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University of Naples Federico II

University of Salerno

Bocconi University, Milan

CSEF - Centre for Studies in Economics and Finance
DEPARTMENT OF ECONOMICS AND STATISTICS – UNIVERSITY OF NAPLES FEDERICO II
80126 NAPLES - ITALY
Tel. and fax +39 081 675372 – e-mail: csef@unina.it
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Giuseppe De Marco*, **Maria Romaniello**†, **and Alba Roviello**‡

Abstract

The literature on strategic ambiguity in classical games provides generalized notions of equilibrium in which each player best responds to ambiguous or imprecise beliefs about hisopponents' strategy choices. In a recent paper, strategic ambiguity has been extended topsychological games, by taking into account ambiguous hierarchies of beliefs and maxmin preferences. Given that this kind of preference seems too restrictive as a general method to evaluate decisions, in this paper we extend the analysis by taking into account α-maxmin preferences in which decisions are evaluated by a convex combination of the worst-case (with weight α) and the best-case (with weight 1−α) scenarios. We give the definition of α-maxmin Psychological Nash Equilibrium; an illustrative example shows that the set of equilibria is affected by the parameter α and the larger is ambiguity the greater is the effect. We also provide a result of stability of the equilibria with respect to perturbations that involve the attitudes toward ambiguity, the structure of ambiguity and the payoff functions: converging sequences of equilibria of perturbed games converge to equilibria of the unperturbed game as the perturbation vanishes. Surprisingly, a final example shows that existence of equilibria is not guaranteed for every value of α.

Keywords: Psychological games, ambiguous beliefs, α-MEU, equilibrium existence.

* University of Naples Parthenope, University of Naples Federico II, and CSEF *(Corresponding author).* Email: giuseppe.demarco@uniparthenope.it

† University Campania Vanvitelli.

‡ University Campania Vanvitelli.

1 Introduction

It is well known that the Nash equilibrium concept for strategic games prescribes that: *i)* each player chooses his best strategy in response to the beliefs he has about his opponents' strategy choices; *ii)* each player's beliefs are correct, that is, each player believes with probability 1 that opponents will follow their equilibrium strategies. The evidence arising from Decision Theory tells us that beliefs cannot always be assumed to be correct. The literature that focuses on the issue *strategic ambiguity* in classical strategic form games provides generalized notions of equilibrium in which each player best responds to *ambiguous* or *imprecise* beliefs about his opponents' strategy choices, that is, beliefs may take the form of a capacity or of a set of probability distributions (see [Dow, Werlang, 1994], [Eichberger, Kelsey, 2000], [Lehrer, 2012], [Riedel, Sass, 2013], [Battigalli et al., 2015], [De Marco, Romaniello, 2015] and references therein). There might be many sources of *strategic ambiguity* in a game; for example, [Lehrer, 2012] focuses on the case in which players do not have precise knowledge of the mixed strategy chosen by each of the other players but rather know only the probability of some subsets of pure strategies, being not aware of the precise subdivision of probabilities within those subsets.

In [De Marco et al., 2022], the study of strategic ambiguity has been extended to *psychological games*, by looking at ambiguous or imprecise *hierarchies of beliefs*. Psychological games provide a generalization of classical games that aims to explicitly take into account emotions, opinions, and intentions of the decision makers in the strategic interaction[2]. This class of games is characterized by the assumption that each player's payoff depends on his hierarchy of beliefs, that is, it depends not only on what every player does but also on what he thinks every player believes, on what he thinks every player believes the others believe, and so on. The main solution concept for psychological games is presented in Geanakoplos et al. [1989] and it is based on the idea that the entire hierarchy of beliefs of each player must be correct in equilibrium.

Since beliefs about opponents' strategy choices can be regarded as first-order beliefs, the literature on strategic ambiguity substantially looks at games in which first-order beliefs are ambiguous. [De Marco et al., 2022], instead, looks at ambiguity regarding the entire hierarchy of beliefs as, for instance, partial knowledge may appear directly in the second (or higher) order beliefs, or strategic ambiguity produces ambiguous higher order beliefs as a natural consequence. Therefore, the function that maps strategy profiles to the *correct* hierarchies of beliefs, that is used in the classical definition of psychological Nash equilibria, is therein replaced by a set-valued map (called *ambiguous belief correspondence*), that maps strategy profiles to the subsets of those hierarchies of beliefs that players perceive to be consistent with the corresponding strategy profile. In the corresponding equilibrium notion presented in [De Marco et al., 2022], players are assumed to be

[2]The literature on psychological games has increased considerably in the past decades; we recall [Battigalli, Dufwenberg, 2009] for further theoretical findings, [Rabin, 1993], [Battigalli, Dufwenberg, 2007], [Attanasi et al., 2010] for some applications, just to quote a few, and [Attanasi, Nagel, 2008] and [Battigalli, Dufwenberg, 2020] for surveys on psychological games and references.

completely pessimistic as they are endowed with *maxmin* preferences (also called *MEU* preferences, see [Gilboa, Schmeidler, 1989]): each player maximizes (with respect to his own strategy) the minimum expected utility computed along the graph of the ambiguous belief correspondence whose values, in turn, depend on the entire strategy profile.

The maxmin approach turns to be analytically convenient; furthermore, it has a clear axiomatic foundation. Nevertheless, it seems to be too restrictive as a general approach because only the "worst-case scenario" is relevant for the evaluation of a decision so that the analysis is limited to an extreme form of pessimism[3]. The restrictiveness of the MEU model can be naturally overcome by considering the so called *α-maxmin preferences* (also called *α-MEU* or *Hurwicz Preferences*), firstly introduced in [Hurwicz, 1951]. In this model, decisions are evaluated by a convex combination of the worst-case (with weight α) and the best-case (with weight $1 - \alpha$) scenarios.

In this paper we extend the analysis of psychological games under ambiguity to α-maxmin preferences and provide the notion of *α-MEU Psychological Nash Equilibrium* (α-PNE) for the situations in which players have Hurwicz preferences. The weights α that characterize the attitudes of the players toward ambiguity turn to be a key tool to understand how equilibria change according to their degree of pessimism/optimism. We present an illustrative example showing not only that the set of equilibria depends on the parameter α but also that differences are emphasized by the amount of ambiguity in the game: the *larger* is ambiguity the greater are the differences. The example highlights another relevant feature: equilibria corresponding to a given value of α cannot always be approached by sequences of equilibria of games in which the parameter α is slightly perturbed, meaning that equilibria are unstable with respect to perturbations on the degree of pessimism/optimism. From the mathematical point of view, this results in a lack of *lower semicontinuity* of psychological Nash equilibria under Hurwicz preferences. The failure of this property is not suprising since lack of lower semicontinuity of the equilibrium correspondence is a common feature in most of the game models. We show, instead, that the α-PNE correspondence satisfies a upper semicontinuity-like stability: converging sequences of equilibria of perturbed games converge to equilibria of the unperturbed game as the perturbation *vanishes*. The issue of the upper semicontinuity properties of equilibria has been largely investigated in the literature for classical games (see for instance [Yu, 1999], [Carbonell-Nicolau, 2010], [Scalzo, 2019] and references therein) and turns out to be a key property to build refinements of equilibria based on stability with respect to *trembles*. In this paper, we obtain stability of equilibria under general perturbations that involve the attitudes toward ambiguity, the structure of ambiguity and the payoff functions.

The most surprising feature of α-PNE is, however, a negative result. Although for psychological Nash equilibria and psychological Nash equilibria under maxmin preferences an existence result was obtained under standard assumptions, in this paper we provide a counterexample in which a game has no α-PNE. This negative result comes from the fact that the best reply correspondence of the summary utility function (that is used to obtain equilibrium existence) does not have convex images and therefore fixed points, in general.

[3] Optimistic and intermediate attitudes actually have a strong empirical support (see for example [Ivanov, 2011]).

The paper is organized as follows: Section 2 defines the game and the equilibrium concept. Section 3 presents the illustrative example while Section 4 is dedicated to the upper-semicontinuity property of equilibria. In Section 5 the issue of the lack of existence of equilibria is studied.

2 Model and Equilibria

We consider a finite set of players $I = \{1, \dots n\}$, and, for each player i, we denote with $A_i = \{a_i^1, \dots, a_i^{k(i)}\}$ the (finite) pure strategy set of player i. As usual, the set of strategy profiles A is the cartesian product of the strategy sets of each player, that is $A = A_1 \times \dots \times A_n = \prod_{i \in I} A_i$, and $A_{-i} = A_1 \times \dots \times A_{i-1} \times A_{i+1} \times \dots \times A_n = \prod_{j \neq i} A_j$. Let Σ_i be the set of mixed strategies of player i, where each mixed strategy $\sigma_i \in \Sigma_i$ is a nonnegative vector $\sigma_i = (\sigma_i(a_i))_{a_i \in A_i} \in \mathbb{R}_+^{k(i)}$ such that $\sum_{a_i \in A_i} \sigma_i(a_i) = 1$. Denote also with $\Sigma = \prod_{i \in I} \Sigma_i$ and with $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$. We use (σ_i, σ_{-i}) with $\sigma_i \in \Sigma_i$ and $\sigma_{-i} \in \Sigma_{-i}$ to represent $\sigma \in \Sigma$.

Hierarchies of beliefs

The beliefs structure is constructed following [Geanakoplos et al., 1989]. Recall that, for any set S, $\Delta(S)$ denotes the set of probability measures on S. For every player i and for every $k \in \mathbb{N}$, $k > 1$, the $k-$th order beliefs set is defined recursively as follows:

$$B_i^1 = \Delta(\Sigma_{-i}),\ B_i^2 = \Delta(\Sigma_{-i} \times B_{-i}^1), \dots,\ B_i^k := \Delta(\Sigma_{-i} \times B_{-i}^1 \times B_{-i}^2 \times \dots \times B_{-i}^{k-1}), \dots$$

where $B_{-i}^k := \prod_{j \neq i} B_j^k$. The set of all *hierarchies of beliefs* of player i is $B_i = \prod_{k=1}^{\infty} B_i^k$. Note that for every k, B_i^k is compact and can be metrized as a separable metric space. Consequently, since B_i is a countable product of separable and compact metric spaces, it is also a separable and compact metric space[4].

