\xi)$.

To prove condition 2., let us suppose that $\gamma' > \gamma$ and $\gamma \in \mathcal{R}^{\alpha}_S(\xi)$. Then there exists $y' \in Y(S, \xi)$ such that $\gamma' > \gamma = x'(S) - y'$ and $(x'_S, z_{S^c}) \succsim_i x$, for each

$i \in S$ and for each $z_{S^c} \in \Phi_S(\xi, y')$. Let us fix a vector $z_{S^c} \in \Phi_S(\xi, y')$. Then according to (SGM) there exist vectors $\zeta_i \geq 0$, $i \in S$, such that $\zeta(S) = \gamma' + y' > x'(S)$ and invoking (SGS) and transitivity, one easily shows that $\gamma' = \zeta(S) - y'$ belongs to $\mathcal{R}^\alpha_S(\xi, y')$ and then to $\mathcal{R}^\alpha_S(\xi)$. ∎

Proof of Lemma 29. Let $\omega(S) \in \mathcal{R}^\alpha_S(\xi)$. So, we have that $\omega(S) = x'(S) - y'$, where $y' \in Y(S, \xi)$, and for any agent $i \in S, (x'_S, z_{S^c}) \succsim_i x$, for any $z_{S^c} \in \Phi_S(\xi, y')$. Therefore, $\omega(S) - 0 = \omega(S) = x'(S) - y' \in \mathcal{R}^\alpha_S(\xi)$ and $0 \in (\{\omega(S)\} + Y(S, \xi)) \cap \mathbb{R}^l_+$ since $0 = \omega(S) - \omega(S)$ and $-\omega(S) \in Y(S, \xi)$ (by Points 3 and 5 of Assumption 3). Thus, $0 \in \Psi^\alpha_S(\xi)$ and consequently $\Psi^\alpha_S(\xi) \neq \emptyset$. Vice-versa, suppose that $\Psi^\alpha_S(\xi) \neq \emptyset$. So, there exists γ such that $\gamma \geq 0$ and $\omega(S) - \gamma = x'(S) - y' \in \mathcal{R}^\alpha_S(\xi)$. If $\gamma = 0$ the proof is complete. If $\gamma > 0$, then by point 2. of Lemma 28, $\omega(S) > \omega(S) - \gamma$ and $\omega(S) - \gamma \in \mathcal{R}^\alpha_S(\xi)$ imply $\omega(S) \in \mathcal{R}^\alpha_S(\xi)$ which completes the proof. ∎

Proof of Lemma 30. In order prove the lemma, it is enough to show the boundedness of $(\omega(S) + Y(S, \xi)) \cap \mathbb{R}^l_+$. Since a translation of a set does not affect its asymptotic cone, then $\mathcal{A}(\omega(S) + Y(S, \xi)) = \mathcal{A}(Y(S, \xi))$. Furthermore, by Points 1, 2 and 3 of Assumption 3, $\mathcal{A}(Y(S, \xi)) \subseteq Y(S, \xi)$, and in particular, by definition of the asymptotic cone, $0 \in \mathcal{A}(Y(S, \xi))$. Since $\mathcal{A}(\mathbb{R}^l_+) = \mathbb{R}^l_+$, then by Points 3 and 4 of Assumption 3 we get $\mathcal{A}(\omega(S) + Y(S, \xi)) \cap \mathcal{A}(\mathbb{R}^l_+) = \{0\}$, which concludes the proof by Point 5. of Proposition 40. ∎

