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Preliminary Version

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Maria Gabriella Graziano[*], Marialaura Pesce[†], and Niccolò Urbinati[‡]

Abstract

We define the equitable bargaining set for exchange economies. Our definition differs from that in Mas-Colell (1989) because it requires that objections and counterobjections must satisfy some equitability conditions. We show that the equitable bargaining set coincides with that of Mas-Colell when the underlying economy is atomless, but not in general. Then we provide two sets of conditions for economies with market imperfections that apply to finite economies and to mixed market economies. In the first case our conditions imply that the equitable bargaining set is a subset of the core, and so it converges to the set of competitive allocations if the economy is replicated. In the second case, we show that all allocations in the equitable bargaining set are competitive, extending the Walras-bargaining equivalence of Mas-Colell (1989) to the framework of mixed markets. All the conditions we use follow from well-established assumptions from the literature in finite and mixed market economies.

JEL Classification: D62; D85; I12; I18.

Keywords: Bargaining set, Core, Equal treatment property, Walrasian objections.

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1 Introduction

A *bargaining set* is a solution concept based on a two-step veto mechanism. It assumes that a proposed outcome is stable if it is either uncontested, or if every objection to it is opposed with a valid counterobjection. In principle, many different definitions of bargaining set are possible, depending on which classes of objections and counterobjections are considered valid from time to time.

Aumann and Maschler (1964) and Davis and Maschler (1963) first introduce the notion of bargaining set as a extension of the core of cooperative games by assuming that objections and counterobjections must be proposed by single agents. Mas-Colell (1989) formulates an alternative definition that dispenses with the idea of proposers and that is insensitive of the personal initiative of negligible individuals. This way, he extends the study of the bargaining set to models of competitive economies.

In the context of exchange economies, there are several advantages in studying the bargaining set over the core. Aumann (1973) observes that the core is inappropriate in describing some economic phenomena involving monopolies or cartels, which in Maschler (1976) find a better explanation in terms of bargaining set. In the case of competitive economies Mas-Colell (1989) shows that the bargaining set characterizes the class of competitive equilibria, and uses this result to provide new insights on the real market power of coalitions and on the behaviour of competitive agents outside the state of equilibrium[1].

This paper presents a variation of Mas-Colell's bargaining set with equitable flavors. In our definition, we assume that an agent accepts to join an objection or a counterobjection only if she is promised a bundle at least as good as those assigned to her peers (i.e. to the agents with her same preferences and endowment). Objections and counterobjections of this type are *equitable*. The *equitable bargaining set* consists of all allocations with the equal treatment property (i.e. those that assign equivalent bundles to identical agents) that cannot be blocked with an equitable objection that is immune to equitable counterobjections. This definition reduces drastically the conflicts between identical agents and has significant properties in terms of envy-freeness: in every stage of the bargaining process, in fact, none of the deviating agents would rather take the place of any of her peers.

A germinal discussion on equitability in the bargaining set solution appears already in (Mas-Colell, 1989, Remark 5). In the case of perfectly competitive economies, in fact, Mas-Colell observes that his bargaining set is entirely described by means of equitable objections without counterobjections[2]. Thus, in this case, the two notions of bargaining set coincide.

We study the properties of the equitable bargaining set beyond the assumption that the

[1] Mas-Colell (1989) shows that the only objections that are immune to counterobjections, and hence relevant to the bargaining set, arise as the spontaneous reaction of agents to a set of prices. These objections can be obtained in a fully decentralized process, without any need of coordination within agents, and so they reflect the purely Walrasian behaviour of competitive agents even outside the state of equilibrium.

[2] Precisely, (Mas-Colell, 1989, Remark 5) observes that in atomless economies, every non-core allocation with the equal treatment property has an objection without counterobjections, and in which all deviating agents of the same type receive equivalent bundles. As it turns out, objections of this type are equitable. See the discussion in paragraph 3.3.

underlying economy is perfectly competitive, with a specific attention to those situations in which the comparisons between agents of the same type are especially relevant. In these scenarios the equitable bargaining set contains all competitive allocations but, in general, it is not comparable with the core nor with Mas-Colell's notion of bargaining set. The difference between the three solution concepts is therefore substantial, and they can be used to study alternative aspects of the bargaining process. In particular, the equitable bargaining set captures the competition across agents of different types, which is invisible to the core and to Mas-Colell's bargaining set.

Our main results provide conditions under which the equitable bargaining set coincides with the core or with the set of competitive allocations. The analysis covers the cases of atomless (or perfectly competitive) economies, finite economies, and mixed markets, which are hybrid situations where a number of influential traders interacts with a mass of negligible ones.

Our first set of results studies the equitable bargaining set through a special class of price-based objections, which we call *weakly-Walrasian*. We introduce them in Section 3.3 and show that finding a weakly-Walrasian objection to an allocation x is sufficient to conclude that x is not in the equitable bargaining set. Section 4.1 proves that, in atomless economies, every non-competitive allocation is blocked by some coalition via a weakly-Walrasian objection, and that this coalition can be taken arbitrarily small in size. This result, which resembles Schmeidler (1972), simplifies the search for valid objections and makes the equitable bargaining particularly ductile and suitable to extensions. A similar property, in fact, does not hold for the classes of objections that describe Mas-Colell's bargaining set; see Schjodt and Sloth (1994) and Hervés-Estévez and Moreno-García (2015).

Section 4.2 addresses the bargaining sets in mixed market economies. In this framework large traders can easily raise counterobjections, and very few objections go unopposed. The bargaining set of Mas-Colell's is therefore pathologically large, even when the core coincides with the set of competitive allocations; see Shitovitz (1989). We suggest that, in some situations, the inefficacy of the bargaining set may be driven by the conflicts between identical agents. To support this idea, we find conditions under which the equitable bargaining set coincides with the set of competitive allocations, and this is strictly contained in Mas-Colell's bargaining set. These conditions are weaker than other assumptions that are common in the literature on mixed markets.

Our last set of results considers economies in which a type of agents is extraordinary influential, to the point that no coalition can raise objections or counterobjections without their participation. We call *leaders* the agents of such type. Leaders appear, for example, when a group of identical agents owns the whole endowment of a commodity that every agent needs to survive. We show that in the presence of finitely many leaders of the same type the equitable bargaining set coincides with the set of core allocations with the equal treatment property. This result has interesting implications: suppose that in a finite economy there is a group of leaders, and suppose that this economy is replicated as in Debreu and Scarf (1963). Then the equitable bargaining set in each replica eventually becomes a subset of the core, and so it converges to the set of competitive allocations. Anderson, Trockel, and Zhou (1997) show that the same convergence result does not hold for Mas-Colell's and others' bargaining set, see also Iñarra, Serrano, and Shimomura (2020) and Hervés-Estévez and Moreno-García (2018). It bears emphasis that a stronger notion of leader (called *veto-player*) is used in Shitovitz (1989) and Aiche (2019) to prove the coincidence of the core with the bargaining set, but these equivalence results cannot be used in replicated economies, since the copies of a veto-player are no longer veto-players. It follows that Mas-Colell's bargaining set may coincide with the core in the original economy but it becomes much larger as soon as this is replicated.

The rest of the paper is organized as follows. Section 2 describes the main model, which is that of an exchange economy with a measure space of agents. All assumptions made at this stage are standard as in Hildenbrand (1974). Section 3 gives the main definition in the paragraph 3.1, compares the equitable bargaining set with the core in 3.2 and then introduces the notion of weakly-Walrasian obejctions in 3.3. Section 4 presents the equivalence results.

All the longer proofs are relegated to the appendix A. The appendix B includes a series of examples that illustrate the differences between the various solution concepts introduced and the necessity of the assumptions we use in the main Theorems.

2 The model

The economy consists of a finite-dimensional commodity space $\mathbb{R}^m_+$ and a finite measure space of consumers (T, Σ, λ). The set T represents all individual traders, while Σ is the collection of all groups that are able or allowed to trade. For $S \in \Sigma$, $\lambda(S)$ is the size (or weight) of the group S. A *coalition* is an economically relevant group of agents, i.e. a set in Σ with positive measure. We allow the presence of atoms in (T, Σ, λ), which are coalitions that cannot be broken in two, smaller subcoalitions. Atoms may represent single agents with significant market power (such as monopolists or oligopolies) or large groups of traders that are forced to act compactly by some binding agreements (such as unions or cartels). As the measure λ is finite, the set $\mathcal{A}$ of all atoms in (T, Σ, λ) is at most countable and so we can partition T into its atomic and atomless components, which are $T_1 = \bigcup \mathcal{A}$ and $T_0 = T \setminus T_1$ respectively.

Every agent $t \in T$ is characterized by a preference relation $\succcurlyeq_t$ on $\mathbb{R}^m_+$ and an endowment bundle $\omega(t) \in \mathbb{R}^m_+$. The irreflexive and symmetric components of $\succcurlyeq_t$ are $\succ_t$ and $\sim_t$ respectively. We make the following assumptions, that are standard in models with a measure space of agents (see e.g. Hildenbrand (1974)): (i) $\omega\colon T \to \mathbb{R}^m_+$ is an integrable function with $\int \omega \, d\lambda \gg 0$; (ii) preferences are strictly monotone, continuous, total preorders on $\mathbb{R}^m_+$; (iii) preferences are measurable in the sense that $\{t : v \succcurlyeq_t w\} \in \Sigma$ for every $v, w \in \mathbb{R}^m_+$; and (iv) $\succcurlyeq_t$ is convex for every $t \in T_1$.

We say that two agents s, t are of the *same type* (and write $s \sim t$) if they have identical preferences and endowments. Under our assumptions, the relation $\sim$ is measurable in the sense that $\{s : t \sim s\} \in \Sigma$ for every $t \in T$ and, in particular, every atom consists only of agents of the same type. We call *type of agents* a class in the quotient $T/_\sim$, i.e. a set formed by all the agents that share a given preference relation and endowment bundle.

An *allocation* is an integrable function of the type $x\colon T \to \mathbb{R}^m_+$. A coalition S *attains* an allocation x if $\int_S x\, d\lambda \leqslant \int_S \omega\, d\lambda$, i.e. if the amount of resources that x assigns to agents in S does not exceed their initial endowments. If x is attained by the grand coalition T we say that x is *feasible*. An allocation x has the *equal treatment property* (ETP for short) on a coalition S if $x(t) \succcurlyeq_t x(s)$ for every $t, s \in S$ of the same type. If x has the ETP on the whole T we simply say that it has the ETP. We write $\mathcal{M}$ for the set of allocations, and $\tilde{\mathcal{M}}$ for the set of allocations with the ETP.

Given a price vector $p \in \mathbb{R}^m_+ \setminus \{0\}$, the *budget set* of consumer t at p is $\beta(t,p) = \left\{x \in \mathbb{R}^m_+ : p \cdot x \leqslant p \cdot \omega(t)\right\}$. A feasible allocation x is *competitive* at the price p if $x(t)$ maximizes $\succcurlyeq_t$ on the set $\beta(t,p)$ for a.e. $t \in T$, i.e. if $x(t) \in \beta(t,p)$ and $x(t) \succcurlyeq_t \beta(t,p)$[3]. The set of competitive allocations is $\mathcal{W}$. Since agents of the same type maximize their preferences on the same budget sets, a competitive allocation always satisfies the ETP and so $\mathcal{W} \subseteq \tilde{\mathcal{M}}$.

Remark 2.1 Given an allocation $x \in \mathcal{M}$ and two agents t and s, we say that t *envies* s at x if t prefers receiving the bundle of s rather than consuming her own bundle, i.e. $x(s) \succ_t x(t)$. The allocation x is called *envy-free* if there is no envy among agents at x, i.e. $x(t) \succcurlyeq_t x(s)$ for almost all $t, s \in T$ (see Foley (1967) and Thomson (2011)). Asking that x has the ETP is less demanding than the absence of envy, as it only requires that agents are not envious when they compare themselves with individuals of their same type.

Remark 2.2 The assumptions of our model are standard and allow to cover a variety of classical situations. Finite economies are obtained when T is finite and λ is the counting measure on 2^T. Competitive economies arise when (T, Σ, λ) is atomless, e.g. when $T = [0,1]$, Σ is the Borel algebra and λ the Lebesgue measure. Last, when both sets T_0 and T_1 have positive measure, an ocean of negligible agents interacts with at most countably many influential agents or oligopolies (the atoms). We refer to this latter situation as mixed market, or mixed economy.

