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WORKING PAPER NO. 653

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Giovanni Angelini, Luca Fanelli, and Marco M. Sorge

September 2022

University of Naples Federico II

University of Salerno

Bocconi University, Milan

CSEF - Centre for Studies in Economics and Finance
DEPARTMENT OF ECONOMICS AND STATISTICS – UNIVERSITY OF NAPLES FEDERICO II
80126 NAPLES - ITALY
Tel. and fax +39 081 675372 – e-mail: csef@unina.it
ISSN: 2240-9696

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Giovanni Angelini[*], Luca Fanelli[†], and Marco M. Sorge[‡]

Abstract

Recently developed models of the business cycle exhibit a recursive timing structure, which enforces delayed propagation of exogenous shocks driving short-run dynamics. We propose a simple empirical strategy to test for the relevance of timing restrictions and ensuing shock transmission delays in general DSGE environments. Based on a bootstrap maximum likelihood estimator, our approach mitigates over-rejection concerns typically arising from conventional tests of non-linear hypotheses that exploit first-order asymptotic approximations. We showcase the empirical usefulness of the testing procedure by means of numerical simulations of a workhorse model of the monetary transmission mechanism.

JEL classification: C1, C3, E3, E5.

Keywords: DSGE models; Timing restrictions; Transmission delays.

Acknowledgements: We wish to thank Efrem Castelnuovo and Chetan Dave for insightful discussions on the main topic of the present work. Any errors are our own.

* Università di Bologna. E-mail: g.angelini@unibo.it

† Università di Bologna. E-mail: luca.fanelli@unibo.it

‡ Università di Salerno, University of Göttingen, and CSEF-DISES. E-mail: msorge@unisa.it *(Corresponding author)*

1 Introduction

Recently developed models of the business cycle feature a recursive timing structure, according to which decision rules of forward-looking (rational) economic agents reflect the presence of delayed shock observability and/or partial information. Two examples from the macroeconomic domain stand out: first, general equilibrium frameworks with a role for fiscal policy often posit that government spending is predetermined with respect to the current state of the economy, implying that the policy instrument cannot react to sources of uncertainty other than fiscal shocks – see Schmitt-Grohé and Uribe (2012) and Kormilitsina and Zubairy (2018) among many others; second, in dynamic settings where monetary authorities exhibit a concern for price stability, policy surprises are oftentimes assumed not to trigger contemporaneous changes in non-policy variables (such as consumption, wages and prices), while allowed to slowly propagate through the underlying economy – see e.g. Rotemberg and Woodford (1997), Christiano et al. (2005), Boivin and Giannoni (2006), Altig et al. (2011).[1]

A clear-cut implication of the recursive timing protocol is the emergence of transmission delays for a subset of the exogenous forces (i.e. the structural shocks) driving short-run dynamics. This structural timing assumption in modern macroeconomic writing has mirrored the diffuse adoption of econometric frameworks for the analysis of dynamic impulse responses that rely on short-run exclusion restrictions for identification purposes, e.g. Sims (1980), Christiano et al. (1999). From a structural point of view, a direct empirical test of the relevance of timing restrictions and the ensuing shock transmission delays in a general dynamic stochastic general equilibrium (DSGE) environments has not been advanced thus far. As a result, whether macroeconomic data favor the recursive timing assumption, as opposed to the conventional (unrestricted) one, still remains an open question.

We fill this gap by developing a (time-domain) bootstrap-based procedure for likelihood ratio (LR) testing of delayed shock transmission in DSGE economies. Specifically, building on recent computational techniques in the realm of DSGE

[1] Further instances of recursive timing assumptions in the DSGE literature include models of factor hoarding, where employment decisions predate the full realization of aggregate uncertainty (e.g. Burnside and Eichenbaum, 1994), and limited participation settings in which households might engage in financial decision-making prior to observing the whole set of current period shocks (e.g. Fuerst, 1992).

models featuring timing restrictions – e.g. Kormilitsina (2013) and Angelini and Sorge (2021), we submit to formal testing the null hypothesis that a subset of endogenous model variables of interests (e.g. the inflation rate and the output gap) simultaneously and/or fully adjust to changes in the current fundamentals of the economy (e.g. the monetary policy innovation), thus providing evidence against the alternative of staggered impulse propagation. Since the solution to *any* restricted DSGE model can be constructed via a uniquely defined linear transformation of the solution to its unrestricted counterpart, however computed, nesting requirements are generically fulfilled. Hence, upon estimating the model-implied set of endogenous responses across timing structures (restricted versus unrestricted) along with other structural model's parameters, information stemming from likelihood-based tests for the rational expectations cross-equation restrictions (CERs) placed on equilibrium reduced forms can be exploited to evaluate the empirical plausibility of the recursive timing assumption in the DSGE context.