Proof of Proposition 31. Suppose that $\mathcal{L}^{g,\alpha}_S(\xi) > 0$ for some $g > 0$. So, there exists $0 < \lambda \leq \mathcal{L}^{g,\alpha}_S(\xi)$ such that $0 < \lambda \cdot g \in \Psi^\alpha_S(\xi)$. Therefore, $\lambda \cdot g \in (\{\omega(S)\} + Y(S, \xi)) \cap \mathbb{R}^l_+$ and $\omega(S) - \lambda \cdot g = x'(S) - y'$, for some $y' \in Y(S, \xi)$ and $(x'_i)_{i \in S} \in \mathbb{R}^{l \cdot |S|}_+$ such that, for any $i \in S$, $(x'_S, z_{S^c}) \succsim_i (x_S, x_{S^c})$ for any $z_{S^c} \in \Phi_S(\xi, y')$. Since $x^{'}(S) \gg 0$, and $\omega(S) + y' > x'(S)$, for a fixed $z_{S^c} \in \Phi_S(\xi, y')$, by (SGM), there exists x'' such that $\omega(S) = x''(S) - y'$, with $x''_i \geq 0$ and $(x''_S, z_{S^c}) \succ_i (x'_S, z_{S^c})$ for any $i \in S$. By (SGS) and transitivity, for any $i \in S$ $(x''_S, z_{S^c}) \succ_i x$ for any $z_{S^c} \in \Phi_S(\xi, y')$. In particular, for any $i \in S$ $(x''_S, 0_{S^c}) \succ_i x$, which implies that the case in which each agent receives a boundary consumption bundle x''_j does not occur. Take any arbitrary reference bundle $g' > 0$ with $g' \neq g$. Consider the two sets $S^\circ_{x''} = \{i \in S \colon x''_i \in \partial \mathbb{R}^l_+\}$ and $S^*_{x''} = \{i \in S \colon x''_i \in \operatorname{Int} \mathbb{R}^l_+\}$. We may have the following two cases: (1) $S^\circ_{x''} = \emptyset$; (2) $S^\circ_{x''} \neq \emptyset$ and $S^*_{x''} \neq \emptyset$.

If $S^\circ_{x''} = \emptyset$, that is, $x''_i \gg 0$ for each agent $i \in S$, then by the continuity of $\succsim_i$, we can choose a scalar $\lambda' > 0$ that is small enough for $\omega(S) - \lambda' \cdot g' = \eta(S) - y'$ and $\eta_i \coloneqq x''_i - \lambda' \cdot \frac{g'}{|S|}$ with $(\eta_S, z_{S^c}) \succ_i x$, for any $z_{S^c} \in \Phi_S(\xi, y')$ and for each $i \in S$[30], i.e., $\omega(S) - \lambda' \cdot g' \in \mathcal{R}^\alpha_S(\xi)$. Notice that $\lambda' \cdot g'$ belongs to

[30] Since the preference relations are continuous and $x''_i \gg 0$ for any $i \in S$, using the sets $\mathrm{K}_{S,\xi}(x''_S, y')$ and $\mathrm{NBT}_S(x)$ and the same argument of Theorem 27 we can find $\varepsilon > 0$ such that for any $z_{S^c} \in \Phi_S(\xi, y')$ and for any ζ which belongs to the

$(\{\omega(S)\} + Y(S,\xi)) \cap \mathbb{R}^l_+$. Indeed, (1) $\lambda' \cdot g' > 0$ by $\lambda' > 0$ and $g' \in \mathbb{R}^l_+$ with $g' \neq 0$; (2) $\lambda' \cdot g' \in (\omega(S) + Y(S,\xi))$ since $\lambda' \cdot g' = \omega(S) + (\lambda' \cdot g' - \omega(S))$ and $\lambda' \cdot g' - \omega(S) = y' - \eta(S) \in \{y'\} - \mathbb{R}^l_+ \subseteq Y(S,\xi)$ by $y' \in Y(S,\xi)$ and Point 5 of Assumption 3. Finally, since $\lambda' \cdot g' \in \Psi^{\alpha}_S(\xi)$, then $\widetilde{\mathcal{L}}^{g',\alpha}_S(\xi) > 0$.

