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The Equitable Bargaining Set

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Abstract
We define the equitable bargaining set for exchange economies. Our definition differs from that in Mas-Colell (1989) because it requires that objections and counterobjections must satisfy some equitability conditions. We show that the equitable bargaining set coincides with that of Mas-Colell when the underlying economy is atomless, but not in general. Then we provide two sets of conditions for economies with market imperfections that apply to finite economies and to mixed market economies. In the first case our conditions imply that the equitable bargaining set is a subset of the core, and so it converges to the set of competitive allocations if the economy is replicated. In the second case, we show that all allocations in the equitable bargaining set are competitive, extending the Walras-bargaining equivalence of Mas-Colell (1989) to the framework of mixed markets. All the conditions we use follow from well-established assumptions from the literature in finite and mixed market economies.

JEL Classification: D62; D85; I12; I18.

Keywords: Bargaining set, Core, Equal treatment property, Walrasian objections.

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1 Introduction

In the study of allocations of goods within a heterogeneous group of agents there is trade-off between normative and strategic considerations. Allocation processes that are socially desirable are often susceptible to manipulations by individuals or coalitions. Conversely, a procedure that is stable from a strategic point of view, because it discourages agents to deviate from the proposed allocations, may fail to satisfy the most elementary criteria of fairness. Ideally, one wishes to find allocation rules that are *equitable* and *stable* in the sense that they obey to specific principles of fairness and, at the same time, are safe from strategic manipulations by individuals and coalitions. In this paper we define the Equitable Bargaining set of an exchange economy as a criterion for the allocation of resources that addresses both the concerns for fairness and stability.

Loosely speaking, an allocation rule qualifies as *equitable* if it "treats equals equally", that is, if it ensures that agents with the same characteristics receive bundles that they find equivalent. This is commonly recognized as the weakest requirement for fair allocation rules, see Moulin (2004) and Thomson (2011). A *stable* allocation, on the other hand, is one that prevents coalitions to act independently and look for alternative ways to allocate the resources in their control.

We consider an exchange economy whose members are selfish and act strategically, but are sensible to egalitarian principles. This means that they will agree on a proposed allocation only if they do not feel discriminated and if they cannot reach better outcomes on their own. What allocations will emerge in this case?

A first observation is that an egalitarian-minded agent accepts an inequitable allocation x only if it puts her in a more favorable position with respect to the others. Precisely, she checks that what she receives from x is at least as good as what is given to all of her peers, and she rejects the allocation if this is not the case.[1] It follows that the only allocations that will be accepted by all agents are those with the so-called *Equal Treatment Property* (ETP), i.e., those that assign equivalent bundles to identical agents.

Another observation is that the society will implement the allocation x only if this is sufficiently stable. If a group of agents S can reach a better allocation y on their own, then they will leave and work independently to realize y, making x unviable. But, in our framework, this is plausible only if all members of S, seen as a society on their own, agree on the allocation y. In other words, the deviation is possible only if y, restricted to S, is itself stable and equitable. This has two implications. The first is that the members of S must not feel discriminated by y or, equivalently, that the restriction of y to S must have the ETP. When this is the case, we say that S raises an *equitable objection* against x. The second implication is that even y must be immune to further deviations. If a second group of agents were to raise an equitable counter-objection against y then S would not be able to realize y and there would be no reason to discard x in the first place.[2] We conclude that the only relevant objections to x are those that are *equitable* and *justified*, in the sense that they are safe from equitable counter-objections.

The *Equitable Bargaining set* is the class of all allocations that treats equals equally and that cannot be opposed with justified equitable objections. This definition rests on a

[1] This is weaker than asking that individuals reject all unequal treatments and follows from the selfish nature of agents. This distinction is irrelevant when all society is involved, since inequitable allocations are opposed by the agents they discriminate, but it becomes significant later in the study of unilateral deviations of coalitions. In this case, in fact, agents outside the coalition do not have a say and may be treated differently from their deviating peers.

[2] One may take this argument even further and say that also the counter-objection, besides being equitable, should be immune to deviations, and that these deviations should in turn be equitable and stable, and so on. We discuss this idea in the next section and explain why, in our framework, considering more than two steps of objections does not add much.

consistency argument: all the limitations in terms of equitability and stability must apply to the allocations proposed to the entire set of agents as well as to every agreement within deviating coalitions, seen as societies on their own.

In general, a bargaining set is a solution concept based on a two-step blocking mechanism: an allocation is in the bargaining set if it is uncontested, or if all its objections are met with counter-objections. In principle, many different definitions of bargaining set are possible, depending on which classes of objections and counter-objections are considered valid from time to time. Aumann and Maschler (1964) and Davis and Maschler (1963) give the first definition of bargaining set as an extension of the core, assuming that objections and counter-objections are proposed by single agents.[3] Mas-Colell (1989) formulates an alternative definition that does not require to specify who proposes the objections, and that is insensitive to the initiatives of negligible individuals. This way, it extends the study of the bargaining set to atomless models of competition.

There are several advantages in studying the bargaining set of an exchange economy over the core. Aumann (1973) observes that the core is inappropriate in describing some economic phenomena involving monopolies or cartels, which in Maschler (1976) find a better explanation in terms of bargaining set. In the case of competitive economies, Mas-Colell (1989) shows that the bargaining set characterizes the class of competitive equilibria, and uses this result to provide new insights on the real market power of coalitions and on the behaviour of competitive agents outside the state of equilibrium. Precisely, Mas-Colell (1989) shows that the only objections that are safe from counter-objections, and thus relevant to the bargaining set, arise as the spontaneous reaction of agents to specific sets of prices. These objections, called *Walrasian*, can be obtained through a fully decentralized process and without any need of coordination within agents, and so they reflect the purely Walrasian behavior of negligible agents even outside the state of equilibrium.

The Equitable Bargaining set can be viewed as a variation of Mas-Colell's bargaining set in which objections and counter-objections must satisfy additional criteria of fairness. This reduces drastically the conflicts between identical agents and has significant properties in terms of envy-freeness: in every stage of the bargaining process none of the deviating agents would rather take the place of anyone with her same characteristics. A germinal discussion on equitability in the bargaining set solution appears already in Mas-Colell (1989, Remark 5). In the case of competitive economies, in fact, Mas-Colell's bargaining set is entirely described by means of equitable objections without counter-objections.[4]

The properties of the Equitable Bargaining set are relevant beyond the assumption that the underlying economy is perfectly competitive, especially in those situations in which there are many agents with similar characteristics and the equitability concerns become more salient. In general, the Equitable Bargaining set contains all competitive allocations but it is not comparable with the core nor with Mas-Colell's notion of bargaining set. The differences between the three solution concepts are therefore substantial and they allow to study alternative aspects of the bargaining process. In particular, the Equitable Bargaining set captures the competition across agents of different types, which cannot be isolated by studying the core and Mas-Colell's bargaining set only.[5]

[3] The core is the set of outcomes without objections and so it is always contained in the bargaining set.

[4] Precisely, Mas-Colell (1989, Remark 5) observes that, in atomless economies, every non-core allocation with the equal treatment property has an objection without counter-objections in which all deviating agents of the same type receive equivalent bundles. As it turns out, objections of this type are equitable and justified. See the discussion in Subsection 3.3.

[5] Two agents are of the same *type* if they have the same preferences and endowment.

1.1 Our contributions

The paper studies the implications of imposing egalitarian restrictions on the collective bargaining process. When the Equitable Bargaining set is compared with other classical solution concepts, such as the core or the class of competitive allocations, these considerations open to novel interpretations of agents' behaviour and the real market power of coalitions. Our main results provide conditions under which such comparisons are possible. The analysis covers the cases of atomless (or perfectly competitive) economies, finite economies, and mixed markets, which represent hybrid scenarios where a number of influential agents interacts with a mass of negligible ones.

Our first set of results studies the Equitable Bargaining set through a special class of price-based objections, called *weakly-Walrasian*, that extend Mas-Colell's notion of Walrasian objections.[6] We introduce them in Section 3.3 and show that finding a weakly-Walrasian objection to an allocation x is enough to conclude that x is not in the Equitable Bargaining set. Precisely, we show that if x has the ETP then every weakly-Walrasian objection to x is equitable (Proposition 3.11) and can be extended to a justified equitable objection to x (Proposition 3.13, see also the discussion in A.6).

We focus on ***atomless economies*** in Section 4.1. In this framework, Mas-Colell (1989) proves that his notion of bargaining set, the core and the set of competitive allocations coincide. We argue that this equivalence holds also for the Equitable Bargaining set (Proposition 4.1) and use this result to provide new intuition and technical observations. In particular, we show that in an atomless economy every non-competitive allocation is blocked by some coalition with a weakly-Walrasian objection, and this coalition can be taken arbitrarily small in size (Proposition 4.3). Thus, when searching for weakly-Walrasian objections it is enough to focus on coalitions that are very small. This property, that resembles Schmeidler (1972), is extremely useful in applications: it simplifies the search for meaningful objections and it makes the Equitable Bargaining set more ductile and easier to study in complex environments. Remarkably, a similar property does not hold for the class of objections that define Mas-Colell's bargaining set; see Schjodt and Sloth (1994) and Hervés-Estévez and Moreno-García (2015).[7]

We move on to studying ***mixed markets*** in Section 4.2. These are models where some influential (large) traders interact with a multitude of negligible ones. In these situations large traders can easily raise counter-objections and very few objections go unopposed. The bargaining set of Mas-Colell is therefore pathologically large and it does not provide insights on the real distribution of market power across participants, even when the core coincides with the set of competitive allocations as in Shitovitz (1989). In our analysis of mixed markets, we suggest that, in some cases, the inefficacy of the bargaining set may be driven by the conflicts among identical agents (Subsection 4.2). To support this idea, we find conditions under which the Equitable Bargaining set coincides with the class of competitive allocations, and this is strictly contained in Mas-Colells bargaining set (Theorem 4.7). These conditions are weaker than other assumptions that are common in the literature on mixed markets (Proposition 4.6).

[6] Intuitively, we can think of a weakly-Walrasian objection as one that is generated through the following mechanism: after an allocation x is proposed, all agents are asked to choose whether to accept the bundle assigned them through x or to trade their endowments in a market regulated by a set of prices p. If the demands of some of the agents who choose to trade at p are mutually compatible, then these can be used to raise an objection to x. This objection is Walrasian (in the sense of Mas-Colell) if it contains all the agents who are willing to trade at p. It is weakly-Walrasian if it contains a "representative group" of these agents, at least one per type.

[7] Similar considerations are carried out in the context of club economies. Bhowmik and Saha (2025a) proves the equivalence of competitive and bargaining set allocations in atomless club economies, and Bhowmik and Saha (2025b) shows that the equivalence holds even with restrictions on the size of counter-objections. Bhowmik, Saha, and Tikader (2026) explores the fairness implications of said restrictions.

With a technique that goes back to Greenberg and Shitovitz (1986), we study the mixed market through an auxiliary atomless economy in which large traders cannot exert their market power and many standard arguments apply. The analysis of this auxiliary economy allows to find, in the original mixed market, small groups of agents who can elude the influence of large traders and oppose effectively any non-competitive allocation with weakly-Walrasian objections (Proposition A.10). This result relies on the great flexibility of weakly-Walrasian objections and on the fact that, unlike Walrasian objections, they can be taken arbitrarily small. A similar approach, in fact, could not be formulated with Walrasian objections and Mas-Colell's notion of bargaining set.

Our last set of results applies to the study of ***finite economies***. In Section 4.3 we consider economies in which a type of agents is extraordinary influential, to the point that no coalition can raise objections or counter-objections without the participation of at least one of them. We call *leaders* the agents of such type. Leaders appear, for example, in bilateral oligopolies or when a group of identical agents controls the whole endowment of a commodity that everyone else needs to survive. The main result of this section is Theorem 4.12, which shows that, in a finite economy with leaders, the Equitable Bargaining set coincides with the set of core allocations with the ETP.[8] The theorem relies on the fact that leaders have the power to extract all the surplus produced in the Equitable Bargaining process and propose deviations that others cannot oppose (Proposition 4.11).[9]

The characterization of the Equitable Bargaining set in economies with leaders has the following interesting implication: suppose that in a finite economy there is a group of leaders, and suppose that this economy is replicated as in Debreu and Scarf (1963). Then the Equitable Bargaining set in the replicas eventually becomes a subset of the core, and so it converges to the set of competitive allocations (Corollary 4.13). This is a remarkable property of the Equitable Bargaining set that typically fails for other notions of bargaining set. For example, Anderson, Trockel, and Zhou (1997) shows that the bargaining sets of Mas-Colell (1989) and Zhou (1994) do not convergence when the underlying economy is replicated; see also Iñarra, Serrano, and Shimomura (2020) and Hervés-Estévez and Moreno-García (2018). These non-convergence results continue to hold even when stronger assumptions on the model ensure that the core and the bargaining set coincide in the original economy, as those considered in Shitovitz (1989) and Bahel (2016); see Remark 4.14.

Our analysis of the Equitable Bargaining set opens to comparisons with notions of bargaining set different from Mas-Colell's. In Remark 4.4 we explore the similarities between the weakly-Walrasian objection mechanism and the one introduced in Vind (1992) to define the so-called *global bargaining set* of atomelss economies. With a different approach, Hervés-Beloso, Hervés-Estévez, and Moreno-García (2018) and Graziano, Pesce, and Urbinati (2020) study a variation of Mas-Colell's bargaining set in finite economies and mixed markets respectively, and use it to restore the equivalence with the competitive allocations. Instead of putting restrictions on the deviations allowed, their results consider a more relaxed blocking mechanism in which objections and counter-objections raised by a more general notion of coalitions. Hervés-Beloso, Hervés-Estévez, and Moreno-García (2018) and Liu (2017) extend the results of Hervés-Estévez and Moreno-García (2018) to the case of production economies. A further refinement of the Mas-Colell's bargaining set in which each objection in a "chain"of objections is tested in the same way as its predecessor is given in Dutta, Ray, Sengupta, and Vohra (1989). It turns out that the consistency property of the bargaining set required by

[8]The theorem applies also to economies with infinitely many agents, provided that there is a leader that has only finitely many agents with her same characteristics.

[9]Precisely, Proposition 4.11 shows that if x is a non-core allocation with the ETP then leaders can propose an equitable objection to x that they prefer to any other equitable objection. But if leaders cannot improve upon this "optimal" objection, then nobody else does by the very definition of leaders. The objection is then justified, and so x is not in the Equitable Bargaining set.