We will restrict the attention to the subset of *collectively coherent beliefs* $\overline{B}_i \subset B_i$, that is, the compact set of beliefs of player i in which he is sure that it is common knowledge that beliefs are coherent. Precisely, a belief $b_i = (b_i^1, b_i^2, \dots) \in B_i$ is said to be *coherent* if, for every $k \in \mathbb{N}$, the marginal probability of b_i^{k+1} on $\Sigma_{-i} \times B_{-i}^1 \times B_{-i}^2 \times \dots \times B_{-i}^{k-1}$ coincides with b_i^k, that is

$$\operatorname{marg}(b_i^{k+1}, \Sigma_{-i} \times B_{-i}^1 \times B_{-i}^2 \times \dots \times B_{-i}^{k-1}) = b_i^k.$$

You can find the construction of the set of collectively coherent beliefs in [Geanakoplos et al., 1989] and the proof of its compactness in [De Marco et al., 2022]. In the remainder of the paper, with an abuse of notation we will denote with $\overline{B}_i$ the set of collectively coherent beliefs or any of its compact subsets.

As in [De Marco et al., 2022], we allow for ambiguity in the beliefs, therefore beliefs are compact subsets $K_i \subseteq \overline{B}_i$. We denote with $\mathscr{K}_i$ the set of all compact subsets of $\overline{B}_i$. This choice enables

[4] See [De Marco et al., 2022] for additional details on the topological and metric structure of the beliefs space.

to consider the ambiguity players come up against during the game, due to the uncertainty about other players' actions and beliefs: the agent does not have a precise belief b_i but knows that the belief can be any $b_i \in K_i$. If K_i is a singleton, then the belief is not ambiguous, leading the theory back to the standard case.

Game and equilibria

Following the model in [Geanakoplos et al., 1989], each agent i is endowed with an utility function of the form

$$u_i : \overline{B}_i \times \Sigma \to \mathbb{R}, \tag{1}$$

depending not only on the mixed strategy profile but also on agent's beliefs: $u_i(b_i, \sigma)$ represents the payoff to player i if he believed b_i and the strategy profile σ is actually played. Indeed, fixing b_i, $u_i(b_i, \cdot)$ can be (but not necessarily) the classical expected utility function as it is assumed in [Geanakoplos et al., 1989]. As agents face set-valued beliefs $K_i \in \mathscr{K}_i$, they have a set-valued payoff $\{u_i(b_i, \sigma)\}_{b_i \in K_i}$ for every given ambiguous belief $K_i \in \mathscr{K}_i$ and strategy profile $\sigma \in \Sigma$. There are several ways in which agents' ambiguity might be solved depending on the agents' attitudes towards ambiguity. In [De Marco et al., 2022] it was considered the case in which players are ambiguity averse, modeling the utility functions as maxmin preferences. In order to include a large spectrum of ambiguity attitude, in this paper we focus on the so called *α-maxmin* preferences, which allow us to range from the ambiguity seeking attitude (as $\alpha = 0$) to the ambiguity aversion attitude (as $\alpha = 1$). In this framework, each agent i has an utility function of the form $U_i^\alpha : \mathscr{K}_i \times \Sigma \to \mathbb{R}$ defined, for $\alpha_i \in [0, 1]$, by

$$U_i^\alpha(K_i, \sigma) = \alpha_i \left[\inf_{b_i \in K_i} u_i(b_i, \sigma) \right] + (1 - \alpha_i) \left[\sup_{b_i \in K_i} u_i(b_i, \sigma) \right] \quad \forall (K_i, \sigma) \in \mathscr{K}_i \times \Sigma, \tag{2}$$

where α denotes the vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$. Now, it is possible to define the game.

Definition 2.1: A *α-MEU normal form psychological game* is defined by

$$G^\alpha = \{A_1, \cdots, A_n, U_1^\alpha, \cdots, U_n^\alpha\}$$

where the utility functions U_i^α are defined as in formula (2) for every $i \in N$.

In the models of strategic ambiguity where players have partial knowledge of the strategies played by their opponents, players' beliefs depend on the actual strategy and take the form of set-valued maps (correspondences) from the set of strategy profiles to the set of probability distributions over opponents' strategies (see [Lehrer, 2012], [Battigalli et al., 2015], [De Marco, Romaniello, 2012]). In [De Marco et al., 2022] this approach is generalized to hierarchies of beliefs: agent i is endowed with a set-valued map $\gamma_i : \Sigma \rightsquigarrow \overline{B}_i$, (called *ambiguous belief correspondence* of player i), where each image $\gamma_i(\sigma)$ is a not empty and compact set, i.e.:

$$\emptyset \neq \gamma_i(\sigma) \in \mathscr{K}_i \quad \forall \sigma \in \Sigma.$$

Each subset $\gamma_i(\sigma) \subseteq \overline{B}_i$ provides the set of hierarchies of beliefs that player i perceives to be consistent given the strategy profile σ. The set-valued maps γ_i are exogenous and have a different structure depending on the specific problem , therefore they can be considered as parameters of the game. In this paper we follow the approach in [De Marco et al., 2022]:

DEFINITION 2.2: A *α-MEU Psychological Nash Equilibrium* (henceforth α-PNE) of the game G^α with belief correspondences $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a pair (K^*, σ^*), where $K^* = (K_1^*, \ldots, K_n^*)$ with $K_i^* \subseteq \overline{B}_i$ and $\sigma^* \in \Sigma$ such that, for every player i:

i) $K_i^* = \gamma_i(\sigma^*)$;

ii) $U_i^\alpha(K_i^*, \sigma^*) \geqslant U_i^\alpha(K_i^*, (\sigma_i, \sigma_{-i}^*))$ for every $\sigma_i \in \Sigma_i$.

In this case, we can also say that $(\gamma(\sigma^*), \sigma^*)$ is a α-MEU Psychological Nash Equilibrium.

We point out that the definition above captures, in a natural way, the main features of the classical equilibrium notion since condition *ii)* requires that the equilibrium strategy of each player is optimal given his beliefs and condition *i)* requires that beliefs must be consistent with the equilibrium strategy profile.

Similarly to [Geanakoplos et al., 1989], α-PNE have a characterization as Nash equilibria. Let $w_i^\alpha : \Sigma \times \Sigma \to \mathbb{R}$ be the *summary utility function* defined by

$$w_i^\alpha(\sigma, \tau) = U_i^\alpha(\gamma_i(\sigma), \tau) = \alpha_i \left[\inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau)\right] + (1 - \alpha_i) \left[\sup_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau)\right] \quad \forall (\sigma, \tau) \in \Sigma \times \Sigma. \quad (3)$$

Then, it immediately follows from the definition that

LEMMA 2.3: *The profile $(\gamma(\sigma^*), \sigma^*)$ is a α-MEU Psychological Nash Equilibrium if and only if, for every player i,*

$$w_i^\alpha(\sigma^*, (\sigma_i^*, \sigma_{-i}^*)) \geqslant w_i^\alpha(\sigma^*, (y_i, \sigma_{-i}^*)) \quad \forall y_i \in \Sigma_i. \quad (4)$$

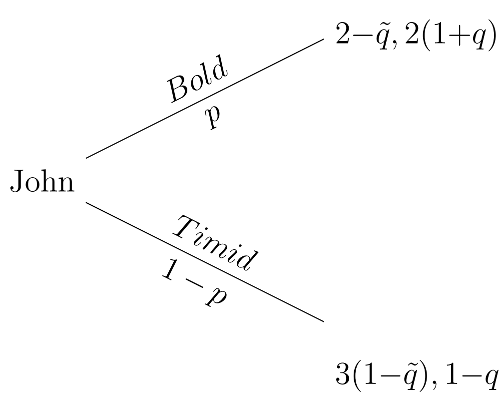
REMARK 2.4: In [Geanakoplos et al., 1989] equilibrium beliefs of each agent i are described by the *correct beliefs function* $\beta_i : \Sigma \to \overline{B}_i$ which, for every $\sigma \in \Sigma$, specifies the unique hierarchy of beliefs of player i that is correct, given σ. Now, if we replace γ_i with β_i in definition 2.2 we get back the definition of classical psychological Nash equilibria. On the other hand, if we replace γ_i with β_i in (3) we obtain the original summary utility function defined [Geanakoplos et al., 1989].

3 An Illustrative Example

In this section, we present an example of a psychological game under ambiguity in which players have Hurwicz preferences. The goal is twofold: on the one hand, we aim to put definitions to work and show how to find psychological Nash equilibria under ambiguity in simple models. On the other hand, the example highlights in which way the equilibria may be sensitive to variations in the amount or the structure of ambiguity of the game and in the attitudes of the players toward ambiguity. More precisely, we consider a specific form of ambiguity: players' beliefs are provided by a perturbation of the correct belief function that takes the form of a ball of radius ε around the correct belief. This approach resembles the *contamination model* approach and allows to analyze the sensitivity of α-PNE with respect to the unique parameter ε. Moreover, as the attitude towards ambiguity of each player i is parametrized by the corresponding value of α_i, we study the sensitivity of equilibria with respect to α_i.

The game considered in the example is the *Bravery Game* that has been firstly analyzed, in the framework of standard psychological games, by [Geanakoplos et al., 1989]. In [De Marco et al., 2022] it has been shown that allowing for ambiguous hierarchies of beliefs may significatively affect the set of equilibria when players are endowed with maxmin preferences. In this work, we study the game with respect to the double parametrization ε and α_i.