3 The equitable bargaining set

This section introduces the main solution concept, which is a variation of Mas-Colell's bargaining set based on a weaker mechanism of objections and counterobjections. Intuitively, we assume that an agent accepts to join an objection (or a counterobjection) only if she is promised a bundle at least as good as those consumed by her peers, i.e. by the agents of her same type. Objections and counterobjections of this type are called *equitable*. The *equitable bargaining set* consists of all feasible allocations with the ETP that cannot be blocked by an equitable objection without triggering some equitable counterobjection.

[3]Throughout, for a $v \in \mathbb{R}^m_+$ and a $C \subset \mathbb{R}^m_+$ we write $v \succcurlyeq_t C$ when $v \succcurlyeq_t w$ for every $w \in C$.

3.1 Definitions

A coalition B *blocks* (or *objects to*) an allocation x if its members can rearrange their own endowments in a way that they all find at least as good as x and that some strictly prefer to x. Formally, (B, y) is a (*standard*) *objection* to x if:

- B attains y, i.e. $\int_B (y - \omega) \leqslant 0$;
- $y(t) \succcurlyeq_t x(t)$ for a.e. $t \in B$;
- $\lambda(\{t \in B : x(t) \succ_t y(t)\}) > 0$.

The set of (standard) objections to x is $Ob(x)$. The *core* is the set $\mathcal{C}$ of all feasible allocations that cannot be blocked, i.e. the feasible $x \in \mathcal{M}$ such that $Ob(x) \neq \emptyset$.

We introduce below a notion of objection with equity flavors, in the sense that members of the objection do not envy any other agents of the same type neither inside the objection nor outside the objection.

Definition 3.1 *An objection (B, y) to x is* ***equitable*** *if for a.e. $t \in B$ and $s \in T$ of the same type, one has:*

$(EO1)$ $y(t) \succcurlyeq_t y(s)$ *if $s \in B$;*

$(EO2)$ $y(t) \succcurlyeq_t x(s)$ *otherwise.*

The set of equitable objections to x is $Ob_e(x)$. We denote by $\mathcal{C}_e$ the set of all feasible allocations with no equitable objections and by $\tilde{\mathcal{C}}_e$ the set of allocations in $\mathcal{C}_e$ with the ETP, i.e. $\tilde{\mathcal{C}}_e = \mathcal{C}_e \cap \tilde{\mathcal{M}}$.

Condition $(EO1)$ rules out the possibility that agents in B of the same type envy each other, and it is equivalent with asking that y has the ETP on B. At the same time, condition $(EO2)$ always holds when x has the ETP because for a.e. $t \in B$ and $s \notin B$ of the same type, we have $y(t) \succcurlyeq_t x(t)$ by the properties of objections and $x(t) \succcurlyeq_t x(s)$ by the ETP of x. We conclude that, when x has the ETP, $(B, y) \in Ob(x)$ is equitable if and only if y has the ETP on B.

We now introduce the notion of *counterobjection* as in Mas-Colell (1989). Let (B, y) be an objection to an allocation x. A (*standard*) *counterobjection* to (B, y) consists of a coalition C and an allocation z such that:

- C attains z, i.e. $\int_C (z - \omega)\, d\lambda \leqslant 0$;
- $z(t) \succ_t y(t)$ for all $t \in C \cap B$;
- $z(t) \succ_t x(t)$ for all $t \in C \setminus B$.

The set of counterobjections to (B, y) is $Cob^x(B, y)$. An objection is *justified* if it has no counterobjections. The (*standard*) *bargaining set* is the class of all feasible allocations that have no justified objections, i.e. the set of all feasible $x \in \mathcal{M}$ such that either $Ob(x) = \emptyset$

or such that $(B, y) \in Ob(x)$ implies that $Cob^x(B, y) \neq \emptyset$. We write BS for the (standard) bargaining set.

On the same line of equitable objections, we say that a counterobjection (C, z) is *equitable* if no agent in C wishes to switch position with any other agent of her same type. Thus, we ask that no agent in the counterobjection envies what her peers in C receive from z, what her peers in B receive from y and what the others receive from x.

Definition 3.2 *A counterobjection $(C, z) \in Cob^x(B, y)$ is* ***equitable*** *if for a.e. $t \in C$ and $s \in T$ of the same type, one has:*

$(EC1)$ $z(t) \succcurlyeq_t z(s)$ *if $s \in C$;*

$(EC2)$ $z(t) \succcurlyeq_t y(s)$ *if $s \in B \setminus C$;*

$(EC3)$ $z(t) \succcurlyeq_t x(s)$ *otherwise.*

The set of equitable counterobjections to (B, y) is $Cob_e^x(B, y)$. An objection (B, y) is said to be e-***justified*** *if it is equitable and there is no equitable counterobjection to it.*

As it was for equitable objections, one observes that condition $(EC1)$ is equivalent with asking that z has the ETP on C and that every counterobjection satisfies condition $(EC3)$ when $x \in \tilde{\mathcal{M}}$.

We can now define a new bargaining set that consists only of allocations with the ETP and that considers only equitable objections and counterobjections.

Definition 3.3 *The* ***equitable bargaining set*** *is the set BS_e of all feasible allocations with the ETP that have no e-justified objections. In formulas:*

$$BS_e = \left\{ x \in \tilde{\mathcal{M}} : x \text{ is feasible and } (B, y) \in Ob_e(x) \Rightarrow Cob_e^x(B, y) \neq \emptyset \right\}.$$

The equitable bargaining set contains every core allocation with the ETP, and hence all competitive allocations. In other words, one has the following chain of inclusions:

$$\emptyset \neq \mathcal{W} \subseteq \tilde{\mathcal{C}}_e \subseteq BS_e. \tag{1}$$

Without any specific restriction, it is possible that in some economy all the inclusions in the equation (1) are strict (see Examples B.1 and B.2 in Section B). Our main results in Section 4 give conditions under which BS_e coincides with $\mathcal{W}$, and conditions under which $BS_e = \tilde{\mathcal{C}}_e$ even when this may be strictly larger than $\mathcal{W}$.

Remark 3.4 The difference between the standard and the equitable notions of objections, counterobjections and bargaining set arises only when agents of the same type form appreciable coalitions. When every type of agents consists of a single individual, in fact, no consumer can compare herself with others and so none is envious. This implies that all allocations have the ETP, that every objection (and counterobjection) is equitable and hence that the equitable bargaining set coincides with the standard one.

Remark 3.5 As discussed in Mas-Colell (1989, Remark 1), the definition of counterobjection can be weakened to just requiring strict preference for a positive measure subset of the counterobjecting coalition. With this change, even if the set of counterobjections is formally larger, the set of justified objections (and hence the bargaining set) remains unaltered. A similar argument does not apply to objections. If one considers only objections in which all the deviating have strict preferences then the core does not change, but the bargaining set may become significantly larger. See Yamazaki (1995) for a formal comparison of the bargaining sets generated by these different classes of objections.

Remark 3.6 We observed that equitable objections and counterobjections to allocations with the ETP have a simpler description. On this line, an equivalent definition of the equitable bargaining set is the following: $x \in BS_e$ if and only if x has the ETP and for every $(B, y) \in Ob(x)$ such that y has the ETP on B there is a $(C, z) \in Cob^x(B, y)$ with the property that z has the ETP on C and $z(t) \succcurlyeq_t y(s)$ for a.e. $t \in C$, $s \in B$ of the same type.

3.2 Comparisons between cores and bargaining sets

The introduction of equitable objections and counterobjections, as well as the focus on allocations with the ETP, defines variations of the notions of core and bargaining set for exchange economies. This section studies how these new solution concepts relate to each other.

By previous considerations, we know that the following series of inclusions hold.

$$\mathcal{W} \subseteq \tilde{\mathcal{C}}_e \subseteq BS_e, \quad \text{and} \quad \mathcal{W} \subseteq \tilde{\mathcal{C}}_e \subseteq \mathcal{C} \subseteq BS.$$

Comparisons between BS_e and the standard notion of core $\mathcal{C}$ are not straightforward: there might be core allocations without the ETP and so outside BS_e (see Example B.2) as well as non-core allocations inside BS_e (see Example B.3). Thus, without further restrictions on the measure space of agents, the core and the equitable bargaining set are not comparable solution concepts

The veto mechanism based on equitable objections is typically very weak and the class $\mathcal{C}_e$ of feasible allocations without equitable objections may be extremely large. It may even include allocations that are not individually rational, i.e. some $x \in \mathcal{M}$ such that $\{t : \omega(t) \succ_t x(t)\}$ is non-null. As an example, think of the economy formed by two identical agents: the allocation that assigns all commodities to one of the two agents is not individually rational and has no equitable objections. Even limiting $\mathcal{C}_e$ to individually rational allocations defines a particularly large set. Example B.4, for instance, describes an atomless economy where $\mathcal{C}_e$ contains a non-competitive, individually rational allocation. Therefore, $\mathcal{C}_e$ may be strictly larger than the core and the bargaining set even when the latter two coincide.

Interestingly, standard and equitable objections are equally effective in blocking allocations with the ETP.

Theorem 1 *Let x be a non-core allocation with the ETP. Then there exists an equitable objection (B, y) to x. Furthermore, y can be chosen so that $y(t) = y(s)$ whenever $t, s \in B$ are agents of the same type with convex preferences.*

For readability, the most technical proofs are relegated to the Appendix A. Theorem 1 is covered in Appendix A.1.

A consequence of Theorem 1 is that the set of all core allocations with the ETP coincides with $\tilde{\mathcal{C}}_e$, the set of feasible allocations with the ETP that have no equitable objections. In symbols, $\mathcal{C} \cap \tilde{\mathcal{M}} = \tilde{\mathcal{C}}_e$.

3.3 Objections and competitive behaviour

When agents act competitively (i.e. as price-takers), objections to a given allocation x may emerge as the result of a fully decentralized, price-based mechanism. This paragraph introduces a special class of price-based objections, called *weakly-Walrasian*, that can be used to test whether an allocation is not in the equitable bargaining set. Precisely, we show that if there is a weakly-Walrasian objection to an allocation x then $x \notin BS_e$. This class of objections generalize the notion of Walrasian objections in Mas-Colell (1989).

Let us recall that for a price vector $p \in \mathbb{R}^m_+ \setminus \{0\}$ and a $t \in T$, the *budget set* of t at p is $\beta(t, p) = \left\{v \in \mathbb{R}^m_+ : p \cdot v \leqslant p \cdot \omega(t)\right\}$. Agent t's *demand set* at p is $\xi(t, p) = \{v \in \beta(t, p) : v \succcurlyeq_t \beta(t, p)\}$. An allocation x is *competitive* at p when it is feasible and $x(t) \in \xi(t, p)$ for a.e. $t \in T$.

Definition 3.7 *An objection (B, y) to x is **weakly-Walrasian** at a price vector $p \gg 0$ if:*

$(WO1)$ *$y(t) \in \xi(t, p)$ for a.e. $t \in B$;*

$(WO2)$ *$x(t) \succcurlyeq_t \beta(t, p)$ for a.e. $t \notin B$ such that $\lambda(\{s \in B : s \sim t\}) > 0$.*

*The objection (B, y) is **Walrasian** if, in addition to $(WO1)$, it satisfies:*

$(WO3)$ *$x(t) \succcurlyeq_t \beta(t, p)$ for a.e. $t \notin B$.*

Clearly, a Walrasian objection is weakly-Walrasian. The definition of Walrasian objection appears in Mas-Colell (1989) in a different but equivalent formulation; see Remark 3.8.

We can imagine that a Walrasian objection (B, y) to x at the price p arises when every $t \in T$ can choose independently whether to accept $x(t)$ or to trade at a price p. Agents in B are those who choose deviate from x because, at the price p, they can afford a bundle at least as good (remember that, being $(B, y) \in Ob(x)$, $y(t) \succcurlyeq_t x(t)$ for a.e. $t \in B$). Agents outside B are those who accept x, as they find it at least at good as anything they can afford at the price p. The difference between Walrasian and weakly-Walrasian objections is that, in the second case, not all agents can freely choose between x and p. Precisely, when (B, y) is only weakly-Walrasian we cannot tell if a $t \notin B$ that is of the same type of members of B prefers p over x.