From an operational perspective, we build on recent contributions by Stoffer and Wall (1991), Bårdsen and Fanelli (2015) and Angelini et al. (2022) on hypothesis testing and estimation in state-space models. Stoffer and Wall (1991) propose a nonparametric Monte Carlo bootstrap that abstracts from distributional assumptions that are hardly valid in small to moderate samples. Bårdsen and Fanelli (2015) develop a frequentist approach to testing sequentially cointegration/common-trend restrictions along with conventional rational expectations CERs in DSGE models, arguing that classical likelihood-based tests are able to handle both long- and short-run restrictions placed by the model on time series data representations. Angelini et al. (2022) formally show that, in the case of 'strong identification', meaning that all the regularity conditions for standard asymptotic inference are at work, the bootstrap maximum likelihood (ML) estimator of the structural parameters replicates the asymptotic distribution of the ML estimator, and prove formally that the restricted bootstrap (i.e. with the null hypothesis under investigation being imposed in estimation) is consistent. In this scenario, the asymptotic distribution of the ML estimator of the structural parameters can be estimated accurately by the bootstrap. Importantly, not only the (either standard or bootstrap) LR test is asymptotically pivotal and chi-square distributed, but the bootstrap tends to reduce the discrepancy between actual and nominal probabilities of type-I error. It turns out that the bootstrap in DSGE models (and, more generally, in frameworks that admit a conventional state space representation) has the potential to mitigate the over-rejection phenomenon that characterizes tests of non-linear hypothesis that rely on first-order asymptotic approximations. Remarkably, our resampling method still improves upon the asymptotic LR test, for the empirical size of the bootstrap-based LR test tends to approach the chosen nominal level.

To showcase the empirical validity of our test, we employ the hybrid New Keynesian model introduced in Benati and Surico (2009), where imposing the recursiveness assumption amounts to embodying timing restrictions on the transmission of the monetary policy innovation, which thereby imparts no impact changes on slow-moving variables (inflation and output gap). Even under these circumstances, unexpected increases in the policy rate entail recessionary and deflationary dynamic effects that prove qualitatively similar across the two information structures (restricted versus unrestricted), and eventually follow the same exact pattern when the effect of the monetary policy shock fades out. Our simulation results indicate that the bootstrap-based approach manages to counterbalance the tendency of the standard LR test to over-reject the hypothesis of structural timing restrictions in small samples, with rejection frequencies close to the 5% nominal level.

The remainder of the paper is organized as follows. Section (2) presents the general state space representation of first-order approximate solution to general DSGE models featuring timing restrictions, with a specific focus on how this representation maps into the one complying with the conventional timing protocol. Section (3) introduces the testing problem and discusses the bootstrap algorithm used to test for the relevance of shock transmission delays in DSGE environments. Section (4) reports the outcome of our Monte Carlo experiment, whose goal is to show how the bootstrap-based LR test performs in the empirical evaluation of the hypothesis under scrutiny. Section (5) concludes.

2 Transmission delays in DSGE models

Equilibrium conditions of DSGE models are generally described by a system of n_F expectational stochastic difference equations of the form

$$E_t[f\left(y_{t+1}, y_t, x_{t+1}, x_t; \sigma, \theta\right)] = 0 \tag{1}$$

where the random processes (y_t) and (x_t) are defined on the same probability space, and E_t is the conditional (rational) expectation operator associated with the underlying probability measure. The n_y-dimensional vector y collects the model's endogenous jump variables, whereas the n_x-dimensional vector x contains n_x^1 endogenous predetermined variables as well as n_x^2 exogenous states ($n_x^1 + n_x^2 = n_x$). Finally, θ denotes the vector of structural parameters and $\sigma \geq 0$ is a parameter capturing the size of aggregate uncertainty surrounding the economy, see Schmitt-Grohé and Uribe (2004).[2]

2.1 Unrestricted timing

To ease notation, let the prime superscript denotes one-step ahead variables. Under the conventional timing assumption, policy functions for all the endogenous variables depend on all the state variables x. Analytic solutions to (1) are in the form

$$y = g(x, \sigma), \quad x' = h(x, \sigma) + \sigma\epsilon' \tag{2}$$

where the elements of the n_x-dimensional vector ϵ are i.i.d. zero-mean, unit variance innovations (e.g. structural shocks).

As shown in Schmitt-Grohé and Uribe (2004), up to first order certainty equivalence holds generically, i.e. one has

$$y = g_x x, \quad x' = h_x x + \sigma\epsilon' \tag{3}$$

where g_x and h_x are conformable matrices of first-order derivatives of the maps $g(x, \sigma)$ and $h(x, \sigma)$ with respect to x, evaluated at the non-stochastic steady state $(\bar{y}, \bar{x})$ that solves (1) when $\sigma = 0$.