Dutta, Ray, Sengupta, and Vohra (1989) is achieved whenever the bargaining set is coincident with the set of competitive allocations. Hence, consistency of our bargaining set notion can be derived as a consequence of Proposition 4.1; Theorems 4.7 and 4.12 and Corollary 4.13.

The rest of the paper is organized as follows. Section 2 describes the main model, which is that of an exchange economy with a measure space of agents. All assumptions made at this stage are standard as in Hildenbrand (1974). Section 3 gives the main definition in the Subsection 3.1, compares the Equitable Bargaining set with the core in 3.2 and then introduces the notion of weakly-Walrasian objections in 3.3. Section 4 presents the equivalence results: in Subsection 4.1 for atomless economies, in 4.2 for mixed markets and in 4.3 for large finite economies. All the longer proofs are relegated to the Appendix A. The Appendix B includes a series of original examples that illustrate the differences between the various solution concepts introduced and the necessity of the assumptions we use in the main Theorems.

2 The model

The economy consists of a finite-dimensional commodity space $\mathbb{R}^m_+$ and a finite measure space of consumers (T, Σ, λ). The set T represents all individual traders, while Σ is the collection of all groups that are able or allowed to trade. For $S \in \Sigma$, $\lambda(S)$ is the size (or weight) of the group S. A *coalition* is an economically relevant group of agents, i.e., a set in Σ with positive measure. We allow the presence of atoms in (T, Σ, λ), which are coalitions that cannot be broken in two, smaller subcoalitions. Atoms may represent single agents with significant market power (such as monopolists or oligopolists) or large groups of traders that are forced to act compactly by some binding agreements (such as unions or cartels). As the measure λ is finite, the set $\mathcal{A}$ of all atoms in (T, Σ, λ) is at most countable and so we can partition T into its atomic and atomless components, which are $T_1 = \bigcup \mathcal{A}$ and $T_0 = T \setminus T_1$ respectively.

Every agent $t \in T$ is characterized by a preference relation $\succcurlyeq_t$ on $\mathbb{R}^m_+$ and an endowment bundle $\omega(t) \in \mathbb{R}^m_+$. The irreflexive and symmetric components of $\succcurlyeq_t$ are $\succ_t$ and $\sim_t$ respectively. We make the following assumptions, that are standard in models with a measure space of agents (see e.g. Hildenbrand (1974)): (*i*) $\omega\colon T \to \mathbb{R}^m_+$ is an integrable function with $\int \omega \, d\lambda \gg 0$; (*ii*) preferences are strictly monotone, continuous, total preorders on $\mathbb{R}^m_+$; (*iii*) preferences are measurable in the sense that $\{t : v \succcurlyeq_t w\} \in \Sigma$ for every $v, w \in \mathbb{R}^m_+$; and (*iv*) $\succcurlyeq_t$ is convex for every $t \in T_1$.

We say that two agents s, t are of the *same type* (and write $s \sim t$) if they have identical preferences and endowments. Under our assumptions, the equivalence relation $\sim$ is measurable in the sense that $\{s : t \sim s\} \in \Sigma$ for every $t \in T$ and, in particular, every atom consists only of agents of the same type. We call *type of agents* an equivalence class in the quotient $T/_\sim$, i.e., a set formed by all the agents that share a given preference relation and endowment bundle.

An *allocation* is an integrable function of the type $x\colon T \to \mathbb{R}^m_+$. A coalition S *attains* an allocation x if $\int_S x \, d\lambda \leqslant \int_S \omega \, d\lambda$, i.e., if the amount of resources that x assigns to agents in S does not exceed their initial endowments. If x is attained by the grand coalition T we say that x is *feasible*. An allocation x has the *equal treatment property* (ETP for short) on a coalition S if $x(t) \succcurlyeq_t x(s)$ for every $t, s \in S$ of the same type. If x has the ETP on the whole T we simply say that it has the ETP. We write $\mathcal{M}$ for the set of allocations, and $\widetilde{\mathcal{M}}$ for the set of allocations with the ETP.

Given a price vector $p \in \mathbb{R}^m_+ \setminus \{0\}$, the *budget set* of consumer t at p is $\beta(t,p) = \left\{x \in \mathbb{R}^m_+ : p \cdot x \leqslant p \cdot \omega(t)\right\}$. A feasible allocation x is *competitive* at the price p if $x(t)$ maximizes $\succcurlyeq_t$ on the set $\beta(t,p)$ for a.e. $t \in T$, i.e., if $x(t) \in \beta(t,p)$ and $x(t) \succcurlyeq_t \beta(t,p)$[10]. The set of competitive allocations is $\mathcal{W}$. Since agents of the same type maximize their preferences on the same budget sets, a competitive allocation always satisfies the ETP and so $\mathcal{W} \subseteq \widetilde{\mathcal{M}}$.

Remark 2.1 Given an allocation $x \in \mathcal{M}$ and two agents t and s, we say that t *envies* s at x if t prefers receiving the bundle of s rather than consuming her own bundle, i.e., $x(s) \succ_t x(t)$. The allocation x is called *envy-free* if there is no envy among agents at x, i.e., $x(t) \succcurlyeq_t x(s)$ for almost all $t, s \in T$ (see Foley (1967) and Thomson (2011)). Asking that x has the ETP is less demanding than the absence of envy, as it only requires that agents are not envious when they compare themselves with individuals of their same type.

Remark 2.2 The assumptions of our model are standard and allow to cover a variety of classical situations. Finite economies are obtained when T is finite and λ is the counting measure on 2^T. Competitive economies arise when (T, Σ, λ) is atomless, e.g. when $T = [0, 1]$, Σ is the Borel algebra and λ is the Lebesgue measure. Last, when both sets T_0 and T_1 have positive measure, an ocean of negligible agents interacts with at most countably many influential agents or oligopolists (the atoms). We refer to this latter situation as mixed market, or mixed economy.

3 The Equitable Bargaining set

This section introduces the main solution concept, which is a variation of Mas-Colell's bargaining set based on a weaker mechanism of objections and counter-objections. Intuitively, we assume that an agent accepts to join an objection (or a counter-objection) only if she is promised a bundle at least as good as those consumed by her peers, i.e., by the agents of her same type. Objections and counter-objections of this type are called *equitable*. The *Equitable Bargaining set* consists of all feasible allocations with the ETP that cannot be blocked by an equitable objection without triggering some equitable counter-objection.

3.1 Definitions

A coalition B *blocks* (or *objects to*) an allocation x if its members can rearrange their own endowments in a way that they all find at least as good as x and that some strictly prefer to x. Formally, (B, y) is a (*standard*) *objection* to x if:

- B attains y, i.e., $\int_B (y - \omega \, d\lambda \leqslant 0$;
- $y(t) \succcurlyeq_t x(t)$ for a.e. $t \in B$;
- $\lambda(\{t \in B : y(t) \succ_t x(t)\}) > 0$.

The set of (standard) objections to x is $Ob(x)$. The *core* is the set $\mathcal{C}$ of all feasible allocations that cannot be blocked, i.e., the feasible $x \in \mathcal{M}$ such that $Ob(x) \neq \emptyset$. It is well known that any competitive allocation is in the core, that is $\mathcal{W} \subseteq \mathcal{C}$.

We introduce below a notion of objection with equity flavors, in the sense that members of the objection do not envy any other agents of the same type neither inside the objection nor outside the objection.

Definition 3.1 *An objection (B, y) to x is* ***equitable*** *if for a.e. $t \in B$ and $s \in T$ of the same type, one has:*

(EO1) $y(t) \succcurlyeq_t y(s)$ if $s \in B$;

[10]Throughout, for a $v \in \mathbb{R}^m_+$ and a $C \subset \mathbb{R}^m_+$ we write $v \succcurlyeq_t C$ when $v \succcurlyeq_t w$ for every $w \in C$.

$(EO2)$ $y(t) \succcurlyeq_t x(s)$ *otherwise.*

The set of equitable objections to x is $Ob_e(x)$. We denote by $\mathcal{C}_e$ the set of all feasible allocations with no equitable objections.

Condition $(EO1)$ rules out the possibility that agents in B of the same type envy each other, and it is equivalent to asking that y has the ETP on B. At the same time, condition $(EO2)$ always holds when x has the ETP because for a.e. $t \in B$ and $s \notin B$ of the same type, we have $y(t) \succcurlyeq_t x(t)$ by the properties of objections and $x(t) \succcurlyeq_t x(s)$ by the ETP of x. We conclude that, when x has the ETP, $(B, y) \in Ob(x)$ is equitable if and only if y has the ETP on B.

The *equitable core* is the set of allocations with the ETP without equitable objections. By the argument above, an allocation x is in the equitable core if it has the ETP and if there is no objection (B, y) to x such that y has the ETP on B.

Definition 3.2 *The* ***equitable core*** *is the set $\widetilde{\mathcal{C}}_e$ of all feasible allocations with the ETP without equitable objections, that is, the set $\widetilde{\mathcal{C}}_e = \mathcal{C}_e \cap \widetilde{\mathcal{M}}$.*

We now introduce the notion of *counter-objection* as in Mas-Colell (1989). Let (B, y) be an objection to an allocation x. A (*standard*) *counter-objection* to (B, y) consists of a coalition C and an allocation z such that:

- C attains z, i.e., $\int_C (z - \omega)\, d\lambda \leqslant 0$;
- $z(t) \succ_t y(t)$ for all $t \in C \cap B$;
- $z(t) \succ_t x(t)$ for all $t \in C \setminus B$.

The set of counter-objections to (B, y) is $Cob^x(B, y)$. An objection is *justified* if it has no counter-objections. The (*standard*) *bargaining set* is the class of all feasible allocations that have no justified objections, i.e., the set of all feasible $x \in \mathcal{M}$ such that either $Ob(x) = \emptyset$ or such that $(B, y) \in Ob(x)$ implies that $Cob^x(B, y) \neq \emptyset$. We write BS for the (standard) bargaining set.

On the same line of equitable objections, we say that a counter-objection (C, z) is *equitable* if no agent in C wishes to switch position with any other agent of her same type. Thus, we ask that no agent in the counter-objection envies what her peers in C receive from z, what her peers in B receive from y and what the others receive from x.

Definition 3.3 *A counter-objection $(C, z) \in Cob^x(B, y)$ is* ***equitable*** *if for a.e. $t \in C$ and $s \in T$ of the same type, one has:*

$(EC1)$ $z(t) \succcurlyeq_t z(s)$ *if* $s \in C$;

$(EC2)$ $z(t) \succcurlyeq_t y(s)$ *if* $s \in B \setminus C$;

$(EC3)$ $z(t) \succcurlyeq_t x(s)$ *otherwise.*

The set of equitable counter-objections to (B, y) is $Cob_e^x(B, y)$. An objection (B, y) is said to be e***-justified*** *if it is equitable and there is no equitable counter-objection to it.*

As it was for equitable objections, one observes that condition $(EC1)$ is equivalent to asking that z has the ETP on C, and that every counter-objection satisfies condition $(EC3)$ when $x \in \widetilde{\mathcal{M}}$.

We can now define a new bargaining set that consists only of allocations with the ETP and that considers only equitable objections and counter-objections.

Definition 3.4 *The **Equitable Bargaining set** is the set BS_e of all feasible allocations with the ETP that have no e-justified objections. In formulas:*

$$BS_e = \left\{ x \in \widetilde{\mathcal{M}} : x \text{ is feasible and } (B, y) \in Ob_e(x) \Rightarrow Cob_e^x(B, y) \neq \emptyset \right\}.$$

The Equitable Bargaining set contains every core allocation with the ETP, and hence all competitive allocations. In other words, one has the following chain of inclusions:

$$\emptyset \neq \mathcal{W} \subseteq \widetilde{\mathcal{C}}_e \subseteq BS_e. \tag{1}$$

Without any specific restriction, it is possible that in some economy all the inclusions in the equation (1) are strict (see the examples in Appendix B). Our main results in Section 4 give conditions under which BS_e coincides with $\mathcal{W}$, and conditions under which $BS_e = \widetilde{\mathcal{C}}_e$ even when this may be strictly larger than $\mathcal{W}$.

Remark 3.5 The difference between the standard and the equitable notions of objections, counter-objections and bargaining set arises only when agents of the same type form appreciable coalitions. When every type of agents consists of a single individual, in fact, no consumer can compare herself with others and nobody is envious. This implies that all allocations have the ETP, that every objection (and counter-objection) is equitable and hence that the Equitable Bargaining set coincides with the standard one.

Remark 3.6 As discussed in Mas-Colell (1989, Remark 1), the definition of counter-objection can be weakened to just requiring strict preference for a positive measure subset of the counter-objecting coalition. With this change, even if the set of counter-objections is formally larger, the set of justified objections (and hence the bargaining set) remains unaltered. A similar argument does not apply to objections: if one considers only objections in which all the deviating have strict preferences then the core does not change, but the bargaining set may become significantly larger. See Yamazaki (1995) for a formal comparison of the bargaining sets generated by these different classes of objections.

Remark 3.7 We observed that equitable objections and counter-objections to allocations with the ETP have a simpler description. On this line, an equivalent definition of the Equitable Bargaining set is the following: $x \in BS_e$ if and only if x has the ETP and for every $(B, y) \in Ob(x)$ such that y has the ETP on B there is a $(C, z) \in Cob^x(B, y)$ with the property that z has the ETP on C and $z(t) \succcurlyeq_t y(s)$ for a.e. $t \in C$, $s \in B$ of the same type.

3.2 Comparisons between core and bargaining sets

The introduction of equitable objections and counter-objections, as well as the focus on allocations with the ETP, defines variations of the notions of core and bargaining set for exchange economies. This section studies how these new solution concepts relate to each other.

By previous considerations, we know that the following series of inclusions hold.

$$\mathcal{W} \subseteq \widetilde{\mathcal{C}}_e \subseteq BS_e, \quad \text{and} \quad \mathcal{W} \subseteq \widetilde{\mathcal{C}}_e \subseteq \mathcal{C} \subseteq BS.$$

Comparisons between BS_e and the standard notion of core $\mathcal{C}$ are not straightforward: there might be core allocations without the ETP and so outside BS_e (see Example B.2) as well as non-core allocations inside BS_e (see Example B.4). Thus, without further restrictions on the measure space of agents, the core and the Equitable Bargaining set are not comparable solution concepts.