Example 3.1: The game is so described: Player 1 (John) has to publicly take a decision, and he is concerned about what Player 2 (Anne) will think about him. He can either be bold, exposing himself to the possibility of danger, or he can opt for a timid decision; therefore John's pure strategy set is $A_1 = \{Bold, Timid\}$. Anne is inactive during the whole interaction but her beliefs about John has an impact on John's behavior; indeed, his payoff depends not only on what he does but also on what he believes Anne thinks he will do. Suppose that John chooses *Bold* with probability p and *Timid* with probability $1-p$. We consider the case in which John cares only about the expectation $\tilde{q}$ of his belief about the expectation q of Anne's first order belief. Moreover, John would rather be timid, unless he thinks Anne is expecting him to be bold, in which case he prefers not to disappoint her. Anne prefers to think of her friend as bold, and it is better for her if he opts for the bold decision. The game and payoffs are described below:



Since Anne is a non-active player, the mixed strategy profile is given only by John's mixed strategy p. With an abuse of notation, the correct belief functions are defined as follows: $\beta_2(p) = p$ tells that the expectation of Anne's first order correct beliefs about John's strategy p must be equal to p; $\beta_1(p) = p$ tells that the expectation of John's correct second order beliefs about Anne's expectation of correct first order belief about John's strategy p must be equal to p as well.

The expected utility of John takes the following form:

$$u_1(\tilde{q}, p) = p(2 - \tilde{q}) + 3(1 - p)(1 - \tilde{q}) = p(2\tilde{q} - 1) + 3(1 - \tilde{q}).$$

In the case of non-ambiguous beliefs, the game has three psychological equilibria, as shown in [Geanakoplos et al., 1989]:

- $p = 1 = \tilde{q} = q$: John chooses to be *Bold*;
- $p = 0 = \tilde{q} = q$: John chooses to be $Timid$;
- $p = 1/2 = \tilde{q} = q$: John randomizes with probability $p = 1/2$.

If John is supposed to have ambiguous belief, then John's belief is represented by the map $\gamma_1^\varepsilon(p) = [p - \varepsilon, p + \varepsilon] \cap [0, 1]$ with $0 < \varepsilon \leq 1$. Let us now look at what happens if one allows player to have different attitude towards ambiguity, considering α-MEU. In order to compute John's summary utility function, we firstly compute, for every pair of John's mixed strategies (p, y) the followings:

$$\arg\min_{\tilde{q} \in \gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) = \left\{ \tilde{q}' \in [0, 1] \,|\, u_1(\tilde{q}', y) = \min_{\tilde{q} \in \gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) \right\},$$

$$\arg\max_{\tilde{q} \in \gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) = \left\{ \tilde{q}' \in [0, 1] \,|\, u_1(\tilde{q}', y) = \max_{\tilde{q} \in \gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) \right\}.$$

We get

$$\arg\min_{\tilde{q} \in \gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) = \arg\min_{\tilde{q} \in [p-\varepsilon, p+\varepsilon] \cap [0,1]} [\tilde{q}(2y - 3) + 3 - y] = \min\{p + \varepsilon, 1\}, \quad \forall\, y \in [0, 1].$$

Similarly,

$$\arg\max_{\tilde{q} \in \gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) = \arg\max_{\tilde{q} \in [p-\varepsilon, p+\varepsilon] \cap [0,1]} [\tilde{q}(2y - 3) + 3 - y] = \max\{p - \varepsilon, 0\}, \quad \forall\, y \in [0, 1].$$

If $p^+ := \min\{p + \varepsilon, 1\}$ and $p^- := \max\{p - \varepsilon, 0\}$, for every pair of John's mixed strategies (p, y) and for every $\alpha \in [0, 1]$, we have that:

$$w_1^\alpha(p, y) = \alpha \left[\min_{\tilde{q} \in \gamma_1^\varepsilon(p)} \tilde{q}(2y - 3) + 3 - y \right] + (1 - \alpha) \left[\max_{\tilde{q} \in \gamma_1^\varepsilon(p)} \tilde{q}(2y - 3) + 3 - y \right] =$$

$$\alpha[p^+(2y - 3) + 3 - y] + (1 - \alpha)[p^-(2y - 3) + 3 - y] =$$

$$y[2\alpha(p^+ - p^-) + 2p^- - 1] - 3\alpha(p^+ - p^-) + 3(1 - p^-).$$

Recall that p gives a psychological Nash equilibrium under ambiguity if and only if

$$w_1^{\alpha}(p,p) \geqslant w_1^{\alpha}(p,y) \quad \forall y \in [0,1],\ \alpha \in [0,1].$$

It is clear that equilibria depend on α and ε, therefore we will discuss different cases.

CASE 1. Suppose $0 < \varepsilon \leqslant \frac{1}{2}$. In this case the summary utility function takes the form:

$$w_{\varepsilon}^{\alpha}(p,y) = \begin{cases} y\left[2\alpha\left(p+\varepsilon\right)-1\right]-3\alpha\left(p+\varepsilon\right)+3 & \text{if } 0 \leqslant p \leqslant \varepsilon, \\ y[4\alpha\varepsilon+2(p-\varepsilon)-1]-6\alpha\varepsilon+3(1-p+\varepsilon) & \text{if } \varepsilon < p < 1-\varepsilon \\ y\left[2\alpha\left(1-p+\varepsilon\right)+2(p-\varepsilon)-1\right]+(3-3\alpha)\left(1-p+\varepsilon\right) & \text{if } 1-\varepsilon \leqslant p \leqslant 1. \end{cases}$$

Let $\tilde{p} = \frac{1}{2} + \varepsilon(1-2\alpha)$, $p^* = \frac{1}{2\alpha} - \varepsilon$, and $\hat{p} = \frac{1-2\alpha}{2-2\alpha} + \varepsilon$.

a) Denote with $h_1(y) := y\left[2\alpha\left(p+\varepsilon\right)-1\right]-3\alpha\left(p+\varepsilon\right)+3$. If $\alpha = 0$, the function $h_1(y)$ is decreasing in the entire interval $[0,1]$. If $\alpha > 0$, $h_1(y)$ is decreasing in the entire interval $[0,1]$ for $p < p^*$, constant in $[0,1]$ for $p = p^*$ and increasing in $[0,1]$ for $p > p^*$. Moreover, $p^* \geqslant 0$ for all $\alpha \in]0,1]$ while $p^* \leqslant \varepsilon$ if and only if $\alpha \in \left[\frac{1}{4\varepsilon}, 1\right]$.

b) Denote with $h_2(y) := y[4\alpha\varepsilon + 2(p-\varepsilon) - 1] - 6\alpha\varepsilon + 3(1-p+\varepsilon)$. The function $h_2(y)$ is decreasing in the entire interval $[0,1]$ for $p < \tilde{p}$, constant in $[0,1]$ for $p = \tilde{p}$ and increasing in $[0,1]$ for $p > \tilde{p}$. Moreover, $\tilde{p} > \varepsilon$ if and only if $\alpha \in \left[0, \frac{1}{4\varepsilon}\right[$ while $\tilde{p} < 1-\varepsilon$ if and only if $\alpha \in \left]1-\frac{1}{4\varepsilon}, 1\right]$.

c) Denote with $h_3(y) := y\left[2\alpha\left(1-p+\varepsilon\right)+2(p-\varepsilon)-1\right]+(3-3\alpha)\left(1-p+\varepsilon\right)$. If $\alpha = 1$ the function $h_3(y)$ is increasing in the entire interval $[0,1]$. If $\alpha < 1$ the function $h_3(y)$ is decreasing in $[0,1]$ for $p < \hat{p}$, constant in $[0,1]$ for $p = \hat{p}$ and increasing in the entire interval $[0,1]$ if $p > \hat{p}$. Moreover, $\hat{p} \leqslant 1$ for every $\alpha \in [0,1[$ while $\hat{p} \geqslant 1-\varepsilon$ if and only if $\alpha \in \left[0, 1-\frac{1}{4\varepsilon}\right]$.

Note that, if $\varepsilon < 1/4$ then $[0,1] \subset \left[1-\frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right]$, if $\varepsilon = 1/4$ then $\left[1-\frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right] = [0,1]$ while if $\varepsilon > 1/4$ then $\left[1-\frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right] \subset [0,1]$. Therefore:

- If $\varepsilon < 1/4$, the function $y \to w_{\varepsilon}^{\alpha}(p,y)$ is decreasing in $[0,1]$ for $p < \tilde{p}$, constant for $p = \tilde{p}$, increasing for $p > \tilde{p}$.

- If $\varepsilon \geqslant 1/4$, the function $y \to w_{\varepsilon}^{\alpha}(p,y)$ is decreasing in $[0,1]$ for $p < P$, constant for $p = P$, increasing for $p > P$, where

$$P = \begin{cases} \hat{p} & \text{if } \alpha \in \left[0, 1-\frac{1}{4\varepsilon}\right], \\ \tilde{p} & \text{if } \alpha \in \left]1-\frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right[, \\ p^* & \text{if } \alpha \in \left[\frac{1}{4\varepsilon}, 1\right]. \end{cases}$$

So equilibria are computed as follows:

SUBCASE 1.1: If $\varepsilon < \frac{1}{4}$, for every $\alpha \in [0,1]$ it follows that:

$$\begin{aligned} w_\varepsilon^\alpha(p,0) > w_\varepsilon^\alpha(p,y) \quad &\forall y \in]0,1], \quad \text{if } p < \tilde{p};\\ w_\varepsilon^\alpha(p,1) > w_\varepsilon^\alpha(p,y) \quad &\forall y \in [0,1[, \quad \text{if } p > \tilde{p};\\ w_\varepsilon^\alpha(p,y) = 3/2 \quad &\forall y \in [0,1], \quad \text{if } p = \tilde{p}. \end{aligned}$$

Therefore, for every $\alpha \in [0,1]$, we have the equilibria: $p = 0$, $p = 1$ and $p = \tilde{p}$.