2.2 Timing restrictions

In the presence of information-based timing restrictions, the general form of the multivariate RE model is

$$\mathcal{E}_t \left[f \left(y_{t+1}, y_t, x_{t+1}, x_t; \sigma, \theta \right) \right] = 0 \tag{4}$$

[2]This section draws on Sorge (2020) and Angelini and Sorge (2021), to which the reader is referred for further details.

where $\mathcal{E}_t$ denotes the collection of (conditional) expectation operators accounting for partial shock observability, which in turn enforces delays in the propagation of exogenous impulses. Following Angelini and Sorge (2021), we can expand (4) as follows

$$\mathcal{E}_t\left[f(y_{t+1}, y_t, x_{t+1}, x_t; \sigma, \theta)\right] = \begin{pmatrix} \mathcal{E}\left[f_1^{(y,x)}(y_{t+1}, y_t, x_{t+1}, x_t; \sigma, \theta) \,\Big|\, \mathcal{I}_{1,t}\right] \\ \mathcal{E}\left[f_2^{(y,x)}(y_{t+1}, y_t, x_{t+1}, x_t; \sigma, \theta) \,\Big|\, \mathcal{I}_{2,t}\right] \\ \vdots \\ \mathcal{E}\left[f_{n_y+n_x^1}^{(y,x)}(y_{t+1}, y_t, x_{t+1}, x_t; \sigma, \theta) \,\Big|\, \mathcal{I}_{n_y+n_x^1,t}\right] \\ f_1^{(x)}(x_{t+1}^2, x_t^2; \sigma, \theta) \\ f_2^{(x)}(x_{t+1}^2, x_t^2; \sigma, \theta) \\ \vdots \\ f_{n_x^2}^{(x)}\left(x_{t+1}^2, x_t^2; \sigma, \theta\right) \end{pmatrix}$$

where $f_k^{(y,x)}$ $(k \leq n_y + n_x^1)$ is the model's equation used to pin down the k-th endogenous variable (y, x^1), conditional on the equilibrium values for the other endogenous variables and the relevant states, for which t-dated optimal projections are framed on the basis of the restricted information set $\mathcal{I}_{k,t}$, $k \leq n_y$; and $f_j^{(x)}$ $(j \leq n_x^2)$, is the possibly nonlinear equation that governs the dynamics of j-th exogenous state variable x_j.[3]

Let us next consider the case where any given time period is split into two fictitious sub-periods, according to the timing of the model's variables. Formally, the control and state vectors are accordingly partitioned as

$$y = [y_u;\, y_r], \quad x = [x_u;\, x_r] \tag{5}$$

where the n_{x_u}-dimensional vector x_u consists of endogenous predetermined as well as exogenous variables realizing in the beginning of the first subperiod, x_r contains n_{x_r} exogenous variables materializing in the second subperiod, y_u is the n_{y_u}-dimensional

[3]The conditioning set $\mathcal{I}_{k,t}$, $k \leq n_y$ is the smallest closed linear subspace spanned by the semi-infinite history of all the observed variables k up to time t.

vector of endogenous variables which respond to the whole set of current time state variables x. Finally, the n_{y_r}-dimensional vector y_r collects endogenous variables selected in the first subperiod, when realizations of only a subset of state variables are observed. In order to apply Kormilitsina (2013)'s perturbation approach, as generalized in Sorge (2020), the RE system (1) is partitioned as follows

$$f = \left[f^0;\ f^1;\ f^{x_r}\right] \tag{6}$$

so that the sub-system f^0 includes n_{y_r} equations pinning down endogenous variables y_r, the sub-system f^1 includes n_{y_u} equations that determine endogenous variables y_u and n_{x_u} equations delivering the dynamics of the states x_u, and the sub-system f^{x_r} describes the evolution of exogenous shocks x_r, represented as a first-order stationary autoregressive process

$$x_r' = Px_r + \sigma\epsilon_{x_r}', \quad \epsilon_{x_r} \sim i.i.d.N(0, V_{\epsilon_{x_r}}) \tag{7}$$

where P is a stable square matrix of autoregressive coefficients, and ϵ_{x_r}' collects the n_{x_r} structural shocks associated with the states x_r.

As shown in Kormilitsina (2013), the first-order approximation to the recursive solution of (4) is

$$\begin{aligned} y_u &= \hat{g}_{x_u}(\theta)x_u + \hat{g}_{x_r}(\theta)x_r + \hat{g}_{x_{r,1}}(\theta)x_{r,-1}, \\ y_r &= \hat{j}_{x_u}(\theta)x_u + \hat{j}_{x_{r,-1}}(\theta)x_{r,-1}, \\ x_u' &= \hat{h}_{x_u}(\theta)x_u + \hat{h}_{x_r}(\theta)x_r + \hat{h}_{x_{r,-1}}(\theta)x_{r,-1} + \sigma\epsilon_{x_u}' \end{aligned} \tag{8}$$

where the dependence of the reduced form matrices on the structural parameters θ has been made explicit. In a more compact form one has

$$y = \hat{g}_x(\theta)\begin{pmatrix} x_u \\ x_r \\ x_{r,-1} \end{pmatrix}, \quad x' = \hat{h}_x(\theta)\begin{pmatrix} x_u \\ x_r \\ x_{r,-1} \end{pmatrix} + \sigma\epsilon' \tag{9}$$

where

$$\hat{g}_x(\theta) = \begin{pmatrix} \hat{g}_{x_u}(\theta) & \hat{g}_{x_r}(\theta) & \hat{g}_{x_{r,-1}}(\theta) \\ \hat{j}_{x_u}(\theta) & 0_{n_{y_r} \times n_{x_r}} & \hat{j}_{x_{r,-1}}(\theta) \end{pmatrix}, \quad \hat{h}_x(\theta) = \begin{pmatrix} \hat{h}_{x_u}(\theta) & \hat{h}_{x_r}(\theta) & \hat{h}_{x_{r,-1}}(\theta) \\ 0_{n_{x_r} \times n_{x_r}} & P(\theta) & 0_{n_{x_r} \times n_{x_r}} \end{pmatrix} \tag{10}$$

