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Bank Diversity and Financial Contagion

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Bank Diversity and Financial Contagion

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Abstract
This paper analyzes financial contagion in a banking system where banks are linked by interbank claims and common assets. We find that asset commonality makes banking systems more vulnerable to idiosyncratic shocks and helps to determine which interbank network structures are resistant to contagion. When the degree of commonality is homogeneous across banks, the most resilient structure is the complete interbank network in which each bank borrows evenly from all the others. However, when the bank most exposed to the defaulting bank is not the one whose portfolio is most similar to it, incomplete interbank networks are more resilient than complete. We also show that the degree and variability of asset commonality between banks and the way this intertwines with the cross-holdings of interbank deposits have important implications for macroprudential regulation.

Keywords: Banking crisis; financial contagion; interbank network; asset commonality.

JEL classification: G01, G21, G28.

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1 Introduction

The role of banks as liquidity providers, the acceptance of bank deposits at par in the payments system and the complex web of mutual interbank exposures make banking systems vulnerable to liquidity shocks at single banks and prone to systemic crisis (Bernanke, 1983; Kindleberger et al., 1996; Allen and Gale, 2009). While the interbank market offers insurance against idiosyncratic liquidity shocks, at the same time cross-holdings of interbank deposits also expose banks to the risk of financial contagion (Bhattacharya and Gale, 1987; Rochet and Tirole, 1996). When a bank defaults on its interbank obligations, the creditor banks suffer from liquidity losses that may, in turn, lead to further payment shortfalls and new defaults. That is, the initial failure can have a domino effect, producing a cascade of bank failures.

Since from the seminal contribution of Allen and Gale (2000), and in reaction to the chain of bank failures and the paralysis of interbank markets worldwide triggered by the subprime crisis and the collapse of Lehman Brothers, an extensive theoretical and empirical literature has explored the resilience of networks of cross-holdings of interbank deposits (or,for simplicity, interbank networks) to idiosyncratic financial shocks to one or a few banks (Summer, 2013; Glasserman and Young, 2016; Allen and Walther, 2021; Jackson and Pernoud, 2021).

The conclusions of this literature reflect the trade-off between the potential for infection and the potential for losses. Allen and Gale (2000) show that the complete, perfectly interconnected interbank network, each bank borrowing evenly from all the others, is the most resilient to an aggregate liquidity shortage. The uniform distribution of interbank claims minimizes adverse spillovers from the failure of one bank, enabling the others to avoid contagion by liquidating only a small fraction of their long-term assets. In a similar multi-bank setting with redistributive liquidity shocks, risky assets and costly information acquisition by depositors, however, Freixas et al. (2000), Brusco and Castiglionesi (2007) and Hasman and Samartín (2008) produce contrary results. The ring network, in which each bank is directly exposed to only one counterparty, is less conducive to financial contagion, because this structure of interbank exposures allows insolvent banks to stay in business by passing on a larger share of their losses to other banks, thus helping to impede systemic failure.

Other studies have noted that complete networks can prove either robust or fragile, depending on the magnitude of the financial shocks to banks (Gai and Kapadia, 2010; Acemoğlu et al., 2015; Glasserman and Young, 2015; Castiglionesi and Eboli, 2018; Eboli, 2019). Drawing on the intuition of Andrew Haldane (2013, p.249), these studies show that interbank networks, like rainforests and other complex adaptive systems, "exhibit a knife-edge, or tipping point, property. Within a certain range, [interbank] connections serve as a shock-absorber. ... But beyond [it, they] ... interconnections serve as shock-amplifiers. ... The system acts not as a mutual insurance device but as a mutual incendiary device".

Our contribution: a numerical illustration. This paper takes up a second conjecture of Haldane, turning on the risk pf contagion engendered by the increasing homogeneity of bank portfolios in recent decades. In Haldane's words, "banks' balance sheets, like Tolstoy's happy families, grew all alike. So too did their risk management strategies" (Haldane, 2013, p. 247). This has been a factor in making "the whole system less resistant to disturbance - mirroring the fortunes of marine eco-systems whose diversity has been steadily eroded and whose susceptibility to collapse has thereby increased" (ibid., p. 245). To explore the nexus between bank diversity and financial contagion, we propose a model in the spirit of Acemoğlu et al. (2015) in which banks, in addition to the cross-holding of interbank claims, are interlinked through overlapping financial asset portfolios.

The systemic risks stemming from the commonality of bank assets are well understood in the literature (Acharya and Yorulmazer, 2007; Ibragimov et al., 2011; Allen et al., 2012). In particular, asset commonality is a vehicle for financial contagion that adds to the interbank market via the feedback mechanisms generated by fire sales at distressed banks (Greenwood et al., 2015; Barucca et al., 2021). When financial markets are not perfectly liquid, massive sales of illiquid banks' assets drive market prices down. Banks that are otherwise sound, unaffected by the liquidity shock but holding the assets sold by the distressed banks, thus also suffer mark-to-market losses which can force them into additional sales, leading to further price reductions (Shleifer and Vishny, 1992; Cifuentes et al., 2005; Ellul et al., 2011; Diamond and Rajan, 2011; Cont and Schaanning, 2019).

We add to the literature by showing that when asset commonality is not uniform among banks, overlapping portfolios may not only amplify the contagion but also impact on the structure (the degree of completeness) of interbank networks that are (most) resilient to the default of a single bank. Precisely, we show that where the commonality of financial assets between the liquidity-strapped bank and the rest of the system is pronounced enough, the complete interbank network can be vulnerable to the losses imposed by the fire sales of the distressed bank's assets. However, incomplete interbank network structures (possibly even the ring network) may still be free from the risk of contagion if each bank lends mainly to one other bank whose financial asset portfolio is sufficiently different. A simple numerical example can illustrate the mechanism behind our result.

Consider a banking system consisting of four banks, one of which (say bank 1) fails. Each of the other banks has a value of equity capital of 100 and will survive the default of bank 1 only if the consequent losses do not result in negative equity. Each bank has overall exposure of 30 to the others. The defaulting bank is totally unable to repay this debt. In the complete network, each non-distressed bank lends 10 to bank 1; in the ring network, bank 2 is exposed to bank 1 for 30, while the other banks have no exposure to it. Figure 1 compares the resilience of these two networks, i.e., their ability to absorb the losses provoked by the default of bank 1 without other banks failing. In Panel A, we assume that banks have no assets in common, so the fire sales of bank 1 do not generate market-to-market losses for the others. In this case, the liquidity losses are small enough with respect to equity so that both networks are resistant to the default of bank 1.[1]

Now, assume that all the banks in the network hold the same "apple" portfolio as bank 1. The latter's fire sales generate mark-to-market losses for the other banks. These losses make the interbank networks more vulnerable but do not alter the relative resilience of the different network structures, in that they increase the liquidity losses by a constant factor. Thus, if the mark-to-market losses are greater than 90, the banks' equity turns negative in both the complete and the ring networks (as in Panel B). However, if these are moderate (71 to 90), the other banks would survive in the case of a complete interbank network. With mark-to-market losses of 70 or less 70, both networks would still be resilient. However, where the commonality of assets varies between banks, the situation is different. In Panel C, we assume that banks 2 and 4 hold a "orange" portfolio, while bank 3 holds a apple portfolio, like bank 1. Suppose the decline in apple asset prices associated with the liquidation of bank 1's portfolio causes moderate losses (less than 90). In this case, the banking system can avoid financial contagion when the mutual exposures between banks form a complete network, or even in a ring network (if the losses are less than 70). However, if the market-to-market losses were large (90-100), the ring network in which orange banks lend to apple banks prove resilient to the default of bank 1. By contrast, in the complete network, the value of bank 3's equity would go negative and this bank would fail.

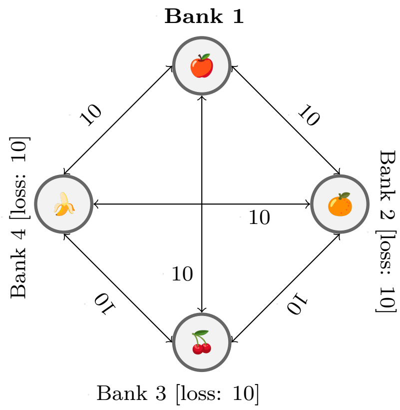
An outline of the model. Our model assumes a banking system in which n banks are linked by unsecured interbank debt and common assets. Banks' other liabilities are demand deposits (senior to interbank debt) and equity. The cross-holdings of interbank claims define the topology of the interbank network. As in Acemoğlu et al. (2015), we consider two extreme connected network topologies – complete and ring – and their linear π-convex combinations, with $\pi \in [0, 1]$. In the ring network, each bank lends to only one other, so they are all linked together through a single cycle. In the complete network, the interbank claims of each bank are distributed evenly over all the other $n - 1$ banks. The banks' assets consist of cash, interbank loans and long-term financial assets. Their asset portfolios overlap, producing a network of pairwise asset commonality.

The maturity mismatch between demand liabilities and long-term assets makes banks vulnerable to unexpectedly large deposit withdrawals (Diamond and Dybvig, 1983). We assume that at time t, bank i faces withdrawals that exceed its cash plus possible net interbank repayments. To meet the depositors' "excess" demand, it has to liquidate its long-term assets. If the financial markets are not perfectly liquid, the sales depress the prices of the divested assets. Thus, bank i incurs liquidation losses and fails, and

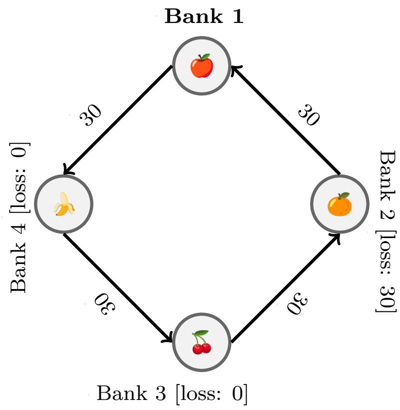
[1]If the banks' equity is less than 30 but greater than 10, the complete network is resilient to the default of bank 1, whereas in the ring network bank 2 would fail. If the equity value is less than 10, no one would survive the default of bank 1, whatever the network structure.

Figure 1: Complete and ring network architectures with 4 banks.

Panel A: No asset commonality

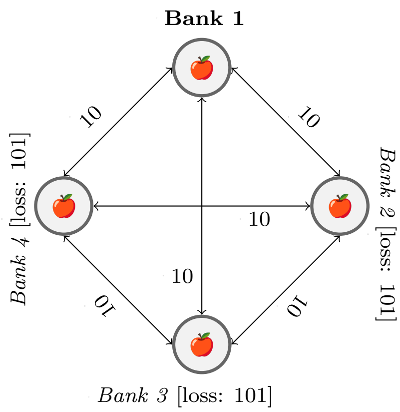


(a) Complete network

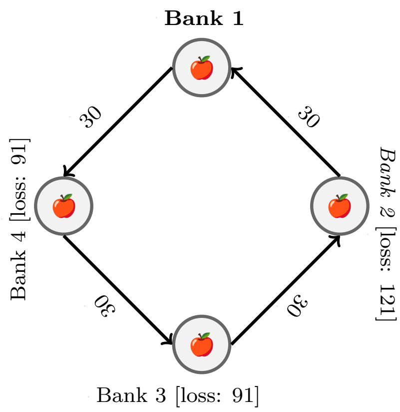


(b) Ring network

Panel B: Homogeneous asset commonality

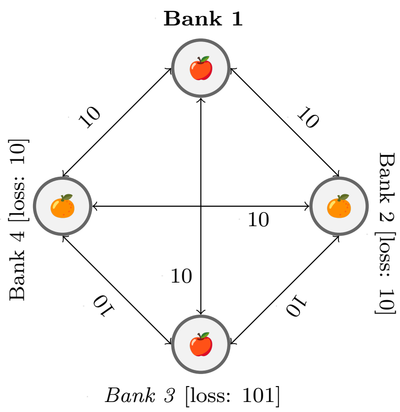


(a) Complete network

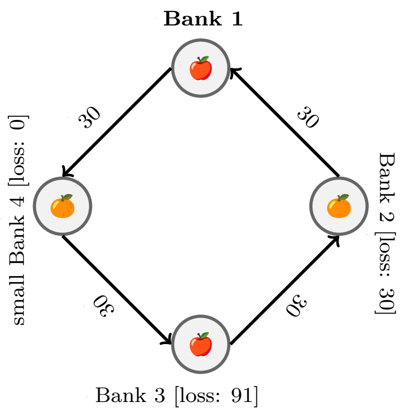


(b) Ring network

Panel C: Heterogeneous asset commonality



(a) Complete network



(b) Ring network

Notes: Arrows point toward the debtor. Defaulting bank is in bold. Banks that go bankrupt by contagion in italics. Fruit symbols in circles denote the type of assets held by banks. The numbers close to the edges denote the exposure to a counterparty in the interbank market. Banks' liquidity losses of banks are assumed equal to their exposure to the defaulting bank, market-to-market losses are assumed equal to 91 and each bank's equity value is assumed equal to 100. In square brackets total losses (i.e., liquidity plus market-to-market losses) induced by the default of bank 1.

the other banks holding the same assets suffer mark-to-market losses on their asset portfolios. Following the default of bank i, retail depositors and creditor banks share the bankruptcy proceeds according to seniority and proportionally to their claims. Therefore, following the first default, other banks suffer from liquidity losses owing to the impossibility of fully recovering their claims on the insolvent bank. The total loss incurred by each bank, and the strength of the contagion effects, depend on three factors: (i) the liquidity of the financial markets (their ability to absorb sell orders without driving asset prices down), (ii) the extent to which the long-term asset portfolio overlaps that of the defaulting bank and (iii) the interbank exposure to the insolvent bank.

In this setting, we show that there exists a continuous set of resilient interbank networks formed by convex combinations between the complete and the ring networks, the set possibly excluding these two extreme architectures. The set of resilient networks shrinks as the maximum degree of asset commonality between the defaulting bank and the others increases and the liquidity of the financial markets decreases. The intuition for this result is simple. As the network structure approaches the ring, the liquidity losses due to the default of bank i weighs mainly on its major interbank creditor. Therefore, there is a threshold of π below which the defaulting bank's main interbank creditor cannot cope with liquidity losses and so also fails. This threshold is lower, the greater the diversity of portfolios between the two banks. By contrast, as the network completeness increases, so do the liquidity losses for all other banks. Their ability to withstand liquidity losses depends on their mark-to-market losses, which are proportional to their degree of asset commonality with the insolvent bank. Thus there is a threshold of π above which the bank suffering the largest mark-to-market losses cannot withstand the losses stemming from the initial default. This threshold gets higher as portfolio diversity between the two banks increases.

