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Ritesh Jain[*], Ville Korpela[†], and Michele Lombardi[‡]

Abstract
The paper characterizes the class of two-player social choice functions implementable in rationalizable strategies. We offer two equivalent conditions, Two-Player Generalized Strict Maskin Monotonicity** and Partition Monotonicity. Similar to Bergemann et al. (2011) and Xiong (2022), Two-Player Generalized Strict Maskin Monotonicity** relies on the existence of a partition of the set of states. However, Partition Monotonicity provides a construction for the partition.

Keywords: Implementation, Two Players, Rationalizability, Complete Information.

JEL classification: C79, D82.

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1. Introduction

The main objective of implementation theory is to study conditions under which we can design a mechanism within which, at every state (of the world), the outcome of players' strategic interaction coincides with the outcome prescribed by a given social choice function (SCF) for that state. Players' strategic interaction is modeled via game-theoretic solution concepts, each giving rise to a different notion of implementation.

Following Palfrey (2002), we can divide the implementation problem into two components. The first component is *incentive compatibility.* The mechanism must be devised so that players' incentives give rise to an outcome that coincides with the goal set by the SCF. The second component is *uniqueness.* The mechanism must be devised so that players' incentives never give rise to an outcome that does not coincide with the goal set by the SCF. There is tension between these two components when there are only two players (see, for instance, Hurwicz and Schmeidler (1978) and Maskin (1999)) or when information is incomplete among players (see, for instance, Jackson (1991), Oury and Tercieux (2012), and Jain and Lombardi (2022)). However, incentive compatibility is not an issue when information is complete and there are three or more players.[1]

The idea of Nash equilibrium is fundamental to much of economic theory. An extensive literature on implementation theory assumes Nash equilibrium as the solution concept.[2] However, Nash equilibrium relies on the assumption that each player correctly predicts the strategic choices of his opponents. If players' conjectures are not correct and players' rationality is common knowledge among players, then (corre-

[1]Information is complete when players preferences and possible outcomes are common knowledge among all the players.

[2]Maskin (1999; circulated since 1977) shows that only Maskin monotonic SCFs are Nash implementable. He also shows that when there are three or more players, an SCF is Nash implementable if it is Maskin monotonic and satisfies the condition of no veto-power. Full characterization results for Nash implementation can be found, for example, in Moore and Repullo (1990), Dutta and Sen (1991), Sjostrom (1991), Saijo et al. (1996), Saijo et al. (2007), Danilov (1992), Yamato (1992), and Lombardi and Yoshihara (2013).

lated) rationalizability (Brandenburger and Dekel (1987)) is the appropriate solution concept.

In complete information environments with three or more players, Bergemann et al. (2011) (BMT, henceforth) address this issue by studying implementation problems in rationalizable strategies. In this setup, Xiong (2022) provides a necessary and sufficient condition for the rationalizable implementation of SCFs. The condition is referred to as Strict Event Monotonicity** (SEM**, henceforth). Their conditions offer powerful insights into ensuring that only outcomes consistent with the SCF are implementable in rationalizable strategies.[3]

However, these studies have two significant limitations. First, since they study rationalizable implementation in complete information environments with more than two players, they avoid incentive compatibility issues arising in two-player cases. Incentive compatibility issues make the two-player implementation problems harder than the many-player problems. Second, although their constructive proofs employ stochastic mechanisms, their necessary and sufficient conditions rely on the existence of a partition of the set of states Θ. This feature contrasts with the Nash implementation literature, where stochastic mechanisms allow getting rid of the existential clauses underlying the necessary and sufficient conditions for Nash implementation (see, for instance, Bochet (2007), and Benoît and Ok (2008)).[4] Finally, from a practical standpoint, their conditions are difficult to check as the number of partitions of Θ grows exponentially with the size of Θ.[5]

[3]For multi-valued social choice correspondences, implementation in rationalizable strategies is studied in Kunimoto and Serrano (2019) and Jain (2021). See Section 7 for more details.

[4]The existential clause also appears in the characterization offered in Jain (2021). For instance, similar to BMTs condition, Θ_F distinguishability of Jain (2021) relies on the existence of a partition of the set of states Θ (see Definition 4.2, p,52.)

[5]In combinatorial mathematics, the number of partitions of a set of size n is referred to as bell number. Bell numbers can be recursively defined as follows: $B(1) = 1$ and for every $n+1$,

$$B(n) = \sum_{k=0}^{n} \binom{n}{k} B(k).$$

Motivated by the above reasoning, the paper aims to answer two questions: What SCFs can we implement if only two players exist? Can we provide a characterization of rationalizable implementation that does not rely on any existential clause, irrespective of the number of players? We offer complete answers to these questions by providing two equivalent characterizations. The first characterization is given using a condition called Two-Player Generalized Strict Maskin Monotonicity** (2P-GSSM**). This condition is directly comparable with SEM**. Similar to SEM**, 2P-GSMM** also relies on an existential clause. The second characterization is obtained by using a condition called Partition Monotonicity. This condition is equivalent to 2P-GSMM** but it does not rely on any existential clause.[6]

The study of two-player implementation problems has always been at the heart of implementation theory. For instance, a classical impossibility result can be found in Hurwicz and Schmeidler (1978) and Maskin (1999), which show that any Pareto-optimal two-player multi-valued SCF that is Nash implementable is dictatorial if the domain of preferences is unrestricted. Laslier et al. (2021)) provides a solution to this classical problem by considering deterministic mechanisms only on the equilibrium path. Moreover, the understanding of two-player problems has a bearing on a wide variety of bilateral contracting and negotiating problems (Moore and Repullo (1988), Moore and Repullo (1990), and Dutta and Sen (1991)). For instance, De Clippel et al. (2014) studies the problem of the selection of arbitrators from the perspective of implementation theory in a setting with complete information and no monetary transfers. Note that arbitrator selection involves only two parties.

Seminal works on two-player Nash implementation are Moore and Repullo (1990),

[6]Xiong (2022) also provides a condition that is equivalent to SEM**, which is referred to as Strict Iterated Elimination Monotonicity (henceforth, SIEM). In contrast to SEM**, SIEM embeds the iterative logic of rationalizability into the implementing condition. Its main role is to clarify the relationship between rationalizability and SEM**. In environments with only two players, we provide in Jain et al. (2021) an iterative implementing condition, termed Iterative Monotonicity. Unlike 2P-GSMM**, Iterative Monotonicity is not directly comparable with SEM** and SMM**. Finally, the underlying algorithm that defines Partition Monotonicity is much more transparent and short than the algorithm that defines Iterative Monotonicity.

Dutta and Sen (1991) and Sjostrom (1991). Moore and Repullo (1990) and Dutta and Sen (1991) provide a full characterization of the class of Nash implementable functions, whereas Sjostrom (1991) provides a constructive way of checking whether or not an SCF can be implemented in Nash equilibria. Their condition comes from recognizing that two-player implementation requires a mechanism to distinguish the true state when the two players report distinct messages. Thus, any implementable SCF needs to satisfy a two-player condition. The condition requires that a "punishment outcome" exists when players report distinct messages. Moreover, it requires that when the punishment outcome is a Nash equilibrium outcome at the true state, it must be consistent with the SCF. The incentive compatibility issues that arise for the two-player case do not have any bite in separable environments, that is, in environments with an outcome that, at every state, every player deems strictly worse than the outcome prescribed by the SCF (see, for instance, Jackson (2001)). However, they have a bite in many important applications, such as in environments with no monetary transfers or in classic exchange economies where free disposal is not allowed.

The two-player condition for Nash implementation does not solve the incentive compatibility issues that arise under rationalizability. The reason is that rationalizability imposes more stringent requirements than Nash equilibrium on selecting the punishment outcome when the two players report distinct messages. Let f be the goal of the designer. Suppose that the actual state is θ^* but player 1 reports θ as the actual state, and player 2 reports θ'. What outcome can be selected as a punishment outcome when

$$SL_1\left(f\left(\theta\right),\theta\right)\bigcap L_2\left(f\left(\theta'\right),\theta'\right)$$

is empty? The two-player condition for Nash implementation allows us to choose any outcome on the indifference curve of player 1 generated by $f(\theta)$ at θ. However, rationalizable implementation forces us to select $f(\theta)$. The reason is that any other point e of player 1's indifference curve becomes a rationalizable outcome at θ but

$e \neq f(\theta)$. This choice is problematic when the two distinct messages constitute a Nash equilibrium at θ^* and $f(\theta) \neq f(\theta^*)$. The reason is that the two distinct messages constitute a bad Nash equilibrium. This bad Nash equilibrium is at the core of the contextualizing example presented in Section 3, where we construct a two-player SCF that is Nash implementable and satisfies BMT's sufficient conditions for rationalizable implementation. However, it is still not implementable in rationalizable strategies. In addition, this example also sheds further light on the relationship between Nash implementation and rationalizable implementation.[7]

Since $f(\theta)$ is to be implemented at θ but $f(\theta) \neq f(\theta^*)$, then $f(\theta)$ must fall in someone's ranking at θ^* to break the bad Nash equilibrium via some deviation. Therefore, the two-player condition for rationalizable implementation must allow preference reversals in these situations to knock out unwanted rationalizable strategy profiles. As discussed above, since the two-player condition for Nash implementation cannot help solve two-player rationalizable implementation problems, we develop our solution in the space of deceptions. Our condition is directly comparable with the BMT's condition and Xiong (2022)'s condition. Furthermore, it does not involve any event-wise strict Maskin monotonicity condition. The reason is that in the two-player case, one player never violates the condition of the no worst alternative (NWA), and the other can violate NWA in some states.[8] Therefore, pairwise comparisons between states conducted in our strict Maskin monotonicity condition are sufficient in a setup where NWA is almost satisfied. Our two-player implementing condition strengthens Xiong (2022)'s condition. We present and discuss our implementing condition in Section 4

[7]BMT's Proposition 2 (see Section II below) implies that in a complete information environment with three or more players, if an SCF satisfies responsiveness and the so-called no-worst alternative condition (NWA), then its Nash implementation is equivalent to its rationalizable implementation. This equivalence result is a conceptual puzzle because the two solutions concepts are very different. In a complete information environment with three or more players, Jain (2021) provide an example showing that the equivalence breaks down when f violates responsiveness, but it satisfies NWA. In the same environment, Xiong (2022) shows that it breaks down when f satisfies responsiveness but violates NWA. Our example shows that the equivalence breaks down in a complete information environment with two players, even when f satisfies responsiveness and NWA.

[8]NWA requires that a player never obtains his worst outcome under the SCF.

and in Section 5. The complete characterization is presented in Section 5, whereas the Appendix contains its proof.

Finally, the necessary and sufficient condition for Nash implementation in abstract environments relies on the existence of certain sets. Thanks to Sjostrom (1991), when we have three or more players, we have a constructive way of checking whether an SCF can be implemented in Nash equilibria. In important papers, Bochet (2007) and Benoît and Ok (2008) show that in environments with three or more players, the necessary and sufficient conditions for Nash implementation simplify considerably by using stochastic mechanisms. However, the situation is very different for rationalizable implementation. The reason is that although the constructive proofs provided so far for rationalizable implementation employ stochastic mechanisms, the necessary and sufficient conditions rely on the existence of a partition of Θ.[9]

Our characterization result does not suffer from this drawback. Section 6 shows how to construct the partition of Θ underlying our two-player implementing condition. The construction is very instructive because we can use a similar algorithm to construct the partition underlying Xiong (2022)'s and Bergemann et al. (2011)'s condition for functions, and that underlying Jain (2021)'s condition for correspondences.

Section 7 discusses how the characterization obtained for SCFs can be used to derive sufficient conditions for the rationalizable implementation of social choice correspondences studied in Jain (2021).

2. Setup

The environment consists of $I = 2$ players (we write $\mathcal{I} = \{1, 2\}$ for the set of players), a finite set of states Θ and a countable set of pure outcomes X. Let $Y \equiv \Delta(X)$ denote the set of lotteries over X. Player i's preferences over lotteries are described

[9] Jain (2021) also shares this feature.

by a utility function $u_i : Y \times \Theta \mapsto \mathbb{R}$, with

$$u_i(y, \theta) = \sum_{x \in X} y_x u_i(x, \theta),$$

where y_x is the probability of pure outcome x. For all $\theta \in \Theta$, $u_i(\cdot, \theta)$ satisfies the expected utility hypothesis. To save writing, for all $i \in \mathcal{I}$, we write $-i$ for player i's opponent.

Given a state $\theta \in \Theta$, a player $i \in \mathcal{I}$, and a lottery $y \in Y$, the lower contour set of $u_i(\cdot, \theta)$ at y is $L_i(y, \theta) = \{y' \in Y | u_i(y, \theta) \geq u_i(y', \theta)\}$; the strict lower contour set of $u_i(\cdot, \theta)$ at y is $SL_i(y, \theta) = \{y' \in Y | u_i(y, \theta) > u_i(y', \theta)\}$; and the strict upper contour set of $u_i(\cdot, \theta)$ at y is $SU_i(y, \theta) = \{y' \in Y | u_i(y', \theta) > u_i(y, \theta)\}$.

A mechanism $\mathcal{M}$ is a pair $\mathcal{M} \equiv (M, g)$, where $M \equiv \prod_{i \in \mathcal{I}} M_i$, with each M_i being a nonempty countable set, and $g : M \longrightarrow Y$. As usual, we refer to M_i as the (pure) strategy space of $i \in \mathcal{I}$, to a member of M, denoted by m, as a (pure) strategy profile, and to g as an outcome function. For all $M' \subseteq M$, let $g[M'] = \{g(m) \in Y | m \in M'\}$.

The environment, when combined with the mechanism, describes a game (of complete information) for all state $\theta \in \Theta$, which is denoted by $(\mathcal{M}, \theta)$. We will use (correlated) rationalizability as a solution concept. Bernheim (1984) and Pearce (1984) provide a definition of rationalizability in which players' conjectures over their opponents' play are independent. In this paper, we follow the convention of some of the recent literature (e.g., Osborne and Rubinstein (1994) in using "rationalizability" for the correlated version of rationalizability (we refer the reader to Brandenburger and Dekel (1987)). Our definition of rationalizability coincides with the standard definition when strategy spaces are compact. However, our definition allows for infinite, non-compact strategy spaces. In this case, our definition is equivalent to one introduced by Lipman (1994).

Formally, let $\mathcal{S}$ be the set of all strategy-set profiles, defined by $\mathcal{S} \equiv \prod_{i \in \mathcal{I}} \mathcal{S}_i$, where $\mathcal{S}_i \equiv 2^{M_i}$ for all $i \in \mathcal{I}$, with $S = (S_i)_{i \in \mathcal{I}}$ as a typical profile of $\mathcal{S}$. The family $\mathcal{S}$ is a lattice with the natural ordering of the set inclusion: $S \leq S'$ if $S_i \subseteq S'_i$ for all $i \in \mathcal{I}$. The smallest element of $\mathcal{S}$ is denoted by $\underline{S} \equiv (\emptyset, ..., \emptyset)$, whereas the largest element is denoted by $\bar{S} \equiv M$.

Fix any game $(\mathcal{M}, \theta)$. The strategy $m_i \in M_i$ is player i's best-response to his belief $\lambda_i \in \Delta(M_{-i})$ at θ if

$$m_i \in \arg\max_{m'_i \in M_i} \sum_{m_{-i} \in M_{-i}} \lambda_i(m_{-i})\, u_i(g(m'_i, m_{-i}), \theta).$$

By following Bergemann et al. (2011), let us define an operator $b^{\mathcal{M},\theta} : \mathcal{S} \longrightarrow \mathcal{S}$, where $b^{\mathcal{M},\theta} \equiv \left(b_i^{\mathcal{M},\theta}\right)_{i \in \mathcal{I}}$ and $b_i^{\mathcal{M},\theta} : \mathcal{S} \longrightarrow \mathcal{S}_i$ is defined, for all $S \in \mathcal{S}$, by

$$b_i^{\mathcal{M},\theta}(S) = \left\{ m_i \in M_i \;\middle|\; \begin{array}{l} \text{there exists } \lambda_i^{m_i,\theta} \in \Delta(M_{-i}) \text{ such that} \\ (1)\ \lambda_i^{m_i,\theta}(m_{-i}) > 0 \implies m_{-i} \in S_{-i}, \\ (2)\ m_i \text{ is a best response to } \lambda_i^{m_i,\theta} \text{ at } \theta \end{array} \right\}.$$

Note that $b^{\mathcal{M},\theta}$ is increasing (that is, $S \leq S' \implies b^{\mathcal{M},\theta}(S) \leq b^{\mathcal{M},\theta}(S')$.

By Tarski's fixed point theorem, there exists a largest fixed point of $b^{\mathcal{M},\theta}$, which is denoted by $S^{\mathcal{M},\theta}$. That is, (1) $b^{\mathcal{M},\theta}\left(S^{\mathcal{M},\theta}\right) = S^{\mathcal{M},\theta}$ and (2) $b^{\mathcal{M},\theta}(S) = S \implies S \leq S^{\mathcal{M},\theta}$. We refer to $m_i \in S_i^{\mathcal{M},\theta}$ as a player i's rationalizable strategy in $\mathcal{M}$ at state θ, and to a member of $S^{\mathcal{M},\theta}$ as a rationalizable strategy profile in $\mathcal{M}$ at state θ.

We say that a profile $S \in \mathcal{S}$ has the best-response property in state θ if $S \leq b^{\mathcal{M},\theta}(S)$, or equivalently, if for all $i \in \mathcal{I}$ and all $m_i \in S_i$, there exists $\lambda_i \in \Delta(M_{-i})$ such that $\lambda_i(m_{-i}) > 0 \implies m_{-i} \in S_{-i}$, and m_i is a best-response to λ_i at θ. It can be checked that $S \leq S^{\mathcal{M},\theta}$ when S has the best-response property in state θ.

A player i's mixed-strategy σ_i is a probability distribution over M_i. The space of player i's mixed-strategies is denoted by Σ_i, where $\sigma_i(m_i)$ is the probability that σ_i assigns to m_i. The space of mixed-strategy profiles is denoted by $\Sigma = \prod_{i \in \mathcal{I}} \Sigma_i$, with element σ as a typical strategy profile. A mixed-strategy may assign probability one to a single strategy m_i, that is, $\sigma_i(m_i) = 1$. In this case, we refer to such a mixed-strategy as a (pure) strategy and denote it by m_i. The support of a mixed-strategy σ_i is the set of pure strategies that are played with positive probability, that is, $supp(\sigma_i) = \{m_i \in M_i | \sigma_i(m_i) > 0\}$. A mixed-strategy profile σ is a Nash equilibrium of $(\mathcal{M}, \theta)$ if for all $i \in \mathcal{I}$,

$$u_i\left(g\left(\sigma_i, \sigma_{-i}\right), \theta\right) \geq u_i\left(g\left(\sigma_i', \sigma_{-i}\right), \theta\right),$$

for all $\sigma_i' \in \Sigma_i$. Write $NE(\mathcal{M}, \theta)$ for the set of Nash equilibrium profiles of $(\mathcal{M}, \theta)$, and write $g(NE(\mathcal{M}, \theta))$ for the set of Nash equilibrium outcomes of $(\mathcal{M}, \theta)$.

An SCF f is a function $f : \Theta \longrightarrow Y$. To avoid trivialities, we focus on non-constant SCFs.[10]

Definition 1. A mechanism $\mathcal{M}$ implements $f : \Theta \longrightarrow Y$ in rationalizable strategies if for all $\theta \in \Theta$, $S^{\mathcal{M},\theta} \neq \emptyset$ and $m \in S^{\mathcal{M},\theta} \implies g(m) = f(\theta)$. If such a mechanism exists, f is said to be rationalizably implementable.

A partition of Θ is a correspondence $P : \Theta \rightrightarrows \Theta$ satisfying the following requirements: (i) $\theta \in P(\theta)$ for all $\theta \in \Theta$, (ii) $\cup_{\theta \in \Theta} P(\theta) = \Theta$, and (iii) $P(\theta) \cap P(\theta') = \emptyset$ if $P(\theta) \neq P(\theta')$. Given an SCF f, P_f is the partition of Θ induced by f, that is, $P_f = \{\Theta_y\}_{y \in f(\Theta)}$ where $\Theta_y = \{\theta \in \Theta | f(\theta) = y\}$. A partition P of Θ is at least as fine as P_f, or equivalently, P_f is coarser than P if $P(\theta) \subseteq P_f(\theta)$ for all $\theta \in \Theta$. Let $\mathcal{P}_f$ denote the set of partitions that are at least as fine as P_f, that is,

$$\mathcal{P}_f = \{P | P \text{ is a partition of } \Theta \text{ such that } P(\theta) \subseteq P_f(\theta) \text{ for all } \theta \in \Theta\}.$$

Let us call any map $\beta_i : \Theta \longrightarrow 2^{\Theta} \setminus \{\emptyset\}$ as player i's deception. A special deception for player i is the truth-telling deception, β_i^t, defined by $\beta_i^t(\theta) = \{\theta\}$ for all $\theta \in \Theta$.

For any β_i and β_i', we write $\beta_i \subseteq \beta_i'$ if $\beta_i(\theta) \subseteq \beta_i'(\theta)$ for all $\theta \in \Theta$. Let $\mathcal{B}_i^t$ denote

[10] An SCF is constant if for all $\theta, \theta' \in \Theta$, $f(\theta) = f(\theta')$.

the set of player i's deceptions containing the truth-telling deception, that is,

$$\mathcal{B}_i^t \equiv \left\{ \beta_i : \Theta \longrightarrow 2^{\Theta} \setminus \{\emptyset\} \middle| \beta_i^t \subseteq \beta_i \right\}. \tag{1}$$

Let $\mathcal{B}^t \equiv \prod_{i \in \mathcal{I}} \mathcal{B}_i^t$, with $\beta = (\beta_i)_{i \in \mathcal{I}}$ as a typical deception profile of $\mathcal{B}^t$. For all $\beta, \beta' \in \mathcal{B}^t$, we write $\beta \subseteq \beta'$ if $\beta_i \subseteq \beta_i'$ for all $i \in \mathcal{I}$. The collection $\mathcal{B}^t$ is a complete lattice with the natural ordering set inclusion: $\beta \leq \beta'$ if $\beta \subseteq \beta'$. The largest element is $\bar{\beta} = (\Theta, ..., \Theta)$. The smallest element is β^t.