The veto mechanism based on equitable objections is typically very weak and the class $\mathcal{C}_e$ of feasible allocations without equitable objections may be extremely large. It may even include allocations that are not individually rational, i.e., some $x \in \mathcal{M}$ such that $\{t : \omega(t) \succ_t x(t)\}$ is non-null. As an example, think of the economy formed by two identical agents: the allocation that assigns all commodities to one of the two agents is not individually rational and has no equitable objections. Even limiting $\mathcal{C}_e$ to individually rational allocations defines a particularly large set. Example B.5, for instance, describes an atomless economy where $\mathcal{C}_e$ contains a non-competitive, individually rational allocation. Therefore, $\mathcal{C}_e$ may be strictly larger than the core and the bargaining set even when the latter two coincide.

Interestingly, standard and equitable objections are equally effective in blocking allocations with the ETP.

Theorem 3.8 *Let x be a non-core allocation with the ETP. Then there exists an equitable objection (B, y) to x. Furthermore, y can be chosen so that $y(t) = y(s)$ whenever $t, s \in B$ are agents of the same type with convex preferences.*

For readability, the most technical proofs are relegated to the Appendix A. Theorem 3.8 is covered in Appendix A.1.

A consequence of Theorem 3.8 is that the set of all core allocations with the ETP coincides with $\widetilde{\mathcal{C}}_e$, the set of feasible allocations with the ETP that have no equitable objections. In symbols, $\mathcal{C} \cap \widetilde{\mathcal{M}} = \widetilde{\mathcal{C}}_e$.

3.3 Objections and competitive behaviour

When agents act competitively (i.e., as price-takers), objections to a given allocation x may emerge as the result of a fully decentralized, price-based mechanism. This paragraph introduces a special class of price-based objections, called *weakly-Walrasian*, that can be used to test whether an allocation is not in the Equitable Bargaining set. Precisely, we show that if there is a weakly-Walrasian objection to an allocation x then $x \notin BS_e$. This class of objections generalize the notion of Walrasian objections in Mas-Colell (1989).

Let us recall that for a price vector $p \in \mathbb{R}^m_+ \setminus \{0\}$ and a $t \in T$, the *budget set* of t at p is $\beta(t, p) = \{v \in \mathbb{R}^m_+ : p \cdot v \leqslant p \cdot \omega(t)\}$. Agent t's *demand set* at p is $\xi(t, p) = \{v \in \beta(t, p) : v \succcurlyeq_t \beta(t, p)\}$. An allocation x is *competitive* at p when it is feasible and $x(t) \in \xi(t, p)$ for a.e. $t \in T$.

Definition 3.9 *An objection (B, y) to x is* ***weakly-Walrasian*** *at a price vector $p \gg 0$ if:*

$(WO1)$ *$y(t) \in \xi(t, p)$ for a.e. $t \in B$;*

$(WO2)$ *$x(t) \succcurlyeq_t \beta(t, p)$ for a.e. $t \notin B$ such that $\lambda(\{s \in B : s \sim t\}) = 0$.*

The objection (B, y) is ***Walrasian*** *if, in addition to $(WO1)$, it satisfies:*

$(WO3)$ *$x(t) \succcurlyeq_t \beta(t, p)$ for a.e. $t \notin B$.*

Clearly, a Walrasian objection is weakly-Walrasian. The definition of Walrasian objection appears in Mas-Colell (1989) in a different but equivalent formulation; see Remark 3.10.

We can imagine that a Walrasian objection (B, y) to x at the price p arises when every $t \in T$ can choose independently whether to accept $x(t)$ or to trade at a price p. Agents in B are those who choose to deviate from x because, at the price p, they can afford a bundle at least as good as x (remember that, being $(B, y) \in Ob(x)$, $y(t) \succcurlyeq_t x(t)$ for a.e. $t \in B$). Agents outside B are those who accept x, as they find it at least as good as anything they can afford at the price p. The difference between Walrasian and weakly-Walrasian objections is that, in the second case, not all agents can freely choose between x and p. Precisely, when (B, y) is only weakly-Walrasian we cannot tell if a $t \notin B$ that is of the same type of members of B prefers p over x.

Remark 3.10 The definitions of Walrasian objections in Mas-Colell (1989) and in 3.9 are formally different, but yet equivalent under the current assumptions. For a $(B, y) \in Ob(x)$, Mas-Colell (1989) asks that there is a price vector p such that, for a.e. $t \in T$ and every $v \in \mathbb{R}^m_+$, (i) $p \cdot v \geqslant p \cdot \omega(t)$ whenever $t \in B$ and $v \succcurlyeq_t y(t)$, and (ii) $p \cdot v \geqslant p \cdot \omega(t)$ whenever $t \notin B$ and $v \succcurlyeq_t x(t)$. Given that B attains y, a standard argument using the continuity and the strict monotonicity of preferences gives that (i) is equivalent to $(WO1)$, whereas (ii) is equivalent to $(WO3)$.

By Mas-Colell (1989, Remark 5), every Walrasian objection to an allocation with the ETP is equitable. Our next proposition shows that a similar result holds for weakly-Walrasian objections. Its proof is in A.4.

Proposition 3.11 *Let $x \in \widetilde{\mathcal{M}}$. Then every weakly-Walrasian objection to x is equitable.*

If $x \in \mathcal{M}$ has the ETP, then all Walrasian objections to x are justified, and hence e-justified. The same cannot be said for a weakly-Walrasian objection to x, for which we may still find equitable counter-objections to it. Nevertheless, we can show that equitable counter-objections to weakly-Walrasian objections are rare and they must satisfy strict conditions. The proof of the following result is in A.5.

Lemma 3.12 *Let $x \in \mathcal{M}$ and let $(B, y) \in Ob(x)$ be weakly-Walrasian at the price p. If $(C, z) \in Cob^x_e(B, y)$, then:*

1. *$z(t) \in \xi(p, t)$ for a.e. $t \in C$;*
2. *$\lambda(B \cap C) = 0$.*

Lemma 3.12 has a series of interesting consequences. The first is that if $(B, y) \in Ob(x)$ is weakly-Walrasian at a price p, then none of the agents in B will accept to join an equitable counter-objection (bullet 2). This means that the society cannot use equitable objections to prevent the agents in B from deviating. The second consequence is that agents participate in an equitable counter-objection (C, z) only if this is at least as convenient as trading their endowments at the price p. In particular, every $t \in C$ prefers trading at p rather than consuming $x(t)$. These considerations, together with Proposition 3.11, are used to prove the following key result. Its proof is in A.6.

Proposition 3.13 *If $x \in \widetilde{\mathcal{M}}$ is blocked with a weakly-Walrasian objection then there exists an e-justified objection to it, and hence $x \notin BS_e$.*

The proof of Proposition 3.13 goes even further than the claim and shows that if $(B, y) \in Ob(x)$ is weakly-Walrasian, then there exists an e-justified $(\tilde{B}, \tilde{y}) \in Ob(x)$ such that $B \subseteq \tilde{B}$ and $\tilde{y}(t) = y(t)$ for a.e. $t \in B$. Thus, if a group of agents accepts to raise a weakly-Walrasian objection to x, then this can be extended to some larger e-justified objection to x.

4 Equivalence results

In general, the set of competitive allocations, the equitable core and the Equitable Bargaining set are distinct. This section introduces additional conditions under which these solution concepts define the same set of allocations. Our assumptions will refine some that are common in the literature and apply to a variety of standard situations: part 4.1 considers atomless economies, i.e., the case $T = T_0$; part 4.2 studies mixed markets that either have infinitely many large traders, all of the same type, or satisfy the "fringe" hypothesis; part 4.3 considers economies with exceptionally influential agents, called leaders, and has interesting applications in the case of finite economies.

4.1 Equivalence in atomless economies

This paragraph studies the Equitable Bargaining set in economies where every individual trader is negligible, i.e., where $T = T_0$. In this framework, we can combine Mas-Colell (1989, Proposition 2) with Proposition 3.13 to obtain the following equivalence.

Proposition 4.1 *Let $T = T_0$. For every allocation $x \notin \mathcal{W}$, if x has the ETP then there exists a weakly-Walrasian objection to x. As a consequence, $\mathcal{W} = \mathcal{C} = BS_e = BS$.*

Proof. Let $x \notin \mathcal{W}$ be an allocation with the ETP. By Mas-Colell (1989, Proposition 2) there exists an objection to x that is Walrasian, and hence weakly-Walrasian. An application of Proposition 3.13 gives that $x \notin BS_e$ and so $BS_e \subseteq \mathcal{W}$. Since $\mathcal{W} \subseteq BS_e$ is always true, and $\mathcal{W} = \mathcal{C} = BS$ by Mas-Colell (1989, Theorem 1), we conclude that $\mathcal{W} = \mathcal{C} = BS_e = BS$. ■

Notice that the result above does not specify how to choose the objection, nor how large the deviating coalition may be. This may become an issue in fully decentralized economies, where it is common to assume that larger groups of agents may not be able to coordinate their actions and that only small coalitions are effective. In this perspective, one asks if a version of Proposition 4.1 holds even when only coalitions under a certain size are allowed to raise objections.

In general, the answer to the question above is negative. In an economy where every agent is of a different type, for example, it could be that, for some $\varepsilon > 0$ and $x \notin \mathcal{W}$ there is no weakly-Walrasian objection (B, y) to x with $\lambda(B) < \varepsilon$. This is so because, in this specific framework, weakly-Walrasian and Walrasian objections coincide and so the arguments in Schjodt and Sloth (1994) apply. Nevertheless, if the economy allows enough comparisons among agents then we may obtain some positive results.

In the following we assume that for a.e. $t \in T$ the set of agents of the same type of t has positive measure, i.e., $\lambda(\{s : s \sim t\}) > 0$. This implies that there are countably many types of agents[11] $(K_n)_n$ such that $\lambda(K_n) > 0$ and $T = \bigcup_n K_n$ up to a null set.

Assumption 4.2 *There are countably many types of agents. We write $(K_n)_n$ for the (possibly finite) sequence of types of agents that have positive measure.*

Under Assumption 4.2, we can associate any coalition S with a set $K(S) = \{n : \lambda(S \cap K_n) > 0\}$ and write $S = \bigcup\{S \cap K_n : n \in K(S)\}$ up to a null set. The set $K(S)$ is then a list of all types of agents that have a representative in S. Notice also that, when $T = T_0$, Assumption 4.2 excludes the possibility that every agent is of a different type.

We now prove that in atomless economies that satisfy Assumption 4.2, it is possible to reduce the size of any objection as much as we want by preserving some properties, such as the types of agents represented in the objection. The proof is in A.7.

Proposition 4.3 *Let $T = T_0$ and assume that Assumption 4.2 holds. Suppose that $x \in \mathcal{M}$ and let $(B, y) \in Ob(x)$. Then, for every $\delta \in (0, 1)$ there exists $B^\delta \subseteq B$ such that:*

(i) $\lambda(B^\delta \cap K_n) = \delta\lambda(B \cap K_n)$ *for every n;*

(ii) $(B^\delta, y) \in Ob(x)$ *and:*

(iia) *if (B, y) is equitable, then (B^δ, y) is equitable too;*

[11] Recall that a *type of agents* is an equivalence class in $T/\sim$ and a measurable set of agents.

(iib) if (B, y) is weakly-Walrasian, then (B^δ, y) is weakly-Walrasian too.

It follows from Proposition 4.3 that a version of 4.1 holds even when we impose a restriction on the size of coalitions that can raise objections, in the same spirit of Schmeidler (1972). Precisely, if (B, y) is a weakly-Walrasian objection to x then there is a weakly-Walrasian objection (B', y) such that (i) $B' \subseteq B$, (ii) every type of agent present in B is present in B' as well and (iii) $\lambda(B') < \varepsilon$ for an arbitrarily small $\varepsilon > 0$.

In proving 4.3, Assumption 4.2 is crucial. This is because the proof applies the Lyapunov theorem to each type of agents, and obtain a subcoalition for each type. Then we consider the union of these subcoalitions. Being Σ a σ-algebra, such a union defines a coalition only if the subcoalitions, and hence the types of agents, are at most countably many.

Remark 4.4 Our notion of bargaining set has some similarities with the *global bargaining set* introduced in Vind (1992) (the terminology we use for this bargaining set is taken from Schjodt and Sloth (1994)). For a given $x \in \mathcal{M}$, a *global objection* to x is a $(B, y) \in Ob(x)$ such that y is feasible. A *global counter-objection* to (B, y) consists of a coalition C and a feasible allocation z such that C attains z and $z(t) \succ_t y(t)$ for every $t \in C$. A feasible allocation is in the global bargaining set if there are no global objection to it without global counter-objections. Therefore, the main difference with Mas-Colell's bargaining is that all the counter-objecting agents improve upon the allocation they counter (i.e., the y) and ignore the original allocation (i.e., the x).

Our notion keeps some features of both, because we require that agents in $C \setminus B$ do not envy only their counterparts in the objection that they are opposing. In other words, we impose a comparison between z and y, as done in Vind (1992), only among agents of the same type and for the rest z is compared with x, as in Mas-Colell (1989). In this perspective, the fact that we can put some restrictions on the size of weakly-Walrasian objections (Proposition 4.3) finds its correspondence in a property of justified global objection, since they can be chosen of any size; see Schjodt and Sloth (1994).

4.2 Equivalence in mixed markets

This section provides conditions that are sufficient to get the equivalence between the Equitable Bargaining set and the class of competitive allocations in mixed markets. As in the previous paragraph, we will work under Assumption 4.2, so that every trader identifies with a non-null coalition of agents. This implies that T coincides with the disjoint union $\bigcup_n K_n$ up to a null set, where the K_n's denote the different type of agents.

We make the following, additional assumption.

Assumption 4.5 *For any sequence $(\mu_n)_n \in (0, 1)$ there exist $\delta \in (0, 1)$ and a sequence of coalitions $(F_n)_n$ such that $F_n \subseteq K_n$ and $\lambda(F_n) = \delta\mu_n\lambda(K_n)$ for any n.*

Despite its technical appearance, Assumption 4.5 weakens two well-known conditions imposed in mixed markets. The first asks that there are infinitely many atoms, and that they are all of the same type. The second assumes a finite number of atoms and requires that to every atom A corresponds an *atomless fringe*, i.e., a coalition in T_0 whose elements are all of the same type of A. The proof is in A.8.