Remark 3.8 The definitions of Walrasian objections in Mas-Colell (1989) and in 3.7 are formally different, but yet equivalent under the current assumptions. For a $(B, y) \in Ob(x)$, Mas-Colell (1989) asks that there is a price vector p such that, for a.e. $t \in T$ and every $v \in \mathbb{R}^m_+$, (i) $p \cdot v \geqslant p \cdot \omega(t)$ whenever $t \in B$ and $v \succcurlyeq_t y(t)$, and (ii) $p \cdot v \geqslant p \cdot \omega(t)$ whenever $t \notin B$ and $v \succcurlyeq_t x(t)$. Given that B attains y, a standard argument using the continuity and the strict monotonicity of preferences gives that (i) is equivalent with $(WO1)$, whereas (ii) is equivalent with $(WO3)$.

By Mas-Colell (1989, Remark 5), every Walrasian objection to an allocation with the ETP is equitable. Our next proposition shows that a similar result holds for weakly-Walrasian objections. Its proof is in A.4.

Proposition 3.9 *Let $x \in \tilde{\mathcal{M}}$. Then every weakly-Walrasian objection to x is equitable.*

If $x \in \mathcal{M}$ has the ETP, then all Walrasian objections to x are justified, and hence e-justified. The same cannot be said for a weakly-Walrasian objection to x, for we may still find equitable counterobjections to it. Nevertheless, we can show that equitable counterobjections to weakly-Walrasian objections are rare and they must satisfy strict conditions. The proof of the following is in A.5.

Lemma 3.10 *Let $x \in \mathcal{M}$ and let $(B, y) \in Ob(x)$ be weakly-Walrasian at the price p. If $(C, z) \in Cob^x_e(B, y)$, then:*

1. *$z(t) \in \xi(p, t)$ for a.e. $t \in C$;*
2. *$\lambda(B \cap C) = 0$.*

Lemma 3.10 has a series of interesting consequences. The first is that if $(B, y) \in Ob(x)$ is weakly-Walrasian at a price p, then none of the agents in B will accept to join an equitable counterobjection (bullet 2). This means that the society cannot use equitable objections to block the agents in B from deviating. The second consequence is that agents participate in an equitable counterobjection (C, z) only if this is at least as convenient as trading their endowments at the price p. In particular, every $t \in C$ prefers trading at p rather than consuming $x(t)$. These considerations, together with Proposition 3.9, are used to prove the following key result. Its proof is in A.6.

Proposition 3.11 *If $x \in \tilde{\mathcal{M}}$ is blocked with a weakly-Walrasian objection then there exists an e-justified objection to it, and hence $x \notin BS_e$.*

The proof of Proposition 3.11 goes even further than the claim and shows that if $(B, y) \in Ob(x)$ is weakly-Walrasian, then there exists an e-justified $(\tilde{B}, \tilde{y}) \in Ob(x)$ such that $B \subseteq \tilde{B}$ and $\tilde{y}(t) = y(t)$ for a.e. $t \in B$. Thus, if a group of agents accepts to raise a weakly-Walrasian objection to x, then this can be extended to some larger e-justified objection to x.

4 Equivalence results

In the generality of the model, the sets of competitive allocations, of core-allocations with the ETP and the equitable bargaining set are distinct. This section introduces additional conditions under which these solution concepts define the same set of allocations. Our assumptions will refine some that are common in the literature and apply to a variety of standard situations: part 4.1 considers atomless economies, i.e. where $T = T_0$; part 4.2 studies mixed markets that either have infinitely many large traders, all of the same type, or satisfy the "fringe" hypothesis; part 4.3 considers economies with exceptionally influential agents, called leaders, and applies to finite economies.

4.1 Equivalence in atomless economies with restricted coalitions

This paragraph studies the equitable bargaining set in economies where every individual trader is negligible, i.e. where $T = T_0$. In this framework, we can combine Mas-Colell (1989, Proposition 1) with Proposition 3.11 to obtain the following equivalence.

Proposition 4.1 *Let $T = T_0$. For every allocation $x \notin \mathcal{W}$, if x has the ETP then there exists a weakly-Walrasian objection to x. As a consequence, $\mathcal{W} = \mathcal{C} = BS_e = BS$.*

Proof. Let $x \notin \mathcal{W}$ be an allocation with the ETP. By Mas-Colell (1989, Proposition 2) there exists an objection to x that is Walrasian, and hence weakly-Walrasian. An application of Proposition 3.11 gives that $x \notin BS_e$ and so $BS_e \subseteq \mathcal{W}$. Since $\mathcal{W} \subseteq BS_e$ is always true, and $\mathcal{W} = \mathcal{C} = BS_e$ by Mas-Colell (1989, Theorem 1), we conclude that $\mathcal{W} = \mathcal{C} = BS_e = BS$. ■

Notice that the result above does not specify how to choose the objection, nor how large the deviating coalition may be. This may become an issue in fully decentralized economies, where it is common to assume that larger groups of agents may not be able to coordinate their actions and that only small coalitions are effective. In this perspective, one asks if a version of 4.1 holds even when only coalitions under a certain size are allowed to raise objections.

In general, the answer to the question above is negative. In an economy where every agent is of a different type, for example, it could be that, for some $\varepsilon > 0$ and $x \notin \mathcal{W}$ there is no weakly-Walrasian objection (B, y) to x with $\lambda(B) < \varepsilon$. This is because weakly-Walrasian and Walrasian objections coincide and so the arguments in Schjodt and Sloth (1994) apply. Nevertheless, if the economy allows enough comparisons among agents then we may obtain some positive results.

In the following we assume that for a.e. $t \in T$ the set of agents of her same type of t positive measure, i.e. that $\lambda(\{s : s \sim t\}) > 0$. This implies that there are countably many types of agents[4] $(K_n)_n$ such that $\lambda(K_n) > 0$ and $T = \bigcup_n K_n$ up to a null set.

Assumption 4.2 *There are countably many types of agents. We write $(K_n)_n$ for the (possibly finite) sequence of types of agents that have positive measure.*

[4] Recall that a *type of agents* is a class in $T/_\sim$ and a measurable set of agents.

Under Assumption 4.2, we can associate any coalition S with a set $K(S) = \{n : \lambda(S \cap K_n) > 0\}$ and write $S = \bigcup\{S \cap K_n : n \in K(S)\}$ up to a null set. The set $K(S)$ is then a list of all types of agents that have a representative in S, up to a null set. Notice also that, when $T = T_0$, Assumption 4.2 excludes the possibility that every agent is of a different type.

We now prove that in atomless economies that satisfies 4.2, it is possible to reduce the size of any objection as much as we want by preserving some properties as the types of agents represented in the objection. The proof is in A.7.

Proposition 4.3 *Let $T = T_0$ and assume that Assumption 4.2 holds. Suppose that $x \in \mathcal{M}$ and let $(B, y) \in Ob(x)$. Then, for every $\delta \in (0, 1)$ there exists $B^\delta \subseteq B$ such that:*

(i) $\lambda(B^\delta \cap K_n) = \delta\lambda(B \cap K_n)$ *for every n;*

(ii) $(B^\delta, y) \in Ob(x)$ *and:*

- *(iia) if (B, y) is equitable, then (B^δ, y) is equitable too;*
- *(iib) if (B, y) is weakly-Walrasian, then (B^δ, y) is weakly-Walrasian too.*

It follows from Proposition 4.3 that a version of 4.1 holds even when we impose a restriction on the size of coalitions that can raise objections, in the same spirit of Schmeidler (1972). Precisely, if (B, y) is a weakly-Walrasian objection to x then there is a weakly-Walrasian objection (B', y) such that (i) $B' \subseteq B$, (ii) every type of agent present in B is present in B' as well and (iii) $\lambda(B') < \varepsilon$ for an arbitrarily small $\varepsilon > 0$.

In proving 4.3, Assumption 4.2 is crucial. This is because the proof applies the Lyapunov theorem to each type of agents, and obtain a subcoalition for each type. Then we consider the union of these subcoalitions. Being Σ a σ-algebra, such a union define a coalition only if it is at most countable.

Remark 4.4 Our notion of bargaining set has some similarities with the *global bargaining set* introduced in Vind (1992) (the terminology we use for this bargaining set is taken from Schjodt and Sloth (1994)). For a given $x \in \mathcal{M}$, a *global objection* to x is a $(B, y) \in Ob(x)$ such that y is feasible. A *global counterobjection* to (B, y) consists of a coalition C and a feasible allocation z such that C attains z and $z(t) \succ_t y(t)$ for every $t \in C$. A feasible allocation is in the global bargaining set if there are no global objection to it without global counterobjections. Therefore, the main difference with Mas-Colell's bargaining is that all the counterobjecting agents improve upon the allocation the counter (i.e. the y) and ignore the original allocation (i.e. the x).

Our notion keeps some features of both, because we require that agents in $C \setminus B$ do not envy only their counterparts in the objection to which they counter. In other words, we impose a comparison between z and y, as done in Vind (1992), only among agents of the same type and for the rest z is compared with x, as in Mas-Colell (1989). In this perspective, the fact that we can put some restrictions on the size of weakly-Walrasian objections (Proposition 4.3) finds its correspondence in a property of justified global objection, since they can be chosen of any size.

4.2 Equivalence in mixed markets

This section provides conditions that are sufficient to get the equivalence between the equitable bargaining set and the class of competitive allocations in mixed markets. As in the previous paragraph, we will work under Assumption 4.2, so that every trader identifies with a non-null coalition of agents. This implies that T coincides with the disjoint union $\bigcup_n K_n$ up to a null set, where the K_n's denote the different type of agents.

We make the following, additional assumption.

Assumption 4.5 *For any sequence $(\mu_n)_n \in (0,1)$ there exist $\delta \in (0,1)$ and a sequence of coalitions $(F_n)_n$ such that $F_n \subseteq K_n$ and $\lambda(F_n) = \delta\mu_n\lambda(K_n)$ for any n.*

Despite its technical appearance, Assumption 4.5 weakens two well-known conditions imposed in mixed markets. The first asks that there are infinitely many atoms, and that they are all of the same type. The second assumes a finite number of atoms and requires that to every atom A corresponds to an *atomless fringe*, i.e. a coalition in T_0 whose elements are all of the same type of A. The proof is at A.8.

Proposition 4.6 *Assumption 4.5 is satisfied if one of the following two conditions holds.*

1. *There are infinitely many atoms and they are all of the same type.*
2. *There are finitely many types of atoms, and for each atom A there is a coalition $S_A \subseteq T_0$ such that every $t \in S_A$ is of the same type of A.*

Notice that Assumption 4.5 rules out the possibility of an economy with finitely many agents. This situation is covered in Section 4.3. Other variations of the bargaining set in finite economies are studied by Hervés-Estévez and Moreno-García (2018), in which a convergence theorem is obtained, and by Graziano, Pesce, and Urbinati (2020) who prove the equivalence between $\mathcal{W}$ and a weakening of the bargaining set through the Aubin (generalized) coalitions.

We can now state the main result of this section.

Theorem 2 *Under Assumptions 4.2 and 4.5, $\mathcal{W} = BS_e$.*

The proof follows the same idea of the equivalence theorem due to Mas-Colell (1989) for atomless economies. It is, in fact, obtained by combining two facts which, on the other hand, rest on a weakening of the Walrasian objection: (1) a non-competitive allocation x with the ETP has a weakly-Walrasian objection to it (Proposition 4.7 below) and (2) if there is a weakly-Walrasian objection to x, then $x \notin BS_e$ (Proposition 3.11).

The proof of the following is in A.11.

Proposition 4.7 *Under Assumptions 4.2 and 4.5, if $x \in \tilde{\mathcal{M}}$ is a non-competitive feasible allocation, there exists a weakly-Walrasian objection to it.*

Note that, contrary to Proposition 4.7, Proposition 3.11 does not need the two assumptions and it holds in more general contexts. Proposition 4.7 is alike of Proposition 2 of Mas-Colell (1989), whereas Proposition 3.11 does not state that any weakly-Walrasian objection is justified as done in Mas-Colell (1989). See the discussion in Section 3.3.

These two propositions combined imply the equivalence $\mathcal{W} = BS_e$.

Proof of Theorem 2. The inclusion $\mathcal{W} \subseteq BS_e$ is always met. For the converse, let $x \in BS_e$, which implies that x is feasible, it satisfies the ETP and there exists no e-justified objection to it. Assume, to the contrary, that $x \notin \mathcal{W}$. Proposition 4.7 ensures the existence of a weakly-Walrasian objection to x and hence, by Proposition 3.11, there exists an e-justified objection to it. This is a contradiction because $x \in BS_e$. ■

The equivalence proved in Theorem 2 is independent from the coincidence of competitive allocations and the standard bargaining set. Example B.5 in Section B shows that it might be the case that, under Assumption 4.5,

$$\mathcal{W} = BS_e \subsetneq BS.$$

This explains why we restrict on weakly-Walrasian objections only. Furthermore, Examples B.2 and B.3 underline the role of Assumption 4.5 as they illustrate mixed economies in which Assumption 4.5 fails and $\mathcal{W} \subsetneq BS_e$.