Provided the rank condition characterized in Sorge (2020) is fulfilled, the solution to the restricted model can be readily constructed via uniquely defined linear transformations of (3), however computed (e.g. exploiting algorithms put forward in Klein (2000), Christiano (2002), King and Watson (2002), Sims (2002)). In fact, upon partitioning the equilibrium coefficient matrices $(g_x(\theta), h_x(\theta))$ in (3) as follows

$$g_x(\theta) = \begin{pmatrix} g_{x_u}(\theta) & g_{x_r}(\theta) \\ j_{x_u}(\theta) & j_{x_r}(\theta) \end{pmatrix}, \quad h_x = \begin{pmatrix} h_{x_u}(\theta) & h_{x_r}(\theta) \\ 0 & P(\theta) \end{pmatrix} \tag{11}$$

we can easily map the coefficient matrices under conventional timing into those appearing in (9), i.e.

$$\hat{g}_x(\theta) = \begin{pmatrix} g_{x_u}(\theta) & g_{x_r}(\theta) + \left[\nabla(f^1)^{-1} f^1_{y_r} j_{x_r}(\theta)\right]_{n_{y_u}} & -\left[\nabla(f^1)^{-1} f^1_{y_r} j_{x_r}(\theta) P(\theta)\right]_{n_{y_u}} \\ j_{x_u}(\theta) & 0_{n_{y_r} \times n_{x_r}} & j_{x_r}(\theta) P(\theta) \end{pmatrix},$$

$$\hat{h}_x(\theta) = \begin{pmatrix} h_{x_u}(\theta) & h_{x_r}(\theta) + \left[\nabla(f^1)^{-1} f^1_{y_r} j_{x_r}(\theta)\right]_{n_{x_u}} & \left[-\nabla(f^1)^{-1} f^1_{y_r} j_{x_r}(\theta) P(\theta)\right]_{n_{x_u}} \\ 0 & P(\theta) & 0 \end{pmatrix}$$

where $\nabla(f^1)$ denotes the Jacobian of the sub-system f^1 with respect to the vector $[x'_u, y_u]$, $f^1_{y_r}$ is the matrix of partial derivatives of f^1 with respect to the slow moving endogenous variables collected in the vector y_r, and $[M]_m$ is used to denote the selection of the first (or last) m rows of some matrix M.

In order to test for the presence of shock transmission delays, we notice that coefficient matrices $\hat{h}_{x_r}$ and $\hat{h}_{x_{r,-1}}$ (and thereby $\hat{g}_{x_r}$ and $\hat{g}_{x_{r,-1}}$) embodying the model's CERs will generally differ from those implied by the counterpart model complying with the standard timing protocol. Timing restrictions in fact enforce an enlarged state space as well as an increased degree of backward dependence in the model's equilibrium representation. As a result, the dynamics of the endogenous variables and their statistics, including the likelihood function, will depend (among other things) on the parameterization of the matrix $\hat{j}_{x_r}$ which characterizes the model-implied transmission of structural shocks driving states x_r to endogenous variables

y_r: when timing restrictions are operative, $\hat{j}_{x_r}$ will necessarily be zero-valued. Information contained in the likelihood function can then be used to derive (classical or Bayesian) inference about the relevance of delayed transmission for the shock(s) of interest.

To frame our bootstrap-based MLE testing procedure, we exploit the structural form in (9) embodying timing restrictions against the following state-space counterpart

$$y = \tilde{g}_x(\phi) \begin{pmatrix} x_u \\ x_r \\ x_{r,-1} \end{pmatrix}, \quad x' = \tilde{h}_x(\phi) \begin{pmatrix} x_u \\ x_r \\ x_{r,-1} \end{pmatrix} + \sigma\epsilon' \tag{12}$$

where $\tilde{g}_x$ and $\tilde{h}_x$ are conformable matrices. We collect the non-zero parameters in $\tilde{h}_x$ and $\tilde{g}_x$ in the vector ϕ (i.e. $\tilde{g}_x = \tilde{g}_x(\phi)$ and $\tilde{h}_x = \tilde{h}_x(\phi)$).