Proposition 4.6 *Assumption 4.5 is satisfied if one of the following two conditions holds.*

1. *There are infinitely many atoms and they are all of the same type.*

2. *There are finitely many types of atoms, and for each atom A there is a coalition $S_A \subseteq T_0$ such that every $t \in S_A$ is of the same type of A.*

Notice that Assumption 4.5 rules out the possibility of an economy with finitely many agents. This situation is covered in Section 4.3.

We can now state the main result of this section.

Theorem 4.7 *Under Assumptions 4.2 and 4.5,* $\mathcal{W} = BS_e$.

The proof follows the same structure used in Mas-Colell (1989) for the Walras-bargaining equivalence in atomless economies, with the only difference that here we consider objections that are only weakly-Walrasian. Precisely, we show that: (1) a non-competitive allocation x with the ETP has a weakly-Walrasian objection to it (Proposition 4.8 below) and (2) if there is a weakly-Walrasian objection to x, then $x \notin BS_e$ (Proposition 3.13).

The proof of the following is in A.11.

Proposition 4.8 *Under Assumptions 4.2 and 4.5, if* $x \in \widetilde{\mathcal{M}}$ *is a non-competitive feasible allocation, there exists a weakly-Walrasian objection to it.*

Note that, contrary to Proposition 4.8, Proposition 3.13 does not need the two assumptions and it holds in more general contexts. Proposition 4.8 is alike of Proposition 2 of Mas-Colell (1989), whereas Proposition 3.13 does not state that any weakly-Walrasian objection is justified as done in Mas-Colell (1989). See the discussion in Section 3.3.

These two propositions combined imply the equivalence $\mathcal{W} = BS_e$.

Proof of Theorem 4.7. The inclusion $\mathcal{W} \subseteq BS_e$ is always met. For the converse, let $x \in BS_e$, which implies that x is feasible, it satisfies the ETP and there exists no e-justified objection to it. Assume, to the contrary, that $x \notin \mathcal{W}$. Proposition 4.8 ensures the existence of a weakly-Walrasian objection to x and hence, by Proposition 3.13, there exists an e-justified objection to it. This is a contradiction because $x \in BS_e$. ■

The equivalence proved in Theorem 4.7 is independent from the coincidence of competitive allocations and the standard bargaining set. Variations of Example B.3 in the Appendix B shows that it might be the case that, under Assumption 4.5,

$$\mathcal{W} = BS_e \subsetneq BS.$$

This explains why we restrict on weakly-Walrasian objections only. Furthermore, Examples B.1 and B.4 underline the role of Assumption 4.5 as they illustrate economies in which Assumption 4.5 fails and $\mathcal{W} \subsetneq BS_e$.

Remark 4.9 The proof of Proposition 4.8 studies the mixed market economy via an auxiliary, atomless economy in the manner of Greenberg and Shitovitz (1986). We sketch here the main ideas, then address it formally in Appendix A.4.

Imagine that every atom is not as a single large trader but a coalition of small, negligible agents forced to act together (as in a cartel or a syndicate). The auxiliary economy is defined by removing these constraints and letting all individuals act independently: the set of agents remains the same, but they can form many more coalitions and find new ways of allocating goods among themselves. In particular, in this relaxed environment the existence Theorems of Mas-Colell and the results of the Subsection 4.1 apply.

The idea of the proof is to think of any non-competitive $x \in \widetilde{\mathcal{M}}$ as an allocation in the auxiliary economy (this can be done because the sets of agents are essentially the same). As we are ignoring atoms, we can apply Proposition 4.1 to find a weakly-Walrasian objection (S, f) to x in the auxiliary economy. The coalition S may not be one that agents can implement in the original mixed market, but under Assumption 4.5 we can apply Proposition 4.3 to modify (S, f) into a smaller, weakly-competitive objection to x that has a counterpart in the original mixed market (Proposition A.10). This will be weakly-Walrasian in the original mixed market too (Lemma A.9), concluding the proof.

4.3 Equivalence in economies with leaders

This section studies economies in which there is a type of agents, called *leaders*, whose contribution is necessary to reach any fruitful transaction.[12] It assumes that coalitions can deviate profitably from an allocation only with the participation of at least some leaders, without whom the coalition is worthless. Leaders play a dominant role in the bargaining process, for they can exert a sort of veto-power on the formation of objections and counter-objections.

We formalize this intuition as follows.

Definition 4.10 *An agent s is a **leader** if for every allocation $x \in \mathcal{M}$ and every objection $(B, y) \in Ob(x)$ the set $\{t \in B : t \sim s\}$ is non-null.*

It is evident that if s is a leader then so is every other agent of her type. We can therefore speak of *types of leaders* to denote a type of agents that consists of leaders. Notice that different types of leaders can coexist at the same time, in which case any objecting coalition must contain representatives from each type of leaders.

Economies with leaders are fairly common in many models of exchange. As a way of illustration, consider a market with many sellers endowed with different goods and no money, and many identical buyers with only money. If sellers' utility depends exclusively on the money they receive for their goods then it is clear that any profitable exchange must include at least some buyers. In other words, buyers are *leaders* in the economy. On a more general level, leaders occur whenever there exists a commodity j that is owned only by a single type of agents, and every other agent weakly prefers her own endowment to any bundle that does not contain any amount of commodity j. In this case, in fact, the agents that control commodity j are leaders. Shitovitz (1989) considers a special case of a market of this type, where a single agent, called *veto-player*, controls the whole endowment of a commodity that everyone else finds necessary to survive. Thus, a veto-player is just a leader with no other agents of her same type. Other examples of economies with leaders are common in the literature on bilateral market games. In the model of bilateral oligopolies formalized in Gabszewicz and Michel (1997), for instance, one typically has that there are only two types of agents and every agent is a leader.

An important feature of economies with leaders is that it is possible to rank all the equitable objections to a given allocation x by the way they treat the leaders of a given type. Suppose that K is a type of leaders, that x is a non-core allocation and that (S, y) and (S', y') are equitable objections to x. As agents in K are leaders, some of them must participate in S and some other (possibly the same ones) in S'. We can then say that agents in K find (S, y) at least as good as (S', y') if their representatives in $K \cap S$ find that what they receive from y at least as good as what the members of $K \cap S'$ receive from y'. In formulas, it must be that:

$$y(t) \succcurlyeq_t y'(s) \qquad \text{for almost every } t \in K \cap S \text{ and } s \in K \cap S'.$$

Notice that this comparison is always possible by the equitability of the two objections and by the fact that agents in K are of the same type (and so they have identical preferences). The resulting ranking is then a total order relation: if agents in K do not find (S, y) at least as good as (S', y') then they strictly prefer the latter to the former.

[12] In the literature on bargaining sets the word "leader" is often used with a different interpretation. In Aumann and Maschler (1964) or Geanakoplos (1978) a leader is an agent (or a group of agents) that proposes the objection and that must be excluded from any counter-objecting coalition. Despite the formal differences, it turns out that our notion of leaders preserves the same intuition. Under the assumptions of Proposition 4.11, in fact, the relevant equitable objections are proposed by some leaders, and all agents of their same type are excluded from every equitable counter-objection.

Our first result shows that if a type of leaders consists of finitely many agents then some of them can impose their most preferred equitable objection. Its proof is in A.12.

Proposition 4.11 *Let K^* be a type of leaders formed by finitely many atoms. Then for every $x \in \widetilde{\mathcal{M}} \setminus \mathcal{C}$ there exists $(B^*, y^*) \in Ob_e(x)$ with the property that, for every $(B, y) \in Ob_e(x)$:*

$$y^*(t) \succcurlyeq_t y(s), \qquad \text{for every } t \in K^* \cap B^* \text{ and } s \in K^* \cap B.$$

The proposition extends a lemma in Shitovitz (1989), where a similar result is proved for standard objections assuming that there is only one leader; see Remark 4.14.

Proposition 4.11 has remarkable consequences. If a leader s raises her most preferred equitable objection (as shown possible by the proposition) then none of her peers can find a better proposal to counter-object. But being s a leader, this implies that no equitable counter-objection is possible at all and so the objection is e-justified. This intuitive argument is the key to the following equivalence theorem. See A.13.

Theorem 4.12 *Let there be a type of leaders formed by finitely many atoms. Then BS_e coincides with the equitable core $\widetilde{\mathcal{C}}_e$.*

In formulas, Theorem 4.12 establishes that:

$$\mathcal{W} \subseteq \widetilde{\mathcal{C}}_e = BS_e \subseteq \mathcal{C} \subseteq BS. \tag{2}$$

The equivalence $BS_e = \widetilde{\mathcal{C}}_e$ opens to interesting insights in terms of replicated economies. Consider a finite economy and use the process in Debreu and Scarf (1963) to construct from it a sequence of replicas. If an agent in the original economy is a leader, then so are all her clones in the corresponding replicas and Theorem 4.12 applies to every economy in the sequence. The Equitable Bargaining set of each replica is then a subset of the core, and so it shrinks to the set of competitive allocations by the core-convergence Theorem in Debreu and Scarf (1963). This convergence result is non-trivial and typically fails for the other notions of bargaining set, including Mas-Colell's; see Anderson, Trockel, and Zhou (1997), Iñarra, Serrano, and Shimomura (2020) and Hervés-Estévez and Moreno-García (2018).

In conclusion, notice that the inclusions in Equation (2) may be strict. Yet, if there are two leaders of the same type, and they are the only atoms in the economy, a direct application of Shitovitz (1973) gives that $\mathcal{C}$ coincides with $\mathcal{W}$, even when this is strictly contained in BS. This translates into the following corollary, whose proof is in A.14.

Corollary 4.13 *Let there be a type of leaders that consists of all atoms in the economy and assume that the number of atoms is finite and at least 2. Then $\mathcal{W} = \widetilde{\mathcal{C}}_e = BS_e = \mathcal{C}$, and these may be strictly contained in the standard bargaining set.*

The assumptions needed in Corollary 4.13 require that the number of atoms is finite, that they are all leaders of the same type, and that there is no coalition of negligible agents with their same characteristics (the so-called "fringe"). As such, they are more restrictive than those typically used to prove the core-Walras equivalence. These assumptions, however, are necessary to combine Shitovitz (1973) with Theorem 4.12. Notice that the case of countably many atoms or the presence of atoms' fringe is covered in Section 4.2.

Remark 4.14 Shitovitz (1989) shows that if a type of leaders consists of a single trader, the so-called veto-player, then she can object to any non-core allocation by proposing her preferred reallocation of goods.[13] Since no coalition can contest her choices, such objection

[13]Formally, the results in Shitovitz (1989) require that agents' preferences reach their minimum on the boundaries of $\mathbb{R}^m_+$, in contrast with our assumption of strict monotonicity of preferences. For what concerns this section, however, this difference is purely formal since all the results hold identically if translated in the framework of Shitovitz (1989).

is justified, and so the core and the (standard) bargaining set coincide. The same result does not hold with many leaders of the same type, because the competition between them makes the core collapse to a much smaller set; see Shitovitz (1973). Theorem 4.12 restores some form of core-bargaining equivalence in this latter case. By allowing only equitable objections and counter-objections, it reduces the bargaining power of individual agents and prevents the competition between leaders of the same type.

Differently from Theorem 4.12, the equivalence in Shitovitz (1973) cannot be used to obtain convergence results in replicated economies. While leaders remain such in every replica, all veto-players loose their status as soon as they are replicated and so the core-bargaining equivalence based on veto-players does not apply to replicated economies. It follows that the standard bargaining set may coincide with the core in the original economy but it may become much larger in the replicas.

Remark 4.15 The notion of leader has an immediate translation in the language of cooperative games and there is a significant literature that studies how the presence of leaders affects the core and the bargaining set of a game. Bahel (2016) considers games with veto-players (leaders with no other agents of their same type) and prove that, in this case, the core and the bargaining set of Aumann-Maschler coincide. This result resembles in many ways the equivalence of Shitovitz (1989) for Mas-Colell's bargaining set; see Remark 4.14. Aiche (2019) extends the results from Shitovitz (1989) to a case of market-games representing oligopolies, that is, to situations with finitely many leaders of the same type and no veto-players. In this context, it provides some conditions under which every allocation in Mas-Colell's bargaining set with the ETP is a core allocation, proving an equivalence similar to that of Theorem 4.12.

In a different scenario, Apartsin and Holzman (2003) proves that the core coincides with Aumann-Maschler's bargaining set in every finite *unitary glove market-game*, a kind of games derived from special exchange economies in which every agent is a leader and there are only finitely many types of agents. Amarante and Montrucchio (2010) extends this class of games to models with a measure space of agents and proves that, in these games, the core coincides also with Mas-Colell's bargaining set.

Remark 4.16 Hervés-Estévez and Moreno-García (2018) discusses a notion of bargaining set for finite economies based on the idea of "endogenous leaders". Objections are raised by an entire type of agents (the leaders) and the only counter-objections allowed must satisfy the following conditions: (i) they shall not include any of the leaders; (ii) an agent can participate to a counter-objection only if this makes her better off than her peers in the objection.

This notion of bargaining set shares some common traits with ours. If x is an allocation with the ETP and (B, y) is an objection to x, then (C, z) is a counter-objection to (B, y) in the sense of Hervés-Estévez and Moreno-García (2018) only if it is equitable in the sense of Definition 3.3. The converse is not true, for in our definition we only ask that agents in C do not envy their peers in the objection and so we allow that $z(t) \sim_t y(s)$ for some $t \in C$, $s \in B$ of the same type.

5 Concluding remarks

The paper introduces the notion of equitable objections, counter-objections and bargaining set, thereby incorporating fairness considerations into collective bargaining dynamics. In a non-equitable objection (or counter-objection), in fact, there are deviating agents who envy the bundle received by someone of their same type, either within or outside the objecting coalition. These agents would switch position with some of their peers rather than accepting to implement the objection. Our solution is close in spirit to Dutta and Ray (1989) where a new solution concept is proposed for TU games under the assumption that individuals value equality as a desirable social goal, although they are endowed with private selfish preferences. It bears emphasis that alternative notions of equitable objections (and counter-objections) could have been obtained by considering different criteria of fairness and envy-freeness from the literature (see Thomson (2011) for a survey on the topic). For instance, Hara (2002) considers objections that are "anonymous" in the sense that no agent (both in and out of the objection) envy the net-trade of others. In general, it is not immediate how the bargaining set changes when one imposes alternative equitability restrictions on objections and counter-objections.