BMT shows that Maskin monotonicity fully identifies the class of rationalizable functions when there are three or more players and f satisfies the following two auxiliary conditions.

Definition 2. $f : \Theta \to Y$ satisfies NWA provided that for all $\theta \in \Theta$ and all $i \in \mathcal{I}$,

$$SL_i\left(f\left(\theta\right), \theta\right) \neq \emptyset.$$

The condition requires that a player never obtains his worst outcome under the SCF. The condition is due to Cabrales and Serrano (2011).

Definition 3. $f : \Theta \mapsto Y$ is responsive provided that for all $\theta, \theta' \in \Theta$,

$$\theta \neq \theta' \implies f\left(\theta\right) \neq f\left(\theta'\right).$$

Responsiveness requires that the SCF "responds" to a change in the state with a change in the socially optimal outcome.

Definition 4. $f : \Theta \mapsto Y$ satisfies Maskin monotonicity provided that for all $\theta, \theta' \in \Theta$,

$$f\left(\theta\right) \neq f\left(\theta'\right) \implies \exists i \in I : L_i\left(f\left(\theta\right), \theta\right) \bigcap SU_i\left(f\left(\theta\right), \theta'\right) \neq \emptyset.$$

Maskin monotonicity states that in the case the socially optimal outcome differs at θ and θ', there exists a player i who, if the actual state is θ' and all other players claims that it is θ, could be offered a outcome y that would give him a strict incentive to "announce" θ', where y does not give any incentive when θ is the actual state.

BMT's Proposition 2. (BMT, p.1261) Suppose that there are more than two players. If $f : \Theta \mapsto Y$ is responsive and it satisfies NWA and Maskin Mononotonicity, then f is rationalizable implementable.

Among the three assumptions in BMT's Proposition 2, Maskin monotonicity is necessary. BMT propose ways to relax responsiveness while keeping NWA, and assuming more than two agents. To relax responsiveness, they introduce a strengthening of Maskin monotonicity, called strict Maskin monotonicity*. Xiong (2022) introduced a weakening of strict Maskin monotonicity*, termed strict Maskin monotonicity (SMM**, henceforth), and he shows that SMM** is a necessary and sufficient condition for rationalizable implementation when NWA is satisfied and there are more than two players. His condition can be stated as follows.

Definition 5. $f : \Theta \mapsto Y$ satisfies SMM** if there exists $P \in \mathcal{P}_f$ such that for all $(\theta, \theta') \in \Theta \times \Theta$,

$$\left[\begin{array}{l} \forall i \in \mathcal{I},\ \exists \hat{\theta} \in P(\theta) \text{ such that} \\ SL_i\left(f(\theta), \hat{\theta}\right) \subseteq L_i(f(\theta), \theta') \end{array} \right] \implies P(\theta') = P(\theta).$$

The condition is based on a partition $P \in \mathcal{P}_f$ and requires that that for any states θ' and θ, with $P(\theta') \neq P(\theta)$, there exists a whistle-blower i who, at true state θ', can find for each $\hat{\theta} \in P(\theta)$, a state-contingent blocking plan $y^{\hat{\theta}}$ that works for this $\hat{\theta}$–that is, $y^{\hat{\theta}} \in SL_i\left(f(\theta), \hat{\theta}\right) \cap SU_i(f(\theta), \theta')$.

3. Contextualizing Example

In this section, we show through an example that the incentive compatibility issues arising in the two-player rationalizable implementation problems are more severe than those arising in the two-player Nash implementation problems. However, these issues do not have any bite in rationalizable implementation problems with three or more players or in environments with an outcome that, at every state, every player deems strictly worse than the outcome prescribed by the SCF (Jackson (2001)).

In particular, we construct a two-player SCF f such that (a) f satisfies all sufficient conditions of BMT's Proposition 2 except the requirement of three or more players; (b) f is Nash implementable; and (c) f is not implementable in rationalizable strategies.

Let us present our example as follows. Suppose that $X = \{a, b, b', c, d, e, f, g\}$ and $\Theta = \{\theta, \theta', \theta''\}$. Players' utilities from pure outcomes are summarized in the table below, where $\varepsilon \in \left(\frac{1}{2}, 1\right)$.

	$u_1(\cdot,\theta)$	$u_2(\cdot,\theta)$	$u_1(\cdot,\theta')$	$u_2(\cdot,\theta')$	$u_1(\cdot,\theta'')$	$u_2(\cdot,\theta'')$
a	1	$-(1-\varepsilon)$	1	-1	1	-1
b	0	0	0	0	0	0
b'	0	0	0	0	0	0
c	-1	1	$-(1-\varepsilon)$	1	-1	1
d	1	-2	-2	-1	1	-1
e	2	$-(2-\epsilon)$	2	-2	-2	-2
f	3	-3	-3	-3	3	-3
g	0	0	0	0	-3	-3

The planner wants to implement f, which is defined over Θ by

$$f(\theta) = \{b\}, f(\theta') = \{b'\}, \text{ and } f(\theta'') = \{a\}.$$

Remark 1. The main feature of the example that will be used in the following claims is that $SL_1(f(\theta), \theta) \cap L_2(f(\theta'), \theta') = \emptyset$. To see, it suppose that there exists $z \in SL_1(f(\theta), \theta) \bigcap SL_2(f(\theta'), \theta')$. Then, $u_1(z, \theta) < u_1(f(\theta), \theta) = 0$ and $u_2(z, \theta') < u_2(f(\theta'), \theta') = 0$, and so $u_1(z, \theta) + u_2(z, \theta') < 0$. However, since $u_1(\cdot, \theta) = -u_2(\cdot, \theta')$ by construction, it follows that $u_1(z, \theta) + u_2(z, \theta) = 0$, which is a contradiction.

We break the message of our example in three claims. The first claim can be stated as follows.

Claim 1. f is responsive and it satisfies NWA and Maskin monotonicity.

Proof. It can be checked that f is responsive and it satisfies NWA. Moreover, it can be checked that f satisfies Maskin monotonicity. Indeed, it can be checked that $\frac{1}{2}a + \frac{1}{2}c \in L_2\left(f\left(\theta'\right), \theta'\right) \cap SU_2\left(f\left(\theta'\right), \theta\right)$, $\frac{1}{2}a + \frac{1}{2}c \in L_1\left(f\left(\theta\right), \theta\right) \cap SU_1\left(f\left(\theta\right), \theta'\right)$, $\frac{1}{2}a + \frac{1}{2}d \in L_1\left(f\left(\theta'\right), \theta'\right) \cap SU_1\left(f\left(\theta'\right), \theta''\right)$, $e \in L_1\left(f\left(\theta''\right), \theta''\right) \cap SU_1\left(f\left(\theta''\right), \theta'\right)$, $e \in L_1\left(f\left(\theta''\right), \theta''\right) \cap SU_1\left(f\left(\theta''\right), \theta\right)$ and $\frac{2}{3}c + \frac{1}{3}d \in L_2\left(f\left(\theta\right), \theta\right) \cap SU_2\left(f\left(\theta\right), \theta''\right)$. ∎

When f satisfies NWA and stochastic mechanisms can be employed, it can be shown that Moore and Repullo (1990)'s necessary and sufficient condition for Nash implementation, called condition $\mu 2$, simplifies as follows.[11]

Definition 6. $f : \Theta \mapsto Y$ satisfies condition $\mu 2$ provided that there exists $e : \Theta \times \Theta \to Y$ such that for all $\theta, \theta' \in \Theta$, (a) $e\left(\theta, \theta'\right) = f\left(\theta\right)$ if $\theta = \theta'$; (b) $e(\theta, \theta') \in L_1\left(f(\theta'), \theta'\right) \cap L_2\left(f(\theta), \theta\right)$ if $\theta \neq \theta'$; and (c) for all $\theta^* \in \Theta$, $f(\theta^*) = e(\theta, \theta')$ if

$$L_1\left(f\left(\theta'\right), \theta'\right) \subseteq L_1\left(e\left(\theta, \theta'\right), \theta^*\right) \text{ and } L_2\left(f\left(\theta\right), \theta\right) \subseteq L_2\left(e\left(\theta, \theta'\right), \theta^*\right). \tag{2}$$

Part (b) of condition $\mu 2$ is a *self-selection* constraint due to incentive compatibility issues. It requires that when players report different states, a feasible outcome exists that can punish both players simultaneously. Part (a) of condition $\mu 2$ requires that the punishment outcome is consistent with f when players report the same state. Part (c) of condition $\mu 2$ states that if such a punishment outcome is a Nash equilibrium outcome at θ^*, then it should be selected by f at θ^*. Note that part (c) of condition $\mu 2$ implies Maskin monotonicity when $\theta = \theta'$. The challenge to satisfy condition $\mu 2$ consists in finding a feasible outcome $e : \Theta \times \Theta \to Y$ such that parts (a)–(c) are satisfied *simultaneously*.

[11]It can be checked that when f satisfies NWA, it is without loss of generality to verify condition $\mu 2$ (condition β by Dutta and Sen (1991) or condition $M2$ of Sjostrom (1991)) under the specifications that the set $B = \Delta\left(X\right)$ and $C_i\left(f\left(\bar{\theta}\right), \bar{\theta}\right) = L_i\left(f\left(\bar{\theta}\right), \bar{\theta}\right)$ for all $i \in \mathcal{I}$.

Claim 2. f satisfies condition $\mu 2$.

Proof. To see that f satisfies condition $\mu 2$, let the mapping $e : \Theta \times \Theta \to Y$ be defined as in Table 1, where the row player is player 1 and the column player is player 2.

	θ	θ'	θ''
θ	$f(\theta)$	$0.1d + 0.9g$	g
θ'	g	$f(\theta')$	g
θ''	$0.1c + 0.9g$	$0.1d + 0.9g$	$f(\theta'')$

Table I: $e : \Theta \times \Theta \mapsto Y$

It is clear from the construction of $e(\cdot, \cdot)$ that it satisfies part (a) of condition $\mu 2$. To check part (b), we have to consider six cases reported in Table II, where θ_1 is player 1's report and θ_2 is player 2's report. It can be checked that $e(\theta_1, \theta_2) \in L_1(f(\theta_2), \theta_2) \cap L_2(f(\theta_1), \theta_1)$ for all $\theta_1, \theta_2 \in \Theta$ with $\theta_1 \neq \theta_2$.

θ_1	θ_2	$e(\theta_1, \theta_2)$	$u_1(e(\theta_1, \theta_2), \theta_2)$	$u_1(f(\theta_2), \theta_2)$	$u_2(e(\theta_1, \theta_2), \theta_1)$	$u_2(f(\theta_1), \theta_1)$
θ	θ'	$0.1d + 0.9g$	-0.2	0	-0.2	0
θ	θ''	g	-3	1	0	0
θ'	θ	g	0	0	0	0
θ'	θ''	g	-3	1	0	0
θ''	θ	$0.1c + 0.9g$	-0.1	0	-2.6	-1
θ''	θ'	$0.1d + 0.9g$	-0.2	0	-2.8	-1

Table II: Part (b)

θ	θ'	θ''
	$e(\theta,\theta') = 0.1d + 0.9g$	
$u_1(f(\theta'),\theta') \geq u_1(d,\theta')$ $u_1(d,\theta) > u_1(e(\theta,\theta'),\theta)$	$u_1(f(\theta'),\theta') \geq u_1(g,\theta')$ $u_1(g,\theta') > u_1(e(\theta,\theta'),\theta')$	$u_1(f(\theta'),\theta') \geq u_1(d,\theta')$ $u_1(d,\theta'') > u_1(e(\theta,\theta'),\theta'')$
	$e(\theta,\theta'') = g$	
$u_1(f(\theta''),\theta'') \geq u_1(e,\theta'')$ $u_1(e,\theta) > u_1(e(\theta,\theta''),\theta)$	$u_1(f(\theta''),\theta'') \geq u_1(e,\theta'')$ $u_1(e,\theta') > u_1(e(\theta,\theta''),\theta')$	$u_1(f(\theta''),\theta'') \geq u_1(d,\theta'')$ $u_1(d,\theta'') > u_1(e(\theta,\theta''),\theta'')$
	$e(\theta',\theta) = g$	
$u_2(f(\theta'),\theta') \geq u_2\left(\frac{2}{3}c + \frac{1}{3}e,\theta'\right)$ $u_2\left(\frac{2}{3}c + \frac{1}{3}e,\theta\right) > u_2(e(\theta',\theta),\theta)$	$u_1(f(\theta'),\theta') \geq u_1\left(\frac{1}{2}a + \frac{1}{2}c,\theta'\right)$ $u_1\left(\frac{1}{2}a + \frac{1}{2}c,\theta'\right) > u_1(e(\theta',\theta),\theta')$	$u_1(f(\theta),\theta) \geq u_1(c,\theta)$ $u_1(c,\theta'') > u_1(e(\theta',\theta),\theta'')$
	$e(\theta',\theta'') = g$	
$u_1(f(\theta''),\theta'') \geq u_1(e,\theta'')$ $u_1(e,\theta) > u_1(e(\theta',\theta''),\theta)$	$u_1(f(\theta''),\theta'') \geq u_1(e,\theta'')$ $u_1(e,\theta') > u_1(e(\theta',\theta''),\theta')$	$u_1(f(\theta''),\theta'') \geq u_1(d,\theta'')$ $u_1(d,\theta'') > u_1(e(\theta',\theta''),\theta'')$
	$e(\theta'',\theta) = 0.1c + 0.9g$	
$u_1(f(\theta),\theta) \geq u_1(g,\theta)$ $u_1(g,\theta) > u_1(e(\theta'',\theta),\theta)$	$u_1(f(\theta),\theta) \geq u_1(c,\theta)$ $u_1(c,\theta') > u_1(e(\theta'',\theta),\theta')$	$u_1(f(\theta),\theta) \geq u_1(c,\theta)$ $u_1(c,\theta'') > u_1(e(\theta'',\theta),\theta'')$
	$e(\theta'',\theta') = 0.1d + 0.9g$	
$u_1(f(\theta'),\theta') \geq u_1(d,\theta')$ $u_1(d,\theta) > u_1(e(\theta'',\theta'),\theta)$	$u_1(f(\theta'),\theta') \geq u_1(g,\theta')$ $u_1(g,\theta') > u_1(e(\theta'',\theta'),\theta')$	$u_1(f(\theta'),\theta') \geq u_1(d,\theta')$ $u_1(d,\theta'') > u_1(e(\theta'',\theta'),\theta'')$

Table III: Part (c)

Finally, let us check that f satisfies part (c) of condition $\mu 2$. Since f satisfies Maskin monotonicity by Claim 1, we need to check that f satisfies part (c) of condition $\mu 2$ for the cases where the punishment outcome $e(\theta_1, \theta_2)$ is given by $\theta_1 \neq \theta_2$. Since $\Theta = \{\theta, \theta', \theta''\}$, there are six cases to be checked for each possible state in Θ, which are reported in Table III. For example, when player 1 reports θ and player 2 reports θ', the punishment outcome is $e(\theta, \theta') = 0.1d + 0.9g$, as reported in Table I and in the second row of Table III. Suppose that the actual state is θ. Then, since $f(\theta) \neq f(\theta')$, to satisfy part (c) of condition $\mu 2$ for $e(\theta, \theta') = 0.1d+0.9g$, Table 3 states that outcome

d is such that $d \in L_1(f(\theta'), \theta') \cap SU_1(e(\theta, \theta'), \theta)$—see the box corresponding to the third row and the first column of Table III. ■

Though f is Nash implementable and satisfies responsiveness, NWA and Maskin monotonicity, our next claim shows that f is not rationalizable implementable.

Claim 3. f is not rationalizably implementable.

Proof. Assume, to the contrary, that there exists a mechanism $\mathcal{M}$ that implements f in rationalizable strategies. For each state $\bar{\theta} \in \Theta$, let $S^{\mathcal{M},\bar{\theta}} = \prod_{i \in \mathcal{I}} S_i^{\mathcal{M},\bar{\theta}}$ be the set of rationalizable strategy profiles for the game $(\mathcal{M}, \bar{\theta})$. Since for each $\theta, S^{\mathcal{M},\theta} \neq \emptyset$, it follows that for each $i \in \mathcal{I}$, there exists $\lambda_i^\theta \in \Delta(S_{-i}^{\mathcal{M},\theta})$ such that $S_i^{\mathcal{M},\theta} = \underset{m_i'}{argmax} \sum_{m_{-i}} \lambda_i^\theta(m_{-i})\, u_i(g(m_i', m_{-i}), \theta) \neq \emptyset$.

Since $\mathcal{M}$ rationalizable implements f, it follows that for each $(m_1, m_2) \in S_1^{\mathcal{M},\theta'} \times S_2^{\mathcal{M},\theta}$, it holds that $g(m_1, m_2) \in L_1(f(\theta), \theta) \bigcap L_2(f(\theta'), \theta')$. Since $SL_1(f(\theta), \theta) \cap L_2(f(\theta'), \theta')$ is empty, by construction, it follows that if $g(m_1, m_2) \neq f(\theta)$, then $g(m_1, m_2)$ becomes a rationalizable outcome at θ, yielding a contradiction. Thus, by rationalizable implementation, it must be the case that $g(m_1, m_2) = f(\theta)$. Since $L_1(f(\theta), \theta) \subseteq L_1(f(\theta), \theta'')$, by construction, it holds that

$$S_1^{\mathcal{M},\theta'} \subseteq argmax \sum_{m_2 \in M_2} \lambda_1^\theta(m_2)\, u_1(g(m_1, m_2), \theta'') \neq \emptyset. \tag{3}$$

Similarly, since $L_2(f(\theta), \theta') \subseteq L_2(f(\theta), \theta'')$, by construction, it holds that

$$S_2^{\mathcal{M},\theta} \subseteq argmax \sum_{m_1 \in M_1} \lambda_2^{\theta'}(m_1)\, u_2(g(m_2, m_1), \theta'') \neq \emptyset. \tag{4}$$

Thus, we have that $S_1^{\mathcal{M},\theta} \times S_2^{\mathcal{M},\theta'} \subseteq S^{\mathcal{M},\theta''}$. Since $\mathcal{M}$ implements f in rationalizable strategies, it follows that $f(\theta) = f(\theta'')$, which is a contradiction. ■

From the arguments provided in the proof of Claim 3, it is clear that rationalizable implementation imposed more restrictions than Nash implementation in selecting the punishment outcome. Indeed, when player 1 reports θ' and player 2 reports θ, Nash implementation allowed us to select g as a punishment outcome and so to satisfy part (c) of condition $\mu 2$. However, this option is not available in the case of rationalizable implementation. The reason is that $SL_1(f(\theta), \theta) \cap L_2(f(\theta'), \theta')$ is empty and rationalizable implementation forced us to select $f(\theta)$ as a punishment outcome. Therefore, any necessary condition for two-player rationalizable implementation needs to select as a punishment outcome $f(\theta)$ when player 1 reports θ' and player 2 reports θ and $SL_1(f(\theta), \theta) \cap L_2(f(\theta'), \theta')$ is empty. One such condition can be obtained by strengthening condition $\mu 2$ of Definition 6 as follows.

Definition 7. $f : \Theta \mapsto Y$ satisfies strong condition $\mu 2$ provided that there exists $e : \Theta \times \Theta \to Y$ such that for all $\theta, \theta' \in \Theta$, (a) $e(\theta', \theta) = f(\theta)$ if either $\theta = \theta'$ or $\theta \neq \theta'$ and $SL_1(f(\theta), \theta) \cap L_2(f(\theta'), \theta') = \emptyset$; (b) $e(\theta', \theta) \in L_1(f(\theta), \theta) \cap L_2(f(\theta'), \theta')$ if $\theta \neq \theta'$ and $SL_1(f(\theta), \theta) \cap L_2(f(\theta'), \theta') \neq \emptyset$; and (c) for all $\theta^* \in \Theta$, $f(\theta^*) = e(\theta', \theta)$ if

$$L_1(f(\theta), \theta) \subseteq L_1(e(\theta', \theta), \theta^*) \text{ and } L_2(f(\theta'), \theta') \subseteq L_2(e(\theta', \theta), \theta^*). \tag{5}$$

When f satisfies condition strong $\mu 2$, the punishment function $e : \Theta \times \Theta \to Y$ induces the following set-valued deception $\beta : \Theta \to \Theta \times \Theta$ defined by

$$\beta(\bar{\theta}) \equiv e^{-1}(f(\bar{\theta})) \tag{6}$$

for all $\bar{\theta} \in \Theta$. In other words, for each $\bar{\theta} \in \Theta$, β identifies profiles of states (θ, θ') that are outcome equivalent to $f(\bar{\theta})$ according to the punishment function $e : \Theta \times \Theta \to Y$. Clearly, the deception identified by condition $\mu 2$ can be a proper subset of that identified by strong condition $\mu 2$. To see it, observe that in our example the deception induced by strong condition $\mu 2$ includes pair of states (θ', θ) such that $\theta' \neq \theta$, whereas the deception induced by condition $\mu 2$ includes pair of states (θ', θ) such that $\theta' = \theta$.

The language of deceptions allows us to formulate the constraints that arise due to rationalizable implementation beyond those arising from condition $\mu 2$. Our two-player implementing condition builds on this insight, and it is defined on the space of deceptions. Moreover, since we are using rationalizability as a solution concept, the deception β has a product structure—i.e., $\beta(\bar{\theta}) = \beta_1(\bar{\theta}) \times \beta_2(\bar{\theta})$, for every $\bar{\theta} \in \Theta$. The next section formalizes our implementing condition.[12]

4. Two-Player Generalized Maskin Monotonicity** (2P-GSMM**)

In this section, we present our implementing condition and relate it to Xiong (2022)'s implementing condition.

What prevented us from rationalizably implementing the two-player SCF presented in the previous section was the fact that the intersection $SL_1(f(\theta), \theta) \cap L_2(f(\theta'), \theta')$ was empty. This implies that any necessary condition for two-player implementable SCFs has to take care of these situations. We define below a deception β_i^P, which explicitly contemplates these situations.