When asset commonality is uniform, the resilience set certainly includes the complete network. Indeed, it is the most resilient network to the diminution of financial market liquidity. By contrast, in the more general case where the bank most exposed to the defaulting bank in the interbank market is not necessarily the one with the most assets in common, the complete network may not be resistant to financial contagion, while incomplete networks are.

Suppose mark-to-market losses are so great that no π-convex network is resilient. In that case, a non-convex redistribution of the defaulter's liabilities from the infected to uninfected banks can restore the network's resilience. Depending on the average level and variability of asset commonality in the banking system, the structure of the network produced by the redistribution of liabilities can be either more or less concentrated than the non-resilient convex network.

Although stylized, our model offers insights into interbank networks and regulation designed for financial stability. First, it allows for a realistic assessment of financial contagion and systemic risk in real-world banking systems. Information asymmetries and transaction costs lead banks to have a small group of privileged partners; at the same time, the desire for diversification may induce them to distribute their loans among a number of banks in the network.[2] As a result, cross-holdings of interbank debt are likely to be incomplete. Our model suggests that network incompleteness does not threaten financial stability if a bank's primary interbank borrower is not the bank with the most pronounced overlap of portofolio assets. However, where interbank lending is more concentrated among banks with similar asset portfolios, incomplete networks are more vulnerable, and the resilience of a network structure increases with its degree of completeness.[3]

Second, our model furnishes new insights into the ways in which regulation of bank liquidity can be effective in sustaining financial stability and managing banking crises. Liquidity requirements have ambiguous effects on systemic risk, depending on the degree of asset commonality and the liquidity of financial markets. Strict requirements reduce the value of banks' portfolios and undercut their ability to sustain the losses generated by the failure of other banks. At the same time, high liquidity buffers mean that fire sales by the defaulting banks produce smaller mark-to-market losses for the rest of the network. The first effect prevails when banks' asset commonality is low – a stricter liquidity regulation may increase financial instability risks. Conversely, where the commonality of assets between banks is high, the fire sales produce severe mark-to-market losses at other banks – limiting long-term balance-sheet assets by increasing liquidity requirements contributes to system stability.

Finally, our model suggests that the combination of cross-held interbank claims and common assets is critical to inform policy action involving crisis management and bank closure. In particular, whether the regulator should act as a lender of last resort (injecting liquidity into the insolvent bank) or as a buyer of last resort (injecting liquidity into the financial market and bolstering asset prices) depends on the web of interbank debt and the degree of asset commonality between banks.

Relationship to the literature. There is an extensive literature on the instability engendered by overlapping bank portfolios. Common asset holdings have been considered a threat to financial stability because of the likelihood of joint distress (Acharya and Yorulmazer, 2007; Wagner, 2008, 2010; Ibragimov et al., 2011), information spillover (Kodres and Pritsker, 2002; Allen et al., 2012) and the effects of fire sales mechanisms (Shleifer and Vishny, 1992; Diamond and Rajan, 2011; Ellul et al., 2011; Caccioli et al., 2014; Greenwood et al., 2015; Cont and Schaanning, 2019). In particular, our paper relates to works on the effects of fire sales on the stability of banking systems (Cifuentes et al., 2005; Nier et al., 2007; Gai and

[2] Empirical studies have shown that real-world interbank lending operations networks are far from complete, and sometimes have a ring structure, each bank lending to few or just one counterparty (Boss et al., 2004; Soramäki et al., 2007; Bech and Atalay, 2010; Craig and von Peter, 2014; Langfield et al., 2014).

[3] Elliott et al. (2021) show that while the socially efficient interbank network that minimizes systemic risk requires that banks have different real exposures to other banks, limited liability prompt them to lend to other banks with similar loan portfolios, in order to increase their market value when they do not fail. Our model indicates that, in this case, the greater systemic risk could be mitigated by increasing the completeness of the interbank network.

Kapadia, 2010; Glasserman and Young, 2015; Weber and Weske, 2020; Chiba, 2020; Shen and Li, 2020). These works commonly posit that banks all hold the same assets. Given the limited absorption capacity of the market, the price at which banks can liquidate their assets varies inversely with the total amount sold by all banks. Simulation models show that the probability and extent of contagion are not monotone in the degree of connectivity of the interbank market. Fire sales amplify financial contagion, both the risk and the scope of financial contagion for any degree of interconnectedness of banks; but they do not affect the ranking of the network structures in resilience.[4] We supplement this literature by exploring the case of bank portfolios with varying degrees of asset commonality. The association between the web of bilateral interbank exposures and the web of bilateral asset commonality proves to affect the topology of the most resilient interbank networks and the degree of completeness of contagion-free networks.

The rest of the paper is organized as follows. In Section 2, we present the model. In Section 3 and 4, we study the existence of resilient interbank networks, characterize the degree completeness of resilient networks and verify which network is most resilient resilient to deterioration in the financial markets' liquidity. Section 5, extends the analysis to non-convex networks. Section 6 discusses the policy implications and Section 7 concludes. The proofs of the propositions are given in the Appendix.

2 The model

2.1 Banks

Consider a banking system consisting of $N = \{1, \dots, n\}$ banks, with $n \geq 3$. In each period, the balance sheets of banks are exogenously given. Each bank collects one unit of money in the form of demand deposits, d, and equity, $E = 1 - d$; in addition, it borrows an amount Y_B on the interbank market. On the asset side, banks hold a portfolio of long-term assets A, interbank claims Y_L and cash reserves c smaller than deposits, but greater than interbank claims ($Y_L < c < d$).[5] Table 1 describes a balance sheet in a given period where loan assets (A_V) and equity (E_V) are marked to market.

[4]Empirical contributions have quantified the potential for contagion of bilateral asset commonality and fire sales (Greenwood et al., 2015; Gualdi et al., 2016). In particular, Caccioli et al. (2013), Poledna et al. (2021) and Siebenbrunner (2021) document the combined contagion effects of interbank exposures and overlapping portfolios; their work suggests that considering only on the former will result in a significant underestimation of the systemic risk.

[5]In this way, for simplicity and without loss of generality, we assume that banks can experience a deposit run but not an interbank run.

Table 1: Bank balance sheet

Assets	Liabilities
Asset portfolio: A_V	Demand deposits: d
Cash: c	Equity: E_V
Interbank lending: Y_L	Interbank borrowing: Y_B

2.2 Interbank market

Interbank borrowing consists of a one-period standard debt contract at zero interest rate. Let y_{ij} and y_{ji} be respectively the amount of money that bank i borrows from and lends to another bank $j \in N \setminus \{i\}$. Formally, the interbank market can be represented using a directed weighted graph in which each node is a bank and the edge starting from i and pointing towards j corresponds to the total claims of bank i on bank j. The weight assigned to each edge is equal to the nominal value of debt. Therefore, the adjacency matrix describing the graph – or the interbank network – is given by:

$$\mathcal{Y} = \begin{bmatrix} 0 & \dots & y_{1,n} \\ \vdots & \vdots & \vdots \\ y_{n,1} & \dots & 0 \end{bmatrix} \tag{1}$$

The total interbank credit and debt of bank i are denoted by $Y_{i_L} = \sum_{j \neq i} y_{j,i}$ and $Y_{i_B} = \sum_{j \neq i} y_{i,j}$. Following Acemoğlu et al. (2015), we restrict our attention to regular and connected interbank networks. In this case: (i) there are no net creditors in the interbank market, and the total amount of interbank claims is the same for all banks, $Y_{i_L} = Y_{i_B} = Y_i = Y \quad \forall i \in N$; (ii) every bank is directly or indirectly linked to all the other banks in the network.[6]

We can define two extreme topologies in the set of connected interbank networks: the complete network and the ring network. In the complete network, denoted by $\mathcal{Y}^C$, each bank lends the same amount to all the others so that the weight assigned to every edge is equal to $Y/(n-1)$. In the ring network $\mathcal{Y}^R$, each bank does all its lending to a single other bank, say the bank $i+1$, and the weight of every edge is thus Y, the total exposure of every banks to the interbank market. A linear convex combination between the complete and the ring networks is defined as follows:

Definition 1. (Acemoğlu et al., 2015) An architecture $\mathcal{Y}(\pi)$ is a linear convex combination of the

[6]Formally, a graph is defined as "connected" if for any two nodes there is always a path connecting them (Jackson, 2010).

complete and the ring networks if there exists a $\pi \in [0,1]$ such that:

$$\mathcal{Y}(\pi) = \pi\mathcal{Y}^C + (1-\pi)\mathcal{Y}^R. \tag{2}$$

As π moves away from zero, each bank i borrows from all the other banks. The amount $Y[1-\pi(n-2/(n-1)]$ is borrowed from the bank $i+1$ and the remaining amount is split evenly among the other $n-2$ banks, each of which is exposed to bank i for $Y\pi/(n-1)$. To illustrate, suppose that $Y = 100$ and $n = 4$. Adjacency matrices for $\pi = \{0, 0.3, 1\}$ are the following:

$$\begin{matrix}\mathcal{Y}(1)\\ \equiv \\ \mathcal{Y}^C\end{matrix} = \begin{bmatrix} 0 & 33.3 & 33.3 & 33.3 \\ 33.3 & 0 & 33.3 & 33.3 \\ 33.3 & 33.3 & 0 & 33.3 \\ 33.3 & 33.3 & 33.3 & 0 \end{bmatrix};\ \mathcal{Y}(0.3) = \begin{bmatrix} 0 & 80 & 10 & 10 \\ 10 & 0 & 80 & 10 \\ 10 & 10 & 0 & 80 \\ 80 & 10 & 10 & 0 \end{bmatrix};\ \begin{matrix}\mathcal{Y}(0)\\ \equiv \\ \mathcal{Y}^R\end{matrix} = \begin{bmatrix} 0 & 100 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 100 \\ 100 & 0 & 0 & 0 \end{bmatrix}$$

where $\mathcal{Y}(1)$ is the complete network and $\mathcal{Y}(0)$ is the ring network.

2.3 Asset commonality

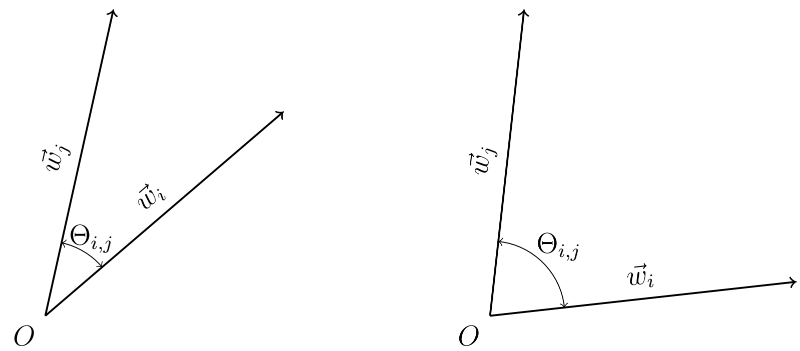
Banks can invest various long-term assets. Let $M = \{1, \ldots, m\}$ be the set of those available. Assets are riskless perpetuities with a common fundamental value $V > 1$ and annuities normalized to zero. Since we assume that there are no net interbank creditors, banks' asset portfolios have a common size $1 - c$, but their asset composition is bank specific. Let $w_{i,k} \in [0,1]$ be the fraction of the portfolio invested in the asset k by bank i. Therefore, the set of all bank portfolios can be described by the matrix:

$$\mathcal{W} = (1-c)\begin{bmatrix} w_{1,1} & \ldots & w_{1,m} \\ \vdots & \vdots & \vdots \\ w_{n,1} & \ldots & w_{n,m} \end{bmatrix} \tag{3}$$

where the i-th row $\vec{w}_i = [w_{i,1}, \ldots, w_{i,m}]$ indicates the composition of bank i's portfolio.

The different portfolios can overlap with one another to varying degrees. An intuitive measure of mutual diversity between the portfolios of bank i and bank j is the solid angle $\Theta_{i,j} \in [0°, 90°]$ between the vectors $\vec{w}_i$ and $\vec{w}_j$, while $\cos\Theta_{i,j} \in [0,1]$ is the corresponding measure of asset commonality between the two. As $\Theta_{i,j}$ shrinks, $\vec{w}_i$ tends to overlap with $\vec{w}_j$ and asset commonality between banks i and j increases. By contrast, when $\Theta_{i,j}$ increases, the two vectors tend toward the orthogonal, meaning that the two banks have few assets in common. Figure 2 offers a graphical representation for the case of two assets.

Figure 2: Bank diversity and asset commonality



(a) Low bank diversity
High asset commonality

(b) High bank diversity
Low asset commonality

2.4 The banking system

In our setting, a banking system is fully characterized by the number of banks, the total interbank exposure of each bank to other banks, the degree of the interbank network completeness, and the portfolios' matrix. Accordingly, the following definition applies:

Definition 2. A banking system $\mathcal{B}$ is given by the quadruplet $\langle n, Y, \pi, \mathcal{W} \rangle$.

2.5 Liquidity shocks, fire sales and asset prices

In tranquil periods, when all assets are marked to market at their fundamental value, the value of bank portfolios is $V(1-c)$. However, a bank z can be hit by a run on deposits υ, sufficiently severe to generate a liquidity shortage: $\upsilon_z \in [c, d]$. To recover liquidity, the bank sells long-term assets on the financial markets. The liquidity of these markets is less than perfect, so that the selling prices are below fundamental values. To be specific, the inverse demand curve for a long-term asset k is assumed:

$$p_k = \max\left\{ V\left(1 - \frac{q_k}{\lambda}\right), 0 \right\} \tag{4}$$

where $q_k = \sum_{i \in N} q_{i,k}$ is the total amount of the asset k to be liquidated, $\lambda > 0$ the liquidity of financial markets (i.e., the ability to absorb the supply of the asset at its fundamental value), and asset prices are non-negative. We assume that long-term assets are sold outside the banking system, so that the size and composition of the asset portfolios of the other banks in the network are unchanged.