SUBCASE 1.2: If $\varepsilon \geqslant \frac{1}{4}$,

i) For $\alpha \in \left[0, 1 - \frac{1}{4\varepsilon}\right]$,

$$\begin{aligned} w_\varepsilon^\alpha(p,0) > w_\varepsilon^\alpha(p,y) \quad &\forall y \in]0,1], \quad \text{if } p < \hat{p};\\ w_\varepsilon^\alpha(p,1) > w_\varepsilon^\alpha(p,y) \quad &\forall y \in [0,1[, \quad \text{if } p > \hat{p};\\ w_\varepsilon^\alpha(p,y) = 3/2 \quad &\forall y \in [0,1], \quad \text{if } p = \hat{p}. \end{aligned}$$

Therefore, we get the equilibria: $p = 0$, $p = 1$, and $p = \hat{p}$.

ii) For $\alpha \in \left]1 - \frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right[$,

$$\begin{aligned} w_\varepsilon^\alpha(p,0) > w_\varepsilon^\alpha(p,y) \quad &\forall y \in]0,1], \quad \text{if } p < \tilde{p};\\ w_\varepsilon^\alpha(p,1) > w_\varepsilon^\alpha(p,y) \quad &\forall y \in [0,1[, \quad \text{if } p > \tilde{p};\\ w_\varepsilon^\alpha(p,y) = 3/2 \quad &\forall y \in [0,1], \quad \text{if } p = \tilde{p}. \end{aligned}$$

Therefore, we get the equilibria: $p = 0$, $p = 1$, and $p = \tilde{p}$.

iii) For $\alpha \in \left[\frac{1}{4\varepsilon}, 1\right]$,

$$\begin{aligned} w_\varepsilon^\alpha(p,0) > w_\varepsilon^\alpha(p,y) \quad &\forall y \in]0,1], \quad \text{if } p < p^*;\\ w_\varepsilon^\alpha(p,1) > w_\varepsilon^\alpha(p,y) \quad &\forall y \in [0,1[, \quad \text{if } p > p^*;\\ w_\varepsilon^\alpha(p,y) = 3/2 \quad &\forall y \in [0,1], \quad \text{if } p = p^*. \end{aligned}$$

Therefore, we get the equilibria: $p = 0$, $p = 1$, and $p = p^*$.

CASE 2. Suppose $\frac{1}{2} < \varepsilon \leqslant 1$. In this case the summary utility function takes the form:

$$w_\varepsilon^\alpha(p,y) = \begin{cases} y\left[2\alpha\left(p+\varepsilon\right)-1\right] - 3\alpha\left(p+\varepsilon\right)+3) & \text{if } 0 \leqslant p \leqslant 1-\varepsilon \\ y[2\alpha-1] - 3\alpha + 3 & \text{if } 1-\varepsilon < p < \varepsilon. \\ y\left[(p-\varepsilon)(2-2\alpha)+2\alpha-1\right] + (3-3\alpha)(1-p+\varepsilon) & \text{if } \varepsilon \leqslant p \leqslant 1, \end{cases}$$

Note that $0 < 1 - \frac{1}{2\varepsilon} \leqslant \frac{1}{2} \leqslant \frac{1}{2\varepsilon} < 1$; consider again $p^* = \frac{1}{2\alpha} - \varepsilon$ and $\hat{p} = \frac{1-2\alpha}{2-2\alpha} + \varepsilon$.

a) Denote with $h_1(y) := y\,[2\alpha\,(p+\varepsilon)-1] - 3\alpha\,(p+\varepsilon)+3)$. If $\alpha=0$, the function $h_1(y)$ is decreasing in the entire interval $[0,1]$. If $\alpha>0$, $h_1(y)$ is decreasing $[0,1]$ if $p<p^*$, constant in $[0,1]$ if $p=p^*$ and increasing in $[0,1]$ if $p>p^*$. Moreover, $p^*\geqslant 0$ if and only if $\alpha\in\left]0,\frac{1}{2\varepsilon}\right]$ while $p^*\leqslant 1-\varepsilon$ if and only if $\alpha\in\left[\frac{1}{2},1\right]$.

b) Denote with $h_2(y) := y[2\alpha-1]-3\alpha+3$; the function $h_2(y)$ is decreasing in the entire interval $[0,1]$ if $\alpha<\frac{1}{2}$, constant in the interval $[0,1]$ if $\alpha=\frac{1}{2}$ and increasing in the entire interval $[0,1]$ if $\alpha>\frac{1}{2}$.

c) Denote with $h_3(y) := y\,[(p-\varepsilon)(2-2\alpha)+2\alpha-1]+(3-3\alpha)(1-p+\varepsilon)$. If $\alpha=1$ the function $h_3(y)$ is increasing in the entire interval $[0,1]$. If $\alpha<1$ the function $h_3(y)$ is decreasing in $[0,1]$ for $p<\hat{p}$, constant in $[0,1]$ for $p=\hat{p}$ and increasing in the entire interval $[0,1]$ if $p>\hat{p}$. Moreover, $\hat{p}\leqslant 1$ if and only if $\alpha\in\left[1-\frac{1}{2\varepsilon},1\right[$ while $\hat{p}\geqslant\varepsilon$ if and only if $\alpha\in\left[0,\frac{1}{2}\right]$.

It follows that:

- If $\alpha\in\left[0,1-\frac{1}{2\varepsilon}\right[$, the function $y\to w_\varepsilon^\alpha(p,y)$ is decreasing in $[0,1]$ for every $p\in[0,1]$.

- If $\alpha=1-\frac{1}{2\varepsilon}$ then the function $y\to w_\varepsilon^\alpha(p,y)$ is decreasing in $[0,1]$ for $p<\hat{p}=1$, constant in $[0,1]$ for $p=\hat{p}=1$.

- If $\alpha\in\left]1-\frac{1}{2\varepsilon},\frac{1}{2}\right[$, the function $y\to w_\varepsilon^\alpha(p,y)$ is decreasing in $[0,1]$ for $p<\hat{p}$, constant in $[0,1]$ for $p=\hat{p}$, increasing in $[0,1]$ for $p>\hat{p}$.

- If $\alpha=\frac{1}{2}$, the function $y\to w_\varepsilon^\alpha(p,y)$ is decreasing in $[0,1]$ for $p<1-\varepsilon$, constant in $[0,1]$ for $1-\varepsilon\leqslant p\leqslant\varepsilon$, increasing in $[0,1]$ for $p>\varepsilon$.

- If $\alpha\in\left]\frac{1}{2},\frac{1}{2\varepsilon}\right[$, the function $y\to w_\varepsilon^\alpha(p,y)$ is decreasing in $[0,1]$ for $p<p^*$, constant in $[0,1]$ for $p=p^*$, increasing in $[0,1]$ for $p>p^*$.

- If $\alpha=\frac{1}{2\varepsilon}$ the function $y\to w_\varepsilon^\alpha(p,y)$ is increasing in $[0,1]$ for $p>p^*=0$, constant for $p=p^*=0$.

- If $\alpha\in\left]\frac{1}{2\varepsilon},1\right]$, the function $y\to w_\varepsilon^\alpha(p,y)$ is increasing in $[0,1]$ for every $p\in[0,1]$.

Equilibria are computed as follows:

i) For $\alpha\in\left[0,1-\frac{1}{2\varepsilon}\right[$,

$$w_\varepsilon^\alpha(p,0)>w_\varepsilon^\alpha(p,y)\quad\forall y\in]0,1],\quad\text{for all } p\in[0,1].$$

Therefore, we have only the equilibrium: $p=0$.

ii) For $\alpha \in \left[1 - \frac{1}{2\varepsilon}, \frac{1}{2}\right[$,

$$\begin{aligned} w_\varepsilon^\alpha(p,0) > w_\varepsilon^\alpha(p,y) \quad & \forall y \in]0,1], \quad \text{if } p < \hat{p}; \\ w_\varepsilon^\alpha(p,1) > w_\varepsilon^\alpha(p,y) \quad & \forall y \in [0,1[, \quad \text{if } p > \hat{p}; \\ w_\varepsilon^\alpha(p,y) = 3/2 \quad & \forall y \in [0,1], \quad \text{if } p = \hat{p}. \end{aligned}$$

Therefore we have the equilibria: $p = 0$, $p = 1$, and $p = \hat{p}$. Note that for $\alpha = 1 - \frac{1}{2\varepsilon}$ we get $\hat{p} = 1$.

iii) For $\alpha = \frac{1}{2}$,

$$\begin{aligned} w_\varepsilon^\alpha(p,0) > w_\varepsilon^\alpha(p,y) \quad & \forall y \in]0,1], \quad \text{if } p < 1-\varepsilon; \\ w_\varepsilon^\alpha(p,1) > w_\varepsilon^\alpha(p,y) \quad & \forall y \in [0,1[, \quad \text{if } p > \varepsilon; \\ w_\varepsilon^\alpha(p,y) = 3/2 \quad & \forall y \in [0,1], \quad \text{if } 1-\varepsilon \leqslant p \leqslant \varepsilon. \end{aligned}$$

In this case we have an infinite number of equilibria: $p = 0$, $p = 1$, and every $p \in [1-\varepsilon, \varepsilon]$.

iv) For $\alpha \in \left]\frac{1}{2}, \frac{1}{2\varepsilon}\right]$,

$$\begin{aligned} w_\varepsilon^\alpha(p,0) > w_\varepsilon^\alpha(p,y) \quad & \forall y \in]0,1], \quad \text{if } p < p^*; \\ w_\varepsilon^\alpha(p,1) > w_\varepsilon^\alpha(p,y) \quad & \forall y \in [0,1[, \quad \text{if } p > p^*; \\ w_\varepsilon^\alpha(p,y) = 3/2 \quad & \forall y \in [0,1], \quad \text{if } p = p^*. \end{aligned}$$

Therefore we have again three equilibria: $p = 0$, $p = 1$, and $p = p^*$. Note that for $\alpha = \frac{1}{2\varepsilon}$, $p^* = 1$.

v) For $\alpha \in \left]\frac{1}{2\varepsilon}, 1\right]$,

$$w_\varepsilon^\alpha(p,1) > w_\varepsilon^\alpha(p,y) \quad \forall y \in [0,1[, \quad \text{for all } p \in [0,1].$$

Therefore we have the unique equilibrium: $p = 1$.