The results in Section 4 provide conditions under which the Equitable Bargaining set coincides with the core or with the set of competitive allocations in economies with influential traders. The Subsection 4.2 studies mixed markets with both an ocean of small traders and some large influential ones, and the equivalence result therein requires some restrictions on the large traders. As pointed out in Proposition 4.6, these restrictions follow from some classical assumptions in the literature on mixed markets. In a recent series of works on noncoperative oliogopolies, however, Busetto, Codognato, Ghosal, Julien, and Tonin (2018) showed that some Cournot-Walras equivalence result can be obtained for mixed markets by imposing conditions not on the large traders, but rather on the preferences and endowments of the negligible ones. Similar considerations are also in Busetto, Codognato, Ghosal, Julien, and Turchet (2022) and Busetto, Codognato, Ghosal, and Turchet (2023). It would be of interest to explore if a similar approach could be used also for equivalences regarding the bargaining set. A related line of literature formalizes Cournots theory through strategic market games (see Koutsougeras and Ziros (2008)) and links the bargaining set with the trading norms of strategic market games by allowing agents to use the ShapleyShubik mechanism in order to determine feasible objections and counter-objections (Ziros (2011)).

We also stress that, in the study of the bargaining set, there are similarities between the issues caused by including large traders in the population and by allowing for lumps in the consumption. In the presence of indivisible commodities, for example, the equivalence between the bargaining set and the class of competitive allocation fails for the same non-convexities that arise when there are atoms in the underlying space of agents; see Yamazaki (1995) and the references therein. The presence of indivisible commodities already introduces complexities in the decentralization of core allocations as competitive allocations. Nonetheless, in certain cases, classical core equivalence results can be established in general exchange economies with indivisible commodities as long as money enters the model and the preferences of agents, and initial resources in money are sufficient in a suitable sense; see Quinzii (1984). It remains unclear whether a similar assumption can be applied to prove equivalences of the corresponding bargaining set.

The results in Section 4.3 prove the equivalence between the Equitable Bargaining set and a subset of the core under the assumption that there exist some *leaders*. This, in turn, implies that the Equitable Bargaining set of a finite economies with leaders converges to the set of competitive allocations when the economy is replicated. It remains unclear whether there are conditions under which a similar limit result can be achieved without recurring to the equivalence between the core and the Equitable Bargaining set.

A Appendix: Proofs

A.1 Proofs of the results in Subsection 3.2

The proof of Theorem 3.8 relies on two technical results which we discuss separately.

The first lemma states that it is possible to modify any objection (B, y) to a $x \in \widetilde{\mathcal{M}}$

into an objection that assigns identical bundles to agents of the same type, provided that their preferences are convex. When (B, y) is equitable in the first place, also the modified objection can be taken equitable. This lemma covers the second statement in Theorem 3.8

Lemma A.1 *Let x have the ETP and let $(B, y) \in Ob(x)$. Then B objects to x via a y' that is constant on each type of agents with convex preferences. If, in addition, y has the ETP on B then y' can be chosen with the ETP on B, and such that $y'(t) \succcurlyeq_t y(t)$ for a.e. $t \in B$.*

Proof. Let $\mathcal{K}_1(B)$ be the set of all types of agents K such that agents in K have convex preferences and that $\lambda(B \cap K) > 0$.

For every $K \in \mathcal{K}_1(B)$, let $B_K = B \cap K$ and let y_K be the average bundle assigned by y to the agents in B_K, i.e., the vector:

$$y_K = \frac{1}{\lambda(B_K)} \int_{B_K} y \, d\lambda.$$

Since x has the ETP and y is weakly preferred to x by a.e. $t \in B$, we have $x(t) \sim_t x(s) \preccurlyeq_t y(s)$ whenever $t, s \in B$ are of the same type. An application of García-Cutrín and Hervés-Beloso (1993, Lemma, page 580) gives that $y_K \succcurlyeq_t x(t)$ for a.e. $t \in B_K$, and that $\lambda(\{t \in B_K : y_K \succ_t x(t)\}) > 0$ if $\lambda(\{t \in B_K : y(t) \succ_t x(t)\}) > 0$. Define $y'(t) = y_K$ if $t \in B_K$ for some $K \in \mathcal{K}_1(B)$, and $y'(t) = y(t)$ otherwise. Then B attains y', a.e. agent in B finds y' at least as good as x and a non-null group of agents in B strictly prefers y' to x. We conclude that (B, y') is the desired objection.

For the last part of the statement, observe that if y has the ETP on B then we can apply García-Cutrín and Hervés-Beloso (1993, Lemma, page 580) to y instead of x and find that $y'(t) \succcurlyeq_t y(t)$ for a.e. $t \in B$. ■

Next lemma states that any group of agents within an objection can propose a redistribution of the resources in which they appropriate all the gains from the trades.

Lemma A.2 *Let $x \in \mathcal{M}$, $(B, y) \in Ob(x)$ and let $B' \subseteq B$ be non-null. Then there exists a $\tilde{y}$ such that:*

1. *$(B, \tilde{y}) \in Ob(x)$;*
2. *$\tilde{y}(t) \sim_t x(t)$ for a.e. $t \in B \setminus B'$;*
3. *$\tilde{y}(t) \succ_t y(t)$ for a.e. $t \in B'$ whenever $\lambda(\{t \in B \setminus B' : y(t) \succ_t x(t)\}) > 0$.*

In addition, if y is constant on B' then $\tilde{y}$ can be taken constant on B' too.

Proof. The idea of the proof is to reduce gradually each of the bundles that y assigns to agents in $B \setminus B'$ until we obtain an allocation $y_\infty \leqslant y$ that a.e. $t \in B \setminus B'$ finds equivalent to x. This produces a surplus vector $\tilde{v} = \int_{B \setminus B'} (y - y_\infty) \, d\lambda$, which is strictly positive when $\lambda(\{t \in B \setminus B' : y(t) \succ_t x(t)\}) > 0$. We can then define $\tilde{y}$ as the allocation that assigns y_∞ to all $t \in B \setminus B'$ and redistributes $\tilde{v}$ among the agents in B'. In formulas:

$$\tilde{y}(t) = \begin{cases} y(t) + \frac{1}{\lambda(B')} \tilde{v}, & \text{if } t \in B', \\ y_\infty(t), & \text{otherwise.} \end{cases}$$

Clearly, if y is constant on B' so is $\tilde{y}$. To see that $\tilde{y}$ is the desired allocation observe that B attains $\tilde{y}$, that $\tilde{y}(t) \sim_t x(t)$ for a.e. $t \in B \setminus B'$ and that $\tilde{y}(t) \succcurlyeq_t y(t)$ for a.e. $t \in B'$, with a strict preference if $\lambda(\{t \in B \setminus B' : y(t) \succ_t x(t)\}) > 0$. This last point also implies that a non-null subset of B' finds $\tilde{y}$ strictly better than x, and so $(B, \tilde{y})$ is an objection to x.

We only have to prove that such y_∞ exists. Set $B_1 = B$ and $y_1 = y$. For every $n \geqslant 1$, define recursively a set B_n and an allocation y_n as follows:

$$B_{n+1} = \left\{t \in B \setminus B' : y_n(t) - 2^{-n}y(t) \succcurlyeq_t x(t)\right\},$$

$$y_{n+1}(t) = \begin{cases} y_n(t) - 2^{-n}y(t), & \text{if } t \in B_{n+1}, \\ y_n(t), & \text{otherwise.} \end{cases}$$

The sequence of the y_n's converges pointwise to an allocation y_∞ that a.e. $t \in B$ finds at least as good as x by the continuity of preferences. For a.e. $t \in B \setminus B'$ one has $y(t) \geqslant y_\infty(t)$, with a strict inequality when $y(t) \succ_t x(t)$, and so $\int_{B \setminus B'}(y - y_\infty)\, d\lambda > 0$ if and only if $\lambda\left(\{t \in B \setminus B' : y(t) \succ_t x(t)\}\right) > 0$. We claim that $y_\infty(t) \sim_t x(t)$ for a.e $t \in B \setminus B'$.

Assume by contradiction that $y_\infty(t) \succ_t x(t)$ for some $t \in B \setminus B'$. By the monotonicity and continuity of preferences, $y_\infty(t) - 2^{-n}y(t) \succ_t x(t)$ for some n. This implies that $t \in B_k$ for all $k > n$, because $y_k(t) - 2^{-k}y(t) \geqslant y_\infty(t) - 2^{-n}y(t)$. Let m be the largest index such that $t \notin B_m$. Then $y_{m-1}(t) = y_m(t)$ and $y_{k+1}(t) = y_k(t) - 2^k y(t)$ for every $k \geqslant m$. These equations together give:

$$y_\infty(t) = y_m(t) - \sum_{k=m}^{\infty} 2^{-k}y(t) = y_{m-1}(t) - 2^{-(m-1)}y(t).$$

Having assumed that $y_\infty(t) \succ_t x(t)$, we conclude that $y_{m-1}(t) - 2^{-(m-1)}y(t) \succ_t x(t)$ in contradiction with the fact that $t \notin B_m$. ∎

We now have all the ingredients to prove Theorem 3.8.

Theorem A.3 (Theorem 3.8) *Let x be a non-core allocation with the ETP. Then there exists an equitable objection $(\tilde{B}, \tilde{y})$ to x. Furthermore, $\tilde{y}$ can be chosen so that $\tilde{y}(t) = \tilde{y}(s)$ whenever $t, s \in B$ are agents of the same type with convex preferences.*

Proof. Let $(B, y) \in Ob(y)$. If (B, y) is equitable, then we may apply Lemma A.1 to (B, y) and find a y' with the ETP on B such that $(B, y') \in Ob(x)$ and that $y'(t) = y'(s)$ for every agent $s, t \in B$ with convex preferences. If (B, y) is not equitable, we show that we can modify (B, y) into an equitable objection to x.

If $B \subseteq T_0$, then the restricted economy $\mathcal{E}|_B$, the one that considers only the agents in B, is atomless. If y, thought as an allocation in $\mathcal{E}|_B$, is a competitive allocation then y has the ETP on B and so (B, y) is equitable. Otherwise, Mas-Colell (1985, Proposition 7.3.2(ii)) implies that there exists a $(\tilde{B}, \tilde{y}) \in Ob(y)$ such that $\tilde{B} \subseteq B$ and $\tilde{y}$ is a competitive allocation in the economy $\mathcal{E}|_{\tilde{B}}$. It follows that $(\tilde{B}, \tilde{y})$ is an objection to x too and so, having $\tilde{y}$ the ETP on $\tilde{B}$, it is equitable.

Suppose now that B contains an atom A, and let B' be the set of agents in B of the same type as A. Lemma A.1 ensures that we can replace (B, y) with an objection (B, y') such that y' is constant on B'. An application of Lemma A.2 to (B, y') gives that there is a $(B, \tilde{y}) \in Ob(x)$ such that $\tilde{y}(t) \sim x(t)$ for all $t \notin B'$ and that $\tilde{y}(t)$ is constant on B'. Since said $\tilde{y}$ has the ETP on B', we conclude that $(B, \tilde{y})$ is an equitable objection to x. ∎

A.2 Proofs of the results in Subsection 3.3

Proposition A.4 (Proposition 3.11) *Let $x \in \widetilde{\mathcal{M}}$. Then every weakly-Walrasian objection to x is equitable.*

Proof. Let $(B, y) \in Ob(x)$ be weakly-Walrasian. Since x has the ETP, (B, y) is equitable if and only if y has the ETP on B.

For a.e. $t, s \in B$ of the same type, we have $y(t) \in \xi(t,p)$ and $y(s) \in \xi(s,p)$. In particular, having t and s identical endowments, $y(s) \in \beta(t,p)$, and so $y(t) \succcurlyeq_t y(s)$. We conclude that y has the ETP on B and so, having x the ETP, (B,y) is equitable. ■

Lemma A.5 (Lemma 3.12) *Let $x \in \mathcal{M}$ and let $(B,y) \in Ob(x)$ be weakly-Walrasian at the price p. If $(C,z) \in Cob_e^x(B,y)$, then:*

1. *$z(t) \in \xi(p,t)$ for a.e. $t \in C$;*
2. *$\lambda(B \cap C) = 0$.*

Proof. We first show that for a.e. $t \in C$ we have $z(t) \succcurlyeq_t \xi(p,t)$, with a strict preference if $t \in B$. To this end, being t indifferent between all the bundles in $\xi(t,p)$, it is enough to show that $z(t) \succcurlyeq_t v$ (resp. $z(t) \succ_t v$) for some $v \in \xi(t,p)$.

If $t \in C \cap B$, $z(t) \succ_t \xi(t,p)$ follows from the facts that $z(t) \succ_t y(t)$ by the definition of counter-objections, and that $y(t) \in \xi(t,p)$ by property $(WO1)$ of weakly-Walrasian objections. If $t \in C \setminus B$ we have two possibilities: if the set $B_t = \{s \in B : t \sim s\}$ is null, then $x(t) \succcurlyeq_t \xi(t,p)$ by property $(WO2)$, and $z(t) \succ_t x(t)$ by the properties of counter-objections. Otherwise, if $\lambda(B_t) > 0$, for a.e. $s \in B_t$ we have that $y(s) \in \xi(s,p)$ by $(WO1)$ and that $z(t) \succcurlyeq_t y(s)$ by property $(EC2)$ of equitable counter-objections. But $\xi(s,p) = \xi(t,p)$ because t and s are of the same type, and so $z(t) \succcurlyeq_t \xi(t,p)$.

To prove point (2), let us observe that:

$$\int_{C\cap B} p\cdot(z-\omega)\,d\lambda + \int_{C\setminus B} p\cdot(z-\omega)\,d\lambda = p\cdot\left(\int_C (z-\omega)\,d\lambda\right) \leqslant 0 \tag{3}$$

where the last inequality follows from the fact that (C,z) is a counter-objection (and so $\int_C (z-\omega)\,d\lambda \leqslant 0$) and that $p \gg 0$. On the other hand, the argument above gives that $p\cdot(z(t)-\omega(t)) \geqslant 0$ for a.e. $t \in C$, with a strict inequality for a.e. $t \in C \cap B$. This means that:

$$\int_{C\cap B} p\cdot(z-\omega)\,d\lambda + \int_{C\setminus B} p\cdot(z-\omega)\,d\lambda \geqslant 0. \tag{4}$$

Equations (3) and (4) give that $p\cdot(z(t)-\omega(t)) = 0$ for a.e. $t \in C$, from which we conclude that $\lambda(B \cap C) = 0$ and that $z(t) \in \xi(t,p)$ for a.e. $t \in C$. ■

We are now ready to prove Proposition 3.13. The idea of the proof is the following: we use Zorn's Lemma to find the largest family $\{(C_j, z_j)\}_j$ of pairwise disjoint, equitable counter-objections to a weakly-Walrasian $(B,y) \in Ob(x)$. Such a family must be at most countable, and point (2) in Lemma 3.12 gives that each one of the C_j's is disjoint from (B,y). Therefore, we can "sew" (B,y) with all the (C_j, z_j) into a large objection $(\tilde{B}, \tilde{y})$ and we show that it is itself a weakly-Walrasian objection to x. To conclude the proof, we argue that any equitable counter-objection (C,z) to $(\tilde{B}, \tilde{y})$ is an equitable counter-objection to (B,y) that is disjoint from all the C_j's, thus the existence of such a (C,z) violates the maximality of the family $\{(C_j, z_j)\}_j$.