For all $i \in \mathcal{I}$ and all $P \in P_f$, let us define $\beta_i^P : \Theta \to 2^\Theta \setminus \{\emptyset\}$ by

$$\beta_i^P(\theta) = P(\theta) \bigcup \left(\bigcup_{\theta' \in \Theta} \left\{ P(\theta') \;\middle|\; \begin{array}{l} \exists \left(\bar{\theta}, \hat{\theta}\right) \in P(\theta) \times P(\theta') \\ SL_i\left(f\left(\bar{\theta}\right), \bar{\theta}\right) \bigcap L_{-i}\left(f\left(\hat{\theta}\right), \hat{\theta}\right) = \emptyset \end{array} \right\} \right) \quad (7)$$

It is clear that $\beta^P \in \mathcal{B}^t$.

Based on the definition of the deception profile β^P, we show below that f satisfies a weakening of the nonempty lower intersection property due to Moore and Repullo (1990). It is a weakening because it requires that for any states $\theta', \theta'' \in \Theta$, the intersection $SL_i(f(\theta''), \theta'') \cap SL_{-i}(f(\theta'), \theta')$ is nonempty if $(\theta', \theta'') \notin \beta_i^P\left(\bar{\theta}\right) \times \beta_{-i}^P\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \Theta$. The property of Moore and Repullo (1990) requires that the intersection

[12] Traditionally, the idea of deceptions is used to study implementation problems with incomplete information. However, even under the assumption of complete information, formulating the condition in the space of deceptions has proved pivotal in characterizing social choice rules that are repeatedly Nash implementable and in making the connection with static Nash implementation transparent. For instance, Mezzetti and Renou (2017) show that dynamic monotonicity, a nontrivial but natural generalization of Maskin monotonicity defined on the space of deceptions, is necessary and almost sufficient for repeated Nash implementation.

is nonempty for all $(\theta', \theta'') \in \Theta \times \Theta$. Formally, we have the following intersection property.

Lemma 1. For all $P \in \mathcal{P}_f$ and all $(\theta', \theta'') \in \Theta \times \Theta$,

$$\left[\begin{array}{l} (\theta', \theta'') \notin \beta_i^P\left(\bar{\theta}\right) \times \beta_{-i}^P\left(\bar{\theta}\right) \\ \forall \bar{\theta} \in \Theta \end{array}\right] \implies SL_i\left(f\left(\theta''\right), \theta''\right) \bigcap SL_{-i}\left(f\left(\theta'\right), \theta'\right) \neq \emptyset.$$

Proof. Fix any (θ', θ'') such that $(\theta', \theta'') \notin \beta_i^P\left(\bar{\theta}\right) \times \beta_{-i}^P\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \Theta$. Assume, to the contrary, that $SL_i\left(f\left(\theta''\right), \theta''\right) \bigcap SL_{-i}\left(f\left(\theta'\right), \theta'\right) = \emptyset$.

Since $P\left(\theta''\right) \subseteq \beta_{-i}^P\left(\theta''\right)$, we obtain a contradiction if we show that $P\left(\theta'\right) \subseteq \beta_i^P\left(\theta''\right)$. Assume, to the contrary, that $P\left(\theta'\right) \nsubseteq \beta_i^P\left(\theta''\right)$. By definition of β_i^P, it follows that $SL_i\left(f\left(\bar{\theta}\right), \bar{\theta}\right) \bigcap L_{-i}\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \neq \emptyset$ for all $\left(\bar{\theta}, \hat{\theta}\right) \in P\left(\theta''\right) \times P\left(\theta'\right)$, and so

$$SL_i\left(f\left(\theta''\right), \theta''\right) \bigcap L_{-i}\left(f\left(\theta'\right), \theta'\right) \neq \emptyset. \tag{8}$$

Let us show that $SL_{-i}\left(f\left(\theta'\right), \theta'\right) = \emptyset$. Assume, to the contrary, that $SL_{-i}\left(f\left(\theta'\right), \theta'\right) \neq \emptyset$. Take any $x \in SL_{-i}\left(f\left(\theta'\right), \theta'\right)$. Since the intersection in (8) is also nonempty, take any $y \in SL_i\left(f\left(\theta''\right), \theta''\right) \bigcap L_{-i}\left(f\left(\theta'\right), \theta'\right)$. Let $z = px + (1-p)\, y$ where $p \in (0,1)$. Thus, for some $p \in (0,1)$, we have that $z \in SL_i\left(f\left(\theta''\right), \theta''\right) \bigcap SL_{-i}\left(f\left(\theta'\right), \theta'\right)$, which is a contradiction. Thus, $SL_{-i}\left(f\left(\theta'\right), \theta'\right) = \emptyset$. The definition of β^P in (7) implies that $\beta_{-i}^P(\theta') = \Theta$. Thus, we have that $(\theta', \theta'') \in \beta_i^P(\theta') \times \beta_{-i}^P(\theta')$, which is a contradiction. ■

Remark 2. In environments with transfers or a common bad outcome, $\beta_i^P(\theta) = P(\theta)$ for all $i \in \mathcal{I}$, all $P \in \mathcal{P}_f$ and all $\theta \in \Theta$. Moreover, the nonempty lower intersection property is always satisfied. This implies that the difficulty discussed in the previous section disappears in environments with a common bad outcome, such as in an environment with transfers.

To define our implementing condition, we need additional notation. For all $E \in 2^\Theta \setminus \{\emptyset\}$, let $\mathcal{I}^E = \bigcap_{\theta \in E} \mathcal{I}^\theta$ where $\mathcal{I}^\theta = \{i \in \mathcal{I} | SL_i\left(f\left(\theta\right), \theta\right) \neq \emptyset\}$. Since we are focusing on non-constant SCFs, it follows from Xiong (2022) that $\mathcal{I}^{\Theta} \neq \emptyset$ if f is rationalizably implementable. Thus, throughout the paper, we assume that player 1 always satisfies NWA, i.e., $\mathcal{I}^{\Theta} = \{1\}$. Let

$$\Theta(2) = \{\theta \in \Theta | SL_2(f(\theta), \theta) = \emptyset\} \tag{9}$$

denote the set of states for which player 2 violates NWA.

Based on $\Theta(2)$ and on $P \in \mathcal{P}_f$, let us define Θ^P. Let

$$\Theta^P = \bigcup_{\theta \in \Theta(2)} P(\theta) \tag{10}$$

Let us define the sequence $\{\beta_k^P\}_{k \geq 0}$ as follows. For all $k \geq 0$, all $i \in \mathcal{I}$ and all $\theta \in \Theta^P$,

$$\beta_{i,k}^P(\theta) = \beta_i^P(\theta). \tag{11}$$

For all $k \geq 0$, all $i \in \mathcal{I}$ and all $\theta \in \bar{\Theta}^P \equiv \Theta \setminus \Theta^P$,

$$\beta_{i,0}^P(\theta) = \beta_i^P(\theta)$$

and

$$\beta_{i,k}^P(\theta) = \left\{ \begin{array}{c|l} P(\theta') \subseteq \bar{\Theta}^P & \exists \bar{\theta} \in \bar{\Theta}^P \text{ such that} \\ & P(\theta') \subseteq \beta_{i,k-1}^P(\bar{\theta}) \text{ and } P(\theta) \subseteq \beta_{-i,k-1}^P(\bar{\theta}). \end{array} \right\} \tag{12}$$

It can be checked that sequence $\{\beta_k^P\}_{k \geq 0}$ is an increasing converging sequence. Let β^{*P} be the limit point of the sequence.

The implementing condition can be stated as follows:

Definition 8. $f : \Theta \mapsto Y$ satisfies Two-Player Generalized SMM** (henceforth, 2P-GSMM**) if $\mathcal{I}^{\Theta} \neq \emptyset$ and there exists $P \in \mathcal{P}_f$ such that $\bigcap_{i \in \mathcal{I}} \beta^{*P} = P$ and the following conditions are satisfied for all $\theta, \theta' \in \Theta$ and all $i \in \mathcal{I}$.

1. *Measurability:*

$$f(\theta) \neq f(\theta') \implies \beta^{*P}(\theta) \bigcap \beta^{*P}(\theta') = \emptyset.$$

2. β^{*P}-*GSMM***:

$$\left[\begin{array}{l} \forall i \in \mathcal{I}^{P(\theta')}, \exists \hat{\theta} \in \beta^{*P}_{-i}(\theta) \text{ such that} \\ SL_i\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \subseteq L_i(f(\theta), \theta') \end{array}\right] \implies P(\theta) = P(\theta').$$

Measurability and β^{*P}-GSMM** are based on a common partition P. An SCF f satisfies Measurability provided that for all $\theta, \theta' \in \Theta$, $\beta^{*P}(\theta) \cap \beta^{*P}(\theta')$ is empty whenever $f(\theta) \neq f(\theta')$. When f satisfies Measurability, β^{*P} allows us to pin down the partition of Θ required to rationalizable implement f.

β^{*P}-GSMM** is an extension of SMM** to our framework with deceptions. For example, under NWA, it requires that for any states θ' and θ, with $P(\theta') \neq P(\theta)$, there exists a whistle-blower i who, at true state θ', can find for each $\hat{\theta} \in \beta^{*P}_{-i}(\theta)$, a state-contingent blocking plan $y^{\hat{\theta}}$ that works for this $\hat{\theta}$–that is, $y^{\hat{\theta}} \in SL_i\left(f(\theta), \hat{\theta}\right) \cap SU_i(f(\theta), \theta')$. In contrast to β^{*P}-GSMM**, SMM** requires the existence of a whistle-blower i who, at true state θ', for each $\hat{\theta} \in P(\theta)$–rather than for each $\hat{\theta} \in \beta^{*P}_{-i}(\theta)$, can find a state-contingent blocking plan $y^{\hat{\theta}}$ that works for this $\hat{\theta}$.

Remark 3. β^{*P}-GSMM** is stronger than SMM**. The reason is that for all $\theta \in \Theta$, it holds that $P(\theta) \subseteq \beta^{*P}_i(\theta)$ for all $i \in \mathcal{I}$. It is worth mentioning here that the SCF presented in Section 3 violates β^{*P}-GSMM**, though it satisfies SMM**.

Remark 4. A key feature that distinguishes the two-agent case from the three or more agent case is the fact that for any pair (θ, θ'), it is possible that $SL_i(f(\theta), \theta) \cap L_{-i}(f(\theta'), \theta')$ is empty, although $L_i(f(\theta), \theta) \cap L_{-i}(f(\theta'), \theta')$ is non-empty. The definition of β^P is fundamentally based on this feature. However, there can be environments such that for any pair (θ, θ'), it is possible that $SL_i(f(\theta), \theta) \cap SL_{-i}(f(\theta'), \theta')$

is nonempty. This is the case in an environment with universal bad outcomes, which has been recently studied by Chen et al. (2021). In these settings, we have that for all $i \in \mathcal{I}$ and all $\theta \in \Theta$, $\beta_i^P(\theta) = P(\theta)$, and hence 2P-GSMM** is equivalent to SMM** under NWA.

Remark 5. Both β^{*P}-GSMM** and SMM** require the existence of a fixed point. To see this, β^{*P}-GSMM** can equivalently be written as

$$P(\theta') = \bigcup_{\theta \in \Theta} \left\{ P(\theta) \;\middle|\; \begin{array}{l} \text{for all } i \in \mathcal{I}^{P(\theta')}, \exists \hat{\theta} \in \beta_{-i}^{*P}(\theta) \text{ such that} \\ SL_i\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \subseteq L_i\left(f(\theta), \theta'\right) \end{array} \right\}. \tag{13}$$

Based on this observation, in Section 6, we define a new condition called Partition monotonicity. This condition relies on a recursive sequence defined on the space of partition $\mathcal{P}_f$ that converges to the partition required to satisfy 2P-GSMM**.

It is worth mentioning that Measurability is a measurability-type condition, which is reminiscent of the classical Abreu–Matsushima measurability (Abreu and Matsushima, 1992), which is based on the limit of a recursive sequence.[13]

5. A Full Characterization

In this section, we prove that the class of rationalizably implementable SCFs coincides with the class of SCFs satisfying 2P–GSMM**.

Theorem 1. $f : \Theta \mapsto Y$ is rationalizably implementable if and only if it satisfies 2P-GSMM**.

Proof. See Appendix A and Appendix B. ■

[13]Abreu and Matsushima (1992) proposed a measurability condition, now referred to as Abreu–Matsushima measurability, to characterize virtual rationalizable implementation when there is incomplete information.

*5.1. Connecting 2P-GSMM** with Xiong (2022)'s implementing condition*

Xiong (2022) shows that rationalizable implementation of an SCF is equivalent to the Strict Event monotonicity** (SEM**, henceforth) when there are three or more players. The condition contains two axioms, SEM and Dictator Monotonicity (DM), that are based on a common partition $P \in \mathcal{P}_f$. SEM is a strengthening of SMM**. His condition can be stated as follows for the two-player case.

Definition 9. $f : \Theta \mapsto Y$ satisfies Two-Player SEM** (2P-SEM**, henceforth) if there exists $P \in \mathcal{P}_f$ such that for all $\theta, \theta' \in \Theta$,

$$\underbrace{\begin{bmatrix} \forall i \in \mathcal{I}^{P(\theta')}, \exists \hat{\theta} \in P(\theta) \text{ such that} \\ SL_i\left(f(\theta), \hat{\theta}\right) \subseteq L_i(f(\theta), \theta') \end{bmatrix}}_{\text{PART A: SEM}} \vee \underbrace{\begin{bmatrix} \mathcal{I}^{P(\theta)} = \{i\}, \exists \hat{\theta} \in \Theta \text{ such that} \\ L_i\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \subseteq L_i(f(\theta), \theta') \end{bmatrix}}_{\text{PART B: DM}} \implies P(\theta') = P(\theta).$$

Part A is the premises of SEM, while Part B is the premises of DM.

SEM requires that for any θ and θ', with $P(\theta') \neq P(\theta)$, there exists a whistle-blower i who, at true state θ', can find for each $\hat{\theta} \in P(\theta)$, a state-contingent blocking plan $y^{\hat{\theta}}$ that works for this $\hat{\theta}$–that is, $y^{\hat{\theta}} \in SL_i\left(f(\theta), \hat{\theta}\right) \cap SU_i(f(\theta), \theta')$. Note that SEM requires the whistle-blower i be an active player in $\mathcal{I}^{P(\theta')}$—i.e., $i \in \mathcal{I}^{P(\theta')}$. In contrast to SMM**, SEM requires that player i must be an active player in $\mathcal{I}^{P(\theta')}$. Under NWA, SEM is equivalent to SMM**.

DM requires that at the true state θ' if player i reports θ when he is a dictator–that is, he is the only one active player in $\mathcal{I}^{P(\theta)} = \{i\}$, while the opponent reports $\hat{\theta}$, and $P(\theta) \neq P(\theta')$, then player i can be the only whistle-blower who must have a blocking plan y that must be credible, i.e., $y \in L_i\left(f\left(\hat{\theta}\right), \hat{\theta}\right)$, and strictly profitable, i.e., $y \in SU_i(f(\theta), \theta')$, when the state moves from $\hat{\theta}$ to θ'.

Remark 6. β^{*P}–GSMM** is a strengthening of 2P-SEM**. β^{*P}–GSMM** implies DM. Let its premised be satisfied. Then, suppose that $\mathcal{I}^{P(\theta)} = \{i\}$ and that $\exists \hat{\theta} \in \Theta$ such that $L_i\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \subseteq L_i(f(\theta), \theta')$. The statement follows because by (7),

$\beta^P_{-i}(\theta) = \Theta$, and $\beta^P_{-i} \subseteq \beta^{*P}_{-i}$, by construction. Furthermore, β^{*P}–GSMM** implies SEM. This statement follows by the fact that $P(\theta) \subseteq \beta_i^{*P}(\theta)$ for all $i \in \mathcal{I}^{P(\theta')}$.

Given the discussion provided in the above remark and the example discussed in Section 3, we state without proving that our two-player implementation condition is strictly stronger than 2P-SEM**.

Theorem 2. If $f : \Theta \mapsto Y$ satisfies 2P-GSMM**, then it satisfies 2P-SEM**. The converse implication is false.

6. Endogenizing Partitions For Two-Player Problems

This section shows how to construct the partition P in our implementing condition. We can use similar reasoning to construct the partition in Xiong (2022)'s implementing condition.

Let us first connect our approach with that used by BMT. These authors discuss the role of partition in their characterization result. In particular, they show that the required partition must be as fine as P_f and as coarse as the partition obtained by their Lemma 1, which BMT called "pairwise inclusion property" (see BMT, p. 1266, for a discussion). However, BMT also argues that this property cannot pin down the partition by stating:

> *We finally observe that the partition P may yet have to be coarser than is indicated by the pairwise inclusion property (BMT, p. 1266).*

In what follows, we formalize the approach of BMT and extend it to a two-player case. To this end, we need additional notation.

For all $\theta \in \Theta$, let $P^t(\theta) = \{\theta\}$. It is clear that $P^t \in \mathcal{P}_f$. For all $\theta' \in \Theta$, let the sequence $\left\{P^\ell\right\}_{\ell \geq 0}$ be defined iteratively as follows. Let

$$P^0(\theta') = P^t(\theta'), \tag{14}$$

and, for all $\ell - 1 \geq 0$ such that $P^{\ell-1} \in \mathcal{P}_f$, let $P^\ell(\theta')$ as follows.

1. For all odd positive integer $\ell > 0$,

$$P^{\ell}(\theta') = \bigcap_{i\in\mathcal{I}} \beta_i^{*P^{\ell-1}}(\theta') \tag{15}$$

2. For all even positive integer $\ell > 0$,

$$P^{\ell}\left(\theta'\right) = \bigcup_{\theta\in\Theta} \left\{ \begin{array}{c|l} P^{\ell-1}(\theta) & \text{for all } i \in \mathcal{I}^{P^{\ell-1}(\theta')}, \exists\hat{\theta} \in \beta_{-i}^{*P^{\ell-2}}(\theta) \text{ such that} \\ & SL_i\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \subseteq L_i\left(f\left(\theta\right), \theta'\right). \end{array} \right\} \tag{16}$$

Suppose that the sequence $\left\{P^{\ell}\right\}_{\ell\geq 0}$ is such that $P^{\ell} \in \mathcal{P}_f$ for all $\ell \geq 0$. Then, $P^{\ell} \subseteq P^{\ell+1}$ for all $\ell \geq 0$. Since the sequence is increasing and Θ is finite, the limit of the sequence exists—that is, there exists ℓ^* such that $P^{\ell} = P^{\ell^*}$ for all $\ell \geq \ell^*$. Let us denote the limit of $\left\{P^{\ell}\right\}_{\ell\geq 0}$ by P^* when the sequence $\left\{P^{\ell}\right\}_{\ell\geq 0}$ is such that $P^{\ell} \in \mathcal{P}_f$ for all $\ell \geq 0$.

The following lemmata will be useful for our endogenization.

Lemma 2. For all $P \in \mathcal{P}_f$, all $i \in \mathcal{I}$, all $k \geq 0$, and all $\theta, \theta' \in \Theta$, $P(\theta') \subseteq \beta_{i,k}^P(\theta)$ if $P(\theta') \bigcap \beta_{i,k}^P(\theta) \neq \emptyset$.

Proof. Fix any $P \in \mathcal{P}_f$, any $i \in \mathcal{I}$ and any $\theta, \theta' \in \Theta$. Take any $k \geq 0$. Suppose that $P(\theta') \cap \beta_{i,k}^P(\theta) \neq \emptyset$. Suppose that $\theta \in \Theta^P$. Then, (11) implies that $\beta_{i,k}^P(\theta) = \beta_i^P(\theta)$, where $\beta_i^P(\theta)$ is defined by (7). If $\theta = \theta'$, then (7) implies that $P(\theta') \subseteq \beta_i^P(\theta)$. Suppose that $\theta \neq \theta'$. Since $P(\theta') \cap \beta_i^P(\theta) \neq \emptyset$, then there exists $\tilde{\theta} \in P(\theta') \cap \beta_i^P(\theta)$. It follows from (7) that there exists $\tilde{\theta}^*$ such that $\tilde{\theta} \in P\left(\tilde{\theta}^*\right) \subseteq \beta_i^P(\theta)$. Since $\tilde{\theta} \in P(\theta') \cap P\left(\tilde{\theta}^*\right)$, it follows that $P(\theta') = P\left(\tilde{\theta}^*\right)$, and so $P(\theta') \subseteq \beta_i^P(\theta)$. Suppose that $\theta \in \bar{\Theta}^P$. If $k = 0$, then $\beta_{i,k}^P(\theta) \equiv \beta_i^P(\theta)$, and so $P(\theta') \subseteq \beta_{i,k}^P(\theta)$, by the preceding arguments. Suppose that $k \neq 0$. Then, $\beta_{i,k}^P(\theta)$ is defined by (12). Since $P(\theta') \cap \beta_{i,k}^P(\theta) \neq \emptyset$, then there exists $\tilde{\theta} \in P(\theta') \cap \beta_{i,k}^P(\theta)$. It follows from (12) that there exists $\tilde{\theta}^*$ such that $\tilde{\theta} \in P\left(\tilde{\theta}^*\right) \subseteq \beta_{i,k}^P(\theta)$. Again, since $\tilde{\theta} \in P(\theta') \cap P\left(\tilde{\theta}^*\right)$, it follows that $P(\theta') = P\left(\tilde{\theta}^*\right)$, and so $P(\theta') \subseteq \beta_{i,k}^P(\theta)$. ■

Lemma 3. Suppose that $P \in \mathcal{P}_f$. For all $\ell \geq 0$, if $P^\ell \subseteq P$, then $\beta^{*P^\ell} \subseteq \beta^{*P}$.