3 Testing problem

We consider the testing problem

$$\mathsf{H}_0 : \tilde{h}_x(\phi) = \hat{h}_x(\theta) \textbf{ and } \tilde{g}_x(\phi) = \hat{g}_x(\theta) \quad \textbf{against } \mathsf{H}_1 : \tilde{h}_x(\phi) \neq \hat{h}_x(\theta) \textbf{ or } \tilde{g}_x(\phi) \neq \hat{g}_x(\theta) \tag{13}$$

by a LR test. The null H_0 incorporates the timing restrictions imposed by (4). The DSGE model under H_0 is given by (9), instead, the state space counterpart of the DSGE model under H_1 corresponds to (12). To compute a LR test of these restrictions it is necessary to maximize the likelihood associated with both systems. Let $\ell_T(\phi)$ and $\ell_T(\theta)$ be the log-likelihoods of the DSGE model under H_1 and H_0, respectively, and let $\hat{\phi}_T = \arg\max_{\phi\in\mathcal{P}_\phi} \ell_T(\phi)$ and $\hat{\theta}_T = \arg\max_{\theta\in\mathcal{P}^D} \ell_T(\theta)$ the ML estimators of ϕ and θ. Estimation of the model under the null (H_0) and under the alternative (H_1) is a necessary preliminary step to the computation of the LR test. We start from the representation in (12). The innovation form representation (Anderson and Moore, 2012) associated with system (12) can be written in the form

$$\hat{x}_{t+1|t} = \tilde{h}_x(\phi)\hat{x}_{t|t-1} + K_t + \epsilon_t(\phi) \tag{14}$$

$$y_t = \tilde{g}_x(\phi)\hat{x}_{t|t-1} + \epsilon_t(\phi) \tag{15}$$

where $K_t = K_t(\phi_\theta)$ is the Kalman gain and

$$\epsilon_t = y_t - \tilde{g}_x(\phi)\hat{x}_{t|t-1} \tag{16}$$

are the innovation residuals with covariance matrix

$$\Sigma_{\epsilon,t}\tilde{g}_x(\phi)P_{t|t-1}\tilde{g}_x(\phi)' + J\Sigma_\omega J' \tag{17}$$

and $P_{t|t-1} = E((x_t - \hat{x}_{t|t-1})(x_t - \hat{x}_{t|t-1})' \mid \mathcal{F}^y_{t-1})$, $P_{1|0}$ being given. Imposing the normality of ϵ_t in (16), i.e.

$$y_t \mid \mathcal{F}^y_{t-1} \sim N(\tilde{g}_x(\phi)\hat{x}_{t|t-1} \quad \Sigma_{\epsilon,t})$$

the estimation of ϕ can be accomplished via Gaussian maximum likelihood estimation.[4]

Let $\ell_T(\phi)$ be the Gaussian log-likelihood function associated with the state space model in (14)-(15). The essential part of the log-likelihood $\ell_T(\phi)$, denoted for simplicity by $\ell_{\circ,T}(\phi) := \sum_{t=1}^{T} l(y_t \mid \mathcal{F}^y_{t-1}; \phi)$, is given by

$$\begin{aligned} \ell_{\circ,T}(\phi) &= -\sum_{t=1}^{T} \ell_t(\phi) \qquad (18) \\ \ell_t(\phi) &= l(y_t \mid \mathcal{F}^y_{t-1}; \phi) = \{\log\det(\Sigma_{\epsilon^c,t}(\phi)) + \epsilon_t^0(\phi)' \Sigma_{\epsilon^0,t}(\phi)^{-1} \epsilon_t^0(\phi)\} \end{aligned}$$

where $\epsilon_t(\phi)$ and $\Sigma_{\epsilon^0,t}(\phi)$ are defined above. Given $\ell_{\circ,T}(\phi)$ in (18), the ML estimator of ϕ solves

$$\hat{\phi}_T = \arg\max_{\phi \in \mathcal{P}^D} \ell_{\circ,T}(\phi) \tag{19}$$

and can be computed by combining the Kalman filter with numerical optimization methods. To estimate the structural parameters in θ, we can consider analogs of systems (14)-(15) and replace $\tilde{h}_x(\phi)$ and $\tilde{g}_x(\phi)$ with $\hat{h}_x(\theta)$ and $\hat{g}_x(\theta)$. The ML

[4] It is maintained throughout the paper that the regularity conditions for standard asymptotic inference in the state space representation of the DSGE model are valid both under the null and the alternative. We refer to Angelini et al. (2022) for a comprehensive treatment of how bootstrap resampling can be used to detect deviations from regularity conditions.

estimator of θ is therefore obtained from

$$\hat{\theta}_T = \arg\max_{\theta\in\mathcal{P}_\theta} \ell_{o,T}(\theta), \qquad \ell_{\circ,T}(\theta) = -\sum_{t=1}^{T}\left\{\log\det(\Sigma_{\epsilon^0,t} + \epsilon_t^{0,\prime}\Sigma_{\epsilon^0,t}^{-1}\epsilon_t^0\right\}. \qquad (20)$$

We use the superscript '0' for ϵ_t^0 in (20) and $\Sigma_{\epsilon^0,t}$ to remark that the representation is obtained under the null H_0 which imposes the timing restrictions. The LR test for the timing restrictions is then given by

$$LR_T = -2[\ell_T(\hat{\theta}_T) - \ell_T(\hat{\phi}_T)]. \qquad (21)$$

The asymptotic properties of the tests statistics LR_T are intimately related to the asymptotic properties of $\hat{\theta}_T$ and $\hat{\phi}_T$ and these crucially depend on whether the regularity conditions for inference are valid in the estimated DSGE model.