2.6 Payments equilibrium

Liquidating portfolio assets is costly, so it is optimal for banks to sell the smallest amount necessary to cover any liquidity shortage. The liquidation costs incurred by a bank, $\phi_k = V \min[q_k/\lambda, 1]$, are asset-specific, in that they depend on the quantity offered by banks in the entire market. Therefore, for a bank z the optimal liquidation decision is:

$$\Lambda_z = \max\left\{0, \min\left\{\frac{\upsilon + Y - c - X_z}{\bar{p}_z}, 1 - c\right\}\right\} \tag{5}$$

where $\upsilon + Y - c - X_z$ is the total liquidity shortage (i.e., the liquidity required by depositors and interbank creditors minus available cash and interbank claims, $X_z \leq Y$), and $\bar{p}_z = \sum_{k \in A_z} \omega^\star_{z,k} p_k$ is the average price received by bank z on the liquidated assets weighted by the share of asset k in total sales.[7]

When a bank i defaults, it is fully liquidated, and its creditors - retail depositors and other banks - share the proceeds according to their seniority. Demand deposits are senior to interbank claims, and all creditors receive equal treatment according to their claim seniority, being repaid in proportion to their credits. Therefore, assuming limited liability, the payment of the bank z to a counterparty j is:

$$x_{z,j} = \max\left\{0, \min\left\{y_{z,j}, \frac{y_{z,j}}{Y}\left(\sum_{k=1}^{m} p_k w_{z,k}(1-c) + c + X_z - d\right)\right\}\right\} \tag{6}$$

where $X_z = \sum_{j \neq z} x_{j,z}$ are the interbank claims repaid to bank z by the banks to which it is exposed.

Definition 3. Payments equilibrium for the banking system $\mathcal{B}$ is given by: (i) the vector of banks' liquidity decisions $\Lambda = \{\Lambda_1, \Lambda_2, ..., \Lambda_n\}$ as in (5); and (ii) the $n \times n$ matrix of liability repayments $\mathcal{X}$ where the i-th row indicates the repayment of interbank debts made by bank i as in (6).

3 Systemic resilience

From here on, we restrict the analysis to the case in which the liquidity shock affects only one bank i. Having posited that banks are all of the same size, to see whether or not such a liquidity shock triggers a cascade of defaults owing to cross-holdings of liabilities and commonality of assets, we consider the

[7] Formally, $\omega^\star_{z,k} = q^\star_{z,k} / \sum_k \in A_z q^\star_{z,k}$, where $q^\star_{z,k}$ is the amount of the k-th asset that minimizes total liquidation costs under the non-default and portfolio constraints, taking the liquidation decisions of other banks as given:

$$\begin{aligned} \max_{q_{i,1}, q_{i,2} \cdots} \quad & \sum_{k \in A_i} q_{i,k} \phi_k(q_i, k) \\ s.t. \quad & \sum_{k \in A_z} p_k q_{z,k} = \upsilon + Y - c - X_z \\ & q_{z,k} \leq w_{z,k}(1-c). \end{aligned}$$

case in which i fails. This is tantamount to assuming that the liquidity shock to bank i is so severe that it cannot recover sufficient resources by fire sale of long-term assets in the financial market to offset the liquidity shortfall and repay outside creditors in full.

Assumption 1.

$$v_z = \begin{cases} v > c + V(1-c)\left(1 - \frac{1-c}{m\lambda}\right) & for \;\; z = i \\ 0 & for \;\; any \;\; z \in N \setminus \{i\}. \end{cases}$$

Finally, without loss of generality, we assume that financial market liquidity is good enough to keep the fire sales from driving any asset price to zero. That is, from the price equation (4), recalling that $q_{i,k} \leq w_{i,k}(1-c)$, we assume:

Assumption 2. $\lambda > 1 - c$.[8]

The initial default of bank i is propagated through the banking system via mark-to-market and liquidity losses to other banks. The former correspond to portfolio depreciation prompted by asset price declines due to the fire sales of bank i's assets; the latter are due to the impossibility of fully recovering banks' claims on the insolvent bank once the depositors' claims have been settled. A bank j, not hit directly by the liquidity shock, is resilient to the default of bank i if its mark-to-market and liquidity losses are small enough to keep its total asset value at least equal to its total external liabilities, i.e., the claims of depositors and other banks:

$$\sum_{k=1}^{m} p_k w_{jk}(1-c) + c + X_j \geq d + Y \tag{7}$$

In other words, bank j resists contagion if the market value of its equity remains positive. Extending this notion to the banking system as a whole, we can introduce the following definition.

Definition 4. A banking system $\mathcal{B} = \langle n, Y, \pi, \mathcal{W} \rangle$ and the related interbank network $\mathcal{Y}(\pi))$ are resilient to the default of a bank i if this does not trigger the default of any other bank, i.e., if inequality (7) holds for any $j \in N \setminus \{i\}$. The payments equilibrium for the resilient banking system is given by: (i) $\Lambda_i = V(1-c)$ and $\Lambda_j = 0$ for any $j \neq i$; (ii) $x_{i,j} = \max\left\{0, \frac{y_{i,j}}{Y}\left(\sum_{k=1}^{m} p_k w_{ik}(1-c) + c + Y - d\right)\right\}$ for any $j \neq i$ and $x_{z,j} = y_{z,j}$ for any $z \neq i$ and $j \neq z$.

The aim here is to verify the existence of a parameter space that makes a banking system $\mathcal{B}$ resilient to the failure of a single bank. We start by assuming that the banking system $\mathcal{B}$ is resilient to bank i's default and then determine any value of parameters for which the liquidity and market-to-market losses

[8] In Lemma A.1 in the Appendix, we show that Assumption 1 on v is a sufficient condition for bank i to fail and that it is consistent with $v < d$ and Assumption 2 for a generic set of parameter values.

of all the other banks caused by the default are small enough not to induce any other failures, so that the resilience of $\mathcal{B}$ is confirmed in the payments equilibrium.

The volume of the interbank liquidity losses depends on how much bank i manages to repay its interbank creditors, which is its liquidation value net of payments to senior depositors. The liquidation value of bank i, LV_i, is given by the proceeds from the sale of its assets, its cash reserves, and the repayment received from its interbank debtors, which, as there are no other defaulting banks in the network, is equal to the nominal value of those credits, Y:

$$LV_i = \sum_{k=1}^{m} p_k w_{ik}(1-c) + c + Y \tag{8}$$

Since demand deposits are senior to interbank claims, the other banks obtain a positive repayment from bank i if and only if $LV_i > d$. In this case, the creditor banks receive a fraction of $LV_i - d$ proportional to their claims. Therefore, the liquidity loss of bank j due to the default of bank i is:

$$LL_{i,j} = \min\left\{ y_{i,j}, y_{i,j} - \Big(LV_i - d\Big)\left[\frac{\pi}{n-1} + \Big(1-\pi\Big)\mathbb{1}_{j=i+1}\right]\right\} \tag{9}$$

where $\mathbb{1}_{j=i+1}$ is an indicator taking the value 1 for $j = i+1$, i.e. for the interbank counterparty most exposed to bank i in every π-convex network except the complete, with $\pi = 1$.

For all the banks in $\mathcal{B}$, the fundamental value of the asset portfolio is $V(1-c)$, regardless of bank-specific asset composition. Following the fire sales of bank i, the value of bank j's portfolio decreases to $\sum_k p_k w_{jk}(1-c)$, which is bank-specific in that it depends on the assets that it has in common with bank i. Since bank i is the only one in default and also the only one to sell off its assets, then $q_k = w_{ik}(1-c)$ and the price of the asset k is equal to $p_k = V[1 - w_{ik}(1-c)/\lambda]$. Therefore, the mark-to-market losses incurred by bank j are equal to:

$$\begin{aligned} ML_j &= V(1-c) - \sum_{k=1}^{m} p_k(\lambda) w_{jk}(1-c) \\ &= V\frac{(1-c)^2}{\lambda}\sum_{k=1}^{m} w_{i,k}w_{j,k} = V\frac{(1-c)^2}{\lambda}\,\|\vec{w}_j\|\,\|\vec{w}_i\|\cos\Theta_{i,j} \end{aligned} \tag{10}$$

where $\sum_k w_{j,k}w_{i,k} = \vec{w}_i \bullet \vec{w}_j' = \|\vec{w}_j\|\,\|\vec{w}_i\|\cos\Theta_{i,j}$.

Mark-to-market losses vary inversely with $\Theta_{i,j}$ and directly with the norms of vectors $\vec{w}_i$ and $\vec{w}_j$. Where $\Theta_{i,j} = 90°$, the portfolios of banks i and j are orthogonal, and the former's fire sales produce no market-to-market losses for the latter. The norms of the two vectors can be interpreted as a measure of the portfolio concentration, with $\|\vec{w}\| \in [1/\sqrt{m}, 1]$. Where $\|\vec{w}\| = 1$, banks hold only one asset; where $\|\vec{w}\| = 1/\sqrt{m}$, they hold the equally weighted portfolio. For any given degree of asset commonality, greater concentration of bank portfolios exacerbates mark-to-market losses, because the fire sales are involve only a few assets.[9] For the sake of simplicity and without loss of generality, we assume that portfolio concentration is the same for all banks, $\|\vec{w}_i\| = \|\vec{w}_j\| = \|\vec{w}\|$ for any bank $i, j \in N$.

Regardless of the structure of the interbank network and the associated liquidity losses, asset commonality between the defaulting bank and another bank can be so high that the latter also fails.

Proposition 1. *If $\Theta_{i,j} < \underline{\Theta}$, the market-to-market losses induced by the fire sales of bank i's assets causes the default of bank $j \neq i$ and no resilient interbank networks exists, where $\underline{\Theta}$ is such that:*

$$\cos \underline{\Theta} = \frac{\lambda}{\|\vec{w}\|^2 (1-c)} \left[1 - \frac{d-c}{V(1-c)}\right] \tag{11}$$

holds, with $\cos \underline{\Theta} < 1$.

Proof. See Appendix B. □

The intuition here is straightforward. Independent of the mutual interbank exposures, if the asset commonality between i and j is great enough, the liquidity shocks to one bank induce mark-to-market losses at the other, such as to make it fail. Therefore, where $\Theta_{i,j} < \underline{\Theta}$, no interbank network is free from the risk of contagion or can be resilient to the failure of bank i or j. When the liquidity of financial markets or the diversification of banks' portfolios is low – i.e., when λ is small or $\|\vec{w}\|$ is large –, the smallest degree of diversity between banks, $\underline{\Theta}$, required to avoid contagion increases.

Now, assume that $\Theta_{i,j} \geq \underline{\Theta}$ for any $i, j \in N$. In this case, the existence of a resilient interbank network and its characterization depend on how mutual exposures and asset commonality between banks intertwine. Rearranging inequality (7), the no-default condition for bank j is:

$$\sum_{k=1}^{m} p_k w_{j,k}(1-c) - (d-c) \geq Y - X_j. \tag{12}$$

The left-hand side (LHS) of (12) decreases as the negative price externalities resulting from fire sales of bank i's assets increase. Substituting for the equilibrium asset prices (4), we can express the value of the bank j's asset as $\sum_k p_k w_{j,k}(1-c) = V(1-c)[1-(1-c)\|\vec{w}\|^2 \cos\Theta_{i,j}/\lambda]$. Therefore, using (11), the LHS of the no-default condition is:

$$\frac{V(1-c)^2 \|\vec{w}\|^2 \left[\cos\underline{\Theta} - \cos\Theta_{i,j}\right]}{\lambda} =: H_j, \tag{13}$$

[9] To illustrate, consider the case of two assets and three banks, i,j and h. Assume that the asset portfolios of the three banks are $\vec{w}_i = [2/3, 1/3]$, $\vec{w}_j = [1/2, 1/2]$, and $\vec{w}_h = [7/8, 1/8]$. In this case, bank i has the same degree of asset commonality with bank j and bank h (i.e., $\cos\Theta_{i,j} = \cos\Theta_{i,h}$). However, the portfolio of bank h is more concentrated than that of bank j, and the Euclidean norms of the two portfolios are, respectively, $\|\vec{w}_j\| = 1/\sqrt{2} < \|\vec{w}_h\| = \sqrt{25}/\sqrt{32}$. In this case, the mark-to-market losses incurred by bank h are 25% greater than those of bank j.

where $H_{i,j}$ is increasing in bank diversity $\Theta_{i,j}$ and decreasing in market liquidity λ.

Under the assumption that interbank networks are regular, the right-hand side (RHS) of (12) coincides with the liquidity losses of bank j, which increase with its exposure to bank i. Therefore, substituting for $y_{i,j} = Y\left[\pi/(n-1) + (1-\pi)\mathbb{1}_{j=i+1}\right]$ in equation (9), the RHS of the no-default condition is:

$$Y - X_j = \left[\frac{\pi}{n-1} + \left(1-\pi\right)\mathbb{1}_{j=i+1}\right]Z \tag{14}$$

where $Z := \min[Y, Y - (LV - d)]$ denotes the system-wide liquidity losses generated by bank i's default to the banking system, where $LV = V(1-c)\left[1 - \left(1-c\right)\|\vec{w}\|^2/\lambda\right] + c + Y$ does not depend on bank-specific parameters.[10]

Expression (14) takes its highest value for the bank $i+1$, the one most exposed to bank i in the interbank market. Expression (13) takes its lowest value for the bank with the most asset commonality with the defaulting bank i (from now on bank $j^\star(i)$, or for brevity, if there is no ambiguity, $j^\star$). Hence the necessary and sufficient conditions for systemic resilience are that the no-default condition (12) hold for bank $i+1$ and for bank $j^\star$. Substituting (13) and (14) into (12) and solving for π, we have that banking system $\mathcal{B}$, and its interbank network $\mathcal{Y}(\pi)$ are resilient to the default of a bank i if and only if:

$$\pi \geq \pi_{i+1} = \frac{(n-1)}{(n-2)}\left[1 - \frac{H_{i+1}}{Z}\right] \tag{15}$$

and

$$\pi \leq \pi_{j^\star(i)} = (n-1)\frac{H_{j^\star(i)}}{Z}. \tag{16}$$

Definition 5. The resilience set $\Pi_i \subseteq [0,1]$ is the set of values of π for which the inequalities (15) and (16) hold.

That is, systemic resilience, if attainable, requires a distribution of the interbank claims that is neither too concentrated nor too homogeneous. The economic intuition is as follows. When the distribution of the interbank claims tends to concentration – i.e., as π decreases and the network topology tends towards the ring structure –, the liquidity losses for bank $i+1$ increase. The threshold π_{i+1} marks the maximum degree of claim concentration with which bank $i+1$ can survive the default of bank i. When $\pi < \pi_{i+1}$, the banking system is not resilient to the liquidity shock to bank i because the losses to its primary bank creditor are so large that it will fail. On the other hand, as the distribution of the interbank claims becomes more similar to that of the complete network, the liquidity losses shrink for bank $i+1$ but expand by the same amount for all the other banks $j \neq i+1$. Among the latter, bank $j^\star(i)$, which suffers the highest mark-to-market losses, is the most vulnerable. The threshold $\pi_{j^\star(i)}$ identifies the least

[10] See Appendix B. Since $\Theta_{i,i} = 0$ and the concentration of portfolios is $\|\vec{w}\|$ for any $i \in N$, the liquidation value (8) is $LV_i = LV$, for any $i \in N$.

concentrated network in which bank $j^\star$ and all the others survive the original default.