Remark 4.8 The proof of Proposition 4.7 studies the mixed market economy via an auxiliary, atomless economy a' la manner of Greenberg and Shitovitz (1986). We anticipate here the main ideas, then address it formally in the Appendix A.4.

Imagine that every atom is not as a single large trader but a coalition of small, negligible agents forced to act together (as in a cartel or a syndacate). The auxiliary economy is defined by removing these constraints and letting all individuals act independently: the set of agents remains the same, but they can form many more coalitions and find new ways of allocating goods among themselves. In particular, in this relaxed environment the existence Theorems of Mas-Colell and the results of the paragraph 4.1 apply.

The idea of the proof is to think of any non-competitive $x \in \tilde{\mathcal{M}}$ as an allocation in the auxiliary economy (this can be done because the sets of agents are essentially the same). As we are ignoring atoms, we can apply Proposition 4.1 to find a weakly-Walrasian objection (S, f) to x in the auxiliary economy. The coalition S may not be one that agents can implement in the original mixed market, but under Assumption 4.5 we can apply Proposition 4.3 to modify (S, f) into a smaller, weakly-competitive objection to x that has a counterpart in the original mixed market (Proposition A.10). This will be weakly-Walrasian in the original mixed market too (Lemma A.9), concluding the proof.

4.3 Equivalence in economies with leaders

This section studies conditions under which some particular groups of agents become predominant in the bargaining process. It assumes that a single type of traders owns the whole endowment of a given commodity, one that every agent finds necessary for her survivor. We call *leaders* the agents of said type, because no coalition can raise objections or counterobjections without the participation of some of them; see Remark 4.14.

The main results of the section show that in the presence of finitely many leaders of the same type, the equitable bargaining set coincides with the set of core allocations with the ETP, i.e. $\tilde{\mathcal{C}}_e = BS_e$. If, in addition, these leaders are the only large traders in the economy (and there are at least two of them), then the equitable bargaining set coincides also with the class of competitive allocations, even when this is strictly smaller than the standard bargaining set.

To formalize the definition of leaders, we need to drop the initial assumption that preferences are strictly monotone on the whole orthant $\mathbb{R}^m_+$, and replace it with the following one.

Assumption 4.9 *For every $t \in T$, the preference $\succcurlyeq_t$ is strictly monotone on int $\left(\mathbb{R}^m_+\right)$ and has its minimum on the boundaries of $\mathbb{R}^m_+$.*

This assumption corresponds to Shitovitz (1989, Condition 7.3). It is satisfied, for example, when agents' preferences are represented by Cobb-Douglas utility functions. This new condition is not too restrictive, for the proofs of all main results can be reformulated in these new settings.

Definition 4.10 *An agent s is a **leader** if there exists a $j \leqslant m$ such that the j-th coordinate of $\omega(t)$ is 0 for every t that is not of the same type of s.*

Clearly, if s is a leader then so is every other agent of her type. Under Assumption 4.9, if K^* is a type of leaders then any coalition S with $\lambda(S \cap K^*) = 0$ is powerless, for it cannot improve upon any allocation with her endowment alone. Notice that there may be more than one type of leaders, as long as each said type holds a different commodity.

Our first result shows that if there exists a type of leaders formed by a finite number of traders, then they can block any non-core allocation with the ETP using an equitable objection that maximizes their satisfaction. Its proof is in A.12.

Proposition 4.11 *Under Assumption 4.9, let K^* be a type of leaders formed by finitely many atoms. Then for every $x \in \tilde{\mathcal{M}} \setminus \mathcal{C}$ there exists $(B^*, y^*) \in Ob_e(x)$ with the property that, for every $(B, y) \in Ob_e(x)$:*

$$y^*(t) \succcurlyeq_t y(t), \qquad \textit{for every } t \in K^*.$$

The proposition extends a lemma in Shitovitz (1989), where a similar result is proved for standard objections assuming that there is only one leader; see Remark 4.15.

Under the assumptions of Proposition 4.11, one shows that if x is an allocation with the ETP, then any objection (B^*, y^*) as in the proposition is e-justified. This is the key of the following equivalence theorem. See A.13.

Theorem 4.12 *Under Assumption 4.9, let K^* be a type of leaders formed by finitely many atoms. Then BS_e coincides with the set of core-allocations with the ETP.*

In formulas, Theorem 4.12 establishes that:

$$\mathcal{W} \subseteq \tilde{\mathcal{C}}_e = BS_e \subseteq \mathcal{C} \subseteq BS$$

where the inclusions may be strict. Yet, if there are two leaders of the same type, and they are the only atoms in the economy, a direct application of Shitovitz (1973) gives that $\mathcal{C}$ coincides with $\mathcal{W}$, even when this is strictly contained in BS. This translates into the following corollary.

Corollary 4.13 *Under the assumptions of Theorem 4.12, suppose that there are at least two atoms in the economy and that they are all of type K^*. Then $\mathcal{W} = \tilde{\mathcal{C}}_e = BS_e = \mathcal{C}$, and these may be strictly contained than the standard bargaining set.*

See A.14 for the proof.

The assumptions needed in Corollary 4.13 require that the number of atoms is finite, that they are all leaders of the same type, and that there is no coalition of negligible agents with their same characteristics (the so-called "fringe"). As such, they are more restrictive than those typically used to prove the core-Walras equivalence. This is necessary to combine Shitovitz (1973) with Theorem 4.12. Notice that the case of countably many atoms or the presence of atoms' fringe is covered in Section 4.2.

Remark 4.14 Typically, a leader is defined as an agent (or a group of agents) that proposes the objection and that must be excluded from any counterobjecting coalition; see, for example, Aumann and Maschler (1964) or Geanakoplos (1978). When the society cannot oppose the proposals of some leaders, the core coincides with the bargaining set. Our definition of leader is formally different, but it preserves the same intuition: a leader, in fact, can propose an objection that maximizes her satisfaction, meaning that she will not participate in any counterobjecting coalition.

Remark 4.15 Shitovitz (1989) shows that if a type of leaders consists of a single trader, then she can object to any non-core allocation by proposing her preferred reallocation of goods. Since no coalition can contest her choices, such objection is justified, and so the core and the (standard) bargaining set coincide. The same result does not hold with many leaders of the same type, because the competition between them makes the core collapse to a much smaller set; see Shitovitz (1973). Theorem 4.12 restores some form of core-bargaining equivalence in this latter case. By allowing only equitable objections and counterobjections, it reduces the bargaining power of individual agents and prevents the competition between leaders of the same type.

Remark 4.16 As a corollary of Theorem 4.12, one proves that in a sequence of replicated economies in which there is at least a leader, the equitable bargaining set converges to the class of competitive allocations. Under such hypothesis, in fact, we know that BS_e consists only of core allocations with the ETP, and so it shrinks to $\mathcal{W}$ by the Theorem of Debreu-Scarf on the convergence of the core, Debreu and Scarf (1963). Notice that the same argument could not be used for other core-bargaining equivalence results, as in Shitovitz (1989) or Aiche (2019), which rest on the existence of at least *veto player* in the economy, i.e. a single leader (see Remark 4.15). In the replication process, in fact, all veto players loose her status as soon as she is replicated, and so the theorems do not apply.

Remark 4.17 Hervés-Estévez and Moreno-García (2018) discusses a notion of bargaining set for finite economies based on the idea of "endogenous leaders". Objections are raised by an entire type of agents (the leaders) and the only counterobjections allowed must satisfy the following conditions: (i) they shall not include any of the leaders; (ii) an agent can participate to a counterobjection only if this makes her better off than her peers in the objection.

This notion of bargaining set shares some common traits with ours. If x is an allocation with the ETP and (B, y) is an objection to x, then (C, z) is a counterobjection to (B, y) in the sense of Hervés-Estévez and Moreno-García (2018) only if it is equitable in the sense of Definition 3.2. The converse is not true, for in our definition we only ask that agents in C do not envy their peers in the objection and so we allow that $z(t) \sim_t y(s)$ for some $t \in C$, $s \in B$ of the same type.

5 Concluding remarks

The paper introduces the notion of equitable objections, counterobjections and bargaining set. In this, it opens to fairness considerations in the collective bargaining dynamics. In a non-equitable objection (or counterobjection), in fact, there are deviating agents who are envious of the bundle received by someone of their same type, either outside or inside the objecting coalition. These agents would rather switch position with some of their peers rather than accepting to implement the objection.

It bears emphasis that alternative notions of equitable objections (and counterobjections) could have been obtained by considering different criteria of fairness and envy-freeness from the literature (see Thomson (2011) for a survey on the topic). As an example, Hara (2002) considers objections that are "anonymous" in the sense that no agent (both in and out of the objection) envy the net-trade of others. In general, it is not immediate how the bargaining set changes when one imposes alternative equitability restrictions on objections and counterobjections.

The results in Section 4 provide conditions under which the equitable bargaining set coincides with the core or with the set of competitive allocations in economies with influential traders. The paragraph 4.2 studies mixed markets with both an ocean of small traders and some large influential ones, and the equivalence result therein requires some restrictions on the large traders. As pointed out in Proposition 4.6, these restrictions follow from some classical assumptions in the literature on mixed markets. In a recent series of works on noncoperative oliogopolies, however, Busetto, Codognato, Ghosal, Julien, and Tonin (2018) showed that some Cournot-Walras equivalence result can be obtained for mixed markets by imposing conditions not on the large traders, but on the preferences and endowments of the negligible ones. Similar considerations are also in Busetto, Codognato, Ghosal, Julien, and Turchet (2022) and Busetto, Codognato, Ghosal, and Turchet (2023). It would be of interest to explore if a similar approach could be used also for equivalences regarding the bargaining set.

The results in Section 4.3 prove the equivalence between the equitable bargaining set and a subset of the core under the assumption that there exist some *leaders*. This, in turn, implies that the equitable bargaining set of a finite economies with leaders converges to the set of competitive allocations when the economy is replicated (see Remark 4.16). It remains unclear whether there are conditions under which a similar limit result can be obtained without recurring to the equivalence between the core and the equitable bargaining set.

A Appendix: Proofs

A.1 Proofs of the results in subsection 3.2

The proof of Teorem 1 relies on two technical results which we discuss separately.

The first lemma states that it is possible to modify any objection (B, y) to a $x \in \tilde{\mathcal{M}}$ into an objection that assigns identical bundles to agents of the same type, provided that their preferences are convex. When (B, y) is equitable in the first place, also the modified objection can be taken equitable. This lemma covers the second statement in Theorem 1

Lemma A.1 *Let x have the ETP and let $(B, y) \in Ob(x)$. Then B objects to x via a y' that is constant on each type of agents with convex preferences. If, in addition, y has the ETP on B then y' can be chosen with the ETP on B, and such that $y'(t) \succcurlyeq_t y(t)$ for a.e. $t \in B$.*

Proof. Let $\mathcal{K}_1(B)$ be the set of all types of agents K such that agents in K have convex preferences and that $\lambda(B \cap K) > 0$.

For every $K \in \mathcal{K}_1(B)$, let $B_K = B \cap K$ and let y_K be the average bundle assigned by y to the agents in B_K, i.e. the vector:

$$y_K = \frac{1}{\lambda(B_K)} \int_{B_K} y \, d\lambda.$$

Since x has the ETP and y is weakly preferred to x by a.e. $t \in B$, we have $x(t) \sim_t x(s) \preccurlyeq_t y(s)$ whenever $t, s \in B$ are of the same type. An application of García-Cutrín and Hervés-Beloso (1993, Lemma, page 580) gives that $y_K \succcurlyeq_t x(t)$ for a.e. $t \in B_K$, and that $\lambda(\{t \in B_K : y_K \succ_t x(t)\}) > 0$ if $\lambda(\{t \in B_K : y(t) \succ_t x(t)\}) > 0$. Define $y'(t) = y_K$ if $t \in B_K$ for some $K \in \mathcal{K}_1(B)$, and $y'(t) = y(t)$ otherwise. Then B attains y', a.e. agent in B finds y' at least as good as x and a non-null group of agents in B strictly prefers y' to x. We conclude that (B, y') is the desired objection.