Proof. Suppose that $P \in \mathcal{P}_f$. Fix any $\ell \geq 0$ such that $P^\ell \subseteq P$. Let us proceed by induction. It follows from (7) that $\beta_0^{P^\ell} \subseteq \beta_0^P$. Suppose that $\beta_k^{P^\ell} \subseteq \beta_k^P$ for some $k \geq 0$. Let us show that $\beta_{k+1}^{P^\ell} \subseteq \beta_{k+1}^P$. Fix any $i \in \mathcal{I}$ and any θ, θ' such that $\theta' \in \beta_{i,k+1}^{P^\ell}(\theta)$. We show that $\theta' \in \beta_{i,k+1}^P(\theta)$. The statement is obvious if $SL_i(f(\theta), \theta) = \emptyset$ because $\beta_{i,0}^{P^\ell}(\theta) = \Theta$ and $\beta_0^{P^\ell} \subseteq \beta_0^P \subseteq \beta^{*P}$. Otherwise, since $\theta' \in \beta_{i,k+1}^{P^\ell}(\theta)$, it follows from (12) that there exists $\bar{\theta} \in \bar{\Theta}^{P^\ell}$ such that $P^\ell(\theta') \subseteq \beta_{i,k}^{P^\ell}(\bar{\theta})$ and $P^\ell(\theta) \subseteq \beta_{-i,k}^{P^\ell}(\bar{\theta})$. Since $\beta_k^{P^\ell} \subseteq \beta_k^P$, it follows that there exists $\bar{\theta} \in \bar{\Theta}^{P^\ell} \subseteq \bar{\Theta}^P$ such that $P^\ell(\theta') \subseteq \beta_{i,k}^P(\bar{\theta})$ and $P^\ell(\theta) \subseteq \beta_{-i,k}^P(\bar{\theta})$. Since $P^\ell \subseteq P$ and $P \in \mathcal{P}_f$, Lemma 3 implies that implies that $P(\theta') \subseteq \beta_{i,k}^P(\bar{\theta})$ and $P(\theta) \subseteq \beta_{-i,k}^P(\bar{\theta})$. Thus, $\theta' \in \beta_{i,k+1}^P(\theta)$. The statement follows by the principle of mathematical induction. ■

Based on β^{*P^*}, we say that f satisfies *Measurability with respect to* β^{*P^*} provided that for all $\theta, \theta' \in \Theta$,

$$f(\theta) \neq f(\theta') \implies \beta^{*P^*}(\theta) \bigcap \beta^{*P^*}(\theta') = \emptyset.$$

This allows us to define our second implementing condition.

Definition 10. $f : \Theta \mapsto Y$ satisfies *Partition Monotonicity* (PM) provided $\mathcal{I}^\Theta \neq \emptyset$ that f satisfies Measurability with respect to β^{*P^*}.

We have the following alternative characterization of the class of two-player rationalizably implementable SCFs. The characterization has the advantage that it does not rely on any existential clause.

Theorem 3. $f : \Theta \mapsto Y$ satisfies PM if and only if f satisfies 2P-GSMM**.

Proof. Suppose that f satisfies PM. Then $\mathcal{I}^\Theta \neq \emptyset$ and f satisfies Measurability with respect to β^{*P^*} and $\bigcap_{i \in \mathcal{I}} \beta_i^{*P^*} = P^*$. Let us show that f satisfies 2P-GSMM** with respect to P^*. Then, we need only to show that f satisfies β^{*P^*}-GSMM**. To this end, fix any $\theta, \theta' \in \Theta$. Suppose that for all $i \in \mathcal{I}^{P^*(\theta')}$, there exits $\hat{\theta} \in \beta_{-i}^{*P^*}(\theta)$ such that $SL_i\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \subseteq L_i\left(f\left(\theta\right), \theta'\right)$. We show that $P^*\left(\theta'\right) = P^*\left(\theta\right)$. It follows from (16) that $P^*(\theta) \subseteq P^*\left(\theta'\right)$. Since $P^* \in \mathcal{P}_f$, we have that $P^*\left(\theta'\right) = P^*\left(\theta\right)$. Thus, f satisfies 2P-GSMM** with respect to P^*.

For the converse, suppose that f satisfies 2P-GSMM** with respect to $P = \bigcap_{i \in \mathcal{I}} \beta_i^{*P} \in \mathcal{P}_f$. Then $\mathcal{I}^{\Theta} \neq \emptyset$. We show that f satisfies PM with respect to $P^* \subseteq P$. We show this by showing that $P^\ell \subseteq P$ and $P^\ell \in \mathcal{P}_f$ for all $\ell \geq 0$. Let us proceed by induction. Clearly, $P^0 = P^t \subseteq P$ and $P^0 = P^t \in \mathcal{P}_f$. Then, suppose that there exists $\ell \geq 0$ such that $P^\ell \subseteq P$ and $P^\ell \in \mathcal{P}_f$. Let us show that $P^{\ell+1} \subseteq P$ and $P^{\ell+1} \in \mathcal{P}_f$. We proceed according to whether $\ell + 1$ is even or odd.

Case 1: $\ell + 1$ is odd

Since $\ell + 1$ is odd, $P^{\ell+1} = \bigcap_{i \in \mathcal{I}} \beta_i^{*P^\ell}$. Since $P^\ell \subseteq P$ and $P \in \mathcal{P}_f$, Lemma 3 implies that $\beta^{*P^\ell} \subseteq \beta^{*P}$. Since $P = \bigcap_{i \in \mathcal{I}} \beta_i^{*P}$, it follows that $P^{\ell+1} \subseteq P$. Furthermore, since $P^{\ell+1} \subseteq P$ and $P \in \mathcal{P}_f$, we also have that $P^{\ell+1} \in \mathcal{P}_f$.

Case 2: $\ell + 1$ is even

Fix any $\theta' \in \Theta$. Suppose that $\theta \in P^{\ell+1}\left(\theta'\right)$. We show that $\theta \in P(\theta')$. Since $P^\ell \in \mathcal{P}_f$, it follows from definition of $P^{\ell+1}$ in (16) that for all $i \in \mathcal{I}^{P^\ell(\theta')}$, there exists $\hat{\theta} \in \beta_{-i}^{*P^{\ell-1}}(\theta)$ such that $SL_i\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \subseteq L_i\left(f\left(\theta\right), \theta'\right)$. Since $P^\ell \subseteq P$ and $P^{\ell-1} \subseteq P^\ell$, we have that $P^{\ell-1} \in \mathcal{P}_f$. Lemma 3 implies that $\beta^{*P^{\ell-1}}(\theta) \subseteq \beta^{*P}(\theta)$. Thus, for all $i \in \mathcal{I}^{P(\theta')}$, there exits $\hat{\theta} \in \beta_{-i}^{*P}(\theta)$ such that $SL_i\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \subseteq L_i\left(f\left(\theta\right), \theta'\right)$. 2P-GSMM** implies that $P(\theta) = P(\theta')$, and so $\theta \in P(\theta')$. Since θ and θ' were arbitrary, it follows that $P^{\ell+1} \subseteq P$. Since $P^{\ell+1} \subseteq P$ and $P \in \mathcal{P}_f$, it also follows that $P^{\ell+1} \in \mathcal{P}_f$.

By the principle of mathematical induction, we have that $P^\ell \subseteq P$ for all $\ell \geq 0$. It follows that $P^* \subseteq P$, and so $\beta^{*P^*} \subseteq \beta^{*P}$. Since f satisfies 2P–GSMM**, it follows that it satisfies Measurability with respect to β^{*P}. Since $\beta^{*P^*} \subseteq \beta^{*P}$, we have that f satisfies Measurability with respect to β^{*P^*}. Thus, f satisfies PM. ∎

7. Concluding Remarks: Social Choice Correspondences

We restricted our attention to the study of SCFs. Let us briefly discuss the extension of the results to social choice correspondences (SCC). An SCC defines a set of outcomes for each state, and rationalizability is a set-based solution concept. Jain (2021) and Kunimoto and Serrano (2019) derive partial characterizations under different notions of rationalizable implementation for SCCs in environments with complete information and three or more players.

Using the characterization obtained for SCFs, Jain (2021), in his online appendix, formulates a condition termed r-monotonicity**. r-monotonicity** reduces to Maskin monotonicity** when we focus on SCFs. Under NWA and in an environment with more than three players, r-monotonicity** can be shown to be sufficient for rationalizable implementation of an SCC under (Jain (2021)'s notion of implementation. Following Jain (2021)'s approach, it is straightforward to formulate a two-player sufficient condition for rationalizable implementation of SCCs, which will reduce to 2P-GSMM** for SCFs.

Although a complete characterization of the rationalizable implementation of SCCs is beyond the scope of this paper, SEM** of Xiong (2022) and our 2P-GSMM** must be a theoretical benchmark for any work focussing on rationalizable implementation of SCCs. We conjecture that our framework of deceptions could help provide a complete characterization of the rationalizable implementation of SCCs.

Appendices

A. Proof of "Only If" part of Theorem 1

Proof. Suppose that $\mathcal{M}$ rationalizable implements f. Since we are focusing on non-constant SCFs, it follows from Xiong (2022) that $\mathcal{I}^{\Theta} \neq \emptyset$.[14] Our proof is fundamentally based on the deception profile $\hat{\beta}$, where for all $i \in \mathcal{I}$, $\hat{\beta}_i : \Theta \mapsto 2^{\Theta} \setminus \{\emptyset\}$ is defined

[14]See p. 36 of Xiong (2022) for details.

by

$$\hat{\beta}_i(\theta) = \left\{\hat{\theta} \in \Theta | S_i^{\mathcal{M},\hat{\theta}} \subseteq S_i^{\mathcal{M},\theta}\right\}. \tag{17}$$

Our proof is based on three steps. Step 1 shows that f satisfies 2P-GSMM** with respect to $\hat{\beta}$ and that $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i \in \mathcal{P}_f$. Step 2 shows that $\beta^{*P} \subseteq \hat{\beta}$. Step 3 shows that $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i = \bigcap_{i\in\mathcal{I}} \beta_i^{*P} = P$.

Step 1: f satisfies 2P-GSMM** with respect to $\hat{\beta}$ and that $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i = P \in \mathcal{P}_f$.

To show that $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i = P \in \mathcal{P}_f$, let us first show that for all $\theta \in \Theta$,

$$\bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta) = \left\{\hat{\theta} \in \Theta | S^{\mathcal{M},\hat{\theta}} = S^{\mathcal{M},\theta}\right\}. \tag{18}$$

Fix any $\theta \in \Theta$. It is clear from (17)-(18) that $\left\{\hat{\theta} \in \Theta | S^{\mathcal{M},\hat{\theta}} = S^{\mathcal{M},\theta}\right\} \subseteq \bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta)$. For the converse, suppose that $\hat{\theta} \in \bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta)$. The equality in (18) holds if we show that $S^{\mathcal{M},\hat{\theta}} = S^{\mathcal{M},\theta}$. By definition of $\hat{\beta}_i$, we have that $S^{\mathcal{M},\hat{\theta}} \subseteq S^{\mathcal{M},\theta}$. Since $\mathcal{M}$ rationalizable implements f and since $S^{\mathcal{M},\hat{\theta}} \subseteq S^{\mathcal{M},\theta}$, Lemma 1 of Xiong (2022) (p. 19) implies that $S^{\mathcal{M},\hat{\theta}} = S^{\mathcal{M},\theta}$.

Let us show that $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i \in \mathcal{P}_f$. It is clear that $\theta \in \bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta)$ for all $\theta \in \Theta$ and that $\bigcup_{\theta\in\Theta} \left(\bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta)\right) = \Theta$. Fix any $\theta, \theta' \in \Theta$. Suppose that $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta) \neq \bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta')$. We show that $\left(\bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta)\right) \bigcap \left(\bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta')\right) = \emptyset$. Assume, to the contrary, that there exists $\hat{\theta} \in \left(\bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta)\right) \bigcap \left(\bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta')\right)$. Then, by (18), $S^{\mathcal{M},\hat{\theta}} = S^{\mathcal{M},\theta}$ and $S^{\mathcal{M},\hat{\theta}} = S^{\mathcal{M},\theta'}$, and so $S^{\mathcal{M},\theta} = S^{\mathcal{M},\theta'}$. Since $S_i^{\mathcal{M},\theta} = S_i^{\mathcal{M},\theta'}$ for all $i \in \mathcal{I}$, (18) implies that $\hat{\beta}_i(\theta) = \hat{\beta}_i(\theta')$ for all $i \in \mathcal{I}$, and so $\hat{\beta}(\theta) = \hat{\beta}(\theta')$. This implies that $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta) = \bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta')$, which is a contradiction. Therefore, $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i$ is a partition of Θ. Finally, let us show that $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta) \subseteq P_f(\theta)$ for all $\theta \in \Theta$. Fix any $\theta \in \Theta$. Suppose that $\hat{\theta} \in \bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta)$. Then, (18) implies that $S_i^{\mathcal{M},\hat{\theta}} = S_i^{\mathcal{M},\theta}$. Since $\mathcal{M}$ rationalizable implements f and since $S^{\mathcal{M},\hat{\theta}} = S^{\mathcal{M},\theta}$, it follows that $f(\theta) = f\left(\hat{\theta}\right)$, and so $\hat{\theta} \in P_f(\theta)$. Since the choice of $\theta \in \Theta$ was arbitrary, we have that $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i \in \mathcal{P}_f$.

Let us show that f satisfies Measurability. Take any $\theta, \theta' \in \Theta$ such that $f(\theta) \neq$

$f(\theta')$. We show that $\hat{\beta}(\theta) \bigcap \hat{\beta}(\theta') = \emptyset$. Assume, to the contrary, that $\hat{\beta}(\theta) \bigcap \hat{\beta}(\theta') \neq \emptyset$. Then, there exists $\hat{\theta} \in \hat{\beta}_i(\theta) \bigcap \hat{\beta}_i(\theta')$ for all $i \in \mathcal{I}$. It follows from (17) that $S_i^{\mathcal{M},\theta} \bigcap S_i^{\mathcal{M},\theta'} \neq \emptyset$ for all $i \in \mathcal{I}$, and so $S^{\mathcal{M},\theta} \bigcap S^{\mathcal{M},\theta'} \neq \emptyset$. Since $\mathcal{M}$ rationalizable implements f and since $S^{\mathcal{M},\theta} \bigcap S^{\mathcal{M},\theta'} \neq \emptyset$, it follows that $f(\theta) = f(\theta')$, which is a contradiction.

Let us show that f satisfies $\hat{\beta}$-GSMM**. Take any $\theta, \theta' \in \Theta$. Let us proceed according to whether $\mathcal{I}^{P(\theta')} = \mathcal{I}$ or $\mathcal{I}^{P(\theta')} \neq \mathcal{I}$.

Case 1: $\mathcal{I}^{P(\theta')} = \mathcal{I}$

Suppose that for all $i \in \mathcal{I}$, there exists $\hat{\theta}(i) \in \hat{\beta}_{-i}(\theta)$ such that $SL_i\left(f\left(\hat{\theta}(i)\right), \hat{\theta}(i)\right) \subseteq L_i(f(\theta), \theta')$. We show that $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta) = \bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta')$.

Fix any $i \in \mathcal{I}$. Since $\hat{\theta}(i) \in \hat{\beta}_{-i}(\theta)$, (17) implies that $S_{-i}^{\mathcal{M},\hat{\theta}(i)} \subseteq S_{-i}^{\mathcal{M},\theta}$. The definition of $S^{\mathcal{M},\hat{\theta}(i)}$ and the fact that $\mathcal{M}$ implements f imply that there exists $\lambda_i^{\hat{\theta}(i)} \in \Delta\left(S_{-i}^{\mathcal{M},\hat{\theta}(i)}\right)$ such that for all $m_i \in S_i^{\mathcal{M},\hat{\theta}(i)}$, m_i is a best-response to $\lambda_i^{\hat{\theta}(i)}$ at $\hat{\theta}(i)$.[15] Fix any $m_i^\theta \in S_i^{\mathcal{M},\theta}$. Let us show that m_i^θ is a best-response to $\lambda_i^{\hat{\theta}(i)}$ at θ'. Assume, to the contrary, that there exists $\hat{m}_i \in M_i$ such that

$$\sum_{m_{-i}\in M_{-i}} \lambda_i^{\hat{\theta}(i)}(m_{-i})\, u_i(g(\hat{m}_i, m_{-i}), \theta') > \sum_{m_{-i}\in M_{-i}} \lambda_i^{\hat{\theta}(i)}(m_{-i})\, u_i\left(g\left(m_i^\theta, m_{-i}\right), \theta'\right). \tag{19}$$

Since $\mathcal{M}$ rationalizable implements f and since, moreover, $\lambda_i^{\hat{\theta}(i)} \in \Delta\left(S_{-i}^{\mathcal{M},\hat{\theta}(i)}\right)$, $S_{-i}^{\mathcal{M},\hat{\theta}(i)} \subseteq$

[15] To see this, take any $\theta \in \Theta$ and any $i \in \mathcal{I}$. Since f is rationalizably implementable by $\mathcal{M}$, it follows that $S^{\mathcal{M},\theta} \neq \emptyset$ and $f(\theta) = g(m)$ for all $m \in S^{\mathcal{M},\theta}$. Fix any $m_i \in S_i^{\mathcal{M},\theta}$. Then, m_i is a best-response to some $\lambda_i^{m_i,\theta} \in \Delta\left(S_{-i}^{\mathcal{M},\theta}\right)$ at θ. Let $\lambda_i^{m_i,\theta} = \lambda_i^\theta$. Fix any $m_i^\theta \in S_i^{\mathcal{M},\theta}$. Since f is rationalizably implementable by $\mathcal{M}$, we have that

$$u_i(f(\theta), \theta) = \sum_{m_{-i}\in M_{-i}} \lambda_i^\theta(m_{-i})\, u_i\left(g\left(m_i^\theta, m_{-i}\right), \theta\right) \geq \sum_{m_{-i}\in M_{-i}} \lambda_i^\theta(m_{-i})\, u_i(g(m_i, m_{-i}), \theta)$$

for all $m_i \in M_i$. Thus, m_i^θ is a best-response to λ_i^θ at θ. Since the choice of $m_i^\theta \in S_i^{\mathcal{M},\theta}$ is arbitrary, the statement follows.

$S_{-i}^{\mathcal{M},\theta}$ and $m_i^\theta \in S_i^{\mathcal{M},\theta}$, it follows that

$$\sum_{m_{-i}\in M_{-i}} \lambda_i^{\hat{\theta}(i)}(m_{-i})\, u_i\left(g\left(m_i^\theta, m_{-i}\right), \theta'\right) = \sum_{m_{-i}\in M_{-i}} \lambda_i^{\hat{\theta}(i)}(m_{-i})\, u_i\left(f(\theta), \theta'\right).$$

It follows from (19) that

$$\sum_{m_{-i}\in M_{-i}} \lambda_i^{\hat{\theta}(i)}(m_{-i})\, u_i\left(g\left(\hat{m}_i, m_{-i}\right), \theta'\right) > \sum_{m_{-i}\in M_{-i}} \lambda_i^{\hat{\theta}(i)}(m_{-i})\, u_i\left(f(\theta), \theta'\right). \tag{20}$$

Since for all $m_i^{\hat{\theta}(i)} \in S_i^{\mathcal{M},\hat{\theta}(i)}$, $m_i^{\hat{\theta}(i)}$ is a best-response to $\lambda_i^{\hat{\theta}(i)}$ at $\hat{\theta}(i)$, it follows that

$$\sum_{m_{-i}\in M_{-i}} \lambda_i^{\hat{\theta}(i)}(m_{-i})\, u_i\left(g\left(m_i^{\hat{\theta}(i)}, m_{-i}\right), \hat{\theta}(i)\right) \geq \sum_{m_{-i}\in M_{-i}} \lambda_i^{\hat{\theta}(i)}(m_{-i})\, u_i\left(g\left(\hat{m}_i, m_{-i}\right), \hat{\theta}(i)\right).$$

Moreover, since $\mathcal{M}$ rationalizable implements f, we have that

$$\sum_{m_{-i}\in M_{-i}} \lambda_i^{\hat{\theta}(i)}(m_{-i})\, u_i\left(f\left(\hat{\theta}(i)\right), \hat{\theta}(i)\right) \geq \sum_{m_{-i}\in M_{-i}} \lambda_i^{\hat{\theta}(i)}(m_{-i})\, u_i\left(g\left(\hat{m}_i, m_{-i}\right), \hat{\theta}(i)\right). \tag{21}$$

(20) and (21) imply that $SL_i\left(f\left(\hat{\theta}(i)\right), \hat{\theta}(i)\right) \nsubseteq L_i\left(f(\theta), \theta'\right)$, which is a contradiction. Therefore, for all $m_i^\theta \in S_i^{\mathcal{M},\theta}$, m_i^θ is a best-response to $\lambda_i^{\hat{\theta}(i)}$ at θ'. Moreover, since $\lambda_i^{\hat{\theta}(i)} \in \Delta\left(S_{-i}^{\mathcal{M},\hat{\theta}(i)}\right)$ and since $S_{-i}^{\mathcal{M},\hat{\theta}(i)} \subseteq S_{-i}^{\mathcal{M},\theta}$, it follows that $\lambda_i^{\hat{\theta}(i)}(m_{-i}) > 0 \implies m_{-i} \in S_{-i}^{\mathcal{M},\theta}$.

Since the choice of player $i \in \mathcal{I}$ was arbitrary, we have that for all $i \in \mathcal{I}$, there exists $\lambda_i \in \Delta(M_{-i})$ such that $\lambda_i(m_{-i}) > 0 \implies m_{-i} \in S_{-i}^{\mathcal{M},\theta}$, and for all $m_i^\theta \in S_i^{\mathcal{M},\theta}$, m_i^θ is a best-response to λ_i at θ'. This implies that $S^{\mathcal{M},\theta} \subseteq S^{\mathcal{M},\theta'}$. Lemma 1 of Xiong (2022) (p. 19) implies that $S^{\mathcal{M},\theta} = S^{\mathcal{M},\theta'}$. (18) implies that $\hat{\beta}(\theta) = \hat{\beta}(\theta')$, and so $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta) = \bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta')$. Thus, f satisfies $\hat{\beta}$-GSMM**.