Results are summarized in the following table, that is filled with the corresponding values of the parameter α which ensures the existence of the corresponding equilibrium.

	$p=0$	$p=1$	$p=\frac{1}{2}+\varepsilon(1-2\alpha)$	$p=\frac{1}{2\alpha}-\varepsilon$	$p=\frac{1-2\alpha}{2-2\alpha}+\varepsilon$	$p\in]1-\varepsilon,\varepsilon[$
$\varepsilon < \frac{1}{4}$	$[0,1]$	$[0,1]$	$[0,1]$	$\emptyset$	$\emptyset$	$\emptyset$
$\frac{1}{4} \leqslant \varepsilon \leqslant \frac{1}{2}$	$[0,1]$	$[0,1]$	$\left]1-\frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right[$	$\left[\frac{1}{4\varepsilon}, 1\right]$	$\left[0, 1-\frac{1}{4\varepsilon}\right]$	$\emptyset$
$\frac{1}{2} < \varepsilon \leq 1$	$\left[0, \frac{1}{2\varepsilon}\right]$	$[1-\frac{1}{2\varepsilon}, 1]$	$\emptyset$	$\left[\frac{1}{2}, \frac{1}{2\varepsilon}\right]$	$\left[1-\frac{1}{2\varepsilon}, \frac{1}{2}\right]$	$\{\frac{1}{2}\}$

Note that, for $\varepsilon = 0$, we obtain the three equilibria ($p = 1$, $p = 0$ and $p = \frac{1}{2}$) of the original game in [Geanakoplos et al., 1989] while for $\alpha = 1$ we get same equilibria as the model with maxmin preferences computed in [De Marco et al., 2022]. Note that, as ambiguity increases with ε, the set of equilibria in the two extreme cases $\alpha = 0$ and $\alpha = 1$ shrinks to an unique equilibrium for $\varepsilon > \frac{1}{2}$, but the two equilibria are different (i.e. $p = 1$ and $p = 0$ respectively). More generally, the table above shows that the difference among the different attitudes toward ambiguity becomes sharper as ε increases. In particular, the set of values of α that sustain a given equilibrium generally shrinks as ε converges to 1. There is one exception: for $\varepsilon > \frac{1}{2}$, the value $\alpha = \frac{1}{2}$ is a kind of singularity as it sustains the interval of equilibria $[1 - \varepsilon, \varepsilon]$. As a consequence, when $\alpha = \frac{1}{2}$, for $\varepsilon = 1$ all $p \in [0, 1]$ are α-PNE.

4 A Sensitivity Analysis

The example in the previous section shows some interesting features concerning the sensitivity of equilibria with respect to perturbations on the attitudes toward ambiguity. In particular, we notice that equilibria do not satisfy the *lower semicontinuity*-like stability, that is, an equilibrium cannot always be *approached* by a sequence of equilibria of perturbed games if we consider a perturbation on the parameter α. For example:

Example 4.1: Consider $\varepsilon = \frac{3}{4}$ and $\alpha = \frac{1}{2}$, we get that every $p \in [1 - \varepsilon, \varepsilon] = \left[\frac{1}{4}, \frac{3}{4}\right]$ is a α-PNE. In particular, pick $p = \frac{2}{5} \in \left[\frac{1}{4}, \frac{3}{4}\right]$. Now, fixing $\varepsilon = \frac{3}{4}$, consider a sequence $\{\alpha_\nu\}_{\nu \in \mathbb{N}}$ such that $\alpha_\nu \to \frac{1}{2}$ as $\nu \to \infty$ with $\alpha_\nu \neq \alpha$ for every ν. Since $\varepsilon = \frac{3}{4}$, the only α_ν-PNE, for α_ν sufficiently close to α are $p = 0$, $p = 1$, $p = \frac{1}{2\alpha_\nu} - \frac{3}{4}$ and $p = \frac{1-2\alpha_\nu}{2-2\alpha_\nu} + \frac{3}{4}$. It follows immediately that any converging sequence of α_ν-PNE might converge only to $p = 0$, $p = 1$, $p = \frac{1}{2\frac{1}{2}} - \frac{3}{4} = \frac{1}{4}$ and $p = \frac{1-2\frac{1}{2}}{2-2\frac{1}{2}} + \frac{3}{4} = \frac{3}{4}$. Therefore, $p = \frac{2}{5}$ cannot be approached by any sequence of α_ν-PNE.

Lack of lower semicontinuity-like stability is a common feature for equilibria in games. Example 3.4 in [De Marco et al., 2022] shows that in case of maxmin preferences (that is $\alpha = 1$) a psychological Nash equilibrium under ambiguity cannot always be *approached* by a sequence of equilibria of perturbed games if we consider perturbations on the parameter ε[5].

The previous example, in turn, shows that the set of equilibria satisfy an upper semicontinuity-like stability either if we consider a perturbation on the parameter α or a perturbation on the parameter ε: converging sequences of equilibria of perturbed games converge to equilibria of the unperturbed game as the perturbation *vanishes*. The issue of the upper semicontinuity properties of equilibria is a relevant topic in game theory and it has been largely investigated in the literature, under many different assumptions and for different solution concept (for instance, see [Yu, 1999], [Carbonell-Nicolau, 2010], [Scalzo, 2019] and references therein). Moreover, these properties are

[5] The game considered in Example 3.4 in [De Marco et al., 2022] is different from the one presented in the present paper; however ambiguous hierarchies of beliefs have the same structure.

key to build refinements of equilibria based on stability with respect to *trembles* on the strategies or on payoffs. In [De Marco et al., 2022], the upper semicontinuity property is investigated for equilibria in psychological games under ambiguity in case of maxmin preferences; in particular, the main result therein shows in which way ambiguous belief should converge to correct beliefs so that sequences of psychological equilibria under perturbation converge to psychological equilibria of the unperturbed game. In this paper, we extend this result looking also at the stability with respect to the attitudes toward ambiguity parametrized by the weights α_i.

In order to state the stability problem in a clear way, let us firstly construct a sequence of perturbed games:

DEFINITION 4.2: For every player i and for every $\nu \in \mathbb{N}$, let

a) $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ be a sequence of functions with $u_{i,\nu} : \overline{B}_i \times \Sigma \to \mathbb{R}$;

b) $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$ be a sequence of set-valued maps $\gamma_{i,\nu} : \Sigma \rightsquigarrow \overline{B}_i$;

c) $\{\alpha_\nu\}_{\nu\in\mathbb{N}}$ be a sequence with $\alpha_\nu = (\alpha_{1,\nu}, \dots, \alpha_{1,\nu}) \in [0, 1]^n$;

d) $\{U^{\alpha_\nu}_{i,\nu}\}_{\nu\in\mathbb{N}}$ be the sequence of functions $U^{\alpha_\nu}_{i,\nu} : \mathscr{K}_i \times \Sigma \to \mathbb{R}$ defined by

$$U^{\alpha_\nu}_{i,\nu}(K_i, \sigma) = \alpha_{i,\nu} \left[\inf_{b_i\in K_i} u_{i,\nu}(b_i, \sigma)\right] + (1 - \alpha_{i,\nu}) \left[\sup_{b_i\in K_i} u_{i,\nu}(b_i, \sigma)\right] \quad \forall (K_i, \sigma) \in \mathscr{K}_i \times \Sigma.$$

Then the sequence $\{G^{\alpha_\nu}_\nu\}_{\nu\in\mathbb{N}}$, with $G^{\alpha_\nu}_\nu = \left\{A_1, \cdots, A_n, U^{\alpha_\nu}_{1,\nu}, \cdots, U^{\alpha_\nu}_{n,\nu}\right\}$ for every $\nu \in \mathbb{N}$, is a sequence of α-MEU psychological games.

Therefore:

PROBLEM STATEMENT 4.3: Find conditions under which the sequence $\{G^{\alpha_\nu}_\nu\}_{\nu\in\mathbb{N}}$ converges to the game G^α so that any converging sequence $\{\sigma^*_\nu\}_{\nu\in\mathbb{N}}$ of α_ν-PNE of $G^{\alpha_\nu}_\nu$ has a limit σ^* that is a α-PNE of G^α.

In order to state and prove this limit result, we firstly recall definitions on variational convergence of sequences of functions and set-valued maps.

Preliminary definitions

We referred mainly to the paper [Lignola, Morgan, 1992] for the following definitions and results.

DEFINITION 4.4: Let X be a topological space. Consider a sequence of functions[6] $\{g_\nu\}_{\nu\in\mathbb{N}}$ with $g_\nu : X \subset \mathbb{R}^k \to \overline{\mathbb{R}}$ for every $\nu \in \mathbb{N}$ and a function $g : X \subset \mathbb{R}^k \to \overline{\mathbb{R}}$.

[6]For technical reasons, we consider the case where functions take values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

Then, the sequence $\{g_\nu\}_{\nu\in\mathbb{N}}$ *sequentially converges* (or continuously converges) to the function g if for every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X it follows that:

$$g(x) = \lim_{\nu\to\infty} g_\nu(x_\nu) = \limsup_{\nu\to\infty} g_\nu(x_\nu) = \liminf_{\nu\to\infty} g_\nu(x_\nu). \tag{5}$$

The next definition is devoted to set-valued maps.