For the last part of the statement, observe that if y has the ETP on B then we can apply García-Cutrín and Hervés-Beloso (1993, Lemma, page 580) to y instead of x and find that $y'(t) \succcurlyeq_t y(t)$ for a.e. $t \in B$. ■

Next lemma states that any group of agents within an objection can propose a redistribution of the resources in which they appropriate all the gains from the trades.

Lemma A.2 *Let $x \in \mathcal{M}$, $(B, y) \in Ob(x)$ and let $B' \subseteq B$ be non-null. Then there exists a $\tilde{y}$ such that:*

1. $(B, \tilde{y}) \in Ob(x)$*;*

2. $\tilde{y}(t) \sim_t x(t)$ *for a.e.* $t \in B \setminus B'$*;*

3. $\tilde{y}(t) \succ_t y(t)$ *for a.e.* $t \in B'$ *whenever* $\lambda(\{t \in B \setminus B' : y(t) \succ_t x(t)\}) > 0$.

In addition, if y *is constant on* B' *then* $\tilde{y}$ *can be taken constant on* B' *too.*

Proof. The idea of the proof is to reduce gradually each of the bundles that y assigns to agents in $B \setminus B'$ until we obtain an allocation $y_\infty \leqslant y$ that a.e. $t \in B \setminus B'$ finds equivalent to x. This produces a surplus vector $\tilde{v} = \int_{B \setminus B'} (y - y_\infty)\, d\lambda$, which is strictly positive when $\lambda(\{t \in B \setminus B' : y(t) \succ_t x(t)\}) > 0$. We can then define $\tilde{y}$ as the allocation that assigns y_∞ to all $t \in B \setminus B'$ and redistributes $\tilde{v}$ among the agents in B'. In formulas:

$$\tilde{y}(t) = \begin{cases} y(t) + \frac{1}{\lambda(B')}\tilde{v}, & \text{if } t \in B', \\ y_\infty(t), & \text{otherwise.} \end{cases}$$

Clearly, if y is constant on B' so is $\tilde{y}$. To see that $\tilde{y}$ is the desired allocation observe that B attains $\tilde{y}$, that $\tilde{y}(t) \sim_t x(t)$ for a.e. $t \in B \setminus B'$ and that $\tilde{y}(t) \succcurlyeq_t y(t)$ for a.e. $t \in B'$, with a strict preference if $\lambda(\{t \in B \setminus B' : y(t) \succ_t x(t)\}) > 0$. This last point also implies that a non-null subset of B' finds $\tilde{y}$ strictly better than x, and so $(B, \tilde{y})$ is an objection to x.

We only have to prove that such y_∞ exists. Set $B_1 = B$ and $y_1 = y$. For every $n \geqslant 1$, define recursively a set B_n and an allocation y_n as follows:

$$B_{n+1} = \left\{t \in B \setminus B' : y_n(t) - 2^{-n} y(t) \succcurlyeq_t x(t)\right\},$$

$$y_{n+1}(t) = \begin{cases} y_n(t) - 2^{-n} y(t), & \text{if } t \in B_{n+1}, \\ y_n(t), & \text{otherwise.} \end{cases}$$

The sequence of the y_n's converges pointwise to an allocation y_∞ that a.e. $t \in B$ finds at least as good as x by the continuity of preferences. For a.e. $t \in B \setminus B'$ one has $y(t) \geqslant y_\infty(t)$, with a strict inequality when $y(t) \succ_t x(t)$, and so $\int_{B \setminus B'} (y - y_\infty)\, d\lambda > 0$ if and only if $\lambda(\{t \in B \setminus B' : y(t) \succ_t x(t)\}) > 0$. We claim that $y_\infty(t) \sim_t x(t)$ for a.e $t \in B \setminus B'$.

Assume by contradiction that $y_\infty(t) \succ_t x(t)$ for some $t \in B \setminus B'$. By the monotonicity and continuity of preferences, $y_\infty(t) - 2^{-n} y(t) \succ_t x(t)$ for some n. This implies that $t \in B_k$ for all $k > n$, because $y_k(t) - 2^{-k} y(t) \geqslant y_\infty(t) - 2^{-n} y(t)$. Let m be the largest index such that $t \notin B_m$. Then $y_{m-1}(t) = y_m(t)$ and $y_{k+1}(t) = y_k(t) - 2^k y(t)$ for every $k \geqslant m$. These equations together give:

$$y_\infty(t) = y_m(t) - \sum_{k=m}^{\infty} 2^{-k} y(t) = y_{m-1}(t) - 2^{-(m-1)} y(t).$$

Having assumed that $y_\infty(t) \succ_t x(t)$, we conclude that $y_{m-1}(t) - 2^{-(m-1)} y(t) \succ_t x(t)$ in contradiction with the fact that $t \notin B_m$. ■

We now have all the ingredients to prove Theorem 1.

Theorem A.3 (Theorem 1) *Let x be a non-core allocation with the ETP. Then there exists an equitable objection $(\tilde{B}, \tilde{y})$ to x. Furthermore, $\tilde{y}$ can be chosen so that $\tilde{y}(t) = \tilde{y}(s)$ whenever $t, s \in B$ are agents of the same type with convex preferences.*

Proof. Let $(B, y) \in Ob(y)$. If (B, y) is equitable, then we may apply Lemma A.1 to (B, y) and find a y' with the ETP on B such that $(B, y') \in Ob(x)$ and that $y'(t) = y'(s)$ for every agent $s, t \in B$ with convex preferences. If (B, y) is not equitable, we show that we can modify (B, y) into an equitable objection to x.

If $B \subseteq T_0$, then the restricted economy $\mathcal{E}|_B$, the one that considers only the agents in B, is atomless. If y, thought as an allocation in $\mathcal{E}|_B$, is a competitive allocation then y hat the ETP on B and so (B, y) is equitable. Otherwise, Mas-Colell (1985, Proposition 7.3.2(ii)) implies that there exists a $(\tilde{B}, \tilde{y}) \in Ob(y)$ such that $\tilde{B} \subseteq B$ and $\tilde{y}$ is a competitive allocation in the economy $\mathcal{E}|_{\tilde{B}}$. It follows that $(\tilde{B}, \tilde{y})$ is an objection to x too and so, having $\tilde{y}$ the ETP on $\tilde{B}$, it is equitable.

Suppose now that B contains an atom A, and let B' be the set of agents in B of the same type as A. Lemma A.1 ensures that we can replace (B, y) with an objection (B, y') such that y' is constant on B'. An application of Lemma A.2 to (B, y') gives that there is a $(B, \tilde{y}) \in Ob(x)$ such that $\tilde{y}(t) \sim x(t)$ for all $t \notin B'$ and that $\tilde{y}(t)$ is constant on B'. Since said $\tilde{y}$ has the ETP on B', we conclude that $(B, \tilde{y})$ is an equitable objection to x. ∎

A.2 Proofs of the results in subsection 3.3

Proposition A.4 (Proposition 3.9) *Let $x \in \tilde{\mathcal{M}}$. Then every weakly-Walrasian objection to x is equitable.*

Proof. Let $(B, y) \in Ob(x)$ be weakly-Walrasian. Since x has the ETP, (B, y) is equitable if and only if y has the ETP on B.

For a.e. $t, s \in B$ of the same type, we have $y(t) \in \xi(t, p)$ and $y(s) \in \xi(s, p)$. In particular, having t and s identical endowments, $y(s) \in \beta(t, p)$, and so $y(t) \succsim_t y(s)$. We conclude that y has the ETP on B and so, having x the ETP. ∎

Lemma A.5 (Lemma 3.10) *Let $x \in \mathcal{M}$ and let $(B, y) \in Ob(x)$ be weakly-Walrasian at the price p. If $(C, z) \in Cob^x_e(B, y)$, then:*

1. *$z(t) \in \xi(p, t)$ for a.e. $t \in C$;*

2. *$\lambda(B \cap C) = 0$.*

Proof. We first show that for a.e. $t \in C$ we have $z(t) \succsim_t \xi(p, t)$, with a strict preference if $t \in B$. To this end, being t indifferent between all the bundles in $\xi(t, p)$, it is enough to show that $z(t) \succsim_t v$ (resp. $z(t) \succ_t v$) for some $v \in \xi(t, p)$.

If $t \in C \cap B$, $z(t) \succ_t \xi(t, p)$ follows from the facts that $z(t) \succ_t y(t)$ by the definition of counterobjections, and that $y(t) \in \xi(t, p)$ by property $(WO1)$ of weakly-Walrasian objections. If $t \in C \setminus B$ we have two possibilities: if the set $B_t = \{s \in B : t \sim s\}$ is null, then

$x(t) \succcurlyeq_t \xi(t,p)$ by property $(WO2)$, and $z(t) \succ_t x(t)$ by the properties of counterobjections. Otherwise, if $\lambda(B_t) > 0$, for a.e. $s \in B_t$ we have that $y(s) \in \xi(s,p)$ by $(WO1)$ and that $z(t) \succcurlyeq_t y(s)$ by property $(EC2)$ of equitable counterobjections. But $\xi(s,p) = \xi(t,p)$ because t and s are of the same type, and so $z(t) \succcurlyeq_t \xi(t,p)$.

To prove point (2), let us observe that:

$$\int_{C\cap B} p\cdot(z-\omega)\,d\lambda + \int_{C\setminus B} p\cdot(z-\omega)\,d\lambda = p\cdot\left(\int_C (z-\omega)\,d\lambda\right) \leqslant 0 \tag{2}$$

where the last inequality follows from the fact that (C,z) is a counterobjection (and so $\int_C (z-\omega)\,d\lambda \leqslant 0$) and that $p \gg 0$. On the other hand, the argument above gives that $p\cdot(z(t)-\omega(t)) \geqslant 0$ for a.e. $t \in C$, with a strict inequality for a.e. $t \in C \cap B$. This means that:

$$\int_{C\cap B} p\cdot(z-\omega)\,d\lambda + \int_{C\setminus B} p\cdot(z-\omega)\,d\lambda \geqslant 0. \tag{3}$$

Equations (2) and (3) give that $p\cdot(z(t)-\omega(t)) = 0$ for a.e. $t \in C$, from which we conclude that $\lambda(B\cap C) = 0$ and that $z(t) \in \xi(t,p)$ for a.e. $t \in C$. ■

We are now ready to prove Proposition 3.11. The idea of the proof is the following: we use Zorn's Lemma to find the largest family $\{(C_j,z_j)\}_j$ of pairwise disjoint, equitable counterobjections to a weakly-Walrasian $(B,y) \in Ob(x)$. Such a family must be at most countable, and point (2) in Lemma 3.10 gives that each one of the C_j's is disjoint from (B,y). Therefore, we can "sew" (B,y) with all the (C_j,z_j) into a large objection $(\tilde{B},\tilde{y})$ and we show that it is itself a weakly-Walrasian objection to x. To conclude the proof, we argue that any equitable counterobjection (C,z) to $(\tilde{B},\tilde{y})$ is an equitable counterobjection to (B,y) that is disjoint from all the C_j's, thus the existence of such a (C,z) violates the maximality of the family $\{(C_j,z_j)\}_j$.

Proposition A.6 (Proposition 3.11) *If $x \in \tilde{\mathcal{M}}$ is blocked with a weakly-Walrasian objection then there exists an e-justified objection to it, and hence $x \notin BS_e$.*

Proof. Let $x \in \tilde{\mathcal{M}}$ and (B,y) be a weakly-Walrasian objection to x. Let $\mathcal{F}$ be the set of all coalitions that can raise equitable counterobjections to (B,y). If $\mathcal{F} = \emptyset$ then (B,y) is e-justified. Assume $\mathcal{F} \neq \emptyset$ and let $\mathfrak{A}$ be the class of antichains[5] in $\mathcal{F}$ ordered by inclusion. For any totally ordered subset $(\mathcal{A}_j)_j$ in $\mathfrak{A}$, the union $\bigcup_j \mathcal{A}_j$ is still an antichain, and hence an upperbound for $(\mathcal{A}_j)_j$ in $\mathfrak{A}$. From Zorn's Lemma we conclude that there exists a maximal antichain $\mathcal{A}'$ in $\mathcal{F}$.