Case 2: $\mathcal{I}^{P(\theta')} \neq \mathcal{I}$

Then, $\mathcal{I}^{P(\theta')} = \{1\}$. Suppose that for some $\hat{\theta} \in \hat{\beta}_2(\theta)$, it holds that

$$SL_1\left(f\left(\hat{\theta}\right),\hat{\theta}\right)\subseteq L_1\left(f\left(\theta\right),\theta'\right) \tag{22}$$

Arguing as in *Case 1* above, we can see that $S_1^{\mathcal{M},\theta}\subseteq S_1^{\mathcal{M},\theta'}$. Moreover, since $\mathcal{I}^{P(\theta')}=\{1\}$ and since $\Theta=\beta_2^P(\theta')\subseteq\hat{\beta}_2(\theta')$, it holds that $S_2^{\mathcal{M},\theta}\subseteq S_2^{\mathcal{M},\theta'}=M_2$. Thus, $S^{\mathcal{M},\theta}\subseteq S^{\mathcal{M},\theta'}$. Lemma 1 of Xiong (2022) (p. 19) implies that $S^{\mathcal{M},\theta}=S^{\mathcal{M},\theta'}$. (18) implies that $\hat{\beta}\left(\theta\right)=\hat{\beta}\left(\theta'\right)$, and so $\bigcap_{i\in\mathcal{I}}\hat{\beta}_i\left(\theta\right)=\bigcap_{i\in\mathcal{I}}\hat{\beta}_i\left(\theta'\right)$. Thus, f satisfies $\hat{\beta}$-GSMM**.

Step 2: $\beta^{*P}\subseteq\hat{\beta}$.

The following result will be useful in proving this step.

Lemma 4. For all $(\theta,\theta')\in\Theta\times\bar{\Theta}^P$, all $i\in\mathcal{I}$, and for all $k\geq 0$, if f satisfies Measurability with respect β_k^P and $P\left(\theta\right)\subseteq\beta_{i,k}^P\left(\theta'\right)$, then $f\left(\theta\right)=f(\theta')$.[16]

Proof. Let us proceed by induction over $k\geq 0$.

Let $k=0$. Fix any $(\theta,\theta')\in\Theta\times\bar{\Theta}^P$ and any $i\in\mathcal{I}$. Suppose that f satisfies Measurability with respect $\beta_0^P\equiv\beta^P$ and $P\left(\theta\right)\subseteq\beta_i^P\left(\theta'\right)$. We show that $f\left(\theta\right)=f\left(\theta'\right)$. The proof is obvious if $P(\theta)=P(\theta')$. Thus, suppose that $P(\theta)\neq P(\theta')$.

Since $P\left(\theta\right)\subseteq\beta_i^P\left(\theta'\right)$ and $P(\theta)\neq P(\theta')$, the definition of β_i^P implies that there exist $\bar{\theta}\in P\left(\theta'\right)$ and $\hat{\theta}\in P\left(\theta\right)$ such that

$$SL_i\left(f\left(\bar{\theta}\right),\bar{\theta}\right)\cap L_{-i}\left(f\left(\hat{\theta}\right),\hat{\theta}\right)=\emptyset.$$

Let us first show that $P\left(\theta'\right)\subseteq\beta_{-i}^P\left(\theta\right)$. Assume, to the contrary, that $P\left(\theta'\right)\nsubseteq\beta_{-i}^P\left(\theta\right)$. Then, for all $\left(\bar{\theta},\hat{\theta}\right)\in P\left(\theta'\right)\times P\left(\theta\right)$, it holds that

$$SL_{-i}\left(f\left(\hat{\theta}\right),\hat{\theta}\right)\cap L_i\left(f\left(\bar{\theta}\right),\bar{\theta}\right)\neq\emptyset.$$

It follows that there exists $x\in SL_{-i}\left(f\left(\hat{\theta}\right),\hat{\theta}\right)\cap L_i\left(f\left(\bar{\theta}\right),\bar{\theta}\right)$ for all $\left(\bar{\theta},\hat{\theta}\right)\in$

[16] Observe that this lemma does not rely on the assumption that f is rationalizably implementable.

$P(\theta') \times P(\theta)$. Since $\theta' \in \bar{\Theta}^P$, it follows that $P(\theta') \subseteq \bar{\Theta}^P$ and thus, there exists $y \in SL_i\left(f\left(\bar{\theta}\right), \bar{\theta}\right)$ for all $\bar{\theta} \in P(\theta')$. Fix any $\left(\bar{\theta}, \hat{\theta}\right) \in P(\theta') \times P(\theta)$ and any $\varepsilon \in (0,1)$. Let us define the lottery z^ε by $z^\varepsilon = \varepsilon x + (1-\varepsilon) y$. For ε small enough, we have that

$$z^\varepsilon \in SL_i\left(f\left(\bar{\theta}\right), \bar{\theta}\right) \cap L_{-i}\left(f\left(\hat{\theta}\right), \hat{\theta}\right),$$

which is a contradiction. Thus, $P(\theta') \subseteq \beta^P_{-i}(\theta)$. Since $P(\theta) \subseteq \beta^P_i(\theta')$, it follows from definition of β^P that $P(\theta) \subseteq \beta^P_i(\theta') \cap \beta^P_i(\theta)$. Similarly, since $P(\theta') \subseteq \beta^P_{-i}(\theta)$, we have that $P(\theta') \subseteq \beta^P_{-i}(\theta) \cap \beta^P_{-i}(\theta')$. Measurability with respect to β^P implies that $f(\theta) = f(\theta')$.

Inductive hypothesis: Suppose for some $k \geq 0$, the following statement holds for all $(\theta, \theta') \in \Theta \times \bar{\Theta}^P$ and all $i \in \mathcal{I}$: If f satisfies Measurability with respect to β^P_k and $P(\theta') \subseteq \beta^P_{i,k}(\theta)$, then $f(\theta) = f(\theta')$.

Let us show that the statement holds for $k+1$. Fix any $(\theta, \theta') \in \Theta \times \bar{\Theta}^P$ and any $i \in \mathcal{I}$. Suppose that f satisfies Measurability with respect to β^P_{k+1} and that $P(\theta') \subseteq \beta^P_{i,k+1}(\theta)$. We show that $f(\theta) = f(\theta')$. We proceed according to whether $P(\theta') \subseteq \beta^P_{i,k}(\theta)$ or not.

Suppose that $P(\theta') \subseteq \beta^P_{i,k}(\theta)$. Since f satisfies Measurability with respect to β^P_{k+1} and since $\beta^P_k \subseteq \beta^P_{k+1}$, it follows that f satisfies Measurability with respect to β^P_k. The inductive hypothesis implies that $f(\theta) = f(\theta')$.

Suppose that $P(\theta') \subseteq \beta^P_{i,k+1}(\theta)$ but $P(\theta') \not\subseteq \beta^P_{i,k}(\theta)$. Then, since $P(\theta') \subseteq \beta^P_{i,k+1}(\theta)$, it follows that there exists $\bar{\theta} \in \Theta$ such that $P(\theta') \subseteq \beta^P_{i,k}\left(\bar{\theta}\right)$ and $P(\theta) \subseteq \beta^P_{-i,k}\left(\bar{\theta}\right)$. Since f satisfies Measurability with respect to β^P_{k+1} and since $\beta^P_k \subseteq \beta^P_{k+1}$, it follows that f satisfies Measurability with respect to β^P_k. Since $P(\theta') \subseteq \beta^P_{i,k}\left(\bar{\theta}\right)$, it follows from our inductive hypothesis that $f(\theta') = f\left(\bar{\theta}\right)$. Again, since $P(\theta) \subseteq \beta^P_{-i,k}\left(\bar{\theta}\right)$, it follows from our inductive hypothesis that $f(\theta) = f\left(\bar{\theta}\right)$. We conclude that $f(\theta) = f(\theta')$.

The principle of mathematical induction implies that the statement holds.

$\blacksquare$

Let us now complete the proof of Step 2. Recall that $\bigcap_{i\in\mathcal{I}}\hat{\beta}_i = P$. Let us proceed by induction over k.

Initial Step, $k = 0$. Let us show that $\beta^P \subseteq \hat{\beta}$. Fix any $\theta \in \Theta$ and any $i \in \mathcal{I}$. Since $\bigcap_{i\in\mathcal{I}}\hat{\beta}_i \equiv P$, it is clear that $P(\theta) \subseteq \hat{\beta}_i(\theta) \cap \beta_i^P(\theta)$. Then, take any $\theta' \in \Theta$ such that $P(\theta) \neq P(\theta')$. Suppose that $P(\theta') \subseteq \beta_i^P(\theta)$. We show that $P(\theta') \subseteq \hat{\beta}_i(\theta)$. Since $P(\theta') \subseteq \beta_i^P(\theta)$, the definition of β_i^P in (7) implies that $SL_i\left(f\left(\bar{\theta}\right),\bar{\theta}\right)\bigcap L_{-i}\left(f\left(\hat{\theta}\right),\hat{\theta}\right) = \emptyset$ for some $\left(\bar{\theta},\hat{\theta}\right) \in P(\theta)\times P(\theta')$. Since $\mathcal{M}$ rationalizable implements f, it holds that

$$g\left[S_i^{\mathcal{M},\hat{\theta}} \times S_{-i}^{\mathcal{M},\bar{\theta}}\right] \subseteq L_i\left(f\left(\bar{\theta}\right),\bar{\theta}\right)\bigcap L_{-i}\left(f\left(\hat{\theta}\right),\hat{\theta}\right). \tag{23}$$

Let us show that $S_i^{\mathcal{M},\hat{\theta}} \subseteq S_i^{\mathcal{M},\bar{\theta}}$. Assume, to the contrary, $S_i^{\mathcal{M},\hat{\theta}} \nsubseteq S_i^{\mathcal{M},\bar{\theta}}$. Then, there exists $m_i^{\hat{\theta}} \in S_i^{\mathcal{M},\hat{\theta}}$ such that $m_i^{\hat{\theta}} \notin S_i^{\mathcal{M},\bar{\theta}}$. The definition of $S^{\mathcal{M},\hat{\theta}}$ and the fact that $\mathcal{M}$ implements f imply that there exists $\lambda_i^{\bar{\theta}} \in \Delta\left(S_{-i}^{\mathcal{M},\bar{\theta}}\right)$ such that for all $m_i^{\bar{\theta}} \in S_i^{\mathcal{M},\bar{\theta}}$, $m_i^{\bar{\theta}}$ is a best-response to $\lambda_i^{\bar{\theta}}$ at $\bar{\theta}$.[17] Since $m_i^{\hat{\theta}} \in S_i^{\mathcal{M},\hat{\theta}} \backslash S_i^{\mathcal{M},\bar{\theta}}$, it follows from (23) that there exists $m_{-i}^{\bar{\theta}} \in S_{-i}^{\mathcal{M},\bar{\theta}}$ such that $g\left(m_i^{\hat{\theta}}, m_{-i}^{\bar{\theta}}\right) \in SL_i\left(f\left(\bar{\theta}\right),\bar{\theta}\right) \cap L_{-i}\left(f\left(\hat{\theta}\right),\hat{\theta}\right)$, which is a contradiction. Thus, we have that $S_i^{\mathcal{M},\hat{\theta}} \subseteq S_i^{\mathcal{M},\bar{\theta}}$. Since $\bar{\theta} \in P(\theta)$, it follows from (18) that $S_i^{\mathcal{M},\hat{\theta}} \subseteq S_i^{\mathcal{M},\theta}$. Since $\hat{\theta} \in P(\theta')$, it follows from (18) that $S_i^{\mathcal{M},\hat{\theta}} \subseteq S_i^{\mathcal{M},\theta}$ for all $\hat{\theta} \in P(\theta')$. It follows from (17) that $P(\theta') \subseteq \hat{\beta}_i(\theta)$.

Inductive hypothesis: Suppose that for some $k \geq 0$, $\beta_k^P \subseteq \hat{\beta}$.

Let us show that $\beta_{k+1}^P \subseteq \hat{\beta}$. Fix any $\theta \in \Theta$ and any $i \in \mathcal{I}$. Since $\bigcap_{i\in\mathcal{I}}\hat{\beta}_i \equiv P$, it is clear that $P(\theta) \subseteq \hat{\beta}_i(\theta) \cap \beta_{i,k+1}^P(\theta)$. Then, take any $\theta' \in \Theta$ such that $P(\theta) \neq P(\theta')$. Suppose that $P(\theta') \subseteq \beta_{i,k+1}^P(\theta)$. We show that $P(\theta') \subseteq \hat{\beta}_i(\theta)$. We proceed according to whether $P(\theta') \subseteq \beta_{i,k}^P(\theta)$ or not.

Suppose that $P(\theta') \subseteq \beta_{i,k}^P(\theta)$. The inductive hypothesis implies that $P(\theta') \subseteq \hat{\beta}_i(\theta)$. Thus, suppose $P(\theta') \subseteq \beta_{i,k+1}^P(\theta)$ but $P(\theta') \nsubseteq \beta_{i,k}^P(\theta)$. Then, $\theta \in \bar{\Theta}^P$ and there

[17] See footnote 15.

exists $\bar{\theta} \in \bar{\Theta}^P$ such that $P(\theta') \subseteq \beta^P_{i,k}\left(\bar{\theta}\right)$ and $P(\theta) \subseteq \beta^P_{-i,k}\left(\bar{\theta}\right)$. The inductive hypothesis implies that $P(\theta') \subseteq \hat{\beta}_i\left(\bar{\theta}\right)$ and $P(\theta) \subseteq \hat{\beta}_{-i}\left(\bar{\theta}\right)$. It follows from (17) that $S_i^{\mathcal{M},\hat{\theta}} \times S_{-i}^{\mathcal{M},\theta} \subseteq S^{\mathcal{M},\bar{\theta}}$ for all $\hat{\theta} \in P(\theta')$. Since $\mathcal{M}$ rationalizable implements f, we have that $g\left[S_i^{\mathcal{M},\hat{\theta}} \times S_{-i}^{\mathcal{M},\theta}\right] = f\left(\bar{\theta}\right)$ for all $\hat{\theta} \in P(\theta')$. Since we have already shown that f satisfies Measurability with respect to $\hat{\beta}$ and since, moreover, it follows from the inductive hypothesis that $\beta^P_k \subseteq \hat{\beta}$, we have that f satisfies Measurability with respect to β^P_k.

Since $\bar{\theta} \in \bar{\Theta}^P$ and $P(\theta) \subseteq \beta^P_{-i,k}\left(\bar{\theta}\right)$, Lemma 4 implies that $f(\theta) = f\left(\bar{\theta}\right)$, and so $g\left[S_i^{\mathcal{M},\hat{\theta}} \times S_{-i}^{\mathcal{M},\theta}\right] = f(\theta)$ for all $\hat{\theta} \in P(\theta')$. Since $\mathcal{M}$ rationalizable implements f, it follows that $S_i^{\mathcal{M},\hat{\theta}} \subseteq S_i^{\mathcal{M},\theta}$ for all $\hat{\theta} \in P(\theta')$. The definition of $\hat{\beta}_i$ in (17) implies that $P(\theta') \subseteq \hat{\beta}_i(\theta)$.

By the principle of mathematical induction, it follows that $\beta^P_k \subseteq \hat{\beta}$ for all $k \geq 0$. Since β^{*P} is the limit point of the sequence $\{\beta^P_k\}_{k\geq 0}$, we have that $\beta^{*P} \subseteq \hat{\beta}$

Remark 7. In the above proof of Step 2, we did not distinguish between Θ^P and $\bar{\Theta}^P$. Indeed, in proving that $\beta^P \subseteq \hat{\beta}$ we chose θ arbitrarily from Θ.

Remark 8. For all $\theta \in \Theta^P$, it can easily be shown that $\beta^P(\theta) = \hat{\beta}(\theta)$. To see this, recall that when $\theta \in \Theta^P$, it holds that $\beta^P_2(\theta) = \Theta$. Since $\beta^P_2(\theta) \subseteq \hat{\beta}_2(\theta)$, by Step 2, it holds that $\hat{\beta}_2(\theta) = \Theta$. Thus, $\beta^P_2(\theta) = \hat{\beta}_2(\theta)$. To show that $\hat{\beta}_1(\theta) = \beta^P_1(\theta)$, since we have already proved that $\beta^P_1(\theta) \subseteq \hat{\beta}_1(\theta)$, it suffices to show that $\hat{\beta}_1(\theta) \subseteq \beta^P_1(\theta)$. Since $\hat{\beta}_1(\theta) = P(\theta)$ and $\hat{\beta}_2(\theta) = \Theta$, it holds that $\hat{\beta}_1(\theta) = P(\theta) \subseteq \beta^P_1(\theta)$.

Step 3: $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i = \bigcap_{i\in\mathcal{I}} \beta^{*P}_i$.

Since for all $i \in \mathcal{I}$ and all $\theta \in \Theta$, $\beta^{*P}_i(\theta) \subseteq \hat{\beta}_i(\theta)$, it is clear that $\bigcap_{i\in\mathcal{I}} \beta^{*P}_i \subseteq \bigcap_{i\in\mathcal{I}} \hat{\beta}_i$. Let us show that $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i \subseteq \bigcap_{i\in\mathcal{I}} \beta^{*P}_i$. Fix any $\theta \in \Theta$. Suppose that $\theta' \in \bigcap_{i\in\mathcal{I}} \hat{\beta}_i(\theta)$. Since $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i \equiv P$, it follows that $\theta' \in P(\theta)$. Since $P(\theta) \subseteq \beta^P_i(\theta) \subseteq \beta^{*P}_i(\theta)$ for all $i \in \mathcal{I}$, we have that $\theta' \in \bigcap_{i\in\mathcal{I}} \beta^{*P}_i(\theta)$. Thus, $\bigcap_{i\in\mathcal{I}} \hat{\beta}_i = \bigcap_{i\in\mathcal{I}} \beta^{*P}_i$.

Since $\bigcap_{i\in\mathcal{I}}\hat{\beta}_i = \bigcap_{i\in\mathcal{I}}\beta_i^{*P} = P \in \mathcal{P}_f$ and $\beta^{*P} \subseteq \hat{\beta}$, it follows that f satisfies 2P-GSMM** with respect to P such that $P = \bigcap_{i\in\mathcal{I}}\beta_i^{*P}$.

■

B. Proof of "If" part of Theorem 1

Suppose that f satisfies 2P-GSMM**. The following allocation will be useful to define the mechanism.

$$\underline{y} = \frac{1}{2}\frac{1}{|\Theta|}\sum_{i\in\mathcal{I}}\sum_{\theta'\in\Theta} y_i(\theta') \tag{24}$$

where $y_i(\theta') \in SL_i(f(\theta'),\theta')$ if $SL_i(f(\theta'),\theta') \neq \emptyset$, otherwise, $y_i(\theta') \in L_i(f(\theta'),\theta')$. For all (θ',θ'') such that $SL_i(f(\theta''),\theta'')\bigcap SL_{-i}(f(\theta') \neq \emptyset$, let $e(\theta',\theta'')$ be defined by

$$e(\theta',\theta'') = (1-\epsilon)z(\theta',\theta'') + \epsilon\underline{y}. \tag{25}$$

where $z(\theta',\theta'') \in SL_i(f(\theta''),\theta'')\bigcap SL_{-i}(f(\theta') \neq \emptyset$ and $\epsilon > 0$ is small enough such that $e(\theta',\theta'') \in SL_i(f(\theta''),\theta'')\bigcap SL_{-i}(f(\theta'),\theta')$.

For all $\bar{\theta} \in \Theta$ such that $SL_i(f(\bar{\theta}),\bar{\theta}) \neq \emptyset$, let us define $\underline{x}_i(\bar{\theta},\bar{\theta})$ by

$$\underline{x}_i(\bar{\theta},\bar{\theta}) = (1-\epsilon)z_i(\bar{\theta}) + \epsilon\underline{y} \tag{26}$$

where $z_i(\bar{\theta}) \in SL_i(f(\bar{\theta}),\bar{\theta})$. Since $z_i(\bar{\theta}) \in SL_i(f(\bar{\theta}),\bar{\theta})$, we can find an $\epsilon > 0$ sufficiently small such that $\underline{x}_i(\bar{\theta},\bar{\theta}) \in SL_i(f(\bar{\theta}),\bar{\theta})$.

To define the mechanism, we need to define more allocations. To this end, let us define $\underline{y}_i(\theta)$ by

$$\underline{y}_i(\theta) = \frac{1}{2}\frac{1}{|\Theta|}\Big[\sum_{\theta'\in\Theta} y_{-i}(\theta') + \sum_{\theta'\in\Theta\setminus\{\theta\}} y_i(\theta') + f(\theta)\Big]. \tag{27}$$

Lemma 5. For all $i \in \mathcal{I}$ and all $\theta \in \Theta$, there exists a function $\hat{x}_i^{\theta} : \Theta \to Y$ such that for all $\bar{\theta} \in \Theta$, if $SL_i(f(\theta), \theta) \neq \emptyset$ and $SL_i(f(\bar{\theta}), \bar{\theta}) \neq \emptyset$, then $u_i(\hat{x}_i^{\theta}(\bar{\theta}), \theta) > u_i(\underline{x}_i(\bar{\theta}, \bar{\theta}), \theta)$ and $\hat{x}_i^{\theta}(\bar{\theta}) \in SL_i(f(\bar{\theta}), \bar{\theta})$.

Proof. Fix any $i \in \mathcal{I}$ and any $\theta \in \Theta$. Fix any $\bar{\theta} \in \Theta$. Suppose that $SL_i(f(\theta), \theta) = \emptyset$ or $SL_i(f(\bar{\theta}), \bar{\theta}) = \emptyset$. Then, let $\hat{x}_i^{\theta}(\bar{\theta}) = f(\bar{\theta})$.

Assume that $SL_i(f(\theta), \theta) \neq \emptyset$ and $SL_i(f(\bar{\theta}), \bar{\theta}) \neq \emptyset$. Since $y_i(\theta) \in SL_i(f(\theta), \theta)$, it holds that $u_i(\underline{y}_i(\theta), \theta) > u_i(\underline{y}, \theta)$. Based on $\underline{y}_i(\theta)$, defined in (27), let us define $\hat{x}_i(\bar{\theta}, \theta)$ by

$$\hat{x}_i^{\theta}(\bar{\theta}) = (1 - \gamma) z_i(\bar{\theta}) + \gamma \underline{y}_i(\theta), \tag{28}$$

where $z_i(\bar{\theta}) \in SL_i(f(\bar{\theta}), \bar{\theta})$ is used in (26) to define $\underline{x}_i(\bar{\theta}, \bar{\theta})$ and where $\gamma > 0$ is small enough such that $u_i(\hat{x}_i^{\theta}(\bar{\theta}), \theta) > u_i(\underline{x}_i(\bar{\theta}, \bar{\theta}), \theta)$ and $\hat{x}_i^{\theta}(\bar{\theta}) \in SL_i(f(\bar{\theta}), \bar{\theta})$. ■

Lemma 6. For all $i \in \mathcal{I}$ and all $\theta \in \Theta$, there exists a function $\tilde{x}_i^{\theta} : \Theta \times \Theta \to Y$ such that for all $(\theta', \bar{\theta}) \in \Theta \times \Theta$, if $SL_i(f(\theta), \theta) \neq \emptyset$ and $(\theta', \bar{\theta}) \notin \beta^{*P}(\theta^*)$ for all $\theta^* \in \Theta$, then $u_i(\tilde{x}_i^{\theta}(\theta', \bar{\theta}), \theta) > u_i(e(\theta', \bar{\theta}), \theta)$ and $\tilde{x}_i^{\theta}(\theta', \bar{\theta}) \in SL_i(f(\bar{\theta}), \bar{\theta})$.