Suppose that $S^{\circ}_{x''} \neq \emptyset$, and $S^{*}_{x''} \neq \emptyset$. As in the previous case, by continuity of $\succsim$, there exists $\delta > 0$ such that, for any $z_{S^c} \in \Phi_S(\xi, y')$, if $(z_S, z_{S^c}) \in B_\delta(x''_S, z_{S^c}) \cap \mathbb{R}^l_+$ then $(z_S, z_{S^c}) \succ_i x$, for any $i \in S$. For any $z_{S^c} \in \Phi_S(\xi, y')$, let $\zeta^z = (\zeta_S, z_{S^c})$, with $\zeta_S := (x''_{S^\circ}, (1-\varepsilon)x''_{S^*})$, and notice that $\|(\zeta_S, z_{S^c}) - (x''_S, z_{S^c})\| = \|(0, \varepsilon x''_{S^*}, 0)\| \leq \varepsilon \sum_{i \in S^*} \|x''_i\| < \delta$ if $\varepsilon < \frac{\delta}{\sum_{i \in S^*} \|x''_i\|}$. Thus, $(\zeta_S, z_{S^c}) \geq 0$ and $(\zeta_S, z_{S^c}) \succ_i x$ for any $z_{S^c} \in \Phi_S(\xi, y')$ and for any $i \in S$. By (SBA), for any $z_{S^c} \in \Phi_S(\xi, y')$, one may consider the vector $\widetilde{\zeta}^z = (\widetilde{\zeta}_S, z_{S^c})$, with $\widetilde{\zeta}_S := (\widetilde{x}_{S^\circ_{x''}}, (1-\varepsilon)x''_{S^*_{x''}}) \gg 0$, with $\widetilde{x}_i := x''_i + \frac{\varepsilon x''(S^*_{x''})}{|S^\circ_{x''}|} \gg 0$ for any $i \in S^\circ_{x''}$, in order to have $(\widetilde{\zeta}_S, z_{S^c}) \succsim (\zeta_S, z_{S^c}) \succ_i x$ for any $z_{S^c} \in \Phi_S(\xi, y')$ and for any $i \in S$. Notice that $\widetilde{\xi}(S) - y' = \widetilde{x}(S^\circ_{x''}) + (1-\varepsilon)x''(S^*_{x''}) - y' = x''(S^\circ_{x''}) + \varepsilon x''(S^*_{x''}) + (1-\varepsilon)x''(S^*_{x''}) - y' = x''(S) - y' = \omega(S)$. Therefore, we come back to the previous case since $S^\circ_{\widetilde{\zeta}} = \emptyset$. ■

We conclude the Appendix with a Table presenting the models covered in the paper.

open ball $B_\varepsilon(x''_S, z_{S^c}) \subseteq \operatorname{Int} \mathbb{R}^{l \cdot n}_+$, then $\zeta \succ_i x$ for any $i \in S$. In particular for any $(\eta_S, z_{S^c}) \in B_\varepsilon(x''_S, z_{S^c}) \cap (\mathbb{R}^{l \cdot |S|}_{++} \times \{z_{S^c}\})$ we obtain $(\eta_S, z_{S^c}) \succ_i x$, for any $i \in S$. So, taking $0 < \lambda' < \frac{\varepsilon |S|}{2\|g'\|}$ we get $(\eta_S, z_{S^c}) \in B_\varepsilon(x''_S, z_{S^c})$ for any $i \in S$.

Table 1

$\succsim$	$Y(S,\xi)$	$\sigma_{S,\xi}(y')$	$\Lambda(S,\xi)$	Economic Model
Selfish	Y	$\{0\}$	$\mathbb{R}^l_+$	Production economy in Debreu and Scarf (1963)
Selfish	$\sum_{j\in J}\sum_{i\in S}\theta_{ij}Y_j$	$\left\{\sum_{j\in J}\sum_{i\notin S}\theta_{ij}y'_j \,\vert\, y'_j\in Y_j \; y'=\sum_{j\in J}\sum_{i\in S}\theta_{ij}y'_j\right\}$	$\mathbb{R}^l_+$	Considerate dominance in Private ownership economy
Selfish	$\sum_{j\in\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}Y_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y_j$	$\left\{\sum_{j\in\widetilde{J}(S)}\sum_{i\notin S}\theta_{ij}y'_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\notin S}\theta_{ij}y_j \,\vert\, y'_j\in Y_j,\; y'=\sum_{j\in\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y'_j+\sum_{j\notin\widetilde{J}(S)}\sum_{i\in S}\theta_{ij}y_j\right\}$	$\mathbb{R}^l_+$	Considerate Dominance with Corporate Shares
Other-regarding	$-\mathbb{R}^l_+$	$\{0\}$	$\mathbb{R}^l_+$	Exchange economy with consumption externalities
Other-regarding	General	General	General	This paper

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