The family $\mathcal{A}'$ is formed by disjoint coalitions of positive measure and so it must be at most countable. Enumerate the elements of $\mathcal{A}'$ as a sequence $(C_n)_{n\in\mathbb{N}}$. By construction, each C_n is in $\mathcal{F}$ and so there exists an allocation z_n such that (C_n,z_n) is an equitable

[5]Let $(\mathcal{S},\subseteq)$ be a poset. Two elements A and B of $\mathcal{S}$ are called incompatible if neither $A \subseteq B$ nor $B \subseteq A$, that is if there is no order relation between them. An antichain of $\mathcal{S}$ is a subset $\mathcal{S}'$ of $\mathcal{S}$ in which each pair of different elements is incomparable.

counterobjection to (B, y). Let $\tilde{B} := B \cup (\bigcup_n C_n)$ and let $\tilde{y}$ be an allocation that assigns $y(t)$ to each $t \in B$ and $z_n(t)$ to each $t \in C_n$. The allocation $\tilde{y}$ is well defined because $\lambda(C_n \cap B) = 0$ for each n by Lemma 3.10, and it satisfies $\tilde{y}(t) \in \xi(p, t)$ for a.e. $t \in B$. We claim that $(\tilde{B}, \tilde{y})$ is a e-justified, weakly-Walrasian objection.

First we show that $(\tilde{B}, \tilde{y}) \in Ob(x)$. To prove that $\tilde{B}$ attains $\tilde{y}$ notice that:

$$\int_{\tilde{B}} (\tilde{y} - \omega)\, d\lambda = \int_B (y - \omega)\, d\lambda + \sum_n \int_{C_n} (z_n - \omega)\, d\lambda \leqslant 0.$$

To prove that $\tilde{y}(t) \succcurlyeq_t x(t)$ for a.e. $t \in \tilde{B}$ recall that either $t \in C_n$ for some n, or $t \in B$. In the first case, $\tilde{y}(t) = z_n(t)$ and so $z_n(t) \succ_t x(t)$ because $(C_n, z_n) \in Cob^x(B, y)$; in the latter case $\tilde{y}(t) \succcurlyeq x(t)$ because $(B, y) \in Ob(x)$. Last, observe that $\{t \in \tilde{B} : \tilde{y}(t) \succ_t x(t)\}$ contains $\{t \in B : y(t) \succ_t x(t)\}$, and that this is non-null subset of because $(B, y) \in Ob(x)$.

We show that $(\tilde{B}, \tilde{y})$ is weakly-Walrasian at the price p. To prove $(WO1)$, notice that for a.e. $t \in \tilde{B}$ either $t \in B$, or $t \in C_n$ for some n. In the first case, $\tilde{y}(t) = y(t)$ and $y(t) \in \xi(t, p)$ because (B, y) is weakly-Walrasian at p; in the second case $\tilde{y}(t) = z_n(t)$ and $z_n(t) \in \xi(t, p)$ because $(C_n, z_n) \in Cob_e^x(B, y)$ and so Lemma 3.10 applies. To prove $(WO2)$ observe that a.e. $t \notin \tilde{B}$ such that $\lambda(\{s \in \tilde{B} : s \sim t\}) = 0$ is also such that $\lambda(\{s \in B : s \sim t\}) = 0$. Being (B, y) weakly-Walrasian at the price p, it must be that $x(t) \succcurlyeq_t \xi(t, p)$.

Last, we prove that $(\tilde{B}, \tilde{y})$ is e-justified. Suppose by contradiction that (C, z) is an equitable counterobjection to $(\tilde{B}, \tilde{y})$. Having proved that $(\tilde{B}, \tilde{y})$ is a weakly-Walrasian objection to x, Lemma 3.10 gives that $\lambda(C \cap \tilde{B}) = 0$. But then C is a coalition in $\mathcal{F}'$ that is disjoint from any C_n, violating the maximality of the family $(C_n)_n$. ∎

A.3 Proofs of the results in subsection 4.1

Proposition A.7 (Proposition 4.3) *Let $T = T_0$ and assume that Assumption 4.2 holds. Suppose that $x \in \mathcal{M}$ and let $(B, y) \in Ob(x)$. Then, for every $\delta \in (0, 1)$ there exists $B^\delta \subseteq B$ such that:*

(i) $\lambda(B^\delta \cap K_n) = \delta \lambda(B \cap K_n)$ *for every n;*

(ii) $(B^\delta, y) \in Ob(x)$ *and:*

- *(iia) if (B, y) is equitable, then (B^δ, y) is equitable too;*
- *(iib) if (B, y) is weakly-Walrasian, then (B^δ, y) is weakly-Walrasian too.*

Proof. For every n, consider the (possibly null) set $B_n = K_n$ consisting of the agents of type n in B, then divide B_n in the set $B_n^\succ = \{t \in B_n : y(t) \succ_t x(t)\}$ and $B_n^\sim = \{t \in B_n : y(t) \sim_t x(t)\}$. By construction, B is the disjoint union $(\bigcup_n B_n^\succ) \cup (\bigcup_n B_n^\sim)$ and $\lambda(B) = \sum_n [\lambda(B_n^\succ) + \lambda(B_n^\sim)]$. In particular, there must be a $\bar{n}$ for which $B_{\bar{n}}^\succ$ is not null, otherwise a.e. $t \in B$ would be indifferent between y and x and (B, y) would not be an objection to x.

Consider now the measure $\eta\colon \Sigma \to \mathbb{R}^{m+1}$ that assigns to each $S \in \Sigma$ the vector:

$$\eta(S) = \left(\int_S (y-\omega)\, d\lambda, \lambda(S) \right).$$

Having assumed that $T = T_0$, η is an atomless measure and Lyapunov's Theorem applies. In particular, for every $\delta \in (0,1)$ we can find $S_n^{\succ} \subseteq B_n^{\succ}$ and $S_n^{\sim} \subseteq B_n^{\sim}$ such that $\eta\left(S_n^{\succ}\right) = \delta\eta\left(B_n^{\succ}\right)$ and $\eta\left(S_n^{\sim}\right) = \delta\eta\left(B_n^{\sim}\right)$. Notice that all the $S_n^{\succ}$'s and the $S_n^{\sim}$'s are pairwise disjoint, and that $S_{\bar{n}}^{\succ}$ is non-null.

For every n, let us put $S_n = S_n^{\succ} \cup S_n^{\sim}$. This means that:

$$\lambda(S_n) = \lambda\left(S_n^{\succ}\right) + \lambda(S_n^{\sim}) = \delta\lambda\left(B_n^{\succ}\right) + \delta\lambda\left(B_n^{\sim}\right) = \delta\lambda(B_n) \tag{4}$$

and that:

$$\begin{aligned} \int_{S_n} (y-\omega) d\lambda &= \int_{S_n^{\succ}} (y-\omega) d\lambda + \int_{S_n^{\sim}} (y-\omega) d\lambda = \\ &= \delta \int_{B_n^{\succ}} (y-\omega) d\lambda + \delta \int_{B_n^{\sim}} (y-\omega) d\lambda = \delta \int_{B_n} (y-\omega) d\lambda. \end{aligned} \tag{5}$$

We claim that $B^\delta = \bigcup_n S_n$ is the desired coalition. Condition (i) follows from Equation (4), given that $B_n = B \cap K_n$ for every n. We focus on (ii).

First we show that $(B^\delta, y) \in Ob(x)$. To prove that B^δ attains y, use Equation (5) to write:

$$\int_{B^\delta} (y-\omega)\, d\lambda = \sum_n \int_{S_n} (y-\omega)\, d\lambda = \sum_n \delta \int_{B_n} (y-\omega)\, d\lambda = \delta \int_B (y-\omega)\, d\lambda.$$

Since B attains y, the last term of the equation is smaller or equal to 0, and so B^δ attains y too. The fact that $y(t) \succcurlyeq_t x(t)$ for a.e. $t \in B^\delta$ follows from the inclusion $B^\delta \subseteq B$ and the fact that (B, y) is itself an objection to x. Last, observe that $S_{\bar{n}}$ is a non-null subset of B^δ with the property that every $t \in S_{\bar{n}}$ strictly prefers $y(t)$ to $x(t)$.

To prove (iia) suppose that (B, y) is equitable. Since $B^\delta \subseteq B$, for a.e. $t, s \in B^\delta$ of the same type we have that $y(t) \succcurlyeq_t y(s)$ because $s, t \in B$ and (B, y) is equitable. Thus, (B^δ, y) satisfies $(EO1)$. On the other hand, for a.e. $t \in B^\delta$ and $s \notin B^\delta$ of the same type we have two possibilities: if $s \notin B$ then $(EO2)$ applied to (B, y) gives that $y(t) \succcurlyeq_t x(s)$; if $t \in B \setminus B^\delta$ then $y(t) \succcurlyeq_t y(s)$ because (B, y) is equitable, and $y(s) \succcurlyeq_t x(s)$ because (B, y) is an objection to x and $\succcurlyeq_s = \succcurlyeq_t$. We conclude that (B^δ, y) satisfies also condition $(EO2)$ and so it is equitable.

Last, to prove (iib) suppose that (B, y) is weakly Walrasian at a price p. Being $B^\delta \subseteq B$ we have that $y(t) \in \xi(t, p)$ for a.e. $t \in B^\delta$, and so $(WO1)$ is met. To show that $(WO2)$ is also satisfied observe that $\lambda(B \cap K_n) = \frac{1}{\delta}\lambda(S \cap K_n)$ for every n, meaning that every type of agent that is represented in B is also represented in B^δ. This implies that for a.e. $t \notin B^\delta$, if $\lambda\left(\left\{s \in B^\delta : s \sim t\right\}\right) > 0$ then $\lambda\left(\{s \in B : s \sim t\}\right) > 0$ and so $x(t) \succcurlyeq_t \xi(t, p)$ because (B, y) satisfies $(WO2)$. ∎

A.4 Proofs of the results in subsection 4.2

Proposition A.8 (Proposition 4.6) *Assumption 4.5 is satisfied if one of the following two conditions holds.*

1. *There are infinitely many atoms and they are all of the same type.*
2. *There are finitely many types of atoms, and for each atom A there is a coalition $S_A \subseteq T_0$ such that every $t \in S_A$ is of the same type of A.*

Proof. Fix a sequence $(\mu_n)_n$ in $(0, 1)$.

If (1) is satisfied, there are infinitely many atoms $(A_i)_i$ and they are all subsets of some $K_{\bar{n}}$. This implies that $\lambda(A_i) \to 0$, and so we can choose an atom A such that $\lambda(A) \leqslant \mu_{\bar{n}}\lambda(K_{\bar{n}})$. Let $\delta = \lambda(A)/\mu_{\bar{n}}\lambda(K_{\bar{n}})$, then put $F_{\bar{n}} = A$. For every $n \neq \bar{n}$, the set K_n is atomless, and so there exists $F_n \subseteq K_n$ such that $\lambda(F_n) = \delta\mu_n\lambda(K_n)$. But then $(F_n)_n$ is the desired sequence.

Suppose now that (2) holds. This means that there are finitely many $n_1, \ldots, n_j$ with the property that $T_1 \subseteq \bigcup_{i \leqslant j} K_{n_i}$ and that the set $S_i = K_{n_1} \cap T_0$ is non-null for every $i = 1, \ldots, j$. Put $\delta = \min_{i \leqslant j} \frac{\lambda(S_i)}{\mu_i \lambda(K_{n_i})}$ if this is smaller or equal to 1, otherwise put $\delta = 1$. We can define the sequence $(F_n)_n$ as follows: if $n \notin \{n_1, \ldots, n_k\}$ then K_n is atomless and so there exists a $F_n \subseteq K_n$ such that $\lambda(F_n) = \delta\mu_n\lambda(K_n)$; otherwise, if $n = n_i$ for some $i = 1, \ldots, j$, S_i is atomless and such that $\lambda(S_i) \geqslant \delta\mu_{n_i}\lambda(K_{n_i})$. We can then take $F_{n_i} \subseteq S_i$ such that $\lambda(F_{n_i}) = \delta\mu_{n_i}\lambda(K_{n_i})$. ∎

The splitted economy associated with the mixed market For the proof of Proposition 4.7 we associate to the mixed market an atomless economy a' la manner of Greenberg and Shitovitz (1986). To do this, we replace each atom $A \in \Sigma$ with a coalition A^* with the same measure, formed by a continuum of negligible agents of the same type as A.