However, thresholds (15) and (16) are not limited in the unit interval, and the latter can be lower than the former. For both of these reasons, the resilience set Π_i can be empty. Hence, for an interbank network resilient to the default of a bank i to exist – that is, for $\Pi_i \neq \emptyset$ – two feasibility conditions must hold:

$$\begin{cases} \pi_{j^\star(i)} \geq \pi_{i+1} \\ \left[\pi_{i+1}, \pi_{j^\star(i)}\right] \cap \left[0, 1\right] \neq \emptyset \end{cases} \tag{17}$$

If either of these fails to hold, there is no π-convex connected network such that the bank most highly exposed and the bank with the largest share of long-term assets held in common with the defaulting bank can both survive. Proposition 2 shows that if there are enough banks in the network, a resilient banking system $\mathcal{B}$ and interbank network $\mathcal{Y}(\pi)$ always exist – that is, the feasibility conditions (17) are satisfied for any set of possible model parameters.

Proposition 2. *Let Assumptions 1 and 2 hold true and let $\Theta_{i,j} \geq \underline{\Theta}$, $\forall j \in N \setminus \{i\}$. A sufficient condition for $\Pi_i \neq \emptyset$ is:*

$$n \geq \hat{n} = \max\left[\left(1 + \frac{Z}{H_{i+1}}\right), \left(2 - \frac{H_{i+1}}{H_{j^\star(i)}} + \frac{Z}{H_{j^\star(i)}}\right)\right].$$

Proof. See Appendix C. □

The intuition behind Proposition 2 is straightforward. Where bank portfolios are diverse enough, the fire sales of the defaulting bank i alone do not provoke other defaults. That is, the interbank network is resilient if the liquidity loss for any bank $j \neq i$ is sufficiently low. As n increases, the liquidity loss due to the failure of bank i can be made arbitrarily small, and the resilience set therefore non-empty, by spreading the loss over a larger number of banks in the network.

4 Bank diversity, interbank debts and network resilience

Both Allen and Gale (2000) and Acemoğlu et al. (2015) argue that if an incomplete interbank market is resistant to contagion, increasing the degree of completeness cannot make the banking system vulnerable. Intuitively, the market will be resilient as long as all the banks command enough resources to absorb the liquidity loss provoked by the default of bank i. Since the loss transmitted to each creditor is the smallest possible in the complete structure, if there is no contagion in an incomplete network, there can be none when the network is complete. In the same way, if the liquidity shock to a bank is great enough to overcome the resilience of the complete network, all the other incomplete networks will also be vulnerable to financial contagion.

In this section, however, we show that the foregoing does not always hold, if one takes into account of asset commonality between the insolvent bank i and the other banks, and how this intertwines with their exposures to bank i. Specifically, we show that if asset portfolios do not all overlap to the same degree, and if the bank with the greatest commonality of assets with the defaulter is not the one most exposed to it, the resilience set may not include the complete interbank network. In such circumstances, in any case, the most resilient network structure is not the complete one.

In order to characterize the resilience set and the interbank network most resilient to financial contagion, we first derive a set of comparative static results.

Lemma 1.

A. The threshold π_{i+1} is strictly decreasing in market liquidity and strictly increasing in asset commonality and portfolio concentration:

$$\frac{\partial \pi_{i+1}}{\partial \lambda} < 0; \quad \frac{\partial \pi_{i+1}}{\partial \Theta_{i,i+1}} < 0; \quad \frac{\partial \pi_{i+1}}{\partial \|\vec{w}\|} > 0.$$

B. The threshold $\pi_{j^\star(i)}$ is strictly increasing in market liquidity and strictly decreasing in asset commonality and portfolio concentration:

$$\frac{\partial \pi_{j^\star}}{\partial \lambda} > 0; \quad \frac{\partial \pi_{j^\star}}{\partial \Theta_{i,j^\star}} > 0; \quad \frac{\partial \pi_{j^\star}}{\partial \|\vec{w}\|} < 0.$$

C. The measure of the resilience set $\mu(\Pi_i)$ is weakly increasing in market liquidity and weakly decreasing in asset commonality and portfolio concentration:

$$\frac{\partial \mu(\Pi_i)}{\partial \lambda} \geq 0; \quad \frac{\partial \mu(\Pi_i)}{\partial \Theta_{i,i+1}} \geq 0; \quad \frac{\partial \mu(\Pi_i)}{\partial \Theta_{i,j^\star}} \geq 0; \quad \frac{\partial \mu(\Pi_i)}{\partial \|\vec{w}\|} \leq 0$$

Proof. See Appendix D. □

Lower market liquidity and greater portfolio concentration both lower the liquidation value of the defaulting bank. At the same time, when market liquidity is lower, or portfolio concentration and asset commonality are greater, the mark-to-market losses for other banks caused by bank i's fire sales increase, and their capacity to absorb liquidity losses accordingly diminishes. As a result, the maximum exposure to bank i that both bank $i+1$ and bank $j^\star$ can have without defaulting themselves tends to be lower – that is, the π_{i+1} threshold is higher and the $\pi_{j^\star(i)}$ threshold is lower.

The measure of the resilience set $\mu(\Pi_i)$ indicates the fraction of linear convex combinations between the complete and the ring network topologies that are resilient to contagion from the failure of bank

i. Since the resilience set is a subset of the unit interval given by its intersection with the interval $[\pi_{i+1}, \pi_{j^\star(i)}]$, where Π_i is non-empty, a diminution in $\mu(\Pi_i)$ implies that the resilience set shrinks and the network topologies that are close to $\mathcal{Y}(\pi_{i+1})$ and $\mathcal{Y}(\pi_{j^\star(i)})$ shift from resilience to fragility. Accordingly, we can define the most resilient interbank network.

Definition 6. A π-convex connected interbank network $\mathcal{Y}(\pi^\star)$ is the most (or one of the most) resilient interbank network to the default of bank i if $\pi^\star \in \Pi_i$ for any $\Pi_i \neq \emptyset$.

Insofar as some model parameters can vary randomly, the most resilient network topologies are the last ones to become vulnerable to the default of bank i in the event of adverse development in financial markets and banking system. In particular, from Lemma 1, it follows immediately that when financial liquidity market deteriorates, diversity between banks diminishes, or the concentration of bank portfolios increases, the measure of the resilience set, if this is not empty, decreases monotonically. As a result, there is only one interbank network (i.e., a value of π) that remains resilient to the default of bank i; all the others turn out to be vulnerable to contagion.

To characterize the resilience set and identify the most resistant networks, we have to distinguish two cases.

A. $\Theta_{i,i+1} = \Theta_{i,j^\star(i)}$

Where the commonality of assets is homogeneous across all the banks in the network, or anyway the bank most exposed to bank i is also the one holding the most assets in common with it, the main conclusion of Allen and Gale (2000) and Acemoğlu et al. (2015) is confirmed: if the resilience set is not empty, the complete interbank network is certainly one of its elements.

Proposition 3. *Let $\Theta_{i,i+1} = \Theta_{i,j^\star(i)} \geq \underline{\Theta}$. If not empty, the resilience set is $\Pi_i = [\pi_{i+1}, 1]$. The topology most resilient to the default of a bank i is the complete network $\mathcal{Y}^C$, with $\pi^\star = 1$.*

Proof. See Appendix E. □

In this case, bank $i+1$ suffers the greatest liquidity and mark-to-market losses. Thus, if bank $i+1$ can absorb the losses induced by the default of bank i, so can all the other banks. In addition, since the liquidity losses of bank $i+1$ decrease with π, if bank $i+1$ survives the default of bank i when the interbank network is incomplete (i.e., when $\pi_{i+1} \leq \pi < 1$), it will survive it also when the network is complete (i.e., when $\pi = 1$), and its liquidity losses will be smaller. Therefore, to the extent that the commonality of assets is the same for all the banks in the network, the defaulter's fire sales make financial contagion more likely. However, the complete network is still the most resilient architecture; it becomes vulnerable to contagion only if the financial market liquidity and bank diversity are severely limited.

B. $\Theta_{i,i+1} > \Theta_{i,j^\star(i)}$

In the more general case in which mutual diversity is not homogeneous, the asset commonality with the defaulting bank combines with the interbank exposures to it and contributes to determining not only the size of the resilience set but also the most resilient topology. In particular, the resilience set may be non-empty and yet not comprise the complete network. Furthermore, the complete network is never the topology most resilient to worsening financial market liquidity or diminishing bank diversity.

Proposition 4. *Let $\Theta_{i,i+1} > \Theta_{i,j^\star(i)} \geq \underline{\Theta}$. If not empty, the resilience set is:*

$$\Pi_i = \begin{cases} [\pi_{i+1}, 1] & if \quad \Theta_{i,j^\star} \geq \widetilde{\Theta} > \underline{\Theta} \\ [\pi_{i+1}, \pi_{j\star(i)}] & if \quad \widetilde{\Theta} > \Theta_{i,j^\star} \geq \underline{\Theta}. \end{cases} \tag{18}$$

For any $\Theta_{i,j^\star} > \underline{\Theta}$, the most resilient interbank network $\mathcal{Y}(\pi^\star)$ is incomplete, where:

$$\pi^\star = \frac{(n-1)H_{j^\star}}{H_{i+1} + (n-2)H_{j^\star}} < 1. \tag{19}$$

When $\Theta_{i,j^\star} = \underline{\Theta}$, the most resilient network is the ring network $\mathcal{Y}^R$, with $\pi^\star = 0$.

Proof. See Appendix F. □

The intuition behind this result is easy to grasp. By construction, bank $j^\star$ incurs the largest mark-to-market losses due to bank i's fire sales, greater than those of bank $i+1$. At the same time, for bank $j^\star$ the liquidity loss produced by the failure is greater in the complete network than in any other π-convex network, whereas for bank $i+1$ the complete network is the architecture in which its liquidity loss is minimized. Thus, if the asset commonality between banks i and $j^\star$ is little enough to make the complete network resilient, then by continuity, the resilient set Π_i includes some $\pi < 1$, that is, there are also resilient incomplete. More importantly, following the same argument, it is never possible for the resilience set to consist solely of the complete network. In other words, the complete interbank network can never be the most resilient topology. By contrast, the resilience set does not necessarily include the complete network. When the asset commonality between banks i and $j^\star$ is great – i.e., when $\cos\underline{\Theta} > \cos\Theta_{i,j^\star} > \cos\widetilde{\Theta}$ –, the mark-to-market losses of bank $j^\star$ are large enough so that it cannot absorb the liquidity losses when they are evenly distributed as in the complete network. In the incomplete networks, however, its liquidity losses are smaller than in the complete network, while those of bank $i+1$ are greater. Once again, since the mark-to-market losses of bank $i+1$ are smaller than those of bank $j^\star$, by continuity there is a range of values of π for which bank $j^\star$'s liquidity losses are small enough that it can survive the default of bank i and the liquidity losses of bank $i+1$ are not great enough to cause it to fail. Under these circumstances, only incomplete networks can be resistant to financial contagion.

In the extreme case where the diversity between i and $j^\star$ is at the lowest possible level compatible with resilience, the market-to-market-losses of the bank $j^\star$ are so great that it is unable to absorb any liquidity losses at all. In this case, the ring network, where the debts of bank i are held exclusively by bank $i+1$, is the only topology in the resilience set.

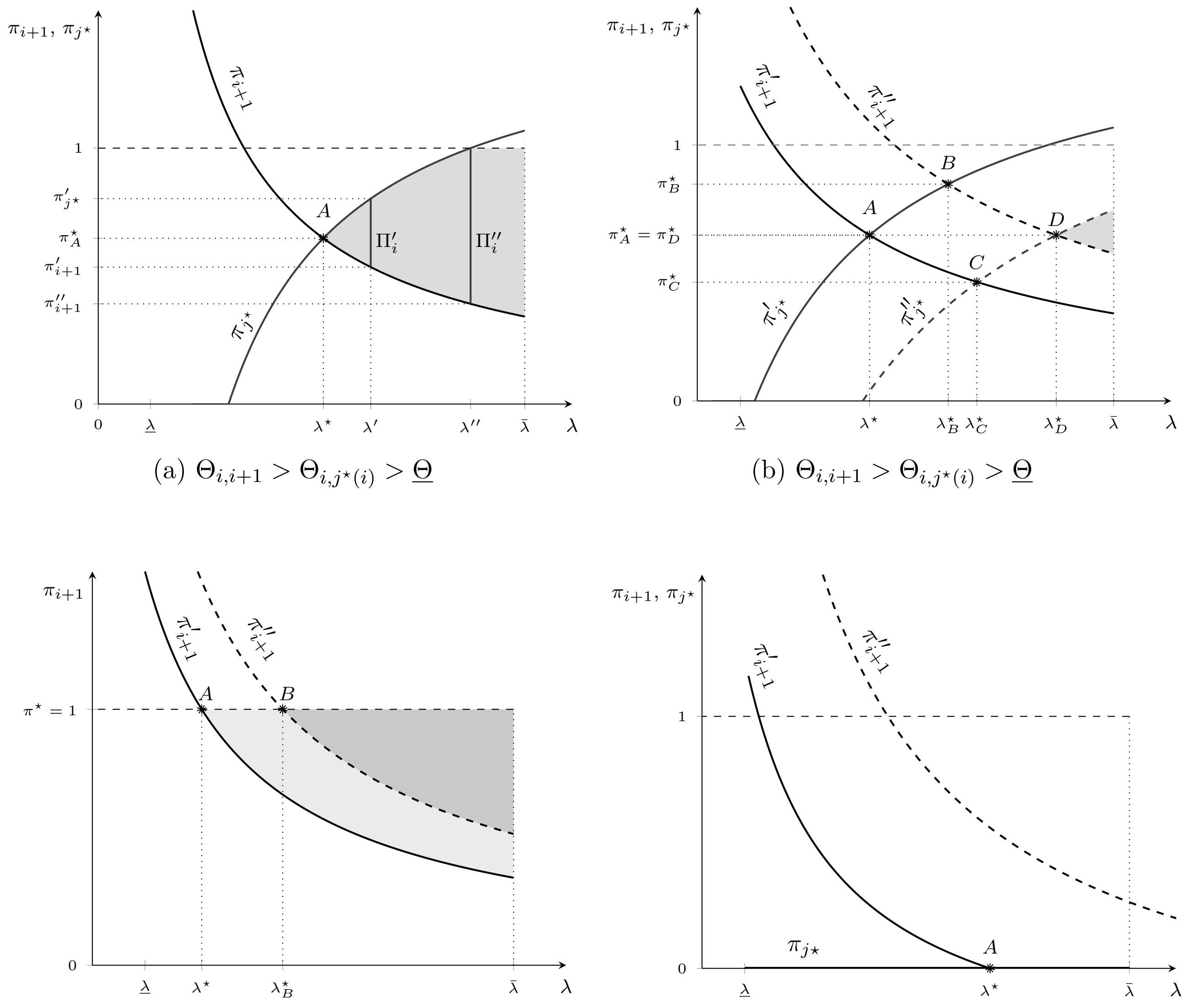
Proposition 5. *Let $\Theta_{i,i+1} > \Theta_{i,j^\star(i)} \geq \underline{\Theta}$. The degree of completeness of the most resilient topology, $\pi^\star$, is the lower, the less (resp., more) diverse are bank i and bank $j^\star$ and the more diverse are bank i and bank $i+1$, or the less liquid are financial markets.*

Proof. See Appendix G. □

When the diversity between bank $i+1$ and the defaulting bank i is greater, the maximum share of the liquidity loss generated by bank i's default that bank $i+1$ can absorb is also greater. As a result, the network topologies closer to the ring structure turn out to be less vulnerable to contagion, and the most resilient degree of completeness, $\pi^\star$, is smaller. Similarly, when the maximum level of asset commonality is higher or when the absorption capacity of financial markets is lower, the market-to-market-losses induced by the defaulter's fire sales drive $j^\star$ closer to insolvency. In this case, only if bank $j^\star$'s liquidity losses are negligible - that is, if its exposure to bank i is very low, as it is in markedly incomplete networks - can it survive the default of bank i.