3.1 Bootstrap algorithm

We employ a nonparametric 'restricted bootstrap' algorithm (see e.g. Davidson and MacKinnon, 1999), where the bootstrap samples are generated using the the parameter estimates $\hat{\theta}_T$ obtained under H_0. The LR test statistic, $LR_T(\hat{\theta}_T)$, computed as in (21) is stored, along with $\hat{\theta}_T$. Our procedure is adapted from Stoffer and Wall (1991) and Angelini et al. (2022), and is described by the following algorithm. Here, steps 1–4 define the bootstrap sample, the bootstrap parameter estimators and related bootstrap LR statistic; steps 5–7 describe the numerical computation of the bootstrap p-value associated to the bootstrap LR test.

Algorithm (Restricted bootstrap)

1. Given the innovation residuals $\hat{\epsilon}_t^0 = y_t - \hat{g}_x(\hat{\theta}_T)\hat{x}_{t|t-1}$ and the estimated covariance matrices $\hat{\Sigma}_{\epsilon^0,t}$ produced by the estimation of the DSGE model under the precesence of timing restrictions (H_0), construct the standardized innovations as

$$\hat{e}_t^0 = \hat{\Sigma}_{\epsilon^0,t}^{-1/2}\hat{\epsilon}_t^{0,c}, \quad t = 1, ..., T, \qquad (22)$$

where $\hat{\Sigma}_{\epsilon^0,t}^{-1/2}$ is the inverse of the square-root matrix of $\hat{\Sigma}_{\epsilon^0,t}$ and $\hat{\epsilon}_t^{0,c}$, $t = 1, ..., T$, are the centered residuals $\hat{\epsilon}_t^{0,c} = \hat{\epsilon}_t^0 - T^{-1}\sum_{t=1}^{T}\hat{\epsilon}_t^0$;

2. Sample, with replacement, T times from $\hat{e}_1^0, \hat{e}_2^0, ..., \hat{e}_T^0$ to obtain the bootstrap sample of standardized innovations $e_1^*, e_2^*, ..., e_T^*$;

3. Mimicking the innovation form representation of the DSGE model in (14)-(15)), the bootstrap sample $y_1^*, y_2^*, ..., y_T^*$ is generated recursively by solving, for $t = 1, ..., T$, the system

$$\begin{pmatrix} \hat{x}_{t+1|t}^* \\ y_t^* \end{pmatrix} = \begin{pmatrix} \hat{h}_x(\hat{\theta}_T) & 0_{n_m \times n_y} \\ \hat{g}_x(\hat{\theta}_T) & 0_{n_y \times n_y} \end{pmatrix} \begin{pmatrix} \hat{x}_{t|t-1}^* \\ y_{t-1}^* \end{pmatrix} + \begin{pmatrix} K_t(\hat{\theta}_T)\hat{\Sigma}_{\epsilon^0,t}^{1/2} \\ \hat{\Sigma}_{\epsilon^0,t}^{1/2} \end{pmatrix} e_t^* \tag{23}$$

with initial condition $\hat{x}_{1|0}^* = \hat{x}_{1|0}$;

4. From the generated pseudo-sample $y_1^*, y_2^*, ..., y_T^*$, estimate the DSGE model under H_0 obtaining the bootstrap estimator $\hat{\theta}_T^*$ and the associated log-likelihood $\ell_T^*(\hat{\theta}_T^*)$, and estimate the DSGE model under H_1 obtaining the bootstrap estimator $\hat{\phi}_T^*$ and the associated log-likelihood $\ell_T^*(\hat{\phi}_T^*)$; the bootstrap LR test for the CER is defined as:

$$LR_T^*(\hat{\theta}_T^*) = -2[\ell_T^*(\hat{\theta}_T^*) - \ell_T^*(\hat{\phi}_T^*)]; \tag{24}$$

5. Steps 2-4 are repeated B times in order to obtain B bootstrap realizations of $\hat{\theta}_T$ and $\hat{\phi}_T$, say $\{\hat{\theta}_{T:1}^*, \hat{\theta}_{T:2}^*, ..., \hat{\theta}_{T:B}^*\}$ and $\{\hat{\phi}_{T:1}^*, \hat{\phi}_{T:2}^*, ..., \hat{\phi}_{T:B}^*\}$, and the B bootstrap realizations of the associated bootstrap LR test, $\{LR_{T:1}^*,\ LR_{T:2}^*, ..., LR_{T:B}^*\}$, where $LR_{T:b}^* = LR_T^*(\hat{\theta}_{T:b}^*)$, $b = 1, ..., B$;

6. The bootstrap p-value of the test of the timing restrictions is computed as

$$\widehat{p}_{T,B}^* = \hat{G}_{T,B}^*(LR_T(\hat{\theta}_T)) \quad , \quad \hat{G}_{T,B}^*(\delta) = B^{-1}\sum_{b=1}^{B}\mathbb{I}\{LR_{T:b}^* > \delta\}, \tag{25}$$

$\mathbb{I}\{\cdot\}$ being the indicator function;

7. The bootstrap LR test for the timing restrictions at the $100\eta\%$ (nominal) significance level rejects H_0 if $\widehat{p}_{T,B}^* \leq \eta$.