Formally, we build the atomless measure space of agents $(T^*, \Sigma^*, \lambda^*)$ as follows. For every atom $A \in \Sigma$ let $(A^*, \Sigma_{A^*}, \lambda_{A^*})$ be an atomless measure space with $\lambda_{A^*}(A^*) = \lambda(A)$. Then define the set T as the union $T_0 \cup (\bigcup_A A^*)$, the algebra Σ^* as the product of Σ_{T_0} and each of the Σ_{A^*}'s, and λ^* as the product of λ restricted to T_0, and each of the λ_{A^*}'s. Being every $t \in T_0$ an agent in the mixed marked as well as in the original economy, she is already given a preference relation and an endowment bundle. For $t \in A^*$, let $\succcurlyeq_t$ and $\omega(t)$ coincide with $\succcurlyeq_s$ and $\omega(s)$ for any $s \in A$ (recall that all agents in A have the same preferences and endowments).

The atomless economy we defined is called the *splitted economy* associated with the mixed market. It meets all the main assumptions of the model, and satisfies Assumption 4.2 when the original mixed market does. Let $\mathcal{M}^*$ be the set of allocations in the splitted economy, and call $\mathcal{W}^*$ and BS_e^* the corresponding set of competitive allocations and equitable bargaining set.

There is a natural correspondence between allocations in the mixed market and in the splitted economy. Precisely, each $x \in \mathcal{M}$ defines an allocation $x^* \in \mathcal{M}^*$ by:

$$x^*(t) = \begin{cases} x(t) & if\ t \in T_0, \\ x(A) & if\ t \in A^*,\ \text{forsomeatom}\ A \in \Sigma \end{cases}$$

where $x(A)$ is the bundle that x assigns to any agent in A (being x measurable and A an atom, this must be the same for all $t \in A$). Conversely, to each $f \in \mathcal{M}^*$ we can associate a $x_f \in \mathcal{M}$ defined by:

$$x_f(t) = \begin{cases} f(t) & if\ t \in T_0, \\ \frac{1}{\lambda^*(A^*)} \int_{A^*} f(t) d\lambda^* & if\ t \in A \text{ for som atom } A \in \Sigma. \end{cases}$$

Clearly, $x_{x^*} = x$. Greenberg and Shitovitz (1986) shows that x is competitive or has the ETP if and only if x^* is competitive or has the ETP respectively. On a similar line, if f is competitive or has the ETP then so does x_f.

To every coalition $S \in \Sigma$ we associate a *splitted coalition* $S^* \in \Sigma^*$ defined by $S^* = \left(\bigcup_{A \in S} A^*\right) \cup (S \cap T_0)$. This relationship can be used to transfer objections from the mixed market to the splitted economy, and viceversa.

Lemma A.9 *Let $x, y \in \mathcal{M}$ and $B \in \Sigma$. Then $(B, y) \in Ob(x)$ if and only if $(B^*, y^*) \in Ob(x^*)$. Furthermore:*

(i) (B, y) Walrasian at a price p if and only if (B^*, y^*) is Walrasian at p;

(ii) (B, y) weakly-Walrasian at a price p if and only if (B^*, y^*) is weakly-Walrasian at p.

Proof. By construction, B attains y if and only if B^* attains y^*. Furthermore, for a generic $S \in \Sigma$, every $t \in S$ finds $y(t) \succ_t x(t)$ (resp. $y(t) \succcurlyeq_t x(t)$) if and only if every $t \in S^*$ finds $y^*(t) \succ_t x^*(t)$ (resp. $y^*(t) \succcurlyeq_t x^*(t)$). Thus, a.e. agents in B weakly prefer y to x (and some strictly prefer) if and only if a.e. agents in B^* weakly prefer y^* to x^* (and some strictly prefer). We conclude that $(B, y) \in Ob(x)$ if and only if $(B^*, y^*) \in Ob(x^*)$.

To prove the other two claims it is enough to observe that, if $S = \{t \in T : y(t) \in \xi(t, p)\}$ then $S^* = \{t \in T^* : y^*(t) \in \xi(t, p)\}$. ■

According to the lemma above, if there exists a Walrasian objection to a $x \in \mathcal{M}$ this induces a Walrasian objection to x^* in the splitted economy, and so $x^* \notin BS_e^*$ by Proposition 3.11. The converse may not be true, for there may be Walrasian objections to x^* in the splitted economy that are not induced by any objection in the original mixed market. This is because, given $S \in \Sigma^*$ and $f \in \mathcal{M}^*$ such that $(S, f) \in Ob(x^*)$, it is possible that no $B \in \Sigma$ is such that $B^* = S$. Next lemma shows that, under our additional Assumptions, it is possible to obtain a partial converse to this argument: if there exists a Walrasian objection in the splitted economy, then there exists a weakly-Walrasian one in the original mixed market.

Proposition A.10 *Under Assumptions 4.2 and 4.5, let $x \in \tilde{\mathcal{M}}$. If there exists a Walrasian objection (S, f) to x^*, then there exists a weakly-Walrasian objection (B, y) to x.*

Proof. TBA ■

Proposition A.11 (Proposition 4.7) *Under Assumptions 4.2 and 4.5, if $x \in \tilde{\mathcal{M}}$ is a non-competitive feasible allocation, there exists a weakly-Walrasian objection to it.*

Proof. Let $x \in \tilde{M}$ be a feasible, non-competitive allocation and let $x^* \in \mathcal{M}^*$ be the corresponding allocation in the splitted economy. Then x^* is feasible, non-competitive and has the ETP. By Mas-Colell (1989, Proposition 2) there exists a $S \in \Sigma^*$ and a $f \in \mathcal{M}^*$ such that (S, f) is a Walrasian objection to x^* in the splitted economy. Apply Proposition A.10 to (S, f) to find a $B \in \Sigma$ and $y \in \mathcal{M}$ such that (B^*, y^*) is a weakly-Walrasian objection to x^* in the splitted economy, then observe that (B, y) is a weakly-Walrasian objection to x in the original mixed market, by Proposition A.9. We conclude that $x \notin BS_e$, by Proposition 3.11. ■

A.5 Proofs of the results in subsection 4.3

Proposition A.12 (Proposition 4.11) *Under Assumption 4.9, let K^* be a type of leaders formed by finitely many atoms. Then for every $x \in \tilde{\mathcal{M}} \setminus \mathcal{C}$ there exists $(B^*, y^*) \in Ob_e(x)$ with the property that, for every $(B, y) \in Ob_e(x)$:*

$$y^*(t) \succcurlyeq_t y(t), \qquad \text{for every } t \in K^*.$$

Proof. Consider the binary relation $\geqslant$ on $Ob_e(x)$ defined by:

$$(B_1, y_1) \geqslant (B_2, y_2) \iff y_1(t) \succcurlyeq_t y_2(t) \text{ for every } t \in K^*.$$

We claim that $(Ob_e(x), \geqslant)$ is a totally preordered set, separable in the sense of Debreu[6]. First notice that $Ob_e(x)$ is non-empty, for x is a non-core allocation with the ETP, and so Theorem 1 applies. Then observe that for every two (B_1, y_1), $(B_2, y_2) \in Ob_e(x)$, the assignments y_i's have the ETP and so either all agents in K^* find y_1 at least as good as y_2, or they all prefer y_2 to y_1. Last, the separability of $\geqslant$ follows from that of $\succcurlyeq_{K^*}$. We need to prove that there exists a maximal element in $(Ob_e(x), \geqslant)$.

Let (B_n, y_n) be a sequence in $Ob_e(x)$ that is cofinal in the following sense: for every $(B, y) \in Ob_e(x)$ there exists a $n \in \mathbb{N}$ such that $(B_n, y_n) \geqslant (B, y)$. Said sequence exists because $(Ob_e(x), \geqslant)$ is separable. Without loss of generality, we may assume that there are $\alpha, \beta \in \mathbb{R}^m_+$ such that:

$$\lim \int_{B_n} y_n(t)\, dt = \alpha, \quad \lim \int_{B_n} \omega(t)\, dt = \beta.$$

Since each B_n contains at least an atom of type K^* (and there are only finitely many of them), the $\int_{B_n} \omega(t)\, dt$'s are bounded away from 0, and so $\beta > 0$. For each n, let F_n be the

[6] A preordered set $(Z, \leqslant)$ is separable in the sense of Debreu if there is a countable $Q \subset Z$ with the following property: for every $x, y \in Z$ with $z < y$ there is a $q \in Q$ with $z \leqslant q \leqslant y$.

map defined by:

$$F_n(t) = (y_n(t)\chi_{B_n}(t), \omega(t)\chi_{B_n}(t)) \in \mathbb{R}^{2m}.$$

The sequence (F_n) is integrable in its first coordinate, uniformly integrable in its second one, and it is such that $\lim \int F_n \, dt = (\alpha, \beta)$. By Fatou Lemma (see Hildenbrand (1974, Lemma D.3)) there is a subsequence of (F_n) (which we do not re-label) that converges pointwise to an integrable function $F(t) = (f_1(t), f_2(t))$ with the property that $\int f_1 \, dt \leqslant \alpha$ and $\int f_2 \, dt = \beta$.

Let $B^* = \{t \in T : f_2(t) = \omega(t)\}$ and observe that $t \in B^*$ if and only if $t \in B_n$ for n sufficiently large. This implies that $f_1(t) = \lim y_n(t)$ for a.e. $t \in B^*$. Let y^* be an assignment with the ETP that coincides with f_1 on B^* (such y^* exists because f_1, restricted to B^*, is the pointwise limit of functions with the ETP, and so it assigns equivalent bundles to agents of the same type). We claim that (B^*, y^*) is the desired objection.

We show that $(B^*, y^*) \in Ob(x)$. First observe that B^* is non-null, because $\int_{B^*} \omega \, dt = \int f_2 \, dt = \beta > 0$. Second, notice that B^* attains y^* because:

$$\int_{B^*} (y^* - \omega) \; dt = \int (f_1 - f_2) \; dt \leqslant \lim \int_{B_n} (y_n - \omega) \; dt \leqslant 0.$$

Third, recall that $y^*(t) = \lim y_n(t)$ for a.e. $t \in B^*$, and so $y^*(t) \succcurlyeq_t x(t)$ by the continuity of preferences. In particular, when $t \in B^* \cap K^*$ it must be that $y^*(t) \succ_t x(t)$ (because the y_n are increasingly desirable to agents in K^*). This proves that (B^*, y^*) is an objection to x. The fact that (B^*, y^*) is equitable follows from the ETP of y^* and x^*.

Since the sequence of the (B_n, y_n) is cofinal and $(B^*, y^*) \geqslant (B_n, y_n)$, leaders in K^* find (B^*, y^*) at least as good as any other equitable objection to x. ■

Theorem A.13 (Theorem 4.12) *Under Assumption 4.9, let K^* be a type of leaders formed by finitely many atoms. Then BS_e coincides with the set of core-allocations with the ETP.*

Proof. $BS_e(\mathcal{E})$ contains all core-allocations with the ETP by Theorem 1. To prove the other inclusion, we fix an individually rational $x \notin \mathcal{C}$ with the ETP, and show that $x \notin BS_e$.

Let $(B^*, y^*) \in Ob_e(x)$ be as in Proposition 4.11. We claim that (B^*, y^*) is e-justified. Suppose by contradiction that this was not the case, i.e. that there is a $(C, z) \in Cob_e^x(B^*, y^*)$. Being K^* a type of leaders, $C \cap K^*$ is non-null and so $z(t) \succcurlyeq_t y^*(t)$ for a.e. $t \in K^*$ (because (C, z) is equitable). At the same time, the maximality of (B^*, y^*) gives that $y^*(t) \succcurlyeq_t z(t)$ for every $t \in K^*$. It follows that all leaders of type K^* in C find y^* equivalent to z. Without loss of generality, we may assume that z is constant on $C \cap K^*$ (otherwise apply Lemma A.1).

The coalition C blocks x, which is individually rational, and so it cannot consist only of identical agents with convex preferences. It must be that $\lambda(C \setminus K^*) > 0$.