Figure 3 illustrates how the resilience set and the most resilient network vary with financial market liquidity under the different assumptions on the distribution of asset commonality between banks analyzed in propositions 3 and 4. The graphs in the four panels depict the shape of the π_{i+1} and $\pi_{j^\star}$ thresholds as a function of market liquidity λ. In panel 3a, we display the general case, in which asset commonality is heterogeneous and $\Theta_{i,i+1} > \Theta_{i,j^\star}$. The vertical lines in the shaded area show the size of the resilience set, i.e. the values of π for which the distribution of liquidity losses is consistent with that of the market-to-market losses enabling all banks to survive the default of bank i. The resilience set shrinks monotonically as λ decreases. In our context, therefore, λ plays a role similar to that played in the model of Acemoğlu et al. (2015) by the shock to banks' short-term investment returns. As long as the market liquidity and the resulting liquidation value of bank i's long-term assets are high enough, the complete network is resilient to financial contagion. However, whereas in Acemoğlu et al. (2015) when the magnitude of the negative shock to returns is above a critical threshold there are no π-convex resilient networks, in our model when $\lambda < \lambda''$ the complete network becomes vulnerable to contagion, but incomplete interbank networks are still resilient. The intersection A of the two curves indicates the degree of completeness π of the most resilient topology, i.e. the one that remains resilient when all the others have become vulnerable to financial contagion owing the tightening of financial market liquidity. When $\lambda < \lambda^\star$ the resilience set is empty, and there is no π-convex interbank network that allows all banks to survive the initial default of a bank i.

Figure 3: Resilience set and most resilient network topology



(a) $\Theta_{i,i+1} > \Theta_{i,j^\star(i)} > \underline{\Theta}$

(b) $\Theta_{i,i+1} > \Theta_{i,j^\star(i)} > \underline{\Theta}$

(c) $\Theta_{i,i+1} = \Theta_{i,j^\star(i)} > \underline{\Theta}$

(d) $\Theta_{i,i+1} > \Theta_{i,j^\star(i)} = \underline{\Theta}$

A reduction in the diversity of banks $i+1$ and $j^\star$ determines an upward shift of the threshold π_{i+1} and a downward shift of the threshold $\pi_{j^\star}$ (see figure 3b). This unambiguously diminishes the space of non-empty resilience sets (the shaded area between the two curves) and lowers their measures at any level of market liquidity λ. However, when the asset portfolios of banks $i+1$ and i are more similar, the degree of completeness of the most resilient network rises to $\pi^\star_B$. By contrast, when the commonality of assets between the bank $j^\star$ and the defaulting bank is greater, the degree of completeness that is most resilient falls to $\pi_C^\star$. Clearly, when both $\Theta_{i,j^\star}$ and $\Theta_{i,i+1}$ diminish, the net effect on the topology of the most resilient interbank structure is ambiguous and may even remain unchanged, as plotted in figure 3b.

Panel 3c represents the case where asset commonality between banks is homogeneous or else where the bank most exposed to the defaulting bank i also has the greatest asset commonality with it. In this case, the only relevant threshold is π_{i+1}. The resilience set, if not empty, always includes the complete network, which is also the least vulnerable to liquidity deterioration. These results hold whatever the degree of bank diversity in the banking system, the only effect of a lower Θ being to narrow the resilience set's space.

Finally, panel 3d depicts the case in which the asset commonality of bank $j^\star$ is the greatest consistent with a non-empty resilience set, $\Theta_{i,j^\star} = \underline{\Theta}$. The $\pi_{j^\star}$ threshold coincides with the x-axis, whatever the liquidity of financial markets. In this case, if the π_{i+1} threshold crosses zero at a $\lambda \in]\underline{\lambda}, \bar{\lambda}[$, the only resilient network topology is the ring, where bank $i+1$ is the only bank exposed to bank i.

5 Non-convex resilient networks

In the foregoing, analysis was restricted the analysis to π-convex interbank networks between the ring and the complete structures; in this case, when a resilient interbank network exists, equal distribution of liabilities can produce greater financial fragility if the mutual diversity of long-term asset portfolios differs among banks. We now extend the analysis to the case in which the set of π-convex resilient interbank networks is empty, exploring how the defaulter's liabilities can be redistributed to make the network resilient. This issue has both positive and normative implications: it extends the analysis to a broader set of interbank network structures and suggests a possible intervention to defuse latent crises.

In the spirit of the notion of network majorization introduced by Acemoğlu et al. (2015), we consider a broad class of possible redistributions of the defaulting bank's liabilities among its counterparties.

Definition 7. Let z be a bank in N and A and B a partition of $N \setminus \{z\}$. The interbank network $\widehat{\mathcal{Y}}(\pi)$ is a $\{z, A, B, a\}$-transformation of the π-convex interbank network $\mathcal{Y}(\pi)$ if:

$$y_{i,j} = \begin{cases} \widehat{y}_{i,j} = y_{i,j} + a_{i,j}Y & if\ i = z\ and\ j \in A, with\ a_{i,j} < 0 \\ \widehat{y}_{i,j} = y_{i,j} + a_{i,j}Y & if\ i = z\ and\ j \in B, with\ a_{i,j} \geq 0 \\ \widehat{y}_{i,j} = y_{i,j} & if\ i \neq z \\ \sum_{j \neq i} a_{i,j} = 0. \end{cases} \tag{20}$$

The interbank network $\widehat{\mathcal{Y}}(\pi)$ and the π-convex network $\mathcal{Y}(\pi)$ differ in the z-th row of the corresponding adjacency matrices that describe them. Precisely, a $\{z, A, B, a\}$-transformation of $\mathcal{Y}(\pi)$ redistributes the liabilities of one bank z across its counterparties from the set of banks in A to those in B, while all the mutual exposures between the $N \setminus \{z\}$ banks in the network remain equal to $[1 - \pi(n-2)/(n-1)]Y$ or $[\pi(n-2)/(n-1)]Y$ as in $\mathcal{Y}(\pi)$. The redistribution weights $a_{i,j}$ can take any value, the only restriction being that the total debt of bank z is unchanged; accordingly, the pattern of interbank liabilities can turn out to be either more or less diversified than in a π-convex network.

As in the previous section, consider the case in which only bank i fails, but assume now that the interbank network $\mathcal{Y}(\pi)$ is not resilient to contagion from the default. Again, the average degree of asset commonality between bank i and the other banks, as well as the cross-bank variability of mark-to-market losses, matter in order for there to exists a $\{z, A, B, a\}$-transformation of $\mathcal{Y}(\pi)$ that restores the network's resilience. First, when the commonality of asset assets with the defaulting bank is the same for all the banks, it is impossible to restore resilience by redistributing interbank liabilities.

Proposition 6. *If $\Theta_{i,j} = \Theta$ for any $j \in N \setminus \{z\}$ and $\Pi_i = \emptyset$, there does not exist any $\{z, A, B, a\}$-transformation of $\mathcal{Y}(\pi)$ that makes the interbank network resilient to bank i's default.*

Proof. See Appendix H. □

This proposition is easily understood. When asset commonality is homogeneous and the resilience set is empty, the mark-to-market losses incurred by the banks not hit by the liquidity shock are all equal, and large enough to make them susceptible to contagion. In this case, the total liquidity losses generated by the failure of bank i are more than the system can absorb. Thus there is no way to reallocate bank i's liabilities so as to make the other banks' liquidity losses small enough for all of them to survive. More formally, there is no partition of $N \setminus \{i\}$ such as to obtain a resilient $\widehat{\mathcal{Y}}(\pi)$ network.

Let us now turn to the case where the mark-to-market losses caused by bank i's fire sales differ between banks. Suppose the average degree of asset commonality of the banks in the network with the defaulting bank is not very great. In that case, it is possible to reallocate the liabilities of bank i from the set of the infected banks, $I \subset N \setminus \{i\}$, to the set of sound banks that survive the first default $S = N \setminus \{i, I\}$, by reducing the liquidity losses of the former and increasing those of the latter.

Proposition 7. *If the aggregate liquidity and market-to-market losses induced by the default of bank i are not greater than the equity market value of non-distressed banks*

$$Z + \frac{V(1-c)^2 \left\| \vec{w} \right\|^2}{\lambda} \sum_{j \neq i} \cos \Theta_{i,j} \leq (n-1)[V(1-c) + c - d] \tag{21}$$

a redistribution-weight vector $\widehat{a} \in \mathbb{R}^{n-1}$ exists, such that the $\{i, I, S, \widehat{a}\}$-transformation of the interbank network $\mathcal{Y}(\pi)$ makes the resulting interbank network $\widehat{\mathcal{Y}}(\pi)$ resilient to the default of bank i, where $I \cup S = N \setminus \{i\}$, $I \neq \emptyset$ and $S \neq \emptyset$.

Proof. See Appendix I. □

Essentially, inequality (21) boils down to a condition relating to bank diversity. By rearranging it, we have:

$$\mathcal{M}_{\mathcal{C}}(i) \leq \frac{\lambda}{V(1-c)^2 \left\|\vec{w}\right\|^2}\left[V(1-c)+c-d-\frac{Z}{n-1}\right] \tag{22}$$

where $\mathcal{M}_{\mathcal{C}}(i) = \sum_{j\neq i} \cos\Theta_{i,j}/(n-1)$ is the average asset commonality between bank i and the other banks. Basically, when the asset portfolio of the defaulter is sufficiently different from those of other banks to allow the "average bank" to absorb liquidity losses, the interbank exposures to bank i can be redistributed in such a way that no other bank fails.

In principle, given the continuity of interbank exposures, there is an infinite number of possible $\widehat{a}$ vectors that support the resilience of network $\mathcal{Y}(\pi)$. Therefore, in order to study whether the $\{i, I, S, \widehat{a}\}$-transformation of the π-convex network $\mathcal{Y}(\pi)$ leads to a more or to a less diversified pattern of interbank liabilities compared to the π-convex pattern we focus on the special case in which condition (21) holds as equality. In this case, the redistribution-weight vector $\widehat{a}$ is unique, and we can characterize the concentration of the non-convex resilient network $\widehat{\mathcal{Y}}(\pi)$. Specifically, measuring the concentration of bank i's interbank liabilities in the network $\mathcal{Y}$ by the Herfindahl-Hirschman index, as $\mathcal{H}_i(\mathcal{Y}) = \sum_{j\neq i}(y_{i,j}/Y)^2$, we have:

Proposition 8. *Let $\mathcal{V}_{\mathcal{C}}(i) = \sum_{j\neq i}(\cos\Theta_{i,j} - \mathcal{M}_{\mathcal{C}}(i))^2/(n-1)$ be the variance of asset commonality between the defaulting bank i and the other banks in the network and let condition (21) be satisfied as an equality. If $i+1 \in S$, i.e. if bank $i+1$ belongs to the set of the sound banks, $\mathcal{H}_i(\widehat{\mathcal{Y}}) > \mathcal{H}\big(\mathcal{Y}(\pi)\big)$ holds. If $i+1 \in I$, , i.e. if bank $i+1$ belongs to the set of the infected banks, an upward-sloping isoconcentration curve $\mathcal{H}\big(\mathcal{M}_{\mathcal{C}}(i), \mathcal{V}_{\mathcal{C}}(i)\big) : \mathcal{H}_i(\widehat{\mathcal{Y}}) = \mathcal{H}\big(\mathcal{Y}(\pi)\big)$ exists such that above it $\mathcal{H}_i(\widehat{\mathcal{Y}}) < \mathcal{H}\big(\mathcal{Y}(\pi)\big)$ holds, while below it $\mathcal{H}_i(\widehat{\mathcal{Y}}) > \mathcal{H}\big(\mathcal{Y}(\pi)\big)$ holds.*

Proof. See Appendix J. □

Hence, the pattern of mutual interbank exposures in the resilient non-convex interbank network $\widehat{\mathcal{Y}}(\pi)$ can be either more or less diversified than in the otherwise fragile π-convex interbank network $\mathcal{Y}(\pi)$, depending on the level and variability of asset commonality. When $\mathcal{M}_{\mathcal{C}}(i)$ is large, the mark-to-market losses provoked by the fire sales of bank i's assets are also large, on average. As a result, if a resilient $\{i, I, S, \widehat{a}\}$-transformation of $\mathcal{Y}(\pi)$ exists, it necessitates a broader spreading of liquidity losses across the banks unaffected by the liquidity shock. Likewise, when there is little variability in the commonality of assets, ensuring resilience requires that liquidity losses be reallocated more evenly from infected to sound banks. Finally, an increase in the degree of completeness of the convex network $\mathcal{Y}(\pi)$ shifts the $\mathcal{H}$ curve up and makes the $\{i, I, S, \widehat{a}\}$-transformation more (less) likely to lead to a more diversified pattern of interbank liabilities, and conversely for a decrease in completeness.