4 Simulation experiment

4.1 NK model

We showcase the usefulness of our testing procedure by running a series of Monte Carlo simulation experiments based on the hybrid New Keynesian model put forward by Benati and Surico (2009) in their VAR analysis of the so-called Great Moderation period.

The model is given as follows

$$g_t = \gamma E_t g_{t+1} + (1-\gamma) g_{t-1} - \delta^{-1}(i_t - E_t \pi_{t+1}) + \omega_t^g \tag{26}$$

$$\pi_t = \frac{\beta}{1+\beta\alpha} E_t \pi_{t+1} + \frac{\alpha}{1+\beta\alpha}\pi_{t-1} + \kappa g_t + \omega_t^\pi \tag{27}$$

$$i_t = \rho i_{t-1} + (1-\rho)(\varphi_\pi \pi_t + \varphi_g g_t) + \omega_t^i \tag{28}$$

where

$$\omega_t^j = \rho_j \omega_{t-1}^j + \epsilon_t^j, \quad |\rho_j| < 1, \quad \epsilon_t^j \sim \mathrm{WN}(0, \sigma_j^2) \ \ , \ \ j = g, \pi, i \tag{29}$$

and expectations are conditional on the information set $\mathcal{I}_t$, i.e. $E_t \cdot := E(\cdot \mid \mathcal{I}_t)$. The variables g_t, π_t, and i_t stand for the output gap, inflation, and the nominal interest rate, respectively; γ is the weight of the forward-looking component in the dynamic IS curve; α is price setters' extent of indexation to past inflation; δ is the the households' intertemporal elasticity of substitution in consumption; κ is the slope of the Phillips curve; ρ, φ_π, and φ_g are the interest rate smoothing coefficient, the long-run coefficient on inflation, and that on the output gap in the monetary policy rule, respectively; finally, ω_t^g, ω_t^π and ω_t^i in eq. (29) are the mutually independent, AR(1) exogenous shock processes and ϵ_t^g, ϵ_t^π and ϵ_t^i are the structural innovations.

The model (26)-(27)-(28) is submitted to timing restrictions according to which (i) the monetary policy shocks are orthogonal to the non-policy variables (g_t, π_t), and (ii) these non-policy variables are thus predetermined with respect to the policy instrument, i.e. the nominal interest rate. The information partition enforcing the above mentioned timing restrictions requires the following assignment of variables:

$$\begin{aligned} y_u = i, \quad y_r = [g,\, \pi]' \\ x_u = [g_{-1},\, \pi_{-1},\, \omega^g,\, \omega^\pi]', \quad x_r = \omega^i \end{aligned} \tag{30}$$

As is well known, the model (26)-(27)-(28) can admit a continuum of asymptotically stable equilibria (*equilibrium indeterminacy*) depending on the strength of the monetary authority's response to inflation. Under these circumstances, short-run dynamics for the endogenous variables can be arbitrarily driven by both structural and non-structural (sunspot) shocks (e.g. Lubik and Schorfheide, 2003). Building upon Kormilitsina (2013), Sorge (2020) formally shows that the determinacy/indeterminacy properties of DSGE models complying with the conventional, unrestricted timing protocol are generically (in a measure-theoretic sense) inherited by their restricted counterparts. That is, when a given, unrestricted model exhibits saddle-path stability, so will its analogue featuring timing restrictions: the model's fundamentals (parameters and shocks) will therefore fully characterize the equilibrium representation of the model. In our Monte Carlo simulation experiment, we explicitly restrict attention to the determinate equilibrium version of Benati and Surico (2009)'s model, so that variation in the likelihood across the two information structures (restricted vs. unrestricted) is to be ascribed to the presence of timing restrictions solely, on the assumption that the structural model is correctly specified.[5]

4.2 Monte Carlo simulation

In this section we investigate the empirical performance of the bootstrap test using the New Keynesian structure (26)-(27)-(28) as our data generating process (DGP). More specifically, we consider two DSGE-based equilibrium state space representations, denoted as *DGP under timing restrictions* and *DGP with unrestricted timing*, respectively. In the former, it is assumed that the data are generated by the determinate equilibrium representation that emerges in the presence of structural timing restrictions embodied in (30); in the latter, artificial series are rather generated by allowing for contemporaneous effects of policy innovations on the inflation rate and the output gap (i.e. when no informational constraints are at work), again imposing equilibrium determinacy.

To keep our analysis focused on the testing problem, we calibrate all structural parameters to Benati and Surico (2009)'s posterior median estimates over the Great

[5]See Fanelli (2012) and Dave and Sorge (2021) for an analysis of identification and estimation issues arising in Benati and Surico (2009)'s model in the presence of equilibrium indeterminacy; and Angelini and Sorge (2021) for a discussion of the implications of the co-existence of timing restrictions and equilibrium indeterminacy.