DEFINITION 4.5: Let X and Y be metric spaces. Let $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ be a sequence of set-valued maps with $\Gamma_\nu : X \rightsquigarrow Y$ for every $\nu \in \mathbb{N}$ and let $\Gamma : X \rightsquigarrow Y$ be a set-valued map. Let $S(y, \varepsilon)$ be the ball in Y with center in y and radius ε and

$$\operatorname*{Lim\,inf}_{\nu\to\infty} \Gamma_\nu(x_\nu) = \{y \in Y \mid \forall\varepsilon > 0,\ \exists\overline{\nu} \text{ s.t. for all } \nu \geq \overline{\nu},\ S(y,\varepsilon) \cap \Gamma_\nu(x_\nu) \neq \emptyset\},$$

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \Gamma_\nu(x_\nu) = \{y \in Y \mid \forall\varepsilon > 0,\ \forall\overline{\nu},\ \exists\nu \geq \overline{\nu} \text{ s.t. } S(y,\varepsilon) \cap \Gamma_\nu(x_\nu) \neq \emptyset\}.$$

Then, the sequence $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ is *sequentially convergent* to Γ if, for every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X, it follows that:

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \Gamma_\nu(x_\nu) \subseteq \Gamma(x) \subseteq \operatorname*{Lim\,inf}_{\nu\to\infty} \Gamma_\nu(x_\nu).$$

The stability result

Now we can state the limit theorem.

THEOREM 4.6: *Let $G^\alpha = \{A_1, \cdots, A_n, U_1^\alpha, \cdots, U_n^\alpha\}$ be a α-MEU psychological game and $\{G_\nu^{\alpha_\nu}\}_{\nu\in\mathbb{N}}$ be a sequence of α-MEU psychological games as constructed in Definition 4.2. Assume that, for every player i,*

i) the sequence $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ sequentially converges to the function u_i; [7]

ii) each function $u_{i,\nu}$ and the function u_i are continuous in $\overline{B}_i \times \Sigma$;

iii) the sequence $\{\alpha_\nu\}_{\nu\in\mathbb{N}}$ converges to $\alpha = (\alpha_1, \ldots, \alpha_n)$;

iv) the sequence $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$ sequentially converges to the set-valued map γ_i. Suppose additionally that each $\gamma_{i,\nu}$ and γ_i have compact and not-empty values for every $\sigma \in \Sigma$.

If the sequence $\{\sigma_\nu^*\}_{\nu\in\mathbb{N}} \subset \Sigma$ converges to $\sigma^* \in \Sigma$ and $(\gamma_\nu(\sigma_\nu^*), \sigma_\nu^*)$ is a α-MEU psychological Nash equilibrium of $G_\nu^{\alpha_\nu}$, for every $\nu \in \mathbb{N}$, then it follows that $(\gamma(\sigma^*), \sigma^*)$ is a α-MEU psychological Nash equilibrium of G^α.

[7] The function u_i is the one appearing in the construction of U_i (see equation (2)).

Proof. For every player i and every $\nu \in \mathbb{N}$, let $w_{i,\nu}^{\alpha_\nu}$ be the summary utility function of the game $G_\nu^{\alpha_\nu}$, that is

$$w_{i,\nu}^{\alpha_\nu}(\sigma, \tau) := \alpha_{i,\nu} \left[\inf_{b_i \in \gamma_{i,\nu}(\sigma)} u_{i,\nu}(b_i, \tau) \right] + (1 - \alpha_{i,\nu}) \left[\sup_{b_i \in \gamma_{i,\nu}(\sigma)} u_{i,\nu}(b_i, \tau) \right] \quad \forall (\sigma, \tau) \in \Sigma \times \Sigma,$$

and w_i^α be the summary utility function of the game G^α, that is

$$w_i^\alpha(\sigma, \tau) := \alpha_i \left[\inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau) \right] + (1 - \alpha_i) \left[\sup_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau) \right] \quad \forall (\sigma, \tau) \in \Sigma \times \Sigma.$$

The continuous convergence of the sequence of functions $\{w_{i,\nu}^{\alpha_\nu}\}_{\nu \in \mathbb{N}}$ to the function w_i^α, for every $i \in I$, guarantees the result. In fact, if $\{\sigma_\nu^*\}_{\nu \in \mathbb{N}} \subset \Sigma$ is a sequence converging to $\sigma^* \in \Sigma$ such that, for every $\nu \in \mathbb{N}$, $(\gamma_\nu(\sigma_\nu^*), \sigma_\nu^*)$ is a α-MEU psychological Nash equilibrium of $G_\nu^{\alpha_\nu}$, then it follows that, for every player i,

$$w_{i,\nu}^{\alpha_\nu}(\sigma_\nu^*, \sigma_\nu^*) \geqslant w_{i,\nu}^{\alpha_\nu}(\sigma_\nu^*, (y_i, \sigma_{-i,\nu}^*)) \quad \forall y_i \in \Sigma_i.$$

Applying the continuous convergence of $\{w_{i,\nu}^{\alpha_\nu}\}_{\nu \in \mathbb{N}}$ to w_i^α we get

$$w_i^\alpha(\sigma^*, \sigma^*) = \lim_{\nu \to \infty} w_{i,\nu}^{\alpha_\nu}(\sigma_\nu^*, \sigma_\nu^*) \geqslant \lim_{\nu \to \infty} w_{i,\nu}^{\alpha_\nu}(\sigma_\nu^*, (y_i, \sigma_{-i,\nu}^*)) = w_i^\alpha(\sigma^*, (y_i, \sigma_{-i}^*)) \quad \forall y_i \in \Sigma_i.$$

This latter inequality implies that $(\gamma(\sigma^*), \sigma^*)$ is a α-MEU psychological Nash equilibrium of G^α. Therefore, the proof reduces in verifying the continuous convergence of $\{w_{i,\nu}^{\alpha_\nu}\}_{\nu \in \mathbb{N}}$ to w_i^α. That is, we need to check that for every $(\sigma, \tau) \in \Sigma \times \Sigma$ and for every sequence $\{(\sigma_\nu, \tau_\nu)\}_{\nu \in \mathbb{N}}$ converging to (σ, τ) we get the inequalities

$$\limsup_{\nu \to \infty} w_{i,\nu}^{\alpha_\nu}(\sigma_\nu, \tau_\nu) \leqslant w_i^\alpha(\sigma, \tau) \leqslant \liminf_{\nu \to \infty} w_{i,\nu}^{\alpha_\nu}(\sigma_\nu, \tau_\nu). \tag{6}$$

Denote with

$$w_{i,\nu}^m(\sigma, \tau) = \inf_{b_i \in \gamma_{i,\nu}(\sigma)} u_{i,\nu}(b_i, \tau), \; w_{i,\nu}^M(\sigma, \tau) = \sup_{b_i \in \gamma_{i,\nu}(\sigma)} u_{i,\nu}(b_i, \tau),$$

and

$$w_i^m(\sigma, \tau) = \inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau), \; w_i^M(\sigma, \tau) = \sup_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau).$$

Consider $(\sigma, \tau) \in \Sigma \times \Sigma$ and take a sequence $\{(\sigma_\nu, \tau_\nu)\}_{\nu \in \mathbb{N}}$ converging to (σ, τ). Now we prove that

$$\limsup_{\nu \to \infty} w_{i,\nu}^m(\sigma_\nu, \tau_\nu) \leqslant w_i^m(\sigma, \tau) \leqslant \liminf_{\nu \to \infty} w_{i,\nu}^m(\sigma_\nu, \tau_\nu),$$

and

$$\limsup_{\nu \to \infty} w_{i,\nu}^M(\sigma_\nu, \tau_\nu) \leqslant w_i^M(\sigma, \tau) \leqslant \liminf_{\nu \to \infty} w_{i,\nu}^M(\sigma_\nu, \tau_\nu).$$

Firstly, we show that

$$w_i^m(\sigma,\tau) \leqslant \liminf_{\nu\to\infty} w_{i,\nu}^m(\sigma_\nu,\tau_\nu) \quad \left(\text{resp. } w_i^M(\sigma,\tau) \geqslant \limsup_{\nu\to\infty} w_{i,\nu}^m(\sigma_\nu,\tau_\nu)\right).$$

Suppose by contradiction that

$$w_i^m(\sigma,\tau) > \liminf_{\nu\to\infty} w_{i,\nu}^m(\sigma_\nu,\tau_\nu) \quad \left(\text{resp. } w_i^M(\sigma,\tau) < \limsup_{\nu\to\infty} w_{i,\nu}^m(\sigma_\nu,\tau_\nu)\right). \tag{7}$$

This means that along a subsequence $\{(\sigma_{\nu_k},\tau_{\nu_k})\}_{k\in\mathbb{N}}$ we get

$$\lim_{k\to\infty} w_{i,\nu_k}^m(\sigma_{\nu_k},\tau_{\nu_k}) < w_i^m(\sigma,\tau) \quad \left(\text{resp. } \lim_{k\to\infty} w_{i,\nu_k}^M(\sigma_{\nu_k},\tau_{\nu_k}) > w_i^M(\sigma,\tau)\right). \tag{8}$$

Additionally, continuity of u_i and $u_{i,\nu}$ for every ν and compactness of the images of γ_i and $\gamma_{i,\nu}$, for every ν, guarantees that there exist $b_i^m \in \gamma_i(\sigma)$ and $b_{i,\nu}^m \in \gamma_{i,\nu}(\sigma_\nu)$, (resp. $b_i^M \in \gamma_i(\sigma)$ and $b_{i,\nu}^M \in \gamma_{i,\nu}(\sigma_\nu)$), for every ν, such that

$$w_i^m(\sigma,\tau) = u_i(b_i^m,\tau) = \inf_{b_i\in\gamma_i(\sigma)} u_i(b_i,\tau), \quad \left(\text{resp. } w_i^M(\sigma,\tau) = u_i(b_i^M,\tau) = \sup_{b_i\in\gamma_i(\sigma)} u_i(b_i,\tau)\right)$$

and

$$w_{i,\nu}^m(\sigma_\nu,\tau_\nu) = u_{i,\nu}(b_{i,\nu}^m,\tau_\nu) = \inf_{b_{i,\nu}\in\gamma_{i,\nu}(\sigma_\nu)} u_{i,\nu}(b_{i,\nu},\tau_\nu),$$

$$\left(\text{resp. } w_{i,\nu}^M(\sigma_\nu,\tau_\nu) = u_{i,\nu}(b_{i,\nu}^M,\tau_\nu) = \sup_{b_{i,\nu}\in\gamma_{i,\nu}(\sigma_\nu)} u_{i,\nu}(b_{i,\nu},\tau_\nu)\right).$$