6 Policy implications

The cascade of bank bailouts, closures and hidden failures during the global financial crisis triggered by the collapse of Lehman Brothers, and the enormous economic and social costs, led banking supervisory authorities to adopt a regulatory stance increasingly oriented to the assessment and containment of the systemic effects of localized shocks and individual defaults (Brunnermeier et al., 2009; Basel Committee on Banking Supervision, 2009, 2012; World Bank, 2019). Regulators and scholars realized that the traditional microprudential approach – namely, assessing the immediate impact of financial and economic shocks on the balance sheets of individual banks but ignoring the possible contagion deriving from the adjustments of the stressed banks to the shocks – ultimately underestimated the broader risks inherent in local shocks and did not properly consider the systemic nature of banks. The objective of the new macroprudential perspective on bank stability is thus to incorporate into the regulatory framework the risks of contagion and amplification stemming from the interconnections between banks (Anderson et al., 2018; Aymanns et al., 2018; Farmer et al., 2020).

In our model, an inverse measure of the systemic importance of a bank is the breadth of the resilience set associated with its failure. To the extent that the financial market liquidity and the mutual interbank exposures and asset commonality are not pre-determined, the banking system is the more vulnerable to bank i's default, the smaller $\mu(\Pi_i)$ is, and the greater the likelihood that the mark-to-market losses due to the fire sales will trigger more failures. This section discusses possible regulatory actions to contain systemic risks and deal with systemically important banks. First, we consider ex-ante liquidity requirements for banks in order to expand the set Π_i for any bank i. Second, we compare the costs, in terms of the amount of liquidity required, of two alternative ex-post interventions, namely as lender or as buyer of last resort, to mitigate the systemic impact of a bank default. Note, however, that our analysis is not intended to characterize optimal macroprudential policies, which have a series of incentive effects on banks' behavior that cannot be addressed in our accounting framework of clearing payments.

6.1 Liquidity requirements

Minimum liquidity reserve requirements are one of the chief microprudential tools used by regulators to ensure that individual banks can absorb liquidity shocks and cover unexpected cash outflows. However, the macroprudential role and effectiveness of reserve requirements in guaranteeing the stability of the financial system as a whole are controversial (Tovar et al., 2012; Dassatti Camors et al., 2019; Blanco Barroso et al., 2020). In particular, when the banking system is under stress and financial markets are illiquid, the obligation to maintain substantial liquidity buffers can compel fire-sales, driving down the market value of asset portfolios and furthering the propagation of local shocks throughout the system (De Nicolò et al., 2014). Thus whereas during tranquil times the liquidity requirements are suitable for both microprudential and macroprudential purposes, they cannot attain both goals in periods of turbulence. On the one hand, restoring confidence in individual banks and preserving their stability may necessitate maintaining or even tightening liquidity requirements. On the other hand, stabilizing the banking system as a whole when financial markets are under stress may imply easing the requirement, reducing banks' liquidity buffers countercyclically in order to preserve banks' returns and avert pressure for generalized deleveraging (Osiński et al., 2013).

We can study whether liquidity requirements respond to macroprudential concerns in our framework by examining the effects of changes in c on the breadth of the resilience set, which determines the banking system's ability to avoid contagion. The higher $\mu(\Pi_i)$, the more likely it is that the interbank network will be resilient to a bank failure and remain resilient to the possible deterioration of market liquidity *lambda*, to a higher degree of asset commonality, and to changes in the distribution of interbank exposures π.

We show that expanding c has conflicting effects on the breadth of the resilience set. As long as a bank default is still possible, and conditional on its actual occurrence, stronger liquidity buffers reduce fire sales and with them the liquidity and market-to-market losses for the non-distressed banks. But insofar as the long-term assets have higher yields than liquid assets, stricter liquidity requirements also reduce the ability of non-distressed banks to bear the losses, given the default of bank i.

Proposition 9. *If bank i defaults on its senior liabilities (i.e., if $LV \leq d$), the measure of the resilience set $\mu(\Pi_i)$ is maximum at $c = \widehat{c} < 1$, with $\partial\widehat{c}/\partial\lambda < 0$, $\partial\widehat{c}/\partial\Theta_{i,i+1} < 0$ and $\partial\widehat{c}/\partial\Theta_{i,j^\star} < 0$. If $LV > d$, $\partial\mu(\Pi_i)/\partial c \geq 0$ for any c.*

Proof. See Appendix K. □

As in the macroprudential policy literature, whether the liquidity requirements safeguard or endanger the banking system mainly depends on the liquidity of financial markets. When they are highly liquid, stricter liquidity requirements increase resilience, since they allow non-distressed banks to recover a larger part of their exposures to the defaulting bank. On the other hand, when the liquidity of the financial markets is poor, the pressure on asset prices exerted by bank i's fire sales is so strong that it cannot meet even part of its interbank obligations (i.e., when $Z = Y$), and an optimal $\widehat{c}$ maximizes the width of the resilience set, beyond which the macroprundential effects of liquidity requirements becomes adverse. In this case, the detrimental effect of high liquidity buffers on the ability of non-distressed banks to absorb market-to-market losses prevails, weakening the resilience of the network. The optimal level of liquidity requirements for banking system resilience is the lower, the smaller the pressure of bank i's fire sales on asset prices and the less sharp the price declines are for non-distressed banks or, formally, the greater the liquidity of financial markets λ and the lower the degree of asset commonality between the defaulting banks the other, non-distressed banks.

6.2 Lender or buyer of last resort

Handling banking crises is a prime, challenging task for regulators. Ever since the Henry Thornton and Walter Bagehot in the 19th century,[11] central banks have been seen as playing - and indeed they often have played - the key role of lender of last resort (LOLR), implicitly committed to provide emergency liquidity to fundamentally solvent banks when other sources of funds are depleted or unavailable. The necessity and rationale for this function are still debated among scholars of banking, but in any case the implicit subsidy to private banks' risk-taking provided by government-provided liquidity insurance and the encouragement to become too big to fail are universally acknowledged (Goodhart, 1987; Freixas et al., 2004; Calomiris and Haber, 2014).

To avoid distorting banks' incentives to risk-taking and growth in size, the central bank should commit not to act as LOLR and instead refrain from lending to illiquid banks, even at the cost of letting them fail. At the same time, to ensure the stability of the financial system as a whole, it could act as a buyer of last resort (BOLR) through open market purchases of less liquid financial assets, thus containing the possible contagion effects owing to the decline of asset prices and the risk of a general deleveraging. It is in this spirit that the Federal Reserve System refused to bail out Lehman Brothers in September 2008 but launched a series of lending facilities to provide liquidity to a wide range of banks and primary dealers that were unable to obtain sufficient funds from private sources, given their limited liquid financial collateral (Schooner and Taylor, 2010; Ball, 2018). Similarly, it was through various elastic open-market measures that the Eurosystem exercised the role of last resort liquidity provider in the post-Lehman and sovereign debt crises, including the full allotment of refinancing operations at a fixed rate, the expansion of the range of securities accepted as collateral, and the increase in outright purchases of securities on secondary markets (Acharya et al., 2021; Rostagno et al., 2021).

However, strict bank-closure policy accompanied by open market BOLR action is not free of adverse incentive effects on banks' behavior. In particular, as Acharya and Yorulmazer (2007, 2008) point out, there may be an incentive for herding, so that banks tend to lend to the same industries or hold the same assets. In this way, they can create a too-many-to-save problem, which that makes the BOLR function very costly, so that it becomes optimal to bail out troubled banks first.

In our model of clearing payments, we take banks' behavior as exogenous and cannot characterize their response to LOLR and BOLR intervention. However, the model allows us to compare the cost in

[11] However, their primacy is disputed by David Laidler, who observes "it seems to have been Francis Baring (1797) who first referred to the Bank of England as the 'dernier resort' for the other British banks" (Laidler, 1987, p. 61)

terms of liquidity injected into the banking system of acting as lender-of-last-resort intervention to the distressed bank with that of buying its assets in the financial market at prices high enough to avoid contagion.[12]

As in the previous sections, suppose that one bank i is hit by a liquidity shock $\upsilon \in [c, V(1-c)]$, such that it is illiquid but potentially solvent, and that financial market conditions do not allow it to raise the liquidity needed to avoid default by selling long-term assets. Suppose further that the interbank network $\mathcal{Y}(\pi)$ is not resilient to bank i's default, so that a subset of banks $I \subset N \setminus \{i\}$ will be infected and in a state of potential default. A last-resort loan gives bank i the extra liquidity needed to satisfy depositors, in return for the posting of a corresponding amount of long-term assets as collateral. Therefore, the liquidity cost of the LOLR policy is:

$$\mathcal{L}_{LOLR} = \upsilon - c \tag{23}$$

Alternatively, the central bank could let bank i fail and act instead as buyer of last resort for its long-term assets in order to bolster their prices and prevent the equity market value of potentially infected banks, which hold the same assets, from turning negative. Therefore, the cost of this policy, $\mathcal{L}_{BOLR}$, can be calculated as the amount of liquidity that the central bank must inject into the financial markets by buying the assets of bank i at prices high enough to keep the asset value of all other banks' above that of their liabilities.

Proposition 10. *Suppose bank i defaults and $\Pi_i = \emptyset$. Let I be the set of infected banks that would fail by contagion:*

$$\mathcal{L}_{BOLR} \gtrless \mathcal{L}_{LOLR} \Leftrightarrow \begin{cases} \cos\Theta_{i,i+1} \gtrless \widetilde{\rho}_1(\pi) & if \;\; i+1 \in I \; and \; j^\star(i) \notin I \\ \cos\Theta_{i,j^\star} \gtrless \widetilde{\rho}_2(\pi) & if \;\; i+1 \notin I \; and \; j^\star(i) \in I \\ \cos\Theta_{i,i+1} \gtrless \widetilde{\rho}_1(\pi) & if \;\; i+1, j^\star \in I \; and \; \frac{\cos\Theta_{i,j^\star} - \cos\Theta_{i,i+1}}{1-\cos\Theta_{i,j^\star}} < \frac{1-\pi}{1+\pi/(n-1)} \\ \cos\Theta_{i,j^\star} \gtrless \widetilde{\rho}_2(\pi) & if \;\; i+1, j^\star \in I \; and \; \frac{\cos\Theta_{i,j^\star} - \cos\Theta_{i,i+1}}{1-\cos\Theta_{i,j^\star}} > \frac{1-\pi}{1+\pi/(n-1)} \end{cases} \tag{24}$$

where $\widetilde{\rho}_1(\pi)$ and $\widetilde{\rho}_2(\pi)$ are increasing and decreasing with π, respectively.

Proof. See Appendix L. □

Not surprisingly, the cost of a BOLR action is higher the lower the diversity of the defaulting bank to the other banks and, therefore, the greater the mark-to-market losses and the minimum level of liquidity necessary to avoid a cascade of defaults. The structure of the interbank market is also crucial in determining the relative costs of BOLR and LOLR interventions. When bank $i+1$ is the one hardest

[12] Although we do not carry out an explicit welfare analysis, to the extent that the liquidity used by the regulatory authority is financed through a distortionary taxation of income from labor and capital, the liquidity costs of each intervention can be considered proportional to its welfare costs.

hit by bank i's default – either because it is the only one to be infected or because its losses are the greatest –, the cost of ensuring system resilience by providing liquidity through market purchases is lower than that of last-resort lending only if the commonality of assets between banks i and $i+1$ and the latter's mark-to-market losses are small (see inequality (24), lines 1 and 3). Likewise, when bank $j^\star$ is the most infected, the liquidity required by a BOLR intervention depends on the diversity of bank $j^\star$ (inequality (24), lines 2 and 4).

In either case, when the structure of the interbank network approaches its extremes – the ring network, where the bank most affected by bank i is bank $i+1$, or the complete network, where bank $j^\star$ to suffer the greatest losses – LOLR intervention is increasingly likely to require less liquidity than BOLR. Without a commitment technology, the greater the cost of acting as a BOLR, the more likely it is that the central bank will bail out an illiquid but otherwise solvent bank. This incentivizes banks to take more risk, in the belief that the central bank will not let them fail. From this perspective, Proposition 10 suggests that banks can tacitly coordinate to tie the hands of the regulator in two ways, by increasing the commonality of asset portfolios in the system and by affecting the structure of their mutual interbank exposures.

7 Conclusions

We have analyzed the role of diversity between different banks' asset portfolios in determining the relative resilience or vulnerability of banking systems. We show that the distribution of common asset holdings among banks affects not only the probability of financial contagion but also the topology of the interbank networks that are resilient to a bank default. Our main result is that when asset commonality is distributed unevenly or does not go hand in hand with interbank exposures, the complete network can be vulnerable to contagion, while the incomplete networks are not.

The property of the complete network of being robust but fragile has been highlighted in the literature on interbank networks. In particular, as Acemoğlu et al. (2015) argue, while the complete network is the most resilient topology when the shock to banks' asset value is mild, it becomes the least resilient when the shock is severe. In this case, a reallocation of interbank exposures to the defaulting bank that results in more uniform distribution of liabilities in the network cannot increase its resilience. By including the diversity of bank portfolios within the analytical framework, we restrict the validity of these conclusions to the particular case in which the commonality of assets is identical between any pair of banks. We show that when the mark-to-market value of banks is high and the resilience set is not empty, the complete network can be vulnerable to contagion if the bank most exposed to the defaulting bank is not also the bank with the largest volume of assets in common with it. However, when the liquidation value of banks is low and the resilience set is empty, when the commonality of assets between banks is high on average or varies little, a more diversified reallocation of the distressed bank's liabilities among the other banks can make the interbank network resilient.

From a macroprudential standpoint, the recent literature has pointed to the increased asset commonality among banks as a major source of instability (Haldane, 2013; Jackson and Pernoud, 2021). We show that overlapping banking portfolios may be of greater or lesser concern for financial stability depending on the structure of the interbank market and on which banks are more exposed to one other. Similarly, whether liquidity requirements safeguard or threaten financial stability, and which policy is most effective for handling a banking crisis, depends on the level and distribution of asset commonality among banks.