Proof. Fix any $i \in \mathcal{I}$ and any $\theta \in \Theta$. Let us define $\tilde{x}_i^{\theta}(\theta', \bar{\theta})$ for all $(\theta', \bar{\theta})$. Assume that $SL_i(f(\theta), \theta) = \emptyset$ or $(\theta', \bar{\theta}) \in \beta^{*P}(\theta^*)$ for some $\theta^* \in \Theta$. Then, let $\tilde{x}_i^{\theta}(\theta', \bar{\theta}) = f(\bar{\theta})$. Otherwise, assume that $SL_i(f(\theta), \theta) \neq \emptyset$ and $(\theta', \bar{\theta}) \notin \beta^{*P}(\theta^*)$ for all $\theta^* \in \Theta$. Based on $\underline{y}_i(\theta)$, let us define $\tilde{x}_i^{\theta}(\theta', \bar{\theta})$ by

$$\tilde{x}_i^{\theta}(\theta', \bar{\theta}) = (1 - \gamma) e(\theta', \bar{\theta}) + \gamma \underline{y}_i(\theta). \tag{29}$$

where $e(\theta', \bar{\theta}) \in SL_i(f(\bar{\theta}), \bar{\theta})$ and it is defined in (25), and where $\gamma > 0$ is sufficiently small such that $u_i(\tilde{x}_i^{\theta}(\theta', \bar{\theta}), \theta) > u_i(e(\theta', \bar{\theta}), \theta)$ and $\tilde{x}_i^{\theta}(\theta', \bar{\theta}) \in SL_i(f(\bar{\theta}), \bar{\theta})$. ■

Let $\hat{Y} = \bigcup_{i \in \mathcal{I}} \bigcup_{\theta \in \Theta} \underline{y}_i(\theta)$. Observe that $\hat{Y} \neq \emptyset$. By Lemma 5, let us define the set $\hat{X}$ by $\hat{X} = \cup_{i \in \mathcal{I}} \cup_{\theta \in \Theta} \cup_{\bar{\theta} \in \Theta} \hat{x}_i^{\theta}\left(\bar{\theta}\right)$. Finally, by Lemma 6, let us define the set $\tilde{X}$ by $\tilde{X} = \cup_{i \in \mathcal{I}} \cup_{\theta \in \Theta} \cup_{\left(\theta', \bar{\theta}\right) \in \Theta \times \Theta} \hat{x}_i^{\theta}\left(\theta', \bar{\theta}\right)$. Since Θ is a finite set, it follows that $\hat{Y}$, $\hat{X}$, and $\tilde{X}$ are finite sets as well. The following finite set will be used in defining the mechanism. For every $(i, \bar{\theta}, \theta) \in \mathcal{I} \times \Theta \times \Theta$, let $\mathcal{Y}_i(\bar{\theta}, \theta)$ be defined as follows:

$$\mathcal{Y}_i(\bar{\theta}, \theta) = \left\{ f(\bar{\theta}) \bigcup D_i(\bar{\theta}, \theta) \right\} \bigcup \left\{ \left\{ \hat{Y} \cup \hat{X} \cup \tilde{X} \right\} \cap L_i(f(\bar{\theta}), \bar{\theta}) \right\}, \tag{30}$$

where $D_i(\bar{\theta}, \theta)$ is a finite subset of $\{y \in Y | y \in L_i(f(\bar{\theta}), \bar{\theta}) \bigcap SU_i(f(\bar{\theta}), \theta)\}$. By construction, it holds that $\mathcal{Y}_i(\bar{\theta}, \theta) \subseteq L_i(f(\bar{\theta}), \bar{\theta})$ and since $f(\bar{\theta}) \in \mathcal{Y}_i(\bar{\theta}, \theta)$, $\mathcal{Y}_i(\bar{\theta}, \theta)$ is nonempty.[18]

Let us construct $\mathcal{M} = (M, g)$. Each $i \in \mathcal{I}$ plays a strategy $m_i = (m_i^1, m_i^2)$, where $m_i^1 \in \Theta$, $m_i^2 \in \mathbb{Z}_+$. By construction, $M_i = \Theta \times \mathbb{Z}_+$ is a nonempty countable set for player i. For all $m \in M$, the outcome $g(m)$ is defined by the following rules.

Rule 0: If $m_1^1 \in \Theta^P$ and $m_1^2 = 0$, then

$$g(m) = f\left(m_1^1\right).$$

Rule 1: If there exists $\bar{\theta} \in \bar{\Theta}^P$ such that $(m_i^1)_{i \in \mathcal{I}} \in \beta^{*P}\left(\bar{\theta}\right)$ and $m_i^2 = 0$ for all $i \in \mathcal{I}$, then

$$g(m) = f\left(\bar{\theta}\right).$$

Rule 2: If $m_1^2 = m_2^2 = 0$ and $(m_i^1)_{i \in \mathcal{I}} \notin \beta^{*P}\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \Theta$, then

$$g(m) = e\left(m_i^1, m_{-i}^1\right),$$

[18]When the set of pure outcomes X is finite, it is without loss of generality to set $\mathcal{Y}_i(\bar{\theta}, \theta) \equiv L_i(f(\bar{\theta}), \bar{\theta})$, for every $(i, \bar{\theta}, \theta) \in \mathcal{I} \times \Theta \times \Theta$.

where $e\left(m_i^1, m_{-i}^1\right)$ is defined in (25).

Rule 3: If **Rule 0** does not apply and if for some $\bar{\theta} \in \Theta$, $\left(m_{-i}^1, m_{-i}^2\right) = \left(\bar{\theta}, 0\right)$ and $m_i^2 > 0$, then

$$g\left(m\right) = \left(\frac{m_1^2}{m_1^2 + 1}\right) \bar{x}_i\left(\bar{\theta}, m_i^1\right) + \left(\frac{1}{m_i^2 + 1}\right) \underline{x}_i\left(\bar{\theta}, \bar{\theta}\right).$$

where $\bar{x}_i\left(\bar{\theta}, m_i^1\right) \in \underset{y \in \mathcal{Y}_i(\bar{\theta}, m_i^1)}{argmax}\, u_i(y, m_i^1)$, $\mathcal{Y}_i(\bar{\theta}, m_i^1)$ is defined in (30), and $\underline{x}_i\left(\bar{\theta}, \bar{\theta}\right) \in SL_i(f(\bar{\theta}), \bar{\theta})$ is defined in (26).

Rule 4: In all other cases, an integer game is played: we identify a pivotal player i by requiring that $m_i^2 \geq m_{-i}^2$, and that if $m_i^2 = m_{-i}^2$, then $i < -i$. Then,

$$g\left(m\right) = \left(\frac{m_i^2}{m_i^2 + 1}\right) y_i^*(m_i^1) + \left(\frac{1}{m_i^2 + 1}\right) \underline{y},$$

where $y_i^*(m_i^1) \in \underset{y \in \bar{Y}}{argmax}\, u_i(y, m_i^1)$ and where $\bar{Y}$ be a finite subset of Y such that $\emptyset \neq \hat{Y} \subseteq \bar{Y}$.

Lemma 7. The outcome function g is well-defined.

Proof. To check that g is well-defined, we need only to check that **Rule 1**, **Rule 2**, and **Rule 3** are well-defined.

(A) **Rule 1** is well-defined: Assume, to the contrary, that there exists $m \in M$ falling into **Rule 1** such that for some $\bar{\theta}, \bar{\theta}' \in \bar{\Theta}^P$, $(m_i^1)_{i \in \mathcal{I}} \in \beta^{*P}\left(\bar{\theta}\right)$ and $m_i^2 = 0$ for all $i \in \mathcal{I}$, $(m_i^1)_{i \in \mathcal{I}} \in \beta^{*P}\left(\bar{\theta}'\right)$ and $m_i^2 = 0$ for all $i \in \mathcal{I}$, and $f\left(\bar{\theta}\right) \neq f\left(\bar{\theta}'\right)$. Then, $m_i^1 \in \beta_i^{*P}\left(\bar{\theta}\right) \cap \beta_i^{*P}\left(\bar{\theta}'\right)$ for all $i \in \mathcal{I}$, and so $\beta^{*P}\left(\bar{\theta}\right) \cap \beta^{*P}\left(\bar{\theta}'\right) \neq \emptyset$. Since f satisfies Measurability, we have that $f\left(\bar{\theta}\right) = f\left(\bar{\theta}'\right)$, which is a contradiction.

(B) **Rule 2** is well-defined: To see it, suppose that m is such that $g(m)$ falls into **Rule 2**. Lemma 8 below implies $e\left(m_i^1, m_{-i}^1\right) \in SL_i(f(m_{-i}^1), m_{-i}^1) \bigcap SL_{-i}(f(m_i^1), m_i^1)$.

(C) **Rule 3** is well-defined: To show it, we need to show that $\underset{y\in\mathcal{Y}_i(\bar{\theta},m_i^1)}{argmax}\, u_i(y,m_i^1)$ and the lottery $\underline{x}_i(\bar{\theta},\bar{\theta}) \in SL_i(f(\bar{\theta}),\bar{\theta})$ exists. Since $\mathcal{Y}_i(\bar{\theta},m_i^1)$ is a finite nonempty set, $\underset{y\in\mathcal{Y}_i(\bar{\theta},m_i^1)}{argmax}\, u_i(y,m_i^1)$ exists.

To show that the lottery $\underline{x}_i(\bar{\theta},\bar{\theta}) \in SL_i(f(\bar{\theta}),\bar{\theta})$ exists, suppose that m is such that $g(m)$ falls into **Rule 3**. Suppose that i induces **Rule 3** and $m_{-i} = (\bar{\theta}, 0)$. Since we are in **Rule 3**, it holds that $SL_i(f(\bar{\theta}),\bar{\theta}) \neq \emptyset$. To see it, suppose that $i = 1$. Then, since $\mathcal{I}^\Theta = \{1\}$, by assumption, it follows that $SL_1(f(\bar{\theta}),\bar{\theta}) \neq \emptyset$. Suppose that $i = 2$. Since $m_{-i}^2 = 0$ and **Rule 3** applies, so that **Rule 0** does not apply, it is the case that $m_{-i}^1 = \bar{\theta} \in \bar{\Theta}^P$. Since $\bar{\Theta}^P \cap \Theta(2) = \emptyset$, it follows that $SL_2(f(\bar{\theta}),\bar{\theta}) \neq \emptyset$. We conclude that $SL_i(f(\bar{\theta}),\bar{\theta}) \neq \emptyset$. Let us define $\underline{x}_i(\bar{\theta},\bar{\theta})$ as in (26), where $z_i(\bar{\theta}) \in SL_i(f(\bar{\theta}),\bar{\theta})$. Since $z_i(\bar{\theta}) \in SL_i(f(\bar{\theta}),\bar{\theta})$, we can find an $\epsilon > 0$ sufficiently small such that $\underline{x}_i(\bar{\theta},\bar{\theta}) \in SL_i(f(\bar{\theta}),\bar{\theta})$.

(D) **Rule 4** is well-defined: This follows from the fact that $\bar{Y}$ is a finite nonempty set.

■

Lemma 8.

$$\left[\begin{array}{l} (\theta',\theta'') \notin \beta_i^{*P}\left(\bar{\theta}\right) \times \beta_{-i}^{*P}\left(\bar{\theta}\right) \\ \forall \bar{\theta} \in \Theta \end{array}\right] \implies \begin{array}{l} \exists e\left(\theta',\theta''\right) \in Y \text{ as defined in (25) such that} \\ e\left(\theta',\theta''\right) \in SL_i\left(f\left(\theta''\right),\theta''\right) \bigcap SL_{-i}\left(f\left(\theta'\right),\theta'\right). \end{array}$$

Proof. Fix any (θ',θ'') such that $(\theta',\theta'') \notin \beta_i^{*P}\left(\bar{\theta}\right) \times \beta_{-i}^{*P}\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \Theta$. In particular, since $\beta^P \subseteq \beta^{*P}$, $(\theta',\theta'') \notin \beta_i^{P}\left(\bar{\theta}\right) \times \beta_{-i}^{P}\left(\bar{\theta}\right)$ for all $\bar{\theta} \in \Theta$.

We first show that $SL_i\left(f\left(\theta''\right),\theta''\right) \bigcap SL_{-i}\left(f\left(\theta'\right),\theta'\right) \neq \emptyset$. Assume, to the contrary, that

$$SL_i\left(f\left(\theta''\right),\theta''\right) \bigcap SL_{-i}\left(f\left(\theta'\right),\theta'\right) = \emptyset. \tag{31}$$

Since $P\left(\theta''\right) \subseteq \beta_{-i}^P\left(\theta''\right)$, we obtain a contradiction if we show that $P\left(\theta'\right) \subseteq \beta_i^P\left(\theta''\right)$.

Assume, to the contrary, that $P(\theta') \nsubseteq \beta_i^P(\theta'')$. By definition of β_i^P, it follows that $SL_i\left(f\left(\bar{\theta}\right), \bar{\theta}\right) \bigcap L_{-i}\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \neq \emptyset$ for all $\left(\bar{\theta}, \hat{\theta}\right) \in P(\theta'') \times P(\theta')$, and so

$$SL_i\left(f\left(\theta''\right), \theta''\right) \bigcap L_{-i}\left(f\left(\theta'\right), \theta'\right) \neq \emptyset. \tag{32}$$

Let us show that $SL_{-i}\left(f\left(\theta'\right), \theta'\right) = \emptyset$. Assume, to the contrary, that $SL_{-i}\left(f\left(\theta'\right), \theta'\right) \neq \emptyset$. Take any $x \in SL_{-i}\left(f\left(\theta'\right), \theta'\right)$. Since the intersection in (32) is also nonempty, take any $y \in SL_i\left(f\left(\theta''\right), \theta''\right) \bigcap L_{-i}\left(f\left(\theta'\right), \theta'\right)$. Let $z = px + (1-p)y$ where $p \in (0,1)$. Thus, for some $p \in (0,1)$, we have that $z \in SL_i\left(f\left(\theta''\right), \theta''\right) \bigcap SL_{-i}\left(f\left(\theta'\right), \theta'\right)$, which is a contradiction. Thus, $SL_{-i}\left(f\left(\theta'\right), \theta'\right) = \emptyset$. The definition of β^P in (7) implies that $\beta_{-i}^P(\theta') = \Theta$. Thus, we have that $(\theta', \theta'') \in \beta_i^P(\theta') \times \beta_{-i}^P(\theta')$, which is a contradiction. Therefore, it must be the case that $P(\theta') \subseteq \beta_i^P(\theta'')$, which is a contradiction. Thus, $SL_i\left(f\left(\theta''\right), \theta''\right) \bigcap SL_{-i}\left(f\left(\theta'\right), \theta'\right) \neq \emptyset$.

Let $0 < \epsilon < 1$. Let us define $e(\theta', \theta'')$ as in (25). For ϵ small enough we have that $e(\theta', \theta'') \in SL_i\left(f\left(\theta''\right), \theta''\right) \bigcap SL_{-i}\left(f\left(\theta'\right), \theta'\right)$. ■

Lemma 9. For all $\theta \in \Theta$, $\bar{m} = ((\theta, 0), (\theta, 0)) \in NE(\mathcal{M}, \theta)$—that is, for all $i \in \mathcal{I}$,

$$u_i\left(g\left(\bar{m}_i, \bar{m}_{-i}\right), \theta\right) \geq u_i\left(g\left(m_i, \bar{m}_{-i}\right), \theta\right), \tag{33}$$

for all $m_i \in M_i$.

Proof. Suppose that $\theta \in \Theta$. By construction, $\bar{m}$ falls into either **Rule 0** or **Rule 1**. In both cases, it holds that $g(\bar{m}) = f(\theta)$. Fix any $i \in \mathcal{I}$ and any $m_i \in M_i$. Note that no unilateral deviation of i from $\bar{m}$ can induce **Rule 4**. Thus, **Rules 0-3** apply if i changes $\bar{m}_i$ into m_i.

(A) Suppose that $(m_i, \bar{m}_{-i})$ falls into Rule 0. Assume, to the contrary, that it holds that $u_i(g((m_i, \bar{m}_{-i}), \theta) > u_i(g((\bar{m}_i, \bar{m}_{-i}), \theta)$. Then, $g(m_i, \bar{m}_{-i}) \neq g(\bar{m}_i, \bar{m}_{-i})$ and $i = 1$, that is, $m_i = m_1$ and $m_1^2 = 0$. Moreover, $m_1^1 \in \Theta^P$ and $g(m_i, \bar{m}_{-i}) = f(m_1^1)$. Thus, it holds that $u_1(f(\theta), \theta) < u_1(f(m_1^1), \theta)$. This implies that

$P(m_1^1) \neq P(\theta)$. Since $m_1^1 \in \Theta^P$, it follows that there exists $\hat{\theta} \in P(m_1^1)$ such that $SL_2(f(\hat{\theta}), \hat{\theta}) = \emptyset$. By definition of β_2^P in (7), we have that $\beta_2^P(m_1^1) = \beta_2^{*P}(m_1^1) = \Theta$. Since 2P-GSMM** requires that $\bigcap_{i \in \mathcal{I}} \beta^{*P} = P$, we have that $\beta_1^{*P}(m_1^1) = P(m_1^1)$, and so $\hat{\theta} \in \beta_1^{*P}(m_1^1)$. Since f satisfies β^P-GSMM** and since $P(m_1^1) \neq P(\theta)$ and there exists $\hat{\theta} \in \beta_1^{*P}(m_1^1)$ such that $SL_2(f(\hat{\theta}), \hat{\theta}) \subseteq L_2(f(m_1^1), \theta)$, it must be the case that for $\theta \in \beta_2^{*P}(m_1^1) = \Theta$, it holds that $SL_1(f(\theta), \theta) \bigcap SU_1(f(m_1^1), \theta) \neq \emptyset$. Thus, there is an allocation x such that $u_1(f(\theta), \theta) \geq u_1(x, \theta) > u_1(f(m_1^1), \theta)$. Therefore, we have that $u_1(f(\theta), \theta) > u_1(f(m_1^1), \theta)$.

(B) Suppose that $(m_i, \bar{m}_{-i})$ falls into Rule 1. Then, there exists $\bar{\theta}' \in \bar{\Theta}^P$ such that $\bar{m}_{-i}^1 \in \beta_{-i}^{*P}\left(\bar{\theta}'\right)$, $m_i^1 \in \beta_i^{*P}\left(\bar{\theta}'\right)$, $m_i^2 = \bar{m}_{-i}^2 = 0$ and $g\left(m_i, \bar{m}_{-i}\right) = f\left(\bar{\theta}'\right)$. Since $\theta \in \beta_{-i}^{*P}\left(\bar{\theta}'\right)$, and so $P(\theta) \subseteq \beta_{-i}^{*P}\left(\bar{\theta}'\right)$, and since f satisfies Measurability with respect to β^{*P}, Lemma 4 implies that $f(\theta) = f(\bar{\theta}')$.

(C) Suppose that $(m_i, \bar{m}_{-i})$ falls into Rule 2. Then, $g\left(m\right) = e\left(m_i^1, \bar{m}_{-i}^1\right)$, and so $e\left(m_i^1, \theta\right) \in SL_i\left(f\left(\theta\right), \theta\right)$, by construction of g.

(D) Suppose that $(m_i, \bar{m}_{-i})$ falls into Rule 3. Then, $g\left(m_i, \bar{m}_{-i}\right) = \left(\frac{m_i^2}{m_i^2+1}\right) \bar{x}_i\left(\theta, m_{-i}^1\right) + \left(\frac{1}{m_i^2+1}\right) \underline{x}_i\left(\theta, \theta\right)$. Since $\bar{x}_i\left(\theta, m_{-i}^1\right) \in \mathcal{Y}_i(\theta, m_i^1) \subseteq L_i\left(f\left(\theta\right), \theta\right)$ and $\underline{x}_i\left(\theta, \theta\right) \in L_i\left(f\left(\theta\right), \theta\right)$, it follows that $g\left(m_i, \bar{m}_{-i}\right) \in L_i\left(f\left(\theta\right), \theta\right)$.

Since $\theta \in \Theta$, $i \in \mathcal{I}$ and $m_i \in M_i$ are arbitrary, we conclude that the inequality in (33) is satisfied for all $i \in \mathcal{I}$, for all $\theta \in \Theta$, and all $m_i \in M_i$. Thus, $\bar{m} \in NE\left(\mathcal{M}, \theta\right)$ for all $\theta \in \Theta$. ■

Suppose that θ is true state. Lemma 9 implies that $S^{\mathcal{M},\theta}$ is nonempty. According to Definition 1, to complete the proof, we need to show that $m \in S^{\mathcal{M},\theta} \implies g\left(m\right) = f\left(\theta\right)$. To this end, we need additional intermediate results. The following Lemmata are immediate implications of Lemma 5, Lemma 6, and the definition of $\bar{x}_i$ in Rule 3 of the mechanism.

Lemma 10. For all $i \in \mathcal{I}$ and all $\theta, \bar{\theta} \in \Theta$ such that $SL_i(f(\theta), \theta) \neq \emptyset$ and $SL_i(f(\bar{\theta}), \bar{\theta}) \neq \emptyset$, $u_i(\bar{x}_i(\bar{\theta}, \theta), \theta) > u_i(\underline{x}_i(\bar{\theta}, \bar{\theta}), \theta)$.

Lemma 11. For all $i \in \mathcal{I}$ and all $\theta, \bar{\theta}, \theta' \in \Theta$ such that $SL_i(f(\theta), \theta) \neq \emptyset$ and $(\theta', \bar{\theta}) \notin \beta^{*P}(\theta^*)$ for all $\theta^* \in \Theta$, $u_i(\bar{x}_i(\bar{\theta}, \theta), \theta) > u_i(e(\theta', \bar{\theta}), \theta)$.