Consider the sequence of beliefs $\{b_{i,\nu_k}^m\}_{k\in\mathbb{N}}$, (resp. $\{b_{i,\nu_k}^M\}_{k\in\mathbb{N}}$), obtained along the subsequence $\{(\sigma_{\nu_k},\tau_{\nu_k})\}_{k\in\mathbb{N}}$ as in (8). The sequence $\{b_{i,\nu_k}^m\}_{k\in\mathbb{N}}$, (resp. $\{b_{i,\nu_k}^M\}_{k\in\mathbb{N}}$), has a subsequence $\{b_{i,\nu_h}^m\}_{h\in\mathbb{N}}$, (resp. $\{b_{i,\nu_h}^M\}_{h\in\mathbb{N}}$), which converges to a point $\hat{b}_i^m \in \overline{B}_i$, (resp. $\hat{b}_i^M \in \overline{B}_i$), since $\overline{B}_i$ is compact. The point $\hat{b}_i^m$, (resp. $\hat{b}_i^M$), actually belong to $\gamma_i(\sigma)$. In fact, by definition, the upper limit $\operatorname{Lim\,sup}_{\nu\to\infty} \gamma_{i,\nu}(\sigma_\nu)$ contains the limits of every converging subsequence of $\{b_{i,\nu_k}^m\}_{k\in\mathbb{N}}$, (resp. $\{b_{i,\nu_k}^M\}_{k\in\mathbb{N}}$); that is

$$\hat{b}_i^m, \hat{b}_i^M \in \operatorname*{Lim\,sup}_{\nu\to\infty} \gamma_{i,\nu}(\sigma_\nu).$$

Moreover $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$ is sequentially upper convergent to γ_i, meaning that $\operatorname*{Lim\,sup}_{\nu\to\infty} \gamma_{i,\nu}(\sigma_\nu) \subseteq \gamma_i(\sigma)$; therefore, $\hat{b}_i^m, \hat{b}_i^M \in \gamma_i(\sigma)$. By construction $u_i(b_i^m,\tau) \leqslant u_i(\hat{b}_i^m,\tau)$ (resp. $u_i(b_i^M,\tau) \geqslant u_i(\hat{b}_i^M,\tau)$). The sequence $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ sequentially converges to u_i; since $(b_{i,\nu_h}^m,\tau_{\nu_h}) \to (\hat{b}_i^m,\tau)$, (resp. $(b_{i,\nu_h}^M,\tau_{\nu_h}) \to (\hat{b}_i^M,\tau)$), we get :

$$u_i(\hat{b}_i^m,\tau) = \lim_{h\to\infty} u_{i,\nu_h}(b_{i,\nu_h}^m,\tau_{\nu_h}), \quad \left(\text{resp. } u_i(\hat{b}_i^M,\tau) = \lim_{h\to\infty} u_{i,\nu_h}(b_{i,\nu_h}^M,\tau_{\nu_h})\right).$$

Hence

$$w_i^m(\sigma,\tau) = u_i(b_i^m,\tau) \leqslant u_i(\hat{b}_i^m,\tau) = \lim_{h\to\infty} u_{i,\nu_h}(b_{i,\nu_h}^m,\tau_{\nu_h}) = \lim_{h\to\infty} w_{i,\nu_h}^m(\sigma_{\nu_h},\tau_{\nu_h}),$$

$$\left(\text{resp. } w_i^M(\sigma,\tau) = u_i(b_i^M,\tau) \geqslant u_i(\hat{b}_i^M,\tau) = \lim_{h\to\infty} u_{i,\nu_h}(b_{i,\nu_h}^M,\tau_{\nu_h}) = \lim_{h\to\infty} w_{i,\nu_h}^M(\sigma_{\nu_h},\tau_{\nu_h})\right).$$

Then, inequality (8) implies that

$$w_i^m(\sigma,\tau) \leqslant \lim_{h\to\infty} w_{i,\nu_h}^m(\sigma_{\nu_h},\tau_{\nu_h}) < w_i^m(\sigma,\tau),$$

$$\left(\text{resp. } w_i^M(\sigma,\tau) \geqslant \lim_{h\to\infty} w_{i,\nu_h}^M(\sigma_{\nu_h},\tau_{\nu_h}) > w_i^M(\sigma,\tau)\right),$$

which results in a contradiction. So

$$w_i^m(\sigma,\tau) \leqslant \liminf_{\nu\to\infty} w_{i,\nu}^m(\sigma_\nu,\tau_\nu), \quad \left(\text{resp. } w_i^M(\sigma,\tau) \geqslant \limsup_{\nu\to\infty} w_{i,\nu}^M(\sigma_\nu,\tau_\nu)\right).$$

Now we show that

$$w_i^m(\sigma,\tau) \geqslant \limsup_{\nu\to\infty} w_{i,\nu}^m(\sigma_\nu,\tau_\nu), \quad \left(\text{resp. } w_i^M(\sigma,\tau) \leqslant \liminf_{\nu\to\infty} w_{i,\nu}^M(\sigma_\nu,\tau_\nu)\right).$$

Let $b_i^m \in \gamma_i(\sigma)$ (resp. $b_i^M \in \gamma_i(\sigma)$) be such that

$$u_i(b_i^m,\tau) = \inf_{b_i\in\gamma_i(\sigma)} u_i(b_i,\tau) = w_i^m(\sigma,\tau) \quad \left(\text{resp. } u_i(b_i^M,\tau) = \sup_{b_i\in\gamma_i(\sigma)} u_i(b_i,\tau) = w_i^M(\sigma,\tau)\right).$$

The points b_i^m and b_i^M exist because of the continuity of u_i and the compactness of $\gamma_i(\sigma)$ for every $\sigma \in \Sigma$. Since the sequence $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$ is sequentially convergent to γ_i, that is,

$$\gamma_i(\sigma) \subseteq \operatorname*{Lim\,inf}_{\nu\to\infty} \gamma_{i,\nu}(\sigma_\nu),$$

then, by definition, there exists a sequence $\{\hat{b}_{i,\nu}^m\}_{\nu\in\mathbb{N}}$ converging to b_i^m, (resp. $\{\hat{b}_{i,\nu}^M\}_{\nu\in\mathbb{N}}$ converging to b_i^M), such that, for every ν, $\hat{b}_{i,\nu}^m \in \gamma_{i,\nu}(\sigma_\nu)$ (resp. $\hat{b}_{i,\nu}^M \in \gamma_{i,\nu}(\sigma_\nu)$).

The sequence $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ sequentially converges to u_i; it follows that

$$\limsup_{\nu\to\infty} u_{i,\nu}(\hat{b}_{i,\nu}^m,\tau_\nu) \leqslant u_i(b_i^m,\tau), \quad \left(\text{resp. } \liminf_{\nu\to\infty} u_{i,\nu}(\hat{b}_{i,\nu}^M,\tau_\nu) \geqslant u_i(b_i^M,\tau)\right).$$

By construction, for every $\nu \in \mathbb{N}$, it follows that

$$w_{i,\nu}^m(\sigma_\nu,\tau_\nu) \leqslant u_{i,\nu}(\hat{b}_{i,\nu}^m,\tau_\nu), \quad \left(\text{resp. } w_{i,\nu}^M(\sigma_\nu,\tau_\nu) \geqslant u_{i,\nu}(\hat{b}_{i,\nu}^M,\tau_\nu)\right).$$

This finally implies that

$$\limsup_{\nu\to\infty} w_{i,\nu}^m(\sigma_\nu,\tau_\nu) \leqslant \limsup_{\nu\to\infty} u_{i,\nu}(\hat{b}_{i,\nu}^m,\tau_\nu) \leqslant u_i(b_i^m,\tau) = w_i^m(\sigma,\tau),$$

$$\left(\text{resp. } \liminf_{\nu\to\infty} w_{i,\nu}^M(\sigma_\nu,\tau_\nu) \geqslant \limsup_{\nu\to\infty} u_{i,\nu}(\hat{b}_{i,\nu}^M,\tau_\nu) \geqslant u_i(b_i^M,\tau) = w_i^M(\sigma,\tau)\right)$$

So we finally get

$$\limsup_{\nu\to\infty} w_{i,\nu}^m(\sigma_\nu,\tau_\nu) \leqslant w_i^m(\sigma,\tau) \leqslant \liminf_{\nu\to\infty} w_{i,\nu}^m(\sigma_\nu,\tau_\nu)$$

and

$$\limsup_{\nu\to\infty} w_{i,\nu}^M(\sigma_\nu,\tau_\nu) \leqslant w_i^M(\sigma,\tau) \leqslant \liminf_{\nu\to\infty} w_{i,\nu}^M(\sigma_\nu,\tau_\nu).$$