Appendices

A Lemma A.1

Lemma A.1. *Suppose a liquidity shock $\upsilon > c$ affects only one bank i, which is also the only bank selling assets. There exists a parameter space such that if $\lambda \in \left](1-c), \frac{V(1-c)^2}{m\left[V(1-c)-(d-c)\right]}\right[$ then: (i) bank i defaults and (ii) $p_k > 0$ for any $k \in M$.*

Proof. A bank i hit by a liquidity shock $\upsilon > c$ fails if the liquidity that it can recover by selling its long-term assets in the financial market is insufficient to repay outside creditors (depositors and banks). Assume that bank i is the only bank to sell off assets. Given the liquidity of financial markets λ, the price of any asset k and the liquidity that bank i recovers depend only on the volume of its sales. The portfolio that would enable a distressed bank to recover the greatest amount of liquidity is the one that solves the following maximization program:

$$\begin{aligned} \max_{q_1,\dots,q_m} \sum_{k=1}^{m} \left(1 - \frac{q_k}{\lambda}\right) q_k \\ st: \sum_{k=1}^{m} q_k = 1 - c. \end{aligned} \tag{A.1}$$

Since the objective function is globally concave, and the constraint defines a compact set, Lagrange's theorem provides the necessary and sufficient conditions for the solution. The Lagrangian associated with (A.1) is:

$$\mathcal{L}(q_1, \dots, q_k, \dots q_m, \mu) = \sum_{k=1}^{m} \left(1 - \frac{q_k}{\lambda}\right) q_k + \mu \left(1 - c - \sum_{k=1}^{m} q_k\right). \tag{A.2}$$

The first-order conditions are:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial q_k} = 1 - \frac{2q_k}{\lambda} - \mu = 0, \quad \forall k \in M \\ \frac{\partial \mathcal{L}}{\partial \mu} = \sum_{k=1}^{m} q_k - 1 + c = 0. \end{cases} \tag{A.3}$$

By summing the first conditions for all k, and using the portfolio constraint, we get:

$$\mu = 1 - \frac{2(1-c)}{m\lambda}. \tag{A.4}$$

Substituting (A.4) into $\partial \mathcal{L} / \partial q_k$, we have:

$$q_k^\star = \frac{1-c}{m}, \quad \forall k \in M. \tag{A.5}$$

From (A.5), the portfolio that allows the recovery of the maximum amount of liquidity from asset sales is the equally weighted one. Hence, a sufficient condition for bank i to fail is that the funds raised by sales of the assets of an equally weighted portfolio are insufficient to recover the liquidity shortfall:

$$\upsilon > \sum_{k=1}^{m} V\left(1 - \frac{q_k^\star}{\lambda}\right) q_k^\star + c = V\big(1-c\big)\left(1 - \frac{1-c}{m\lambda}\right) + c. \tag{A.6}$$

Condition (A.6) is feasible if $\upsilon < d$, that is if:

$$\lambda < \frac{V(1-c)^2}{m\big[V(1-c) - (d-c)\big]} \tag{A.7}$$

Finally, condition (A.7) is consistent with the sufficient condition for asset prices to remain positive, $\lambda > 1 - c$ (see Assumption (2) in the text), if:

$$m \leq \frac{V(1-c)}{V(1-c) - (d-c)} \tag{A.8}$$

which since $d > c$ is always satisfied for a suitable set of parameters. □

B Proof of Proposition 1

Suppose that $X_j = Y$, and bank j does not incur liquidity losses. A necessary condition for j to fail is:

$$(1-c)\sum_{k=1}^{m} p_k w_{j,k} - (d-c) < 0 \tag{B.1}$$

Substituting for $p_k = V[1 - w_{ik}(1-c)]/\lambda$, and recalling that $\sum_k w_{j,k} w_{i,k} = \vec{w}_i \bullet \vec{w}_j' = \|\vec{w}_j\|^2 \cos\Theta_{i,j}$, (B.1) holds when:

$$\cos\Theta_{i,j} > \frac{\lambda}{\|\vec{w}\|^2 (1-c)}\left[1 - \frac{d-c}{V(1-c)}\right]. \tag{B.2}$$

Hence, bank j fails if $\Theta_{i,j} < \underline{\Theta}$, where $\underline{\Theta}$ solves (B.2) as an equality.

C Proof of Proposition 2

The resilience set is not empty when:

$$\begin{cases} \pi_{j^\star(i)} > \pi_{i+1} \\ \pi_{i+1} < 1 \end{cases} \tag{C.1}$$

Substituting for $\pi_{j^\star(i)}$ and π_{i+1} from equations (15) and (16), and solving for n we have:

$$\begin{cases} n > 2 + \frac{Z - H_{i+1}}{H_{j^\star}} \\ n > 1 + \frac{Z}{H_{i+1}} \end{cases} \tag{C.2}$$

From which, $\Pi_i \neq \emptyset$ if $n \geq \widehat{n} = \max\left[\left(1 + \frac{Z}{H_{i+1}}\right), \left(2 - \frac{H_{i+1}}{H_{j^\star(i)}} + \frac{Z}{H_{j^\star(i)}}\right)\right]$.

D Proof of Lemma 1

From expressions (15) and (16), the way in which the thresholds π_{i+1} and $\pi_{j^\star(i)}$ vary with the model parameters depends on how the functions $H_{i+1}, H_{j^\star(i)}$ and Z vary as the parameters vary. Substituting $\cos \underline{\Theta}$ from (11) into (13) and LV_i from (8) into (14), we have:

$$H_{i+1} = V(1-c) - (d-c) - \frac{V(1-c)^2 \|\vec{w}\|^2}{\lambda} \cos \Theta_{i,i+1} \tag{D.1}$$

$$H_{j^\star}(i) = V(1-c) - (d-c) - \frac{V(1-c)^2 \|\vec{w}\|^2}{\lambda} \cos \Theta_{i,j^\star(i)} \tag{D.2}$$

$$Z = \min\left[Y, \frac{V(1-c)^2 \|\vec{w}\|^2}{\lambda} - \Big(V(1-c) - (d-c)\Big)\right] \tag{D.3}$$

From (D.1)-(D.3) it follows that:

$$\frac{\partial H_z}{\partial \lambda} > 0; \ \frac{\partial H_z}{\partial \Theta_{i,z}} > 0; \ \frac{\partial H_z}{\partial \|\vec{w}\|} < 0; \ \frac{\partial Z}{\partial \lambda} \leq 0; \ \frac{\partial Z}{\partial \|\vec{w}\|} \geq 0 \tag{D.4}$$

with $z = i+1, j^\star(i)$.

A. & B. Differentiating (15) and (16) with respect to λ, $\Theta_{i,i+1}$, $\Theta_{i,j^\star}$ and $\|\vec{w}\|$, and using the signs of the derivatives in (D.4) points A and B are immediately verified.

C. Since $\mu(\Pi_i) = \pi_{j^\star(i)} - \pi_{i+1}$, if conditions (17) hold and either π_{i+1} or $\pi_{j^\star}$ are in the unit interval, from (D.4) it follows immediately that $\partial\mu(\Pi_i)/\partial\lambda > 0$, $\partial\mu(\Pi_i)/\partial\Theta_{i,i+1} > 0$ $\partial\mu(\Pi_i)/\partial\Theta_{i,j^\star} > 0$, $\partial\mu(\Pi_i)/\partial$ $norm\vec{w} < 0$; otherwise, slight changes in the above parameters do not affect the measure of the resilience set and the derivatives are null.

E Proof of Proposition 3

If $\cos\Theta_{i,i+1} = \cos\Theta_{i,j^\star(i)} < \cos\underline{\Theta}$, from equations (15) and (16), it is easy to verify that the necessary and sufficient condition for $\pi_{i+1} \leq 1$ is $\pi_{j^\star(i)} \geq 1$. Hence, when $\Pi_i \neq \emptyset$, it must include the complete network and $\pi^\star = 1$.

F Proof of Proposition 4

Suppose that $\Theta_{i,i+1} > \Theta_{i,j^\star(i)} > \underline{\Theta}$ and the feasibility condition (17) holds. The resilience set includes the complete network if and only if $\pi_{j^\star(i)} \geq 1$ or, from (16), if and only if:

$$(n-1)\frac{H_{j^\star(i)}}{Z} \geq 1 \tag{F.1}$$

We have to demonstrate that a value $\widetilde{\Theta} \in]\underline{\Theta}, \Theta_{i,i+1}[$ exists such that $\pi_{j^\star(i)} \gtreqless 1$ according to whether $\Theta_{i,j^\star} \gtreqless \widetilde{\Theta}$.

From equation (13), we have that when $\Theta_{i,j^\star(i)} \to \underline{\Theta}$, $H_{j^\star(i)} \to 0$ and, from (F.1), $\pi_{j^\star(i)} < 1$. To verify that when $\Theta_{i,j^\star(i)} \to \Theta_{i+1}$, $\pi_{j^\star(i)} > 1$ holds, first we show that the resilience set cannot be a singleton that comprises the complete network. When $\pi = 1$, all banks $j \neq i$ incur the same liquidity losses and since $\Theta_{i,i+1} > \Theta_{i,j^\star(i)}$, the total losses of bank $i+1$ are strictly smaller than the total losses of bank $j^\star(i)$. This implies that incomplete convex interbank networks with values of π slightly less than 1, in which the liquidity losses of bank $j^\star(i)$ decrease slightly and those of bank and $i+1$ increase slightly, are also resilient. From this, it follows that if $\Pi_i \neq \emptyset$, $\pi_{i+1} < 1$. Now, from Proposition 3, we know that when $\Theta_{i,j^\star(i)} = \Theta_{i+1}$, $\pi_{i+1} < 1 \iff \pi_{j^\star(i)} > 1$. By continuity, when $\Theta_{i,j^\star(i)}$ is very close to $\Theta_{i,i+1}$ we have that $\pi_{j^\star(i)} > 1$ holds. Therefore, the conditions for the resilience set in (18) are verified.

Given Definition 6, from Lemma 1, the degree of completeness of the most resilient topology is that for which $\pi_{i+1} = \pi_{j^\star} = \pi^\star \in 0, 1$. Therefore, from (15) and (16), the most resilient topology is such that:

$$\pi^\star = \frac{(n-1)}{(n-2)}\left[1 - \frac{H_{i+1}}{Z}\right] \tag{F.2}$$

$$\pi^\star = (n-1)\frac{H_{j^\star(i)}}{Z}. \tag{F.3}$$

Substituting Z from (F.3) in (F.2) we have:

$$\pi^\star = \frac{(n-1)H_{j^\star(i)}}{H_{i+1} + (n-2)H_{j^\star(i)}} < 1. \tag{F.4}$$

Finally, from (13) and (16), where $\Theta_{i,j^\star(i)} = \underline{\Theta}$, we have that $H_{i,j^\star} = 0$ and $\pi_\star = 0$. Hence, from equation (F.4), we have that $\pi_{j^\star} = 0$, and the only possible resilient interbank network is $\mathcal{Y}^R$.

G Proof of Proposition 5

Differentiating (F.4), from (D.2) and (D.1) it follows immediately that $\partial\pi^\star/\partial\Theta_{i,j^\star} > 0$ and $\partial\pi^\star/\partial\Theta_{i,i+1} < 0$. Moreover:

$$sign\frac{\partial\pi^\star}{\partial\lambda} = sign\Big\{\cos\Theta_{i,j^\star(i)}H_{i+1} - \cos\Theta_{i,i+1}H_{j^\star(i)}\Big\} > 0 \tag{G.1}$$

given that $\Theta_{i,i+1} > \Theta_{i,j^\star(i)}$ by construction.

H Proof of Proposition 6

When all pairs of banks have the same degree of asset commonality (i.e., when $\Theta_{i,j} = \Theta$ for any j), mark-to-market losses are equal across banks. Thus in the complete network, where interbank liabilities are evenly distributed, the total losses of all banks given default of bank i are identical. It follows that if the resilience set is empty (i.e. if $\Pi_i = \emptyset$), and even the complete interbank network is vulnerable to contagion, it is not possible to reallocate the interbank liabilities of the defaulting bank to reduce the exposure of some banks and increase that of others so that all can diminish the liquidity losses generated by the default of bank i and survive.

I Proof of Proposition 7

Suppose that the resilience set is empty, i.e., $\pi_{i+1} > 1$ or $1 \geq \pi_{i+1} > \pi_{j^\star}$. In this case, the default of bank i causes the default of at least one other bank. Let I be the set of infected banks that default by contagion when a shock hits bank i, and S the set of sound banks that survive, where $I \cup S = N \setminus \{i\}$. Which banks are in I or S depends on the degree of completeness π of the interbank market. For the moment, let us assume that $\pi < \pi_{j^\star}$, that is, that I is a singleton that comprises only bank $i+1$. The redistribution-weight vector $\widehat{a}$, if it exists, must be such that:

$$Y\left(1 - \pi\frac{n-2}{n-1}\right) + a_{i+1}Y \leq Y\left(1 - \pi_{i+1}\frac{n-2}{n-1}\right) \tag{I.1}$$

where the right-hand side is the maximum exposure that enables bank $i+1$ to survive the default of bank i. Hence:

$$\widehat{a}_{i+1} = -\frac{n-2}{n-1}\big(\pi_{i+1} - \pi\big) \tag{I.2}$$

In addition, the amount $|\widehat{a}_{i+1}|Y$ can be spread over other banks without causing them to fail. Assume that the liabilities of bank i that are deducted from the bank $i+1$ are reallocated among all banks in the set $S = N \setminus \{i, i+1\}$. Therefore, the vector $\widehat{a}$ must be such that:

$$\frac{\pi}{n-1}Y + a_j Y \leq \frac{\pi_j}{n-1}Y \tag{I.3}$$

for any $j \in S$. Summing for all banks in S and substituting $\pi_j = (n-1)H_j/Z$ from (16) into (I.3) we have:

$$\frac{n-2}{n-1}\pi + \sum_{j\neq i,i+1} a_j \leq \frac{(n-2)}{Z}\big[V(1-c)-(d-c)\big] - \frac{V(1-c)^2\,\|\vec{w}_i\|^2}{\lambda Z}\sum_{j\neq i,i+1}\cos\Theta_{i,j} \tag{I.4}$$

For $\widehat{a}$ to be a redistribution-weight vector, it must be:

$$\sum_{j\neq i,i+1} a_j = |\hat{a}_{i+1}| = \frac{n-2}{n-1}\big(\pi_{i+1}-\pi\big) \tag{I.5}$$

Substituting (I.5) into (I.4), we have that an $\{i, I, S, \widehat{a}\}$-transformation of the π-convex interbank network $\mathcal{Y}(\pi)$ makes it resilient to the default of bank i if condition (21) holds:

$$Z + \frac{V(1-c)^2\,\|\vec{w}\|^2}{\lambda}\sum_{j\neq i}\cos\Theta_{i,j} \leq (n-1)[V(1-c)+c-d]. \tag{I.6}$$

Where the set I includes other banks, by the same steps, it is easy to prove that inequality (I.6) is a necessary and sufficient condition for an $\{i, I, S, \widehat{a}\}$-transformation to exist. Without loss of generality, suppose that the set I comprises bank $i+1$ plus other banks j. Using condition (I.1), the exposure to i of each bank in I must be diminished by a fraction equal to:

$$\widehat{a}_j = \begin{cases} -\frac{n-2}{n-1}\big(\pi_{i+1}-\pi\big) & if\ \ j = i+1 \\ -\frac{\pi-\pi_j}{n-1} & if\ \ j \in I \setminus \{i+1\} \end{cases} \tag{I.7}$$

To have resilience, it must be that for any bank $j \in S$:

$$\frac{\pi}{n-1}Y + a_j Y \leq \frac{\pi_j}{n-1}Y \tag{I.8}$$

Summing condition (I.8) over the n_S sound banks and substituting for π_j from (16) into (I.8), we have:

$$\frac{n_S}{n-1}\pi + \sum_{j\in S} a_j \leq \frac{n_S}{Z}\big[V(1-c)-(d-c)\big] - \frac{V(1-c)^2\,\|\vec{w}_i\|^2}{\lambda Z}\sum_{j\in S}\cos\Theta_{i,j} \tag{I.9}$$

For $\widehat{a}$ be a redistribution-weight vector, it must be

$$\sum_{j\in S} a_j = \sum_{j\in I}|\hat{a}_j| = \frac{n-2}{n-1}\big(\pi_{i+1}-\pi\big) + \sum_{j\in I\setminus\{i+1\}}\frac{\pi-\pi_j}{n-1}. \tag{I.10}$$

Once again, substituting $\sum_{j\in S} a_j$ from (I.10) in (I.9) and recalling that the number of infected banks is $n_I = n-1-n_S$, it is easy to verify that a $\{i, I, S, \widehat{a}\}$-transformation of the π-convex interbank network $\mathcal{Y}(\pi)$ makes it resilient to the default of bank i if condition (21) holds.