For all $i \in \mathcal{I}$ and all $\theta \in \Theta$, let $\hat{m}_i = (\theta, \hat{m}_i^2) \in M_i$. The following lemmata will help us to complete the proof.

Lemma 12. For all $i \in \mathcal{I}$, all $\theta \in \Theta$, all $m_i \in M_i$ and all $\lambda_i^\theta \in \Delta(M_{-i})$, if m_i is a best-response to λ_i^θ at θ, then $m_i^2 = 0$ if $i = 1$, otherwise, $m_i^2 = 0$, or for some $m_{-i} \in supp(\lambda_i^\theta), m_{-i}^1 \in \Theta^P$, or $\theta \in \Theta^P$.

Proof. Fix any $i \in \mathcal{I}$, any $\theta \in \Theta$, any $\lambda_i^\theta \in \Delta(M_{-i})$ and any $m_i \in M_i$ so that m_i is a best-response to λ_i^θ at θ. We proceed according to whether $i = 1$ or $i = 2$.

Case A: $i = 1$

Assume, to the contrary, $m_1^2 > 0$. Since $i = 1$ and since $m_1^2 > 0$, it follows that for all $m_2 \in supp(\lambda_1^\theta)$, (m_1, m_2) falls either into **Rule 3** or into **Rule 4**. Let $\mathfrak{M}_2^3$ be defined by

$$\mathfrak{M}_2^3 = \left\{m_2 \in supp\left(\lambda_1^\theta\right) | m_2^2 = 0\right\}.$$

Similarly, let $\mathfrak{M}_2^4$ be defined by

$$\mathfrak{M}_2^4 = \left\{m_2 \in supp\left(\lambda_1^\theta\right) | m_2^2 > 0\right\}.$$

Clearly, $supp\left(\lambda_1^\theta\right) = \mathfrak{M}_2^3 \cup \mathfrak{M}_2^4$. Fix any $m_2 \in \mathfrak{M}_2^3$. Then, (m_1, m_2) falls into **Rule 3** and

$$g(m_1, m_2) = \left(\frac{m_1^2}{m_1^2 + 1}\right) \bar{x}_1\left(m_2^1, m_1^1\right) + \left(\frac{1}{m_1^2 + 1}\right) \underline{x}_1\left(m_2^1, m_2^1\right).$$

By definition of $\bar{x}_1\left(m_2^1, \cdot\right)$ provided in the definition of **Rule 3**, we have that

$$u_1\left(\bar{x}_1\left(m_2^1, \theta\right), \theta\right) \geq u_1\left(\bar{x}_1\left(m_2^1, m_1^1\right), \theta\right).$$

Moreover, Lemma 10 implies that

$$u_1\left(\bar{x}_1\left(m_2^1,\theta\right),\theta\right) > u_1\left(\underline{x}_1\left(m_2^1,m_2^1\right),\theta\right). \tag{34}$$

Thus, player 1 by changing m_1 into $\hat{m}_1$, with $\hat{m}_1^2 = m_1^2 + k$ for any positive integer k, he induces **Rule 3** and he obtains

$$g\left(\hat{m}_1,m_2\right) = \left(\frac{m_1^2+k}{m_1^2+k+1}\right)\bar{x}_1\left(m_2^1,\theta\right) + \left(\frac{1}{m_1^2+k+1}\right)\underline{x}_1\left(m_2^1,m_2^1\right).$$

Since $u_1\left(\bar{x}_1\left(\bar{\theta},\theta\right),\theta\right) \geq u_1\left(\bar{x}_1\left(\bar{\theta},m_1^1\right),\theta\right)$, since (34) holds, and since $\hat{m}_1^2 = m_1^2 + k > m_1^2$, it follows that

$$u_1\left(g\left(\hat{m}_1,m_2\right),\theta\right) > u_1\left(g\left(m_1,m_2\right),\theta\right). \tag{35}$$

Since the choice of $m_2 \in \mathfrak{M}_2^3$ was arbitrary, it follows that (35) holds for all $m_2 \in \mathfrak{M}_2^3$, and so

$$\sum_{m_2\in\mathfrak{M}_2^3} \lambda_1^\theta\left(m_2^1\right)u_1\left(g\left(\hat{m}_1,m_2\right),\theta\right) > \sum_{m_2\in\mathfrak{M}_2^3} \lambda_1^\theta\left(m_2^1\right)u_1\left(g\left(m_1,m_2\right),\theta\right). \tag{36}$$

Let us partition $\mathfrak{M}_2^4$ by defining $\mathfrak{M}_2^4(1)$ by

$$\mathfrak{M}_2^4(1) = \left\{m_2 \in \mathfrak{M}_2^4 | m_1^2 \geq m_2^2\right\}.$$

Let $\mathfrak{M}_2^4(2)$ be defined by

$$\mathfrak{M}_2^4(2) = \left\{m_2 \in \mathfrak{M}_2^4 | m_2^2 > m_1^2\right\}.$$

Fix any $m_2 \in \mathfrak{M}_2^4$. Then, (m_1,m_2) falls into **Rule 4** and

$$g\left(m_1,m_2\right) = \left(\frac{m_j^2}{m_j^2+1}\right)y_j^*\left(m_j^1\right) + \left(\frac{1}{m_j^2+1}\right)\underline{y} \tag{37}$$

for some $j \in \mathcal{I}$. Fix any $m_2 \in \mathfrak{M}_2^4(1)$ and any $\bar{m}_2 \in \mathfrak{M}_2^4(2)$. Let $a^* = \max\{m_1^2, \bar{m}_2^2\}$. Player 1 by changing m_1 into $\hat{m}_1$, with $\hat{m}_1^2 = a^* + k'$ for any positive integer k', he induces **Rule 4** and he obtains

$$g(\hat{m}_1, m_2) = \left(\frac{a^* + k'}{a^* + k' + 1}\right) y_1^*(\theta) + \left(\frac{1}{a^* + k' + 1}\right) \underline{y}.$$

By definition of $y_1^*(\cdot)$ provided in the definition of **Rule 4**, we have that

$$u_1(y_1^*(\theta), \theta) \geq u_1\left(y_1^*\left(m_1^1\right), \theta\right) \tag{38}$$

and

$$u_1(y_1^*(\theta), \theta) \geq u_1\left(y_2^*\left(m_2^1\right), \theta\right). \tag{39}$$

Moreover, by construction, it holds that[19]

$$u_1(y_1^*(\theta), \theta) > u_1\left(\underline{y}, \theta\right). \tag{40}$$

Since $\hat{m}_1^2 = a^* + k' > a^* = \max\{m_1^2, \bar{m}_2^2\}$ and since (38)-(40) hold, it follows that

$$u_1(g(\hat{m}_1, m_2), \theta) > u_1(g(m_1, m_2), \theta) \tag{41}$$

and

$$u_1(g(\hat{m}_1, \bar{m}_2), \theta) > u_1(g(m_1, \bar{m}_2), \theta). \tag{42}$$

Given that (39) and (42) hold, it follows that

$$\sum_{\tilde{m}_2 \in \mathfrak{M}_2^4} \lambda_1^\theta\left(\tilde{m}_2^1\right) u_1(g(\hat{m}_1, \tilde{m}_2), \theta) > \sum_{\tilde{m}_2 \in \mathfrak{M}_2^4} \lambda_1^\theta\left(\tilde{m}_2^1\right) u_1(g(m_1, \tilde{m}_2), \theta). \tag{43}$$

Let $b^* = \max\{a^* + k', m_1^2 + k\}$ and let $\hat{m}_1^* = (\theta, b)$. It follows from (36) and (43)

[19] To see this, note that $u_1(y_1^*(\theta), \theta) \geq u_1\left(\underline{y}_1(\theta), \theta\right)$, where $\underline{y}_1(\theta)$ is defined in (27). Since $SL_1(f(\theta), \theta) \neq \emptyset$, it holds that $u_1\left(\underline{y}_1(\theta), \theta\right) > u_1\left(\underline{y}, \theta\right)$.

that

$$\sum_{m_2\in\mathfrak{M}_2^3}\lambda_1^\theta\left(m_2^1\right)u_1\left(g\left(\hat{m}_1^*,m_2\right),\theta\right)+\sum_{\tilde{m}_2\in\mathfrak{M}_2^4}\lambda_1^\theta\left(\tilde{m}_2^1\right)u_1\left(g\left(\hat{m}_1^*,\tilde{m}_2\right),\theta\right)$$
$$>\sum_{m_2\in\mathfrak{M}_2^3}\lambda_1^\theta\left(m_2^1\right)u_1\left(g\left(m_1,m_2\right),\theta\right)+\sum_{\tilde{m}_2\in\mathfrak{M}_2^4}\lambda_1^\theta\left(\tilde{m}_2^1\right)u_1\left(g\left(m_1,\tilde{m}_2\right),\theta\right).$$

which contradicts our initial supposition that m_1 is a best-response to λ_1^θ at θ.

Case B: $i=2$

Assume, to the contrary, that $m_2^2>0$, that $m_1^1\notin\Theta^P$ for all $m_1\in supp(\lambda_2^\theta)$ and that $\theta\notin\Theta^P$. It follows that for all $m_1\in supp\left(\lambda_2^\theta\right)$, (m_1,m_2) falls either into **Rule 3** or into **Rule 4**. By using a reasoning similar to that used in the proof of Case A above, we see that m_2 is not a best-response to λ_2^θ at θ, which is a contradiction.

■

In what follows, we partition M_2 into $M_2(0)$ and $M_2(1)$, where $M_2(0)$ is defined by

$$M_2(0)=\{m_2\in M_2|m_2=0\}\tag{44}$$

and $M_2(1)=M_2\setminus M_2(0)$.

Lemma 13. For all $\theta\in\Theta$ and all $m_1\in M_1$, if m_1 is a best-response to $\lambda_1^\theta\in\Delta(M_2)$ at θ, then $supp\left(\lambda_1^\theta\right)\subseteq M_2(0)$.

Proof. Fix any $\theta\in\Theta$ and any $m_1\in M_1$ such that m_1 is a best-response to $\lambda_1^\theta\in\Delta(M_2)$ at θ. Let us denote the utility of m_1 under λ_1^θ by

$$U_1(m_1,\lambda_1^\theta,\theta)=\sum_{m_2\in M_2}\lambda_1^\theta(m_2)u_1(g(m_1,m_2),\theta).\tag{45}$$

Assume. to the contrary, that there exists $\bar{m}_2\in supp\left(\lambda_1^\theta\right)$ such that $\bar{m}_2^2>0$. Using this assumption, we will show that there exists an integer $\hat{m}_1^2 < \infty$ such that

$$U_1(\hat{m}_1, \lambda_1^\theta, \theta) > U_1(m_1, \lambda_1^\theta, \theta). \tag{46}$$

where $\hat{m}_1 = (\theta, \hat{m}_1^2)$ and $\hat{m}_1^2 > 0$.

Fix any $m_2 \in M_2$. Let us proceed according to whether $m_2 \in M_2(0)$ or not.

Case 1: $m_2 \in M_2(0)$.

Then, $g(m_1, m_2)$ falls into **Rules 0-1**. By construction of g, we have that $u_1(\bar{x}_1(m_2^1, \theta), \theta) \geq u_1(g(m_1, m_2), \theta)$. And so,

$$\underset{\hat{m}_1^2 \to \infty}{lim} u_1(g(\hat{m}_1, m_2), \theta) = u_1(\bar{x}_1(m_2^1, \theta), \theta) \geq u_1(g(m_1, m_2), \theta). \tag{47}$$

Case 2: $m_2 \in M_2(1)$.

By definition of $y_1^*(\theta)$ and the fact that $\underline{y} \in supp(g(m_1, m_2))$, it holds that

$$\underset{\hat{m}_1^2 \to \infty}{lim} u_1(g(\hat{m}_1, m_2), \theta) = u_1(y_1^*(\theta), \theta) > u_1(g(m_1, m_2), \theta). \tag{48}$$

Since

$$\underset{\hat{m}_1^2 \to \infty}{lim} u_1(g(\hat{m}_1, m_2), \theta) = u_1(\bar{x}_1(m_2^1, \theta), \theta) \geq u_1(g(m_1, m_2), \theta). \tag{49}$$

for all $m_2 \in supp(\lambda_1)$, and

$$\underset{\hat{m}_1^2 \to \infty}{lim} u_1(g(\hat{m}_1, m_2), \theta) = u_1(y^*(\theta), \theta) > u_1(g(m_1, m_2), \theta) \tag{50}$$

for some $m_2 \in supp(\lambda_1)$, and since $M_2(1) \neq \emptyset$, we have that

$$\underset{\hat{m}_1^2 \to \infty}{lim} U_1(\hat{m}_1, \lambda_1^\theta, \theta) > U_1(m_1, \lambda_1^\theta). \tag{51}$$

Since the utility of $\hat{m}_1$ under λ_1^θ is strictly increasing in $\hat{m}_1^2$, player 1 can change m_1 into $\hat{m}_1$ and induce **Rule 3**. By appropriately choosing $0 < \hat{m}_1^2 < \infty$, he obtains that

$U_1(\hat{m}_1^\theta, \lambda_1^\theta, \theta) > U_1(m_1, \lambda_1^\theta, \theta)$, which contradicts our initial supposition that m_1 is a best-response to λ_1^θ at θ.

■

Lemma 14. For all $\theta \in \bar{\Theta}^P$, all $i \in \mathcal{I}$, and all $m_i \in S_i^{\mathcal{M},\theta}$, $m_i^1 \notin \Theta^P$ and $m_i^2 = 0$.

Proof. Fix any $\theta \in \bar{\Theta}^P$. We proceed according to whether $i = 1$ or $i = 2$.

Case 1: $i = 1$

Suppose that $m_1 \in S_1^{\mathcal{M},\theta}$. Then, m_1 is a best-response to $\lambda_1^\theta \in \Delta(S_2^{\mathcal{M},\theta})$ at θ. Lemma 12 implies that $m_1^2 = 0$. Lemma 13 implies that $m_2 \in supp\left(\lambda_1^\theta\right)$ is such that $m_2^2 = 0$. Assume, to the contrary, that $m_1^1 \in \Theta^P$. Since $m_1^1 \in \Theta^P$ and $\theta \in \bar{\Theta}^P$, $P(\theta) \neq P(m_1^1)$. Since $m_1^1 \in \Theta^P$, it follows that $\mathcal{I}^{P(m_1^1)} = \{1\}$, and so β^{*P}-GSMM** implies that for all $\hat{\theta} \in \beta_2^{*P}(m_1^1) = \Theta$,

$$L_1(f(\hat{\theta}), \hat{\theta}) \bigcap SU_1(f(m_1^1), \theta) \neq \emptyset. \tag{52}$$

Fix any $\hat{m}_1 = (\theta, \hat{m}_1^2)$, where $\hat{m}_1^2 > 0$. For all $m_2 \in supp\left(\lambda_1^\theta\right)$, $(\hat{m}_1, m_2)$ falls into **Rule 3** and

$$g\left(\hat{m}_1, m_2\right) = \left(\frac{\hat{m}_1^2}{\hat{m}_1^2 + 1}\right) \bar{x}_1\left(m_2^1, \theta\right) + \left(\frac{1}{\hat{m}_1^2 + 1}\right) \underline{x}_1\left(m_2^1, m_2^1\right).$$

Since $u_1\left(\bar{x}_1\left(m_2^1, \cdot\right), \theta\right) > u_1\left(f\left(m_1^1\right), \theta\right)$ for all $m_2 \in supp\left(\lambda_1^\theta\right)$, by choosing a sufficiently high integer $\hat{m}_1^2 > 0$, player 1 obtains that

$$U_1\left(\hat{m}_1, \lambda_1^\theta, \theta\right) > \sum_{m_2 \in supp\left(\lambda_1^\theta\right)} \lambda_1^\theta\left(m_2^1\right) u_1\left(g\left(m_1, m_2\right), \theta\right) = u_1\left(f\left(m_1^1\right), \theta\right). \tag{53}$$

Therefore, by changing m_1 into $\hat{m}_1$ and by choosing an appropriate integer $\hat{m}_1^2 > 0$, player 1 can obtain

$$U_1\left(\hat{m}_1, \lambda_1^\theta, \theta\right) > u_1\left(f\left(m_1^1\right), \theta\right),$$

which contradicts our initial supposition that m_1 is a best-response to λ_1^θ at θ.

Case 2: $i = 2$.

Suppose that $m_2 \in S_2^{\mathcal{M},\theta}$. Then, m_2 is a best-response to $\lambda_2^\theta \in \Delta(S_1^{\mathcal{M},\theta})$ at θ. Lemma 12 implies that or $m_2^2 = 0$, or for some $m_1 \in supp(\lambda_2^\theta), m_1^1 \in \Theta^P$, or $\theta \in \Theta^P$. Since $\theta \in \bar{\Theta}^P$, it follows that $m_2^2 = 0$ or for some $m_1 \in supp(\lambda_2^\theta), m_1^1 \in \Theta^P$. Suppose that for some $m_1 \in supp(\lambda_2^\theta), m_1^1 \in \Theta^P$. Since $m_1 \in S_1^{\mathcal{M},\theta}$, Case 1 above implies that $m_1^1 \notin \Theta^P$, which is a contradiction. Thus, it must be the case that $m_2^2 = 0$.

Let us show that $m_2^1 \notin \Theta^P$. Assume, to the contrary, that $m_2^1 \in \Theta^P$. Fix any $m_1 \in supp(\lambda_2^\theta)$. Since $m_2^1 \in \Theta^P$, we have that $\beta_1^{*P}(m_1^1) = P(m_1^1) \subseteq \bar{\Theta}^P$ and that $\beta_2^{*P}(m_2^1) = \Theta$. Since $m_2^1 \in \Theta^P$ and since $m_1^1 \in \bar{\Theta}^P$, and since $m_1^2 = m_2^2 = 0$, it follows that (m_1, m_2) falls into **Rule 2**. Therefore, $g(m_1, m_2) = e\left(m_2^1, m_2^1\right)$. Since $\theta \notin \Theta^P$, Lemma 11 implies that $u_2\left(\bar{x}_2\left(m_2^1, \theta\right), \theta\right) > u_2\left(e\left(m_2^1, m_2^1\right), \theta\right)$. By choosing an appropriate integer $\hat{m}_2^2 > 0$, player 2 can induce **Rule 3**, obtain $g\left(m_1, \hat{m}_2\right)$ and be strictly better off at θ since $u_2\left(\bar{x}_2\left(m_1^1, \theta\right), \theta\right) > u_2\left(g\left(m_1, m_2\right), \theta\right)$, a contradiction to our initial assumption that m_2 is a best-response to λ_2^θ at θ.

■

Lemma 15. For all $i \in \mathcal{I}$, all $\theta \in \bar{\Theta}^P$ and all $m_i \in M_i$, if m_i is a best-response to $\lambda_i^\theta \in \Delta(S_{-i}^{\mathcal{M},\theta})$ at θ, then there exists $m_{-i} \in supp\left(\lambda_i^\theta\right)$ such that m_i is a best-response to m_{-i} at θ.

Proof. Fix any $i \in \mathcal{I}$, any $\theta \in \bar{\Theta}^P$ anf any $\lambda_i^\theta \in \Delta\left(S_{-i}^{\mathcal{M},\theta}\right)$ and any $m_i \in M_i$ such that m_i is a best-response to λ_i^θ at θ. Assume, to the contrary, that m_i is not a best-response to any $m_{-i} \in supp\left(\lambda_i^\theta\right)$ at θ. Fix any $m_{-i} \in supp\left(\lambda_i^\theta\right)$. Since $m \in S^{\mathcal{M},\theta}$, for each $i \in \mathcal{I}$, there exists $\lambda_i^\theta \in \Delta(S_{-i}^{\mathcal{M},\theta})$ such that m_i is a best-response to λ_i^θ at θ. Lemma 14 implies that $(m_i^2, m_{-i}^2) = (0,0)$ and that $m_1^1 \notin \Theta^P$. Then, (m_i, m_{-i}) cannot fall into **Rule 0**. It follows that (m_i, m_{-i}) falls either into **Rule 1** or **Rule 2**.

Case 1: (m_i, m_{-i}) falls into **Rule 1**.

Then, there exists $\bar{\theta} \in \bar{\Theta}^P$ such that $(m_i^1)_{i \in \mathcal{I}} \in \beta^{*P}\left(\bar{\theta}\right)$ and $m_i^2 = 0$ for all $i \in \mathcal{I}$ and $g(m_i, m_{-i}) = f(\bar{\theta})$. It follows that $g(m_i, m_{-i}) = f(m_{-i}^1)$. To see this, since there exists $\bar{\theta}$ such that $(m_i^1, m_{-i}^1) \in \beta^{*P}(\bar{\theta})$, it follows from definition of β^{*P} that $(m_i^1, m_{-i}^1) \in \beta^{*P}(m_{-i}^1)$. Measurability implies that $f(\bar{\theta}) = f(m_{-i}^1)$.[20]

Clearly, m_i is a best-response to m_{-i} when the true state is m_{-i}^1. Since $m_{-i} \in supp\left(\lambda_i^\theta\right)$, it is the case that $m_{-i}^1 \neq \theta$. Since m_i is not a best-response to m_{-i} at θ, by construction of g, it holds that

$$L_i(f(m_{-i}^1), m_{-i}^1) \bigcap SU_i(f(m_{-i}^1), \theta) \neq \emptyset.$$

By definition of $\bar{x}_i\left(m_{-i}^1, \cdot\right)$ provided in **Rule 3**, it follows that $u_i\left(\bar{x}_i\left(m_{-i}^1, \theta\right), \theta\right) > u_i\left(f\left(m_{-i}^1\right), \theta\right) = u_i\left(g\left(m_i, m_{-i}\right), \theta\right)$.

Case 2: (m_i, m_{-i}) falls into **Rule 2**.

Then, $g(m) = e\left(m_i^1, m_{-i}^1\right)$. Since $\theta \notin \Theta^P$, Lemma 11 implies that $u_i\left(\bar{x}_i\left(m_{-i}^1, \theta\right), \theta\right) > u_i\left(e\left(m_i^1, m_{-i}^1\right), \theta\right) = u_i\left(g\left(m_i, m_{-i}\right), \theta\right)$.