Hence, from the properties of upper and lower limits we get

$$\limsup_{\nu\to\infty} w_{i,\nu}^{\alpha_\nu}(\sigma_\nu,\tau_\nu) = \limsup_{\nu\to\infty}\left[\alpha_{i,\nu} w_{i,\nu}^m(\sigma_\nu,\tau_\nu) + (1-\alpha_{i,\nu}) w_{i,\nu}^M(\sigma_\nu,\tau_\nu)\right] \leqslant$$

$$\limsup_{\nu\to\infty} \alpha_{i,\nu} w_{i,\nu}^m(\sigma_\nu,\tau_\nu) + \limsup_{\nu\to\infty}(1-\alpha_{i,\nu}) w_{i,\nu}^M(\sigma_\nu,\tau_\nu) \leqslant$$

$$\left(\limsup_{\nu\to\infty} \alpha_{i,\nu}\right)\left(\limsup_{\nu\to\infty} w_{i,\nu}^m(\sigma_\nu,\tau_\nu)\right) + \left(\limsup_{\nu\to\infty}(1-\alpha_{i,\nu})\right)\left(\limsup_{\nu\to\infty} w_{i,\nu}^M(\sigma_\nu,\tau_\nu)\right) \leqslant$$

$$\alpha_i w_i^m(\sigma,\tau) + (1-\alpha_i) w_i^M(\sigma,\tau),$$

and

$$\alpha_i w_i^m(\sigma,\tau) + (1-\alpha_i) w_i^M(\sigma,\tau) \leqslant$$

$$\left(\liminf_{\nu\to\infty} \alpha_{i,\nu}\right)\left(\liminf_{\nu\to\infty} w_{i,\nu}^m(\sigma_\nu,\tau_\nu)\right) + \left(\liminf_{\nu\to\infty}(1-\alpha_{i,\nu})\right)\left(\liminf_{\nu\to\infty} w_{i,\nu}^M(\sigma_\nu,\tau_\nu)\right) \leqslant$$

$$\liminf_{\nu\to\infty} \alpha_{i,\nu} w_{i,\nu}^m(\sigma_\nu,\tau_\nu) + \liminf_{\nu\to\infty}(1-\alpha_{i,\nu}) w_{i,\nu}^M(\sigma_\nu,\tau_\nu) \leqslant$$

$$\liminf_{\nu\to\infty}\left[\alpha_{i,\nu} w_{i,\nu}^m(\sigma_\nu,\tau_\nu) + (1-\alpha_{i,\nu}) w_{i,\nu}^M(\sigma_\nu,\tau_\nu)\right] = \liminf_{\nu\to\infty} w_{i,\nu}^{\alpha_\nu}(\sigma_\nu,\tau_\nu).$$

Condition (6) is satisfied and $\{w_{i,\nu}^{\alpha_\nu}\}_{\nu\in\mathbb{N}}$ continuously converges to w_i^α. □

5 Existence of Equilibria: A Counterexample

Differently from [Geanakoplos et al., 1989] and [De Marco et al., 2022] in which an existence theorem was proved respectively for psychological Nash equilibria and psychological Nash equilibria under ambiguity (in case of maxmin preferences), here we cannot provide an analogous result for α-PNE. In fact, existence fails in very simple examples as the one shown below. For the sake of simplicity, in the example we consider the extreme form of ambiguity, given by full ignorance, and the extreme form of optimism, that is $\alpha = 0$.

Example 5.1: We consider a two player game: the pure strategy set of Player 1 (Anne) is $A_1 = \{\textit{Accept}, \textit{Reject}\}$ and the pure strategy set of Player 2 (John) is $A_2 = \{\textit{Accept}, \textit{Reject}\}$. We denote with p the mixed strategy of Player 1, where, with an abuse of notation, p is the probability of *Accept* and $1-p$ is the probability of *Reject*. Similarly r is the mixed strategy of Player 2;

again, with an abuse of notation, r is the probability of *Accept* and $1 - r$ is the probability of *Reject*. It is assumed that John's utility does not depend on beliefs while Anne's utility depends on her second-order beliefs. Moreover, as done in the previous example, it is considered the case in which only the expectations of beliefs play a role in Anne's utility function. We denote with $q \in [0, 1]$ the expectation of John's first-order beliefs about Anne's mixed strategy p and $\tilde{q} \in [0, 1]$ the expectation of Anne's second-order beliefs about the expectation q of John's first-order beliefs. The game is represented below:

Anne \ John	*Accept*	*Reject*
Accept	$-2\tilde{q} - 1, 0$	$2, 1$
Reject	$2, 1$	$2\tilde{q} - 3, 0$

A mixed strategy profile is identified by the pair (p, r). The correct belief functions simply map the strategy profiles (p, r) to correct expectation of beliefs; more precisely, $\beta_1(p, r) = p$ tells that the expectation of John's correct first-order beliefs about Anne's strategy p must be equal to p and $\beta_2(p, r) = p$ tells that the expectation of Anne's correct second-order beliefs about John's first-order belief about Anne's strategy p must be equal to p as well.

The expected utility for Anne (Player 1) having second-order belief $\tilde{q}$ and given the mixed strategy profile (p, r) is

$$u_1(\tilde{q}, (p, r)) = -8pr + (5 - 2\tilde{q})p + (5 - 2\tilde{q})r + 2\tilde{q} - 3 = 2\tilde{q}[1 - p - r] - 8pr + 5p + 5r - 3$$

We consider the case in which there is full ambiguity about Anne's second-order beliefs. More precisely, Anne's second order belief is given by $\gamma_1(p, r) = [0, 1]$ for every strategy profile (p, r). Moreover, we focus on the case in which Anne is ambiguity seeking, that is $\alpha_1 = 0$. For every pair of strategy profiles (p, r) and (x, y), we have that:

$$U_1^{\alpha}(\gamma_1(x, y), (p, r)) = \max_{\tilde{q} \in \gamma_1(x,y)} u_1(\tilde{q}, (p, r)).$$

We get

$$\arg\max_{\tilde{q} \in \gamma_1(x,y)} u_1(\tilde{q}, (p, r)) = \arg\max_{\tilde{q} \in [0,1]} u_1(\tilde{q}, (p, r)) =$$

$$\arg\max_{\tilde{q} \in [0,1]} [2\tilde{q}(1 - p - r) - 8pr + 5p + 5r - 3] = \begin{cases} 1 & \text{if } p < 1 - r, \\ [0, 1] & \text{if } p = 1 - r, \\ 0 & \text{if } p > 1 - r. \end{cases}$$

Therefore, given the two strategy profiles (x,y) and (p,r),

$$w_1^\alpha((x,y),(p,r)) = U_1^\alpha(\gamma_1(x,y),(p,r)) =$$

$$\max_{\tilde{q}\in[0,1]}[2\tilde{q}(1-p-r) - 8pr + 5p + 5r - 3] = \begin{cases} -8pr + 3p + 3r - 1 & \text{if } p < 1-r, \\ -8pr + 5p + 5r - 3 & \text{if } p \geqslant 1-r. \end{cases}$$

Now, for every (x,y) it follows that:

- If $r < 3/8$, $w_1^\alpha((x,y),(\cdot,r))$ is strictly increasing in $[0,1]$ and attains the maximum in $p=1$.
- If $3/8 \leqslant r < 1/2$, $w_1^\alpha((x,y),(\cdot,r))$ is weakly decreasing in $[0,1-r]$ and strictly increasing in $[1-r,1]$ and attains the maximum in $p=1$.
- If $r = 1/2$, $w_1^\alpha((x,y),(\cdot,r))$ is strictly decreasing in $[0,1/2]$ and strictly increasing in $[1/2,1]$ and attains the maximum in $p=0$ and $p=1$.
- If $1/2 < r \leqslant 5/8$, $w_1^\alpha((x,y),(\cdot,r))$ is strictly decreasing in $[0,1-r]$ and weakly increasing in $[1-r,1]$ and attains the maximum for $p=0$.
- If $5/8 < r \leqslant 1$, $w_1^\alpha((x,y),(p,r))$ is strictly decreasing in $[0,1]$ and attains the maximum in $p=0$.

Now, let

$$BR_1^\alpha(r) = \{p \in \Sigma_1 \,|\, w_1^\alpha((p,r),(p,r)) \geqslant w_1^\alpha((p,r),(x,r)),\ \forall x \in \Sigma_i\}.$$

It follows that

$$BR_1^\alpha(r) = \begin{cases} 1 & \text{if } r \in [0,1/2[, \\ \{0,1\} & \text{if } r = 1/2, \\ 0 & \text{if } r \in]1/2,1]. \end{cases}$$

On the other hand, the best reply of Player 2 can be easily computed, as there are no psychological effects:

$$BR_2(p) = \begin{cases} 0 & \text{if } p \in [0,1/2[, \\ [0,1] & \text{if } p = 1/2, \\ 1 & \text{if } p \in]1/2,1]. \end{cases}$$

It immediately follows that this game has no equilibria as there are no fixed points for these best reply correspondences.

Remark 5.2: It is natural to imagine that lack of a general existence theorem depends on a general lack of convexity of the images of the best reply correspondences. This is actually true, but it is useful to understand what kind of best reply we refer to. To this purpose, let $w_i(\sigma,\tau)$ be the summary utility function of player i (it can be the one in [Geanakoplos et al., 1989] or in [De Marco et al., 2022] or the general w_i^α considered in this work). Then there are two possible best replies that can be defined for a player i:

(1) $\overline{BR}_i : \Sigma_{-i} \rightsquigarrow \Sigma_i$, where

$$\overline{BR}_i(\sigma^*_{-i}) = \left\{\sigma_i \in \Sigma_i \mid w_i((\sigma_i, \sigma^*_{-i}), (\sigma_i, \sigma^*_{-i})) \geqslant w_i((\sigma_i, \sigma^*_{-i}), (\tau_i, \sigma^*_{-i}))\ \forall \tau_i \in \Sigma_i\right\},$$

(2) $BR_i : \Sigma \rightsquigarrow \Sigma_i$, where

$$BR_i(\sigma^*) = \left\{\sigma_i \in \Sigma_i \mid w_i((\sigma^*_i, \sigma^*_{-i}), (\sigma_i, \sigma^*_{-i})) \geqslant w_i((\sigma^*_i, \sigma^*_{-i}), (\tau_i, \sigma^*_{-i}))\ \forall \tau_i \in \Sigma_i\right\}.$$

It follows from the definition that σ^* is a psychological Nash equilibrium if and only if it is a fixed point for the set-valued map (1) $\overline{BR}_1 \times \cdots \times \overline{BR}_n$, or (2) $BR_1 \times \cdots \times BR_n$. Now, in the examples in [Geanakoplos et al., 1989] or in [De Marco et al., 2022] the set-valued maps $\overline{BR}_i$ do not have convex images even if for these games the equilibrium existence theorem holds. In fact, the existence theorem follows from the convexity of the images of the set-valued maps BR_i, which for the models in [Geanakoplos et al., 1989] or in [De Marco et al., 2022] results to be guaranteed. On the contrary, in the example above it is also the set-valued map BR_i that has not convex images, leading to the nonexistence of equilibria.

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