Proposition A.6 (Proposition 3.13) *If $x \in \widetilde{\mathcal{M}}$ is blocked with a weakly-Walrasian objection then there exists an e-justified objection to it, and hence $x \notin BS_e$.*

Proof. Let $x \in \widetilde{\mathcal{M}}$ and (B,y) be a weakly-Walrasian objection to x. Let $\mathcal{F}$ be the set of all coalitions that can raise equitable counter-objections to (B,y). If $\mathcal{F} = \emptyset$ then (B,y) is e-justified. Assume $\mathcal{F} \neq \emptyset$ and let $\mathfrak{A}$ be the class of antichains[14] in $\mathcal{F}$ ordered by inclusion.

[14] Let $(\mathcal{S}, \subseteq)$ be a poset, that is, a partially ordered set. Two elements A and B of $\mathcal{S}$ are called incompatible if neither $A \subseteq B$ nor $B \subseteq A$, that is if there is no order relation between them. An antichain of $\mathcal{S}$ is a subset $\mathcal{S}'$ of $\mathcal{S}$ in which each pair of different elements is incomparable.

For any totally ordered subset $(\mathcal{A}_j)_j$ in $\mathfrak{A}$, the union $\bigcup_j \mathcal{A}_j$ is still an antichain, and hence an upperbound for $(\mathcal{A}_j)_j$ in $\mathfrak{A}$. From Zorn's Lemma we conclude that there exists a maximal antichain $\mathcal{A}'$ in $\mathcal{F}$.

The family $\mathcal{A}'$ is formed by disjoint coalitions of positive measure and so it must be at most countable. Enumerate the elements of $\mathcal{A}'$ as a sequence $(C_n)_{n\in\mathbb{N}}$. By construction, each C_n is in $\mathcal{F}$ and so there exists an allocation z_n such that (C_n, z_n) is an equitable counter-objection to (B, y). Let $\tilde{B} := B \cup (\bigcup_n C_n)$ and let $\tilde{y}$ be an allocation that assigns $y(t)$ to each $t \in B$ and $z_n(t)$ to each $t \in C_n$. The allocation $\tilde{y}$ is well defined because $\lambda(C_n \cap B) = 0$ for each n by Lemma 3.12, and it satisfies $\tilde{y}(t) \in \xi(p, t)$ for a.e. $t \in B$. We claim that $(\tilde{B}, \tilde{y})$ is a e-justified, weakly-Walrasian objection.

First we show that $(\tilde{B}, \tilde{y}) \in Ob(x)$. To prove that $\tilde{B}$ attains $\tilde{y}$ notice that:

$$\int_{\tilde{B}} (\tilde{y} - \omega)\, d\lambda = \int_B (y - \omega)\, d\lambda + \sum_n \int_{C_n} (z_n - \omega)\, d\lambda \leqslant 0.$$

To prove that $\tilde{y}(t) \succcurlyeq_t x(t)$ for a.e. $t \in \tilde{B}$ recall that either $t \in C_n$ for some n, or $t \in B$. In the first case, $\tilde{y}(t) = z_n(t)$ and so $z_n(t) \succ_t x(t)$ because $(C_n, z_n) \in Cob^x(B, y)$; in the latter case $\tilde{y}(t) \succcurlyeq x(t)$ because $(B, y) \in Ob(x)$. Last, observe that $\{t \in \tilde{B} : \tilde{y}(t) \succ_t x(t)\}$ contains $\{t \in B : y(t) \succ_t x(t)\}$, and that this is a non-null set because $(B, y) \in Ob(x)$.

We show that $(\tilde{B}, \tilde{y})$ is weakly-Walrasian at the price p. To prove $(WO1)$, notice that for a.e. $t \in \tilde{B}$ either $t \in B$, or $t \in C_n$ for some n. In the first case, $\tilde{y}(t) = y(t)$ and $y(t) \in \xi(t, p)$ because (B, y) is weakly-Walrasian at p; in the second case $\tilde{y}(t) = z_n(t)$ and $z_n(t) \in \xi(t, p)$ because $(C_n, z_n) \in Cob_e^x(B, y)$ and so Lemma 3.12 applies. To prove $(WO2)$ observe that a.e. $t \notin \tilde{B}$ such that $\lambda(\{s \in \tilde{B} : s \sim t\}) = 0$ is also such that $\lambda(\{s \in B : s \sim t\}) = 0$. Being (B, y) is weakly-Walrasian at the price p, it must be that $x(t) \succcurlyeq_t \xi(t, p)$.

Last, we prove that $(\tilde{B}, \tilde{y})$ is e-justified. Suppose by contradiction that (C, z) is an equitable counter-objection to $(\tilde{B}, \tilde{y})$. Having proved that $(\tilde{B}, \tilde{y})$ is a weakly-Walrasian objection to x, Lemma 3.12 gives that $\lambda(C \cap \tilde{B}) = 0$. But then C is a coalition in $\mathcal{F}'$ that is disjoint from any C_n, violating the maximality of the family $(C_n)_n$. ∎

A.3 Proofs of the results in Subsection 4.1

Proposition A.7 (Proposition 4.3) *Let $T = T_0$ and assume that Assumption 4.2 holds. Suppose that $x \in \mathcal{M}$ and let $(B, y) \in Ob(x)$. Then, for every $\delta \in (0, 1)$ there exists $B^\delta \subseteq B$ such that:*

(i) $\lambda(B^\delta \cap K_n) = \delta\lambda(B \cap K_n)$ *for every n;*

(ii) $(B^\delta, y) \in Ob(x)$ *and:*

- *(iia) if (B, y) is equitable, then (B^δ, y) is equitable too;*
- *(iib) if (B, y) is weakly-Walrasian, then (B^δ, y) is weakly-Walrasian too.*

Proof. For every n, consider the (possibly null) set $B_n = K_n$ consisting of the agents of type n in B, then divide B_n in the set $B_n^{\succ} = \{t \in B_n : y(t) \succ_t x(t)\}$ and $B_n^{\sim} = \{t \in B_n : y(t) \sim_t x(t)\}$. By construction, B is the disjoint union $(\bigcup_n B_n^{\succ}) \cup (\bigcup_n B_n^{\sim})$ and $\lambda(B) = \sum_n [\lambda(B_n^{\succ}) + \lambda(B_n^{\sim})]$. In particular, there must be a $\bar{n}$ for which $B_{\bar{n}}^{\succ}$ is not null, otherwise a.e. $t \in B$ would be indifferent between y and x and (B, y) would not be an objection to x.

Consider now the measure $\eta \colon \Sigma \to \mathbb{R}^{m+1}$ that assigns to each $S \in \Sigma$ the vector:

$$\eta(S) = \left(\int_S (y - \omega)\, d\lambda, \lambda(S) \right).$$

Having assumed that $T = T_0$, η is an atomless measure and Lyapunov's Theorem applies. In particular, for every $\delta \in (0,1)$ we can find $S_n^{\succ} \subseteq B_n^{\succ}$ and $S_n^{\sim} \subseteq B_n^{\sim}$ such that $\eta(S_n^{\succ}) = \delta\eta(B_n^{\succ})$ and $\eta(S_n^{\sim}) = \delta\eta(B_n^{\sim})$. Notice that all the $S_n^{\succ}$'s and the $S_n^{\sim}$'s are pairwise disjoint, and that $S_{\bar{n}}^{\succ}$ is non-null.

For every n, let us put $S_n = S_n^{\succ} \cup S_n^{\sim}$. This means that:

$$\lambda(S_n) = \lambda\left(S_n^{\succ}\right) + \lambda(S_n^{\sim}) = \delta\lambda\left(B_n^{\succ}\right) + \delta\lambda\left(B_n^{\sim}\right) = \delta\lambda(B_n) \tag{5}$$

and that:

$$\begin{aligned}\int_{S_n}(y-\omega)d\lambda &= \int_{S_n^{\succ}}(y-\omega)d\lambda + \int_{S_n^{\sim}}(y-\omega)d\lambda = \\ &= \delta\int_{B_n^{\succ}}(y-\omega)d\lambda + \delta\int_{B_n^{\sim}}(y-\omega)d\lambda = \delta\int_{B_n}(y-\omega)d\lambda.\end{aligned} \tag{6}$$

We claim that $B^\delta = \bigcup_n S_n$ is the desired coalition. Condition (i) follows from Equation (5), given that $B_n = B \cap K_n$ for every n. We focus on (ii).

First we show that $(B^\delta, y) \in Ob(x)$. To prove that B^δ attains y, use Equation (6) to write:

$$\int_{B^\delta}(y-\omega)\,d\lambda = \sum_n \int_{S_n}(y-\omega)\,d\lambda = \sum_n \delta\int_{B_n}(y-\omega)\,d\lambda = \delta\int_B(y-\omega)\,d\lambda.$$

Since B attains y, the last term of the equation is smaller than or equal to 0, and so B^δ attains y too. The fact that $y(t) \succcurlyeq_t x(t)$ for a.e. $t \in B^\delta$ follows from the inclusion $B^\delta \subseteq B$ and the fact that (B, y) is itself an objection to x. Last, observe that $S_{\bar{n}}$ is a non-null subset of B^δ with the property that every $t \in S_{\bar{n}}$ strictly prefers $y(t)$ to $x(t)$.

To prove (iia) suppose that (B, y) is equitable. Since $B^\delta \subseteq B$, for a.e. $t, s \in B^\delta$ of the same type we have that $y(t) \succcurlyeq_t y(s)$ because $s, t \in B$ and (B, y) is equitable. Thus, (B^δ, y) satisfies $(EO1)$. On the other hand, for a.e. $t \in B^\delta$ and $s \notin B^\delta$ of the same type we have two possibilities: if $s \notin B$ then $(EO2)$ applied to (B, y) gives that $y(t) \succcurlyeq_t x(s)$; if $t \in B \setminus B^\delta$ then $y(t) \succcurlyeq_t y(s)$ because (B, y) is equitable, and $y(s) \succcurlyeq_t x(s)$ because (B, y) is an objection to x and $\succcurlyeq_s = \succcurlyeq_t$. We conclude that (B^δ, y) satisfies also condition $(EO2)$ and so it is equitable.

Last, to prove (iib) suppose that (B, y) is weakly-Walrasian at a price p. Being $B^\delta \subseteq B$ we have that $y(t) \in \xi(t, p)$ for a.e. $t \in B^\delta$, and so $(WO1)$ is met. To show that $(WO2)$ is also satisfied observe that $\lambda(B \cap K_n) = \frac{1}{\delta}\lambda(S \cap K_n)$ for every n, meaning that every type of agent that is represented in B is also represented in B^δ. This implies that for a.e. $t \notin B^\delta$, if $\lambda\left(\left\{s \in B^\delta : s \sim t\right\}\right) = 0$ then $\lambda\left(\{s \in B : s \sim t\}\right) = 0$ and so $x(t) \succcurlyeq_t \xi(t, p)$ because (B, y) satisfies $(WO2)$. ■

A.4 Proofs of the results in Subsection 4.2

Proposition A.8 (Proposition 4.6) *Assumption 4.5 is satisfied if one of the following two conditions holds.*

1. *There are infinitely many atoms and they are all of the same type.*
2. *There are finitely many types of atoms, and for each atom A there is a coalition $S_A \subseteq T_0$ such that every $t \in S_A$ is of the same type of A.*

Proof. Fix a sequence $(\mu_n)_n$ in $(0,1)$.

If (1) is satisfied, there are infinitely many atoms $(A_i)_i$ and they are all subsets of some $K_{\bar{n}}$. This implies that $\lambda(A_i) \to 0$, and so we can choose an atom A such that $\lambda(A) \leqslant \mu_{\bar{n}}\lambda(K_{\bar{n}})$.

Let $\delta = \lambda(A)/\mu_{\bar{n}}\lambda(K_{\bar{n}})$, then put $F_{\bar{n}} = A$. For every $n \neq \bar{n}$, the set K_n is atomless, and so there exists $F_n \subseteq K_n$ such that $\lambda(F_n) = \delta\mu_n\lambda(K_n)$. But then $(F_n)_n$ is the desired sequence.

Suppose now that (2) holds. This means that there are finitely many $n_1, \ldots, n_j$ with the property that $T_1 \subseteq \bigcup_{i \leqslant j} K_{n_i}$ and that the set $S_i = K_{n_1} \cap T_0$ is non-null for every $i = 1, \ldots, j$. Put $\delta = \min_{i \leqslant j} \frac{\lambda(S_i)}{\mu_i \lambda(K_{n_i})}$ if this is smaller than or equal to 1, otherwise put $\delta = 1$. We can define the sequence $(F_n)_n$ as follows: if $n \notin \{n_1, \ldots, n_k\}$ then K_n is atomless and so there exists a $F_n \subseteq K_n$ such that $\lambda(F_n) = \delta\mu_n\lambda(K_n)$; otherwise, if $n = n_i$ for some $i = 1, \ldots, j$, S_i is atomless and such that $\lambda(S_i) \geqslant \delta\mu_{n_i}\lambda(K_{n_i})$. We can then take $F_{n_i} \subseteq S_i$ such that $\lambda(F_{n_i}) = \delta\mu_{n_i}\lambda(K_{n_i})$. ∎

The splitted economy associated with the mixed market For the proof of Proposition 4.8 we associate to the mixed market an atomless economy in the manner of Greenberg and Shitovitz (1986). To do this, we replace each atom $A \in \Sigma$ with a coalition A^* with the same measure, formed by a continuum of negligible agents of the same type as A.