For all $m_{-i} \in supp(\lambda_i^\theta)$, we have

$$\lim_{\hat{m}_i^2 \to \infty} u_i(g(\hat{m}_i, m_{-i}), \theta) = u_i(\bar{x}_i(m_{-i}^1, \theta), \theta) > u_1(g(m_i, m_{-i}), \theta), \tag{54}$$

and so

$$\lim_{\hat{m}_i^2 \to \infty} U_i(\hat{m}_i, \lambda_i^\theta, \theta) > U_i(m_i, \lambda_i^\theta, \theta). \tag{55}$$

Since the utility of $\hat{m}_i$ under λ_i^θ is strictly increasing in $\hat{m}_i^2$, player i can change m_i into $\hat{m}_i$ and induce **Rule 3**. By appropriately choosing $0 < \hat{m}_i^2 < \infty$, he obtains that $U_i(\hat{m}_i, \lambda_i^\theta, \theta) > U_i(m_i, \lambda_i^\theta, \theta)$, which contradicts our initial supposition that m_i is a best-response to λ_i^θ at θ.

■

[20] Since $\left(m_i^1, m_{-i}^1\right) \in \beta^{*P}\left(\bar{\theta}\right)$ for some $\bar{\theta} \in \bar{\Theta}^P$, it holds that $P\left(m_i^1\right) \subseteq \beta_i^{*P}\left(\bar{\theta}\right)$ and $P\left(m_{-i}^1\right) \subseteq \beta_{-i}^{*P}\left(\bar{\theta}\right)$. Again, since β^{*P} is the limit point of the sequence $\left\{\beta_k^P\right\}_{k\geq 0}$, it follows from the definition of β^{*P} that $P\left(m_i^1\right) \subseteq \beta_i^{*P}\left(m_{-i}^1\right)$, and so $m_i^1 \in \beta_i^{*P}\left(m_{-i}^1\right)$.

Lemma 16. For all $\theta \in \Theta^P$ and all $m_1 \in M_1$, if m_1 is a best-response to $\lambda_1^\theta \in \Delta(S_2^{\mathcal{M},\theta})$ at θ, then there exists $m_2 \in supp\left(\lambda_1^\theta\right)$ such that m_1 is a best-response to m_2 at θ.

Proof. Fix any $\theta \in \Theta^P$ and any $m_1 \in M_1$ such that m_1 is a best-response to $\lambda_1^\theta \in \Delta(S_2^{\mathcal{M},\theta})$ at θ. We show that there exists $m_2 \in supp\left(\lambda_1^\theta\right)$ such that m_1 is a best-response to m_2 at θ. Let us denote the utility of m_1 under λ_1^θ by

$$U_1(m_1, \lambda_1^\theta, \theta) = \sum_{m_2 \in M_2} \lambda_1^\theta(m_2) u_1(g(m_1, m_2), \theta). \tag{56}$$

Assume, to the contrary, that for each $m_2 \in supp\left(\lambda_1^\theta\right)$, m_1 is not a best-response to m_2 at θ. Thus, for each $m_2 \in supp\left(\lambda_1^\theta\right)$, there exists $\bar{m}_1$ such that $u_1(g(\bar{m}_1, m_2), \theta) > u_1(g(m_1, m_2), \theta)$. Using this assumption, we will show that there exists $\hat{m}_1^2 < \infty$ such that

$$U_1(\hat{m}_1, \lambda_1^\theta, \theta) > U_1(m_1, \lambda_1^\theta, \theta) \tag{57}$$

where $\hat{m}_1 = (\theta, \hat{m}_1^2)$ and $\hat{m}_1^2 > 0$.

Fix any $m_2 \in supp(\lambda_1^\theta)$. Lemma 13 implies that $m_2^2 = 0$. We proceed by cases.

Case 1: (m_1, m_2) falls into **Rule 0**.

Then, $g(m_1, m_2) = f(m_1^1)$. Since there exists $\bar{m}_1$ such that $u_1(g(\bar{m}_1, m_2), \theta) > u_1(g(m_1, m_2), \theta)$, it must be the case that $u_1(\bar{x}_1(m_2^1, \theta), \theta) > u_1(f(m_1^1), \theta)$.[21] Thus,

$$\lim_{\hat{m}_1^2 \to \infty} u_1(g(\hat{m}_1, m_2), \theta) = u_1(\bar{x}_1(m_2^1, \theta), \theta) > u_1(g(m_1, m_2), \theta). \tag{58}$$

Case 2: (m_1, m_2) falls into **Rule 1**.

Then, $g(m_1, m_2) = f(\bar{\theta})$ for some $\bar{\theta} \in \bar{\Theta}^P$. By definition of β^{*P} and since f satisfies Measurability, it follows that $g(m_1, m_2) = f(m_2^1)$. Since there exists $\bar{m}_1$ such that $u_1(g(\bar{m}_1, m_2), \theta) > u_1(g(m_1, m_2), \theta)$, it must be the case that $u_1(\bar{x}_1(m_2^1, \theta), \theta) >$

[21] Player 1 can only have a profitable deviation by inducing **Rule 3**.

$u_1(f(m_2^1), \theta)$.[22] Therefore,

$$\lim_{\hat{m}_1^2 \to \infty} u_1(g(\hat{m}_1, m_2), \theta) = u_1(\bar{x}_1(m_2^1, \theta), \theta) > u_1(g(m_1, m_2), \theta). \tag{59}$$

Case 3: (m_1, m_2) falls into **Rule 2**.

Then, $g(m) = e\left(m_i^1, m_{-i}^1\right)$. Since there exists $\bar{m}_1$ such that $u_1(g(\bar{m}_1, m_2), \theta) > u_1(g(m_1, m_2), \theta)$, it must be the case that $u_1(\bar{x}_1(m_2^1, \theta), \theta) > u_1(e(m_1^1, m_2^1), \theta)$. Therefore,

$$\lim_{\hat{m}_1^2 \to \infty} u_1(g(\hat{m}_1, m_2), \theta) = u_1(\bar{x}_1(m_2^1, \theta), \theta) > u_1(g(m_1, m_2), \theta). \tag{60}$$

Since the choice of $m_2 \in supp(\lambda_1^\theta)$ was arbitrary, we have that for all $m_2 \in supp(\lambda_1^\theta)$, $\lim_{\hat{m}_1^2 \to \infty} u_1(g(\hat{m}_1, m_2), \theta) > u_1(g(m_1, m_2), \theta)$. It follows that

$$\lim_{\hat{m}_1^2 \to \infty} U_1(\hat{m}_1, \lambda_1^\theta) > U_1(m_1, \lambda_1^\theta). \tag{61}$$

Since the utility of $\hat{m}_1$ under λ_1^θ is strictly increasing in $\hat{m}_1^2$, player 1 can change m_1 into $\hat{m}_1$ and induce **Rule 3**. By appropriately choosing $0 < \hat{m}_1^2 < \infty$, he obtains that $U_1(\hat{m}_1, \lambda_1^\theta, \theta) > U_1(m_1, \lambda_1^\theta, \theta)$, which contradicts our initial supposition that m_1 is a best-response to λ_1^θ at θ. ■

Lemma 17. For all $i \in \mathcal{I}$, all $\theta \in \bar{\Theta}^P$, all $m_i \in S_i^{\mathcal{M},\theta}$ and all $m_{-i} \in S_{-i}^{\mathcal{M},\theta}$, if m_i is a best-response to m_{-i} at θ, then (m_i, m_{-i}) falls into **Rule 1**.

Proof. Fix any $i \in \mathcal{I}$, any $\theta \in \bar{\Theta}^P$, any $m_i \in S_i^{\mathcal{M},\theta}$ and all $m_{-i} \in S_{-i}^{\mathcal{M},\theta}$ such that m_i is a best-response to m_{-i} at θ. Lemma 14 implies that $m_i^2 = 0$ and $m_i^2 \in \bar{\Theta}^P$, for all $i \in \mathcal{I}$. Thus, (m_i, m_{-i}) falls either into **Rule 1** or **Rule 2**. To complete the proof it suffices to show that (m_i, m_{-i}) does not fall into **Rule 2**.

Assume, to the contrary, that (m_i, m_{-i}) falls into **Rule 2**. Then, $g(m_i, m_{-i}) = e\left(m_i^1, m_{-i}^1\right)$. Since $\theta \in \bar{\Theta}^P$, Lemma 11 implies that $u_i\left(\bar{x}_i\left(m_{-i}^1, \theta\right), \theta\right) > u_i\left(e\left(m_i^1, m_{-i}^1\right), \theta\right)$.

[22] Again, player 1 can only have a profitable deviation by inducing **Rule 3**.

Player i can change m_i into $\hat{m}_i$ and induce **Rule 3**. By appropriately choosing $0 < \hat{m}_i^2 < \infty$, he obtains that $u_1(g(\hat{m}_1, m_2), \theta) > u_i\left(e\left(m_i^1, m_{-i}^1\right), \theta\right)$, which contradicts our initial supposition that m_1 is a best-response to m_{-i} at θ. ■

Lemma 18. For all $\theta \in \Theta^P$ and all $m_1 \in S_1^{\mathcal{M},\theta}$, if m_1 is a best response to some $m_2 \in M_2$ at θ, then (m_1, m_2) falls into **Rule 0**.

Proof. Fix any $\theta \in \Theta^P$ and any $m_1 \in S_1^{\mathcal{M},\theta}$ such that m_1 is a best response to some $m_2 \in M_2$ at θ. Lemma 12 implies that $m_1^2 = 0$. Lemma 13 implies that $m_2^2 = 0$. Therefore, (m_1, m_2) does not fall into either **Rule 3** or **Rule 4**. To complete the proof, it suffices to show that (m_1, m_2) does not fall into **Rule 1** and **Rule 2** either. Assume, to the contrary, that (m_1, m_2) falls into **Rule 2**. Then, $g(m_1, m_2) = e(m_1^1, m_2^1)$ and $e(m_1^1, m_2^1) \in SL_1(f(m_2^1), m_2^1)$. Since $SL_1(f(m_2^1), m_2^1) \neq \emptyset$, **Rule 3** is well-defined for $(\hat{m}_1, m_2)$ where $\hat{m}_1^2 > 0$. Since $SL_1(f(\theta), \theta) \neq \emptyset$ and $SL_1(f(m_2^1), m_2^1) \neq \emptyset$, Lemma 11 implies that $u_2\left(\bar{x}_2\left(m_2^1, \theta\right), \theta\right) > u_2\left(e\left(m_1^1, m_2^1\right), \theta\right)$. Player 1 can change m_1 into $\hat{m}_1 = (\theta, \hat{m}_1^2)$ with $\hat{m}_1^2 > 0$ and induce **Rule 3**. By appropriately choosing $0 < \hat{m}_1^2 < \infty$, he obtains that $u_1(g(\hat{m}_1, m_2), \theta) > u_i\left(e\left(m_1^1, m_2^1\right), \theta\right)$, which contradicts our initial supposition that m_1 is a best-response to m_2 at θ.

Assume, to the contrary, that (m_1, m_2) falls into **Rule 1**. Thus, $m_i^2 = 0$ for all $i \in \mathcal{I}$ and $(m_1^1, m_2^1) \in \beta^{*P}\left(\bar{\theta}\right)$ for some $\bar{\theta} \in \bar{\Theta}^P$, and $g(m_1, m_2) = f(\bar{\theta})$. By definition of β^{*P} and since f satisfies Measurability, it follows that $g(m_1, m_2) = f(m_2^1)$. Since m_1 is a best-response to m_2 at θ, it follows that

$$L_1\left(f\left(m_2^1\right), m_2^1\right) \subseteq L_1\left(f\left(m_2^1\right), \theta\right). \tag{62}$$

Since $\theta \in \Theta^P$, it follows that $\mathcal{I}^{P(\theta)} = \{1\}$. Since $m_2^1 \in \beta_2^{*P}(m_2^1)$, β^{*P}–GSMM** implies that $P(m_2^1) = P(\theta)$. Thus, $m_2^1 \in \Theta^P$, which yields a contradiction.

■

To complete the proof, we show the following result.

Lemma 19. For all $i \in \mathcal{I}$, all $\theta \in \Theta$ and all $m_i \in M_i$, $m_i \in S_i^{\mathcal{M},\theta} \implies m_i^1 \in \beta_i^{*P}(\theta)$.

Proof. We proceed according to whether $\theta \in \bar{\Theta}^P$ or $\theta \in \Theta^P$

Case 1: $\theta \in \bar{\Theta}^P$.

Fix any $i \in \mathcal{I}$, any $\theta \in \bar{\Theta}^P$ and any $m_i \in S_i^{\mathcal{M},\theta}$. Let us show that $m_i^1 \in \beta_i^{*P}(\theta)$. Since $m_i \in S_i^{\mathcal{M},\theta}$, there exists $\lambda_i \in \Delta\left(S_{-i}^{\mathcal{M},\theta}\right)$ such that m_i is a best-response to λ_i at θ. Lemma 15 implies that m_i is a best-response to some $m_{-i} \in supp(\lambda_i)$ at θ. Lemma 17 implies that (m_i, m_{-i}) falls into **Rule 1**. Thus, $m_i^2 = 0$ for all $i \in \mathcal{I}$ and $\left(m_i^1, m_{-i}^1\right) \in \beta^{*P}\left(\bar{\theta}\right)$ for some $\bar{\theta} \in \bar{\Theta}^P$, and $g(m_i, m_{-i}) = f(\bar{\theta})$. By definition of β^{*P} and since f satisfies Measurability, it follows that $g(m_i, m_{-i}) = f(m_{-i}^1)$. Since m_i is a best-response to $m_{-i} \in supp(\lambda_i)$ at θ and $m_i^2 = 0$ for all $i \in \mathcal{I}$, it follows that

$$L_i\left(f\left(m_{-i}^1\right), m_{-i}^1\right) \subseteq L_i\left(f\left(m_{-i}^1\right), \theta\right). \tag{63}$$

Since $m_{-i} \in S_{-i}^{\mathcal{M},\theta}$, there exists $\lambda_{-i} \in \Delta\left(S_i^{\mathcal{M},\theta}\right)$ such that m_{-i} is a best-response to λ_{-i} at θ. Lemma 15 implies that m_{-i} is a best-response to some $\hat{m}_i \in supp(\lambda_{-i})$ at θ. Lemma 17 implies that $(\hat{m}_i, m_{-i})$ falls into **Rule 1**. Thus, $\hat{m}_i^2 = m_{-i}^2 = 0$ and $\left(\hat{m}_i^1, m_{-i}^1\right) \in \beta^{*P}\left(\hat{\theta}\right)$ for some $\hat{\theta} \in \bar{\Theta}^P$. Since m_i is a best-response to $m_{-i} \in supp(\lambda_i)$ at θ and $\hat{m}_i^2 = m_{-i}^2 = 0$, it follows that

$$L_{-i}\left(f\left(\hat{m}_i^1\right), \hat{m}_i^1\right) \subseteq L_{-i}\left(f\left(\hat{\theta}\right), \theta\right).$$

Since $\hat{m}_i^1 \in \beta_i^{*P}\left(\hat{\theta}\right)$ and so $P\left(\hat{m}_i^1\right) \subseteq \beta_i^{*P}\left(\hat{\theta}\right)$, Lemma 1 implies that $f\left(\hat{m}_i^1\right) = f\left(\hat{\theta}\right)$, and so

$$L_{-i}\left(f\left(\hat{m}_i^1\right), \hat{m}_i^1\right) \subseteq L_{-i}\left(f\left(\hat{m}_i^1\right), \theta\right). \tag{64}$$

Since $\left(\hat{m}_i^1, m_{-i}^1\right) \in \beta^{*P}\left(\hat{\theta}\right)$, it follows that $P\left(\hat{m}_i^1\right) \subseteq \beta_i^{*P}\left(\hat{\theta}\right)$ and $P\left(m_{-i}^1\right) \subseteq \beta_{-i}^{*P}\left(\hat{\theta}\right)$. Since β^{*P} is the limit point of the sequence $\left\{\beta_k^P\right\}_{k\geq 0}$, it follows from the definition of β^{*P} that $P\left(\hat{m}_i^1\right) \subseteq \beta_i^{*P}\left(m_{-i}^1\right)$, and so $\left(\hat{m}_i^1, m_{-i}^1\right) \in \beta^{*P}\left(m_{-i}^1\right)$. Since $\left(\hat{m}_i^1, m_{-i}^1\right) \in \beta^{*P}\left(m_{-i}^1\right)$ and since (63) and (64) holds, β^{*P}-GSMM** implies that $P\left(\theta\right) = P\left(m_{-i}^1\right)$.

Since $\left(m_i^1, m_{-i}^1\right) \in \beta^{*P}\left(\bar{\theta}\right)$ for some $\bar{\theta} \in \bar{\Theta}^P$, it holds that $P\left(m_i^1\right) \subseteq \beta_i^{*P}\left(\bar{\theta}\right)$ and

$P\left(m_{-i}^1\right) \subseteq \beta_{-i}^{*P}\left(\bar{\theta}\right)$. Again, since β^{*P} is the limit point of the sequence $\left\{\beta_k^P\right\}_{k\geq 0}$, it follows from the definition of β^{*P} that $P\left(m_i^1\right) \subseteq \beta_i^{*P}\left(m_{-i}^1\right)$, and so $m_i^1 \in \beta_i^{*P}\left(m_{-i}^1\right)$. If $\beta_i^{*P}\left(m_{-i}^1\right) \subseteq \beta_i^{*P}\left(\theta\right)$, then $m_i^1 \in \beta_i^{*P}\left(\theta\right)$. Thus, the complete the proof we are left to show that $\beta_i^{*P}\left(m_{-i}^1\right) \subseteq \beta_i^{*P}\left(\theta\right)$. We proceed according to whether $k \neq 0$ or not.

Let $k = 0$. Let us show that $\beta_i^P\left(m_{-i}^1\right) \subseteq \beta_i^P\left(\theta\right)$. Take any $\theta' \in \Theta$ such that $P\left(\theta'\right) \subseteq \beta_i^P\left(m_{-i}^1\right)$. To avoid trivialities, let us suppose that $\theta' \notin \left\{\theta, m_{-i}^1\right\}$.[23] Then, there exists $\left(\bar{\theta}, \hat{\theta}\right) \in P\left(m_{-i}^1\right) \times P\left(\theta'\right)$ such that $SL_i\left(f\left(\bar{\theta}\right), \bar{\theta}\right) \cap L_{-i}\left(f\left(\hat{\theta}\right), \hat{\theta}\right) = \emptyset$. Since $P\left(\theta\right) = P\left(m_{-i}^1\right)$, it follows from definition of β^P in (7) that $P\left(\theta'\right) \subseteq \beta_i^P\left(\theta\right)$. Since the choice of θ' was arbitrary, it follows that $\beta_i^P\left(m_{-i}^1\right) \subseteq \beta_i^P\left(\theta\right)$.

Assume that $k \neq 0$. Let us show that $\beta_{i,k}^P\left(m_{-i}^1\right) \subseteq \beta_{i,k}^P\left(\theta\right)$. Take any $\theta' \in \Theta$ such that $P\left(\theta'\right) \subseteq \beta_{i,k}^P\left(m_{-i}^1\right)$. To avoid trivialities, let us suppose that $\theta' \notin \left\{\theta, m_{-i}^1\right\}$. Since $P\left(\theta'\right) \subseteq \beta_{i,k}^P\left(m_{-i}^1\right)$, then there exists $\bar{\theta} \in \Theta$ such that $P\left(\theta'\right) \subseteq \beta_{i,k-1}^P\left(\bar{\theta}\right)$ and $P\left(m_{-i}^1\right) \subseteq \beta_{-i,k-1}^P\left(\bar{\theta}\right)$. Since $P\left(\theta\right) = P\left(m_{-i}^1\right)$, it follows that there exists $\bar{\theta} \in \Theta$ such that $P\left(\theta'\right) \subseteq \beta_{i,k-1}^P\left(\bar{\theta}\right)$ and $P\left(\theta\right) \subseteq \beta_{-i,k-1}^P\left(\bar{\theta}\right)$. It follows from definition of $\beta_{i,k}^P$ that $P\left(\theta'\right) \subseteq \beta_{i,k}^P\left(\theta\right)$.

Since $\beta_{i,k}^P\left(m_{-i}^1\right) \subseteq \beta_{i,k}^P\left(\theta\right)$ for all $k \geq 0$, it follows that $\beta_i^{*P}\left(m_{-i}^1\right) \subseteq \beta_i^{*P}\left(\theta\right)$.

Case 2: $\theta \in \Theta^P$.

Suppose that $\theta \in \Theta^P$. It is trivial to show that $S_2^{\mathcal{M},\theta} = M_2$. Since $\theta \in \Theta^P$, it follows that $\beta_2^P(\theta) = \Theta$, and so $M_2^1 = \beta_2^{*P}(\theta)$. In what follows, we focus on player 1.

Suppose that $m_1 \in S_1^{\mathcal{M},\theta}$. Let us show that $m_1^1 \in \beta_1^{*P}(\theta)$. 2P-GSSM** implies that $\beta_1^{*P}(\theta) = P(\theta)$. Since $m_1 \in S_1^{\mathcal{M},\theta}$, m_1 is a best-response to $\lambda_1^\theta \in \Delta(S_2^{\mathcal{M},\theta})$ at θ. Lemma 16 implies that there exists $m_2 \in supp\left(\lambda_1^\theta\right)$ such that m_1 is a best-response to m_2 at θ. Lemma 18 implies that (m_1, m_2) falls into **Rule 0**. Then, $m_1^1 \in \Theta^P$. Since m_1 is a best response to m_2 at θ, it holds that

$$L_1(f(m_2^1), m_2^1) \subseteq L_1(f(m_1^1), \theta). \tag{65}$$

[23]Recall that $P\left(\theta\right) = P\left(m_{-i}^1\right)$.

Since $m_2^1 \in \beta_2^{*P}(m_1^1)$ and $\mathcal{I}^{P(\theta)} = \{1\}$, β^{*P}–GSMM** implies that $P(m_1^1) = P(\theta)$. Thus, $m_1^1 \in \beta_1^{*P}(\theta) = P(\theta)$. ■

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