Moderation period – see Table (1). Then, *for both the restricted and the unrestricted versions of the model*, artificial data samples are generated by simulating the model's determinate solution when shock realizations are independently drawn from the assumed mean-zero, unit variance Gaussian densities at any given period. Operationally, for each of the two DGPs we consider $K = 1000$ simulations and a sample size $T \in \{100, 250, 500\}$ with a burn-in of 200 observations. We then investigate the empirical size of the LR test, using the restricted model a the actual DGP (column *DGP under timing restrictions*), and its power, when the unrestricted model is rather serves as the underlying DGP (column *DGP with unrestriced timing*)

Structural parameters										
γ	δ	β	α	κ	ρ	ϕ_π	ϕ_g	ρ_g	ρ_π	ρ_i
0.744	8.062	0.99	0.059	0.044	0.834	1.749	1.146	0.796	0.418	0.404

Table 1: *Elected parameterization of Benati and Surico (2009)'s model for simulation experiment.*

The key parameters that we estimate on the artificial data are the slope of the Phillips curve (κ), a function of the degree of price-stickiness in the economy that filters the relative response of non-policy variables to an unanticipated deviation from the systematic component of policy equation (28); the feedback parameters γ and α, measuring the degree of backward dependence in the intertemporal IS relationship (26) and the Phillips curve (27) respectively, both influencing the inertial properties of the dynamic evolution of non-policy variables (whether or not timing restrictions are over-imposed); and the interest rate smoothing parameter ρ that governs the partial adjustment of the current policy rate to its own lag, a further source of endogenous time series persistence in the dynamic responses of inflation and output (gap) to shocks. The bootstrap-based test for the non-linear cross-equation restrictions obtained from the ML estimation of the determinate reduced form solution to either model (restricted vs. unrestricted) is then implemented.[6]

[6]We are implicitly assuming that state space systems (9) and (12) are in minimal form and also identified (locally). In general, state space representations can be manipulated so as to deliver an identified system in minimal form. This latter can then be readily used as the DGP implied by the structural DSGE model under scrutiny, see e.g. see Komunjer and Ng (2011).

	DGP under timing restrictions	*DGP with unrestricted timing*
	T=100	
$\kappa = 0.044$	0.044(0.045)	0.043(0.044)
$\gamma = 0.744$	0.748(0.753)	0.742(0.755)
$\alpha = 0.059$	0.058(0.059)	0.055(0.056)
$\rho = 0.834$	0.833(0.833)	0.813(0.804)
LR_T	χ^2_x, $RejRate = 0.082[0.052]$	χ^2_x, $RejRate = 0.194[0.106]$
	T=250	
$\kappa = 0.044$	0.045(0.045)	0.044(0.044)
$\gamma = 0.744$	0.746(0.751)	0.741(0.746)
$\alpha = 0.059$	0.062(0.060)	0.057(0.060)
$\rho = 0.834$	0.833(0.831)	0.812(0.808)
LR_T	χ^2_x, $RejRate = 0.068[0.048]$	χ^2_x, $RejRate = 0.464[0.296]$
	T=500	
$\kappa = 0.044$	0.044(0.044)	0.043(0.043)
$\gamma = 0.744$	0.744(0.746)	0.739(0.741)
$\alpha = 0.059$	0.057(0.055)	0.052(0.055)
$\rho = 0.834$	0.834(0.833)	0.813(0.811)
LR_T	χ^2_x, $RejRate = 0.066[0.045]$	χ^2_x, $RejRate = 0.836[0.690]$

Table 2: *Monte Carlo experiment, estimation of the parameters κ, γ, α and ρ, average across $M = 1000$ simulations (bootstrap estimates in parentheses). LR_T is the likelihood ratio test of the restricted model against the unrestricted counterpart (bootstrap p-values in square brackets).*

Results are summarized in Table 2. It is apparent that the bootstrap tends to mitigate the discrepancy between actual and nominal probabilities of type-I error. Indeed, when asymptotic critical values taken from the χ^2_x distribution ($x = \dim(\phi) - \dim(\theta)$) are employed, the rejection frequency of the LR test for the timing restrictions is 8.2%, 6.8% and 6.6% for $T = 100, 250$ and 500 respectively. Therefore, the bootstrap seems to effectively mitigate the tendency of the standard LR test to over-reject the CERs associate with the restricted timing protocol with rejection frequencies close to the 5% level. Interestingly, from the DGP free of timing restrictions, the bootstrap test shows satisfactory power.

5 Conclusion

This paper develops a simple bootstrap-based testing procedure for the relevance of timing restrictions and ensuing shock transmission delays in general DSGE model environments. While here applied to a rather standard New Keynesian model of the monetary transmission mechanism, our approach is designed to handle virtually any linearized DSGE model whose equilibrium conditions can be represented as a system of expectational stochastic difference equations (see e.g. Schmitt-Grohé and Uribe, 2004), to be estimated against real world data via likelihood-based methods. Remarkably, the computer code is consistent with standard MATLAB packages – such as Sims (2002)'s – that are routinely used to compute first-order approximate solutions to dynamic macroeconomic models; and can be straightforwardly adapted to allow for relatively more sophisticated recursive timing structures than those considered herein, e.g. those involving multi-period informational partitions (Kormilitsina, 2013).

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