Formally, we build the atomless measure space of agents $(T^*, \Sigma^*, \lambda^*)$ as follows. For every atom $A \in \Sigma$ let $(A^*, \Sigma_{A^*}, \lambda_{A^*})$ be an atomless measure space with $\lambda_{A^*}(A^*) = \lambda(A)$. Then define the set T as the union $T_0 \cup (\bigcup_A A^*)$, the algebra Σ^* as the product of Σ_{T_0} and each of the Σ_{A^*}'s, and λ^* as the product of λ restricted to T_0, and each of the λ_{A^*}'s. Being every $t \in T_0$ an agent in the mixed market as well as in the original economy, she is already given a preference relation and an endowment bundle. For $t \in A^*$, let $\succcurlyeq_t$ and $\omega(t)$ coincide with $\succcurlyeq_s$ and $\omega(s)$ for any $s \in A$ (recall that all agents in A have the same preferences and endowments).

The atomless economy we defined is called the *splitted economy* associated with the mixed market. It meets all the main assumptions of the model, and satisfies Assumption 4.2 when the original mixed market does. Let $\mathcal{M}^*$ be the set of allocations in the splitted economy, and call $\mathcal{W}^*$ and BS_e^* the corresponding set of competitive allocations and Equitable Bargaining set.

There is a natural correspondence between allocations in the mixed market and in the splitted economy. Precisely, each $x \in \mathcal{M}$ defines an allocation $x^* \in \mathcal{M}^*$ by:

$$x^*(t) = \begin{cases} x(t) & if\ t \in T_0, \\ x(A) & if\ t \in A^*, \quad \text{for some atom } A \in \Sigma \end{cases}$$

where $x(A)$ is the bundle that x assigns to any agent in A (being x measurable and A an atom, this must be the same for all $t \in A$). Conversely, to each $f \in \mathcal{M}^*$ we can associate a $x_f \in \mathcal{M}$ defined by:

$$x_f(t) = \begin{cases} f(t) & if\ t \in T_0, \\ \frac{1}{\lambda^*(A^*)} \int_{A^*} f(t) d\lambda^* & if\ t \in A, \quad \text{for some atom } A \in \Sigma. \end{cases}$$

Clearly, $x_{x^*} = x$. Greenberg and Shitovitz (1986) shows that x is competitive or has the ETP if and only if x^* is competitive or has the ETP respectively. On a similar line, if f is competitive or has the ETP then so does x_f.

To every coalition $S \in \Sigma$ we associate a *splitted coalition* $S^* \in \Sigma^*$ defined by $S^* = \left(\bigcup_{A \in S} A^*\right) \cup (S \cap T_0)$. This relationship can be used to transfer objections from the mixed market to the splitted economy, and viceversa.

Lemma A.9 *Let $x, y \in \mathcal{M}$ and $B \in \Sigma$. Then $(B, y) \in Ob(x)$ if and only if $(B^*, y^*) \in Ob(x^*)$. Furthermore:*

(i) (B, y) is Walrasian at a price p if and only if (B^*, y^*) is Walrasian at p;

(ii) (B, y) *weakly-Walrasian at a price* p *if and only if* (B^*, y^*) *is weakly-Walrasian at* p.

Proof. By construction, B attains y if and only if B^* attains y^*. Furthermore, for a generic $S \in \Sigma$, every $t \in S$ finds $y(t) \succ_t x(t)$ (resp. $y(t) \not\succ_t x(t)$) if and only if every $t \in S^*$ finds $y^*(t) \succ_t x^*(t)$ (resp. $y^*(t) \not\succ_t x^*(t)$). Thus, a.e. agents in B weakly prefer y to x (and some strictly prefer) if and only if a.e. agents in B^* weakly prefer y^* to x^* (and some strictly prefer). We conclude that $(B, y) \in Ob(x)$ if and only if $(B^*, y^*) \in Ob(x^*)$.

To prove the other two claims it is enough to observe that, if $S = \{t \in T : y(t) \in \xi(t, p)\}$ then $S^* = \{t \in T^* : y^*(t) \in \xi(t, p)\}$. ■

According to the lemma above, if there exists a Walrasian objection to a $x \in \mathcal{M}$ this induces a Walrasian objection to x^* in the splitted economy, and so $x^* \notin BS_e^*$ by Proposition 3.13. The converse may not be true, for there may be Walrasian objections to x^* in the splitted economy that are not induced by any objection in the original mixed market. This is because, given $S \in \Sigma^*$ and $f \in \mathcal{M}^*$ such that $(S, f) \in Ob(x^*)$, it is possible that no $B \in \Sigma$ is such that $B^* = S$. Next lemma shows that, under our additional Assumptions, it is possible to obtain a partial converse to this argument: if there exists a Walrasian objection in the splitted economy, then there exists a weakly-Walrasian one in the original mixed market.

Proposition A.10 *Under Assumptions 4.2 and 4.5, let* $x \in \widetilde{\mathcal{M}}$. *If there exists a Walrasian objection* (S, f) *to* x^*, *then there exists a weakly-Walrasian objection* (B, y) *to* x.

Proof. Let (S, f) be a Walrasian objection to x^* in the splitted economy and let $\mathcal{K}_1(S)$ be the set of the types of agents K such that $\lambda^*(S \cap K^*) > 0$ and that each $t \in K^*$ has convex preferences. Without loss of generality, we may assume that f is constant on each K^* with $K \in \mathcal{K}_1(S)$, i.e., that there is a $f_K \in \mathbb{R}^m_+$ such that $f(t) = f_K$ for every $t \in K^*$ (otherwise, replace f with the allocation f' defined as in Lemma A.1). This implies that $f = y^*$ for the allocation $y \in \mathcal{M}$ defined by $y(t) = f_K$ if $t \in K^*$ for some $K \in \mathcal{K}_1(S)$ and $y(t) = f(t)$ otherwise.

By Assumption 4.2, we can partition T into a sequence of coalitions $(K_n)_n$, where each K_n corresponds to different type of agents. We first prove the claim under the following additional assumption:

> **Auxiliary assumption:** For every n such that $K_n \in \mathcal{K}_1(S)$ there is a $B_n \subseteq K_n$ such that $\lambda(B_n) = \lambda^*(S \cap K_n^*)$.

Let $S_0 = S \setminus \bigcup\{K_n^* : K_n \in \mathcal{K}_1(S)\}$. As $S_0 \subseteq T_0$ up to null sets, S_0 is either null or it coincides with a coalition B_0 of Σ up to null sets. Let $B = B_0 \cup \{B_n : K_n \in \mathcal{K}_1(S)\}$. We claim that (B, y) is a weakly-Walrasian objection to x.

First observe that $B^* \in \Sigma$ and that it attains y. This is because:

$$\begin{aligned}
\int_B (y - \omega)\, d\lambda &= \int_{B_0} (y - \omega)\, d\lambda + \sum_{K_n \in \mathcal{K}_1(S)} \int_{B_n} (y - \omega)\, d\lambda = \\
&= \int_{S_0} (f - \omega^*)\, d\lambda^* + \sum_{K_n \in \mathcal{K}_1(S)} \int_{B_n} (f_{K_n} - \omega_{K_n})\, d\lambda = \\
&= \int_{S_0} (f - \omega^*)\, d\lambda^* + \sum_{K_n \in \mathcal{K}_1(S)} \lambda(B_n)(f_{K_n} - \omega_{K_n}) = \\
&= \int_{S_0} (f - \omega^*)\, d\lambda^* + \sum_{K_n \in \mathcal{K}_1(S)} \lambda^*(S \cap K_n^*)(f_{K_n} - \omega_{K_n}) = \int_S (f - \omega^*)\, d\lambda^*
\end{aligned}$$

and the latter is smaller than or equal to 0 as S attains f. Second, observe that each agent in the splitted coalition B^* is either an agent in the original objecting coalition S, or it is a copy of some agent in S. Therefore, (B^*, f) is a weakly-Walrasian objection to x^* for the same reasons that (S, f) is. But then (B, y^*) is a weakly-Walrasian objection to x in the original economy.

Suppose now that the Auxiliary assumption does not hold. By Assumption 4.2, we can partition T into a sequence of coalitions $(K_n)_n$, where each K_n corresponds to different type of agents. For every n, put $\mu_n = \frac{\lambda^*(S \cap K_n^*)}{\lambda^*(K_n^*)}$ if this is positive, otherwise put $\mu_n = \lambda(K_n)$. This way, $(\mu_n)_n$ is a sequence in $(0, 1)$ to which we can apply Assumption 4.5 to find a $\delta \in (0, 1)$ and a sequence of coalitions $(F_n)_n \in \Sigma$ with the properties that $F_n \subseteq K_n$ and $\lambda(F_n) = \delta \mu_n \lambda(K_n)$ for every n. It follows that:

$$\lambda(F_n) = \delta \mu_n \lambda(K_n) = \delta \mu_n \lambda^*(K_n^*) = \delta \lambda^*(S \cap K_n^*), \quad \text{for } n \in \mathbb{N}. \tag{7}$$

Focus on the splitted economy, which is atomless by construction. As (S, f) is a Walrasian objection to x^*, by Proposition 4.3 there exists a $S^\delta \subseteq S$ such that: (i) $\lambda^*(S^\delta \cap K_n^*) = \delta \lambda^*(S \cap K_n^*)$, (iib) (S^δ, f) is a weakly-Walrasian objection to x^*. To conclude the proof we show that S^δ satisfies the Auxiliary assumption.

For every n, define:

$$B_n = \begin{cases} F_n, & \text{if } K_n \in \mathcal{K}_1(S), \\ S^\delta \cap K_n^*, & \text{otherwise.} \end{cases}$$

We make a few observations. First notice that $B_n \in \Sigma$ for every n (this is because $F_n \in \Sigma$ by definition and $K_n^* \notin \mathcal{K}_1(S)$ implies that $K_n^* \cap S^\delta$ is either null or such that $K_n^* \subseteq T_0$). The second one is that, for every $K_n \in \mathcal{K}_1(S)$, $B_n \subseteq K_n$ and $\lambda(B_n) = \lambda^*(S^\delta \cap K_n)$. In other words, S^δ satisfies the Auxiliary assumption and so the argument above applies. ∎

Proposition A.11 (Proposition 4.8) *Under Assumptions 4.2 and 4.5, if $x \in \widetilde{\mathcal{M}}$ is a non-competitive feasible allocation, there exists a weakly-Walrasian objection to it.*

Proof. Let $x \in \widetilde{M}$ be a feasible, non-competitive allocation and let $x^* \in \mathcal{M}^*$ be the corresponding allocation in the splitted economy. Then x^* is feasible, non-competitive and has the ETP. By Mas-Colell (1989, Proposition 2) there exists a $S \in \Sigma^*$ and a $f \in \mathcal{M}^*$ such that (S, f) is a Walrasian objection to x^* in the splitted economy. Apply Proposition A.10 to (S, f) to find a $B \in \Sigma$ and $y \in \mathcal{M}$ such that (B^*, y^*) is a weakly-Walrasian objection to x^* in the splitted economy, then observe that (B, y) is a weakly-Walrasian objection to x in the original mixed market, by Proposition A.9. We conclude that $x \notin BS_e$, by Proposition 3.13. ∎

A.5 Proofs of the results in Subsection 4.3

Proposition A.12 (Proposition 4.11) *Let K^* be a type of leaders formed by finitely many atoms. Then for every $x \in \widetilde{\mathcal{M}} \setminus \mathcal{C}$ there exists $(B^*, y^*) \in Ob_e(x)$ with the property that, for every $(B, y) \in Ob_e(x)$:*

$$y^*(t) \succcurlyeq_t y(s), \quad \textit{for every } t \in K^* \cap B^* \textit{ and } s \in K^* \cap B.$$

Proof. Consider the binary relation $\geqslant$ on $Ob_e(x)$ defined by:

$$(B_1, y_1) \geqslant (B_2, y_2) \iff y_1(t) \succcurlyeq_t y_2(s) \text{ for every } t \in K^* \cap B_1 \text{ and } s \in K^* \cap B_2.$$

We claim that $(Ob_e(x), \geqslant)$ is a totally preordered set, separable in the sense of Debreu.[15] First notice that $Ob_e(x)$ is non-empty, for x is a non-core allocation with the ETP, and

[15] A preordered set $(Z, \leqslant)$ is separable in the sense of Debreu if there is a countable $Q \subset Z$ with the following property: for every $x, y \in Z$ with $z < y$ there is a $q \in Q$ with $z \leqslant q \leqslant y$.

so Theorem 3.8 applies. Then observe that for every two (B_1, y_1), $(B_2, y_2) \in Ob_e(x)$, the assignments y_i's have the ETP and so either all agents in K^* find y_1 at least as good as y_2, or they all prefer y_2 to y_1. Last, the separability of $\geqslant$ follows from that of $\succcurlyeq_{K^*}$. We need to prove that there exists a maximal element in $(Ob_e(x), \geqslant)$.

Let (B_n, y_n) be a sequence in $Ob_e(x)$ that is cofinal in the following sense: for every $(B, y) \in Ob_e(x)$ there exists a $n \in \mathbb{N}$ such that $(B_n, y_n) \geqslant (B, y)$. Said sequence exists because $(Ob_e(x), \geqslant)$ is separable. Without loss of generality, we may assume that there are $\alpha, \beta \in \mathbb{R}^m_+$ such that:

$$\lim \int_{B_n} y_n(t)\, d\lambda = \alpha, \quad \lim \int_{B_n} \omega(t)\, d\lambda = \beta.$$

Since each B_n contains at least an atom of type K^* (and there are only finitely many of them), the $\int_{B_n} \omega(t)\, d\lambda$'s are bounded away from 0, and so $\beta > 0$. For each n, let F_n be the map defined by:

$$F_n(t) = (y_n(t)\chi_{B_n}(t), \omega(t)\chi_{B_n}(t)) \in \mathbb{R}^{2m}.$$

The sequence (F_n) is integrable in its first coordinate, uniformly integrable in its second one, and it is such that $\lim \int F_n\, d\lambda = (\alpha, \beta)$. By Fatou Lemma (see Hildenbrand (1974, Lemma D.3)) there is a subsequence of (F_n) (which we do not re-label) that converges pointwise to an integrable function $F(t) = (f_1(t), f_2(t))$ with the property that $\int f_1\, d\lambda \leqslant \alpha$ and $\int f_2\, d\lambda = \beta$.