J Proof of Proposition 8

When condition (21) is binding, there is a unique $\{i, I, S, \widehat{a}\}$-transformation of the π-convex interbank network that restores resilience. In particular, the transformation reallocates the liabilities of bank i in such a way that for each bank the liquidity losses are the maximum that can be borne without bankruptcy:

$$\widehat{y}_{i,j} = \begin{cases} \left[1 - \frac{(n-2)\pi_{i+1}}{n-1}\right]Y & if \;\; j = i+1 \\ \frac{\pi_j}{n-1}Y & if \;\; j \neq i+1 \end{cases} \tag{J.1}$$

Thus, the concentration of bank i's interbank liabilities in the network $\widetilde{\mathcal{Y}}$ is:

$$\mathcal{H}_i(\widehat{\mathcal{Y}}) = \sum_{j\neq i}\left(\frac{\widehat{y}_{i,j}}{Y}\right)^2 = \frac{n-1}{Z^2}\left[A^2 - 2AB\mathcal{M}_{\mathcal{C}}(i) + B^2\Big(\mathcal{V}_{\mathcal{C}}(i) + \mathcal{M}^2_{\mathcal{C}}(i)\Big)\right] \tag{J.2}$$

where $A = V(1-c) - (d-c)$ and $B = \frac{V(1-c)^2\|\vec{w}\|^2}{\lambda}$. Instead, the concentration of bank i's liabilities in a π-convex interbank market is:

$$\mathcal{H}(\mathcal{Y}(\pi)) = \sum_{j\neq i}\left(\frac{y_{i,j}}{Y}\right)^2 = 1 - 2\pi\frac{n-2}{n-1} + \pi^2\frac{n-2}{n-1}. \tag{J.3}$$

If bank $i+1 \in S$, that is if $\pi > \pi_{i+1}$, from (J.1) it follows that the $\{i, I, S, \widehat{a}\}$-transformation that restores the network resilience cannot decrease the exposure of bank $i+1$ to bank i. Since in a π-convex network, the other $n-2$ banks are all equally exposed to bank i and the distribution of bank i's liabilities among them is fully diversified, $\mathcal{H}_i(\widehat{\mathcal{Y}})$ cannot be lower than $\mathcal{H}(\mathcal{Y}(\pi))$. That is, rewriting (J.2) and (J.3), we have:

$$\mathcal{H}_i(\widehat{\mathcal{Y}}) - \mathcal{H}(\mathcal{Y}(\pi)) = \left(\widehat{y}^2_{i,i+1} - y^2_{i,i+1}\right) + \sum_{j\neq i,i+1}\left(\widehat{y}^2_{i,j} - y^2_{i,j}\right) > 0. \tag{J.4}$$

By contrast, if bank $i+1$ is one of the infected banks (i.e., if $i+1 \in I$), the concentration of bank i's liabilities in the resilient network following the $\{i, I, S, \widehat{a}\}$-transformation can be either higher or lower than in the non-resilient π-convex interbank network. Let $\mathcal{I}\big(\mathcal{M}_{\mathcal{C}}(i), \mathcal{V}_{\mathcal{C}}(i)\big)$ be the isoconcentration curve such that $\mathcal{H}_i\big(\widehat{\mathcal{Y}}\big) = \mathcal{H}\big(\mathcal{Y}(\pi)\big)$. Using the implicit function theorem, the sign of the slope of the isoconcentration curve $\mathcal{I}$ is:

$$sign \frac{\partial \mathcal{V}_{\mathcal{C}}(i)}{\partial \mathcal{M}_{\mathcal{C}}(i)} = sign \; - \frac{\partial \mathcal{I}/\partial \mathcal{M}_{\mathcal{C}}(i)}{\partial \mathcal{I}/\partial \mathcal{V}_{\mathcal{C}}(i)} \tag{J.5}$$

From (J.2) and (J.4), it is immediate that $\partial \mathcal{I}/\partial \mathcal{V}_{\mathcal{C}} > 0$ and that the derivative with respect to $\mathcal{M}_{\mathcal{C}}(i)$ is:

$$\frac{\partial \mathcal{I}}{\partial \mathcal{M}_{\mathcal{C}}(i)} = -2B\Big[A - B\mathcal{M}_{\mathcal{C}}(i)\Big] < 0 \tag{J.6}$$

if $\mathcal{M}_{\mathcal{C}}(i) < A/B$ or, rearranging, if:

$$\mathcal{M}_{\mathcal{C}(i)} < \frac{\lambda}{(1-c)\left\|\vec{w}\right\|^2}\left[1 - \frac{d-c}{V(1-c)}\right] = \cos \underline{\Theta} \tag{J.7}$$

Since $\Theta_{i,j} > \underline{\Theta}$ for any bank j, condition (J.6) is satisfied and the isoconcentration curve is increasing in the plane $(\mathcal{M}_{\mathcal{C}}(i), \mathcal{V}_{\mathcal{C}}(i))$. As $\mathcal{H}_i(\hat{\mathcal{Y}})$ is increasing in $\mathcal{V}_{\mathcal{C}}(i)$ and decreasing in $\mathcal{M}_{\mathcal{C}}(i)$, above the isoconcentration curve we have that $\mathcal{H}_i(\hat{\mathcal{Y}}) < \mathcal{H}(\mathcal{Y}(\pi))$, while below the isoconcentration curve we have that $\mathcal{H}_i(\hat{\mathcal{Y}}) > \mathcal{H}(\mathcal{Y}(\pi))$.

K Proof of Proposition 9

The measure of the resilience set is given by:

$$\mu(\Pi_i) = \pi_{j^\star(i)} - \pi_{i+1} = (n-1)\left[\frac{H_{j^\star}}{Z} + \frac{H_{i+1}}{Z(n-2)} - \frac{1}{n-2}\right] \tag{K.1}$$

When $LV \leq d$, the distressed bank i defaults on its senior liabilities and the total liquidity losses that ensue are $Z = Y$. hence:

$$\frac{\partial \mu(\Pi_i)}{\partial c} = \frac{(n-1)}{Y}\left[\frac{\partial H_{j^\star}}{\partial c} + \frac{\partial H_{i+1}}{\partial c}\frac{1}{n-2}\right] \tag{K.2}$$

After differentiating $H_{j^\star}$ and $H_{j^\star}$ from (D.1) and (D.2), respectively, and rearranging:

$$\frac{\partial \mu(\Pi_i)}{\partial c} = \frac{2V(1-c)\left\|\vec{w}\right\|^2}{\lambda}\left[\cos \Theta_{i,j^\star(i)} + \frac{\cos \Theta_{i,i+1}}{n-2}\right] - (V-1)\frac{n-1}{n-2} \tag{K.3}$$

Hence, the value of c that maximizes the resilience set is:

$$\frac{\partial \mu(\Pi_i)}{\partial c} = 0 \iff c = 1 - \frac{\lambda(V-1)(n-1)}{2V\left\|\vec{w}\right\|^2\left[(n-2)\cos \Theta_{i,j^\star(i)} + \cos \Theta_{i,i+1}\right]} = \widehat{c} \tag{K.4}$$

where, from (K.3), $\partial^2 \mu(\Pi_i)/\partial c^2 < 0$.

When $LV > d$, bank i does not default on its senior liabilities, and the total liquidity losses are $Z =$

$Y-(LV-d)$, or using expressions (D.1), (D.2) and (D.3):

$$Z = \frac{V(1-c)^2 \|\vec{w}\|^2}{\lambda}\Big(1-\cos\Theta_{i,i+1}\Big) - H_{i+1} = \frac{V(1-c)^2 \|\vec{w}\|^2}{\lambda}\Big(1-\cos\Theta_{i,j^\star(i)}\Big) - H_{j^\star(i)} \tag{K.5}$$

To verify that the measure of the resilience set increases monotonically with c, we show that $\partial\pi_{j^\star(i)}/\partial c > 0$ and $\partial\pi_{i+1}/\partial c < 0$. From (16):

$$sign\frac{\partial\pi_{j^\star(i)}}{\partial c} = sign\Big[Z\frac{\partial H_{j^\star}}{\partial c} - H_j\frac{\partial Z}{\partial c}\Big] \tag{K.6}$$

Using equation (D.2) and the second equality in equation (K.5) for Z, and rearranging, we have:

$$\frac{\partial\pi_{j^\star(i)}}{\partial c} > 0 \iff c < \frac{V+1-2d}{V-1} \tag{K.7}$$

which is always verified for any $c \in (0,1)$. Similarly, from equation (15):

$$sign\frac{\partial\pi_{i+1}}{\partial c} = -sign\Big[Z\frac{\partial H_{i+1}}{\partial c} - H_{i+1}\frac{\partial Z}{\partial c}\Big] \tag{K.8}$$

Using equation (D.1) and the first equality in equation (K.5) for Z, and rearranging, we have:

$$\frac{\partial\pi_{i+1}}{\partial c} > 0, \quad \forall c \in [0,1] \tag{K.9}$$

and, from (K.7) and (K.9), $\partial\mu(\Pi_i)/\partial c > 0$ for any c.

L Proof of Proposition 10

When a central bank acts as buyer of last resort, it buys assets from the distressed bank at a price such that the mark-to-market equity value of other banks is non-negative. Therefore, to compute the amount of liquidity required for a BOLR action, first, we first derive the minimum degree of financial market liquidity $\widetilde{\lambda}$ such that even the most vulnerable bank survives the default of bank i. We then compute the value of bank i's portfolio at the market prices associated with $\widetilde{\lambda}$. As a result, the liquidity costs of a BOLR action depends on the bank that incurs the largest losses from the default of bank i.

Suppose that $i+1 \in I$ and $j^\star(i) \notin I$. From the no default condition (12), the minimum level of market liquidity at which bank $i+1$ avoids the default can be derived by solving:

$$H_{i+1} = Z\Big[(1-\pi) + \frac{\pi}{n-1}\Big] \tag{L.1}$$

Without loss of generality, assume that $Z = Y - (LV - d)$:[1]

$$\widetilde{\lambda}_1 = \frac{V(1-c)^2\,\|\vec{w}\|^2}{V(1-c)-(d-c)}\left[\frac{(1-\pi)+\frac{\pi}{n-1}+\cos\Theta_{i,i+1}}{(2-\pi)+\frac{\pi}{n-1}}\right] \tag{L.2}$$

From equation (4), $\widetilde{p}_k = V(1 - w_{i,k}(1-c)/\widetilde{\lambda}_{i+1})$. Assuming that $\widetilde{\lambda}_1 < \bar{\lambda}$,[2] at these prices bank i still defaults.

The liquidity cost of a BOLR policy is:

$$\mathcal{L}_{BOLR} = \sum_k \widetilde{p}_k w_{i,k}(1-c) \tag{L.3}$$

where from (4), $\widetilde{p}_k = V(1-w_{i,k}(1-c)/\widetilde{\lambda}_{i+1})$. Hence, $\mathcal{L}_{BOLR} \gtrless \mathcal{L}_{LOLR}$ according to whether $\sum_k \widetilde{p}_k w_{i,k}(1-c) \gtrless v - c$. By expanding the price term and rearranging, we have that $\mathcal{L}_{BOLR} \gtrless \mathcal{L}_{LOLR}$ if and only if:

$$\cos\Theta_{i,i+1} \gtrless \frac{V(1-c)-(d-c)}{V(1-c)-(v-c)} - \frac{d-v}{V(1-c)-(v-c)}\left[(1-\pi)+\frac{\pi}{n-1}\right] \equiv \widetilde{\rho}_1(\pi) \tag{L.4}$$

where $\tilde{\rho}$ increases with π.

When $j^\star(i) \in I$ and $i+1 \notin I$, following the same steps as above, we have that the market liquidity for which $H_{j^\star} = Z\pi/(n-1)$ is:

$$\widetilde{\lambda}_2 = \frac{V(1-c)^2\|\vec{w}\|^2}{V(1-c)-(d-c)}\left[\frac{\cos\Theta_{i,j^\star(i)}+\pi/(n-1)}{1+\pi/(n-1)}\right] \tag{L.5}$$

Hence, $\mathcal{L}_{BOLR} \gtrless \mathcal{L}_{LOLR}$ if and only if:

$$\cos\Theta_{i,j^\star(i)} \gtrless \frac{V(1-c)-(d-c)}{V(1-c)-(v-c)} - \frac{\pi}{n-1}\left[\frac{d-v}{V(1-c)-(v-c)}\right] \equiv \widetilde{\rho}_2(\pi) \tag{L.6}$$

where $\widehat{\rho}$ is decreasing in π.

When both $i+1$ and $j^\star(i)$ belong to the set of infected banks, the minimum amount of market liquidity that is sufficient to save both is equal to the greater between $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ where:

$$\widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \iff \frac{\cos\Theta_{i,j^\star(i)} - \cos\Theta_{i,i+1}}{1-\cos\Theta_{i,j^\star(i)}} \leq \frac{1-\pi}{1+\pi/(n-1)}. \tag{L.7}$$

[1]When $Z = Y$, all the steps are the same, and the results are qualitatively the same.

[2]By comparing equations (L.2) and (A.7), it is easy to verify that a parameter space such that $\bar{\lambda} > \tilde{\lambda}$ always exists.

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