Let $B^* = \{t \in T : f_2(t) = \omega(t)\}$ and observe that $t \in B^*$ if and only if $t \in B_n$ for n sufficiently large. This implies that $f_1(t) = \lim y_n(t)$ for a.e. $t \in B^*$. Let y^* be an assignment with the ETP that coincides with f_1 on B^* (such y^* exists because f_1, restricted to B^*, is the pointwise limit of functions with the ETP, and so it assigns equivalent bundles to agents of the same type). We claim that (B^*, y^*) is the desired objection.

We show that $(B^*, y^*) \in Ob(x)$. First observe that B^* is non-null, because $\int_{B^*} \omega\, d\lambda = \int f_2\, d\lambda = \beta > 0$. Second, notice that B^* attains y^* because:

$$\int_{B^*} (y^* - \omega)\ d\lambda = \int (f_1 - f_2)\ d\lambda \leqslant \lim \int_{B_n} (y_n - \omega)\ d\lambda \leqslant 0.$$

Third, recall that $y^*(t) = \lim y_n(t)$ for a.e. $t \in B^*$, and so $y^*(t) \succcurlyeq_t x(t)$ by the continuity of preferences. In particular, when $t \in B^* \cap K^*$ it must be that $y^*(t) \succ_t x(t)$ (because the y_n are increasingly desirable to agents in K^*). This proves that (B^*, y^*) is an objection to x. The fact that (B^*, y^*) is equitable follows from the ETP of y^* and x^*.

Since the sequence of the (B_n, y_n) is cofinal and $(B^*, y^*) \geqslant (B_n, y_n)$, leaders in K^* find (B^*, y^*) at least as good as any other equitable objection to x. ■

Theorem A.13 (Theorem 4.12) *Let there be a type of leaders formed by finitely many atoms. Then BS_e coincides with the equitable core $\widetilde{\mathcal{C}}_e$.*

Proof. $BS_e(\mathcal{E})$ contains all core-allocations with the ETP by Theorem 3.8. To prove the other inclusion, we fix an individually rational $x \notin \mathcal{C}$ with the ETP, and show that $x \notin BS_e$.

Let $(B^*, y^*) \in Ob_e(x)$ be as in Proposition 4.11. We claim that (B^*, y^*) is e-justified. Suppose by contradiction that this was not the case, i.e., that there is a $(C, z) \in Cob^x_e(B^*, y^*)$. Being K^* a type of leaders, $C \cap K^*$ is non-null and so $z(t) \succcurlyeq_t y^*(t)$ for a.e. $t \in K^*$ (because (C, z) is equitable). At the same time, the maximality of (B^*, y^*) gives that $y^*(t) \succcurlyeq_t z(t)$ for every $t \in K^*$. It follows that all leaders of type K^* in C find y^* equivalent to z. Without loss of generality, we may assume that z is constant on $C \cap K^*$ (otherwise apply Lemma A.1).

The coalition C blocks x, which is individually rational, and so it cannot consist only of identical agents with convex preferences. It must be that $\lambda(C \setminus K^*) > 0$.

Notice that (C, z) is an objection to x that assigns identical bundles to agents in $C \cap K^*$. By Lemma A.2, there exists a z^* such that $(C, z^*) \in Ob_e(x)$ and $z^*(t) \succ_t z(t)$ for every $t \in C \cap K^*$. This last part follows from the fact that all $t \in C \setminus K^*$ strictly prefer z to x by definition of counter-objection, and that $\lambda(C \setminus K^*) > 0$ by the argument above, so point (3) in Lemma A.2 applies. This implies that $z^*(t) \succ_t y^*(t)$ for all $t \in K^*$, violating the maximality of (B^*, y^*). ■

Corollary A.14 (Corollary 4.13) *Under the assumptions of Theorem 4.12, suppose that there are at least two atoms in the economy and that they are all of type K^*. Then $\mathcal{W} = \widetilde{\mathcal{C}}_e = BS_e = \mathcal{C}$, and these may be strictly contained in the standard bargaining set.*

Proof. The inclusion $\mathcal{W} \subseteq BS_e$ always hold. For the converse inclusion, notice that $BS_e \subseteq \mathcal{C}$ by Theorem 4.12 and that $\mathcal{C} \subseteq \mathcal{W}$ by Shitovitz (1973, Theorem B). ■

B Appendix: Examples

Section 3 introduced new solution concepts based on equitable restrictions on the allowed objections and counterobjections: the *equitable core* $\widetilde{\mathcal{C}}_e$ and the *equitable bargaining set* BS_e. Subsection 3.2 observes that two series of inclusions hold:

$$\mathcal{W} \subseteq \widetilde{\mathcal{C}}_e \subseteq BS_e, \quad \text{and} \quad \mathcal{W} \subseteq \widetilde{\mathcal{C}}_e \subseteq \mathcal{C} \subseteq BS.$$

Some (or all) of the solution concepts in the listed above may coincide under specific assumptions, but not in general. In this section of the Appendix we provide examples to show that: (1) all of the inclusions in the two series can be strict; and (2) the equitable bargaining set and the Core are, in general, not comparable. For completeness, we present an example for each relation, even though some are trivial or already discussed in the literature. All examples consider finite economies only, but they can be formulated for mixed markets.

The following table anticipates which relations are addressed in every example.

$\mathcal{W} \subsetneq \widetilde{\mathcal{C}}_e$	$\widetilde{\mathcal{C}}_e \subsetneq BS_e$	$\widetilde{\mathcal{C}}_e \subsetneq \mathcal{C}$	$\mathcal{C} \subsetneq BS$	$BS_e \not\subseteq \mathcal{C}$	$\mathcal{C} \not\subseteq BS_e$
Ex. B.1	Ex. B.4	Ex. B.2	Ex. B.3	Ex. B.4	Ex. B.2

The last part of this section includes some considerations on the set $\mathcal{C}_e$ of all allocations that are not blocked with equitable objections, with or without the ETP. It has already been observed that such set may even include allocations that are not individually rational (see page 9), but even considering only individually rational allocations can lead to a pathologically large set. Example B.5 considers a perfectly competitive economy with an atomless measure space of agents, where the different notions of core and bargaining set coincide, and still finds an individually rational allocation in $\mathcal{C}_e$ that is not competitive.

Example B.1 (A non-competitive allocation in the equitable core) *There are two commodities and three agents: a, b and c. Agents are endowed with the same initial bundle $\omega = (2, 2)$ and their preferences are given by the following utility functions:*

$$u_a(x, y) = u_b(x, y) = x + y, \quad u_c(x, y) = 2x + y.$$

So there are two types of agents: type one consists of a and b, type two consists of c only. Consider the allocation x defined by:

$$x_a = \left(\frac{3}{2}, 3\right), \ x_b = \left(\frac{3}{2}, 3\right), \ x_c = (3, 0).$$

It follows that x is a non-competitive allocation with the equal treatment property. Furthermore, computations show that it is not blocked by any coalition, and so it is in the core. By Theorem 3.8, any core allocation with the equal treatment property is in the equitable core.

Example B.2 (A core allocation without the ETP) *There are two commodities and three agents: a, b and c. Agents are endowed with the same initial bundle $\omega = (2, 2)$ and their preferences are given by the following utility functions:*

$$u_a(x, y) = u_b(x, y) = x + y, \quad u_c(x, y) = 2x + y.$$

So there are two types of agents: type one consists of a and b, type two consists of c only. Consider the allocation x defined by:

$$x_a = (3, 2)\,, \; x_b = (0, 4)\,, \; x_c = (3, 0)\,.$$

The allocation x does not satisfy the ETP, as it assigns a a higher utility than b, but there is no coalition that can object to x. We conclude that $x \in \mathcal{C} \setminus \widetilde{\mathcal{C}}_e$, which also implies that $x \in \mathcal{C} \setminus BS_e$.

Example B.3 (An allocation in the standard bargaining set that is not in the core) *There are two commodities and five agents: a, b, c, d and e. All agents are endowed with the same initial bundle $\omega = (2, 2)$, while their preferences are given by the utility functions:*

$$u_a(x, y) = u_b(x, y) = u_c(x, y) = 3x + y, \quad \text{and} \quad u_d(x, y) = u_e(x, y) = x + 3y.$$

Thus, there are only two types of agents: type one is formed by agents a, b and c, while type two is formed by d and e. Consider the allocation x defined by:

$$x_a = x_b = x_c = (3, 0)\,, \quad x_d = x_e = \left(\frac{1}{2}, 5\right).$$

Computations show that x is not in the core, and that the only coalitions that object to it are formed by 2 agents of type one and 1 agent of type two.[16] *In particular, there is always an agent of each type that is left out of the objection.*

Let (B, y) be a generic objection to x and let $j \in B$ be such that $u_j(y_j) > u_j(x_j)$ (such agent j exists by the definition of objection). By continuity, we can reduce y_j and obtain a smaller bundle $\overline{y}_j < y_j$ that is still strictly preferred to x_j by j. Let k be the agent of the same type of j that is left out of B and let $C = (B \setminus \{j\}) \cup \{k\}$ be the coalition obtained by replacing j with k in B, then define z as the allocation that assigns to each $i \in C$ the bundle:

$$z_i = \begin{cases} \overline{y}_j, & \text{if } i = k, \\ y_i + \frac{(y_j - \overline{y}_j)}{2}, & \text{otherwise.} \end{cases}$$

It follows that C attains z, that agent k strictly prefers z to x, and that every $i \in C \setminus \{k\}$ strictly prefers z to y. In other words, (C, z) is a counterobjection to (B, y).

By the generality of (B, y), we conclude that every objection to x is met with a counterobjection, meaning that x is in the bargaining set.

The Example B.3 considers a non-core allocation x with the ETP, and shows that every objection to x is met with a coutneronbjection. The counterobjection it finds, however, are not equitable, as they ground on the assumption that an objecting agent may be replaced with an identical one who is willing to accept a strictly worse bundle (the bundle $\overline{y}_j$ in the

[16] An example of objection to x is the pair (B, y) where $B = \{a, b, d\}$ and $y_a = y_d = (3, 0)$ while $y_d = (0, 6)$.

example). This is not a case: in the economy of Example B.3 every type of agents consists of multiple leaders, and so the equitable bargaining set is a subset of the (standard) core by Theorem 4.12. As x is not in the core, some equitable objection to x is not opposed with equitable counterobjections. In other words, Example B.3 shows that the (standard) bargaining set may strictly contain the equitable bargaining set even when the latter lays entirely in the core of the economy.

The following example builds on Example B.3.

Example B.4 (An allocation in the equitable bargaining set that is not in the core) *There are three commodities and five agents: a, b, c, d and e. Agents' endowments are given by:*

$$\omega_a = \left(\frac{3}{2}, \frac{1}{2}, 2\right), \ \omega_b = \left(\frac{1}{2}, \frac{3}{2}, 2\right), \ \omega_c = (2,2,2), \ \omega_d = \left(\frac{3}{2}, \frac{1}{2}, 2\right), \ \omega_e = \left(\frac{1}{2}, \frac{3}{2}, 2\right).$$

Agents' preferences are given by the utility functions:

$$u_a(x,y,z) = u_b(x,y,z) = u_c(x,y,z) = 3(x+y)+z, \quad \textit{and}$$

$$u_d(x,y,z) = u_e(x,y,z) = (x+y)+3z.$$

As every agent in the economy is of a different type, the core coincides with the equitable core and the (standard) bargaining set with the equitable bargaining set. Notice, however, that agents a, b and c (and agents d and e) differ only in their endowment of the first two commodities, which are perfect substitutes. As all agents own the same total amount of these two commodities, agents a, b and d play interchangeable roles in the bargaining process as if they were of the same type, and so do agents d and e.

The same arguments of Example B.3 apply, and one proves that the map x defined by:

$$x_a = x_b = x_c = \left(\frac{3}{2}, \frac{3}{2}, 0\right), \quad x_d = x_e = \left(\frac{1}{4}, \frac{1}{4}, 5\right)$$

is a non-core allocation in the bargaining set. As there are no differences between equitable and standard core, and between equitable and standard bargaining set, we conclude that $x \in BS_e \setminus \widetilde{\mathcal{C}}_e$.

Example B.5 *There are two commodities and a continuum of agents represented by the Lebesgue unit interval $T = [0,1]$. All agents are identical: their preferences are derived from the utility function $u(x,y) = x^2 + y^2$ and they are all endowed with the bundle $\omega = (1,1)$. It follows that the set of competitive allocations coincides with all notions of core and bargaining set. We claim that there is an individually rational allocation in $\mathcal{C}_e$ that is not competitive (and hence it does not belong to the cores or the bargaining sets).*

Let x be the allocation that assigns to each $t \in T$ the bundle:

$$x(t) = \begin{cases} (3,0), & \textit{if } t \le \frac{1}{3}, \\ \left(0, \frac{3}{2}\right), & \textit{otherwise.} \end{cases}$$

The allocation x is feasible and individually rational, but it does not satisfy the ETP and so it is not competitive. In particular, there is an objection (B,y) to x.

We make two observations. (i) As the coalition B attains y, $\int_B y\, d\lambda \le \int_B \omega\, d\lambda = (\lambda(B), \lambda(B))$ and so $\frac{1}{\lambda(B)} \int_B y\, d\lambda \le (1,1)$. (ii) $\frac{1}{\lambda(B)} \int_B y\, d\lambda \in \overline{co}\,\{y(t) : t \in B\}$ by the Mean *value Theorem for Bochner integrable functions (here $\overline{co}K$ denotes the closed convex hull of the set K). These two observations combined give that:*

$$\frac{1}{\lambda(B)}\int_B y\,d\lambda \in \left\{v \in \mathbb{R}^2 :\ v \leq (1,1)\right\} \cap \overline{co}\,\{y(t) :\ t \in B\} \neq \emptyset. \tag{8}$$

Now, assume by contradiction that (B, y) is equitable. As all agents are of the same type, it must be that $u(y(t)) \geqslant u(3,0) = 9$ for almost every $t \in B$, or there would be some objecting agents that are envious of what their peers in $[0, 1/3]$ receive from x. We can write:

$$\{y(t) :\ y \in B\} \subseteq \left\{(x,y) \in \mathbb{R}^2 :\ x^2 + y^2 \geqslant 9\right\} \subseteq \left\{(x,y) \in \mathbb{R}^2 :\ x + y \geqslant 2.5\right\}.$$

We conclude that $\{y(t) :\ t \in B\}$ is a subset of the closed halfspace $\left\{(x,y) \in \mathbb{R}^2 :\ x + y \geqslant 2.5\right\}$, which in turn is disjoint from $\left\{v \in \mathbb{R}^2 :\ v \leq (1,1)\right\}$. This contradicts Equation (8).

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