Notice that (C, z) is an objection to x that assigns identical bundles to agents in $C \cap K^*$. By Lemma A.2, there exists a z^* such that $(C, z^*) \in Ob_e(x)$ and $z^*(t) \succ_t z(t)$ for every $t \in C \cap K^*$. This last part follows from the fact that all $t \in C \setminus K^*$ strictly prefer z to x by definition of counterobjection, and that $\lambda(C \setminus K^*) > 0$ by the argument above, so point

(3) in Lemma A.2 applies. This implies that $z^*(t) \succ_t y^*(t)$ for all $t \in K^*$, violating the maximality of (B^*, y^*). ■

Corollary A.14 (Corollary 4.13) *Under the assumptions of Theorem 4.12, suppose that there are at least two atoms in the economy and that they are all of type K^*. Then $\mathcal{W} = \tilde{C}_e = BS_e = \mathcal{C}$, and these may be strictly contained than the standard bargaining set.*

Proof. The inclusion $\mathcal{W} \subseteq BS_e$ always hold. For the converse inclusion, notice that $BS_e \subseteq \mathcal{C}$ by Theorem 4.12 and that $\mathcal{C} \subseteq \mathcal{W}$ by Shitovitz (1973, Theorem B). ■

B Appendix: Examples

We present now a series of example. Example B.1 considers an economy where there exists a core allocation without the ETP, proving that $\mathcal{W} \subsetneq \tilde{C}_e$. In the same economy, Example B.2 describes a core allocation without the ETP, proving that BS_e may not contain the core, in which case it is a strict subset of BS. Last, Example B.3 considers an economy where the equitable bargaining set contains non-core allocations with the ETP, proving that: (i) BS_e may not be a subset of the core and (ii) $\tilde{C}_e$ may be strictly contained in BS_e.

Example B.1 *There are two commodities and only three large traders A, B and C with identical weights (i.e. T is formed by three atoms of measure $\frac{1}{3}$ each). Agents' preferences are all derived from the same utility function $u(a,b) = \sqrt{ab}$, while the initial endowments is the function:*

$$\omega(A) = (9,1)\,, \quad \omega(B) = (9,1)\,, \quad \omega(C) = (2,18)\,.$$

There are only two types of agents in the economy: one formed by A and B, and one by C alone. The only competitive allocation is the function z that assigns to agents in A, B and C the bundles:

$$x_A = (5,5)\,, \quad x_B = (5,5)\,, \quad x_C = (10,10)\,.$$

Define y as the function:

$$y_A = (6,6), \quad y_B = (6,6), \quad y_C = (8,8).$$

Then y is a non-competitive allocation with the ETP. However, y is not objected and so it belongs to $\tilde{C}_E$. We conclude that $\mathcal{W} \subsetneq \tilde{C}_e$.

Example B.2 *Let $\mathcal{E}$ and x be as in Example B.1. We claim that there exists a core allocation z without the ETP[7]. This will prove that $C \not\subseteq BS_e$ and so $BS_e \subsetneq BS$.*

Notice that Agents A and C alone cannot achieve the same utility they obtain in equilibrium. To see this, let ξ be the maximum utility that A can reach in the restricted economy

[7] The arguments used here follow directly those in Green (1972).

$\mathcal{E}_{\{A,C\}}$ *while granting* C *the same utility she receives under* x*. Then* ξ *is the solution of the following maximization problem:*

$$\xi := \max_{(a,b)} u(9+a, 1+b) \text{ subject to } \begin{cases} -9 \leqslant a \leqslant 2, \\ -1 \leqslant b \leqslant 18, \\ u(2-a, 18-b) \geqslant u(x_C). \end{cases}$$

Computations show that $u(\omega_A) = 3 < \xi < 5 = u(x_A)$*, which means that there exists individually rational allocation in* $\mathcal{E}_{\{A,C\}}$ *that give* C *the same utility as* x*, but in those allocation* A *is worse off than under* x*. Similar arguments hold for the coalition* $\{B, C\}$*.*

Let $\varepsilon > 0$ *be such that* $\xi + \varepsilon < 5$*. Define* z *as the allocation:*

$$z_A = (5-\varepsilon, 5-\varepsilon)\,, \quad z_B = (5+\varepsilon, 5+\varepsilon)\,, \quad z_C = (10, 10)\,.$$

Notice that z *does not have the ETP, for* $u(z_B) > u(z_A)$*. To prove that* z *is a core allocation we look at coalitions' possibilities of objecting* z*.*

The allocation z *assigns to* A *and* B *a level of utility that is strictly higher than* ξ*, while leaving* C *the same utility as* x*. This implies that* z *is individually rational (and hence it is not blocked by individuals, nor by* $\{A, B\}$*) and that coalitions* $\{A, C\}$ *or* $\{B, C\}$ *cannot improve upon* z *by the argument above. Last, observe that* z *is Pareto-efficient and so it is not blocked by the grand coalition.*

The proof of Shitovitz (1989, Theorem 3) describes an economy where core and competitive allocations coincides and form a strict subset of the (standard) bargaining set. In that example, however, the presence of many agents of the same type causes the equitable bargaining set to coincide with the core. To adapt Shitovitz' result to our purpose we modify the economy in a way reproduces the same structure of coalitions while ensuring that each trader has a different agent type. This will imply that BS_e and BS coincide, because no agents can envy others, and the latter is strictly larger than the core by the same arguments in Shitovitz.

Example B.3 *The space of agents* T *consists of an interval* $(0, 1)$ *of negligible traders (considered with the Lebesgue measure) and two atoms* A_1, A_2 *such that* $\lambda(A_i) = \frac{1}{2}$ *for* $i = 1, 2$*. There are three commodities. Agents have the same utility function* $u(a, b, c) = \sqrt{(a+b)c}$ *and the initial endowment is the allocation:*

$$\omega(t) := \begin{cases} (2t, 2(1-t), 6)\,, & \text{if } t \in (0,1), \\ (4, 2, 2)\,, & \text{if } t = A_1, \\ (2, 4, 2)\,, & \text{if } t = A_2. \end{cases}\,.$$

Since every agent has a different endowment there are no consumers of the same type. This implies that all allocations have the ETP, that all objections (and counterobjections) are *equitable and therefore $BS = BS_e$. We claim that BS, and hence BS_e, is strictly larger than the core.*

Let $\tilde{\mathcal{E}}$ be an exchange economy with two commodities, where T is the measure space of agents, consumers have utility $\tilde{u}(v,c) = \sqrt{vc}$ and the initial endowment is $\tilde{\omega}(t) := (\omega_a(t) + \omega_b(t), \omega_c(t))$. In $\tilde{\mathcal{E}}$ there are two types of agents: those in the interval $(0,1)$ and the two atoms.

Every $x \in \mathcal{M}$ defines an allocation $\tilde{x} \in \mathcal{M}(\tilde{\mathcal{E}})$ by:

$$\tilde{x}(t) = (x_a(t) + x_b(t), x_c(t)) .$$

The map $\phi\colon x \mapsto \tilde{x}$ is surjective and preserves agents' utility, i.e. it is such that $u(x(t)) = \tilde{u}(\tilde{x}(t))$ for every x. Furthermore, ϕ extends to objections and counterobjections in the sense that $(B,y) \in Ob(x)$ if and only if $(B,\tilde{y}) \in Ob(\tilde{x})$, and $(C,z) \in Cob^x(B,y)$ if and only if $(C,\tilde{z}) \in Cob^{\tilde{x}}(B,\tilde{y})$. This implies that $x \in BS$ if and only if $\tilde{x} \in BS(\tilde{\mathcal{E}})$.

Consider now the function x defined by:

$$x(t) := \begin{cases} \left(2 - \frac{1}{20}, 2 - \frac{1}{20}, 4 - \frac{1}{10}\right), & \text{if } t \in (0,1), \\ \left(2 + \frac{1}{20}, 2 + \frac{1}{20}, 4 + \frac{1}{10}\right), & \text{if } t \in \{A_1, A_2\}. \end{cases}$$

Then x is associated with the allocation:

$$\tilde{x}(t) := \begin{cases} \left(4 - \frac{1}{100}, 4 - \frac{1}{10}\right), & \text{if } t \in (0,1), \\ \left(4 + \frac{1}{10}, 4 + \frac{1}{10}\right), & \text{if } t \in \{A_1, A_2\}. \end{cases}$$

Similar computations to those in the proof of Shitovitz (1989, Theorem 3) prove that $\tilde{x} \in BS(\tilde{\mathcal{E}}) \setminus C(\tilde{\mathcal{E}})$. But then the arguments above show that x is a non-core allocation in the standard (and hence equitable) bargaining set.

Example B.4 *There are only two commodities. Agents are the points of the Lebesgue unit interval $T = [0,1]$ and they are all identical: their preferences are derived from the utility function $u(a,b) = a^2 + b^2$ and they are all endowed with the bundle $\omega = (1,1)$.*

Consider the coalitions $A := \left[0, \frac{1}{\sqrt{2}}\right)$ and $B := \left[\frac{1}{\sqrt{2}}, 1\right]$, then define x as the function that assigns to agents in A and B the bundles:

$$x_A := \left(\sqrt{2}, 0\right), \quad x_B := \left(0, \frac{\sqrt{2}}{\sqrt{2}-1}\right).$$

Computations show that x is a feasible allocation and that it gives the utility $u(x_A) = 2$ to agents in A, and $u(x_B) = 6 + 4\sqrt{2}$ to those in B. It follows that x is an individually rational allocation that does not have the ETP, and so it is not competitive nor in the core. We claim that $x \in \mathcal{C}_e$.

Suppose by contradiction that there exists a $(C,y) \in Ob_e(x)$. We may assume that $u(y(t)) \geqslant u(x_B)$ for every $t \in C$, for otherwise there would be a significant share of objecting *agents that envy their peers in B. Let v be:*

$$v := \frac{1}{\lambda(C)} \int_C y(t)\, dt.$$

The vector v belongs to the closed convex hull of $\{y(t) : t \in C\}$ which, in turn, is a subset of $\{(a,b) : \; a+b \geqslant 3\}$. This last inclusion follows from the fact that $u(y(t)) \geqslant u(x_B)$ for every $t \in C$ and hence $y_a(t)^2 + y_b(t)^2 \geqslant 6 + 4\sqrt{2}$. But then v cannot be smaller or equal than $(1,1)$, implying that:

$$\int_C y(t)\, dt \not\leqslant \lambda(C)(1,1) = \int_C \omega(t)\, dt.$$

We conclude that C does not attain y, so (C,y) cannot be an objection to x.

Example B.5 *There are two commodities. The space of agents is $T = \left[0, \frac{3}{5}\right] \cup \{A_1, A_2\}$, where A_1, A_2 are two atoms, each one of size $\frac{1}{5}$, while $\left[0, \frac{3}{5}\right]$ is considered with the Lebesgue measure (and hence is the atomless component T_0 of T). All agents have the same initial endowment $\omega = (1,1)$ and preferences derived from the utility functions:*

$$u_t(a,b) = \begin{cases} 3a + b, & \text{if } t \in \left[0, \frac{1}{2}\right], \\ a + b, & \text{if } t \in \left(\frac{1}{2}, \frac{3}{5}\right] \cup \{A_1, A_2\}. \end{cases}$$

Notice that there are only two types of agents: type 1 is the interval $\left[0, \frac{1}{2}\right]$, while type 2 consists of the two atoms and the coalition of small traders $\left(\frac{1}{2}, \frac{3}{5}\right]$. Since the economy $\mathcal{E}$ meets the assumptions of our main Theorem, we have $\mathcal{W} = BS_e$.

Consider the function x that assigns to agents of type 1 and 2 the bundles:

$$x_1 = \left(\frac{3}{2}, 0\right), \qquad x_2 = \left(\frac{1}{2}, 2\right).$$

Function x defines an allocation with the ETP that is non competitive and so it does not belong to BS_e. To show that $x \in BS$ we use the following auxiliary Lemma.

Lemma B.6 *For a given coalition F, let α_F be the ratio $\lambda(F_1)/\lambda(F_2)$, where F_i denotes the agents in F of type $i = 1,2$. Then, when the economy is restricted to the agents in F, the utility that consumers of type $i = 1,2$ receive in equilibrium is a function $\hat{u}_i$ of α_F. Furthermore:*

(P1) $\hat{u}_1$ decreases with α and it is strictly monotone if and only if $1 \leqslant \alpha \leqslant 3$;

(P2) $\hat{u}_2$ increases with α and it is strictly monotone if and only if $1 \leqslant \alpha \leqslant 3$;

(P3) $\hat{u}_1 \geqslant u_1(x_1)$ if and only if $\alpha_F \leqslant 2$ and $\hat{u}_2 \geqslant u_2(x_2)$ if and only if $\alpha_F \geqslant \frac{3}{2}$.

In particular, F objects x via some $z \in \mathcal{W}(\mathcal{E}_C)$ if and only if $\frac{3}{2} \leqslant \alpha \leqslant 2$.

We are now ready to prove that $x \in BS$. Suppose by contradiction that there exists a justified $(B,y) \in Ob(x)$. Since agents have convex preferences, we may assume that y assigns the same bundle y_i to agents of type $i = 1,2$. Notice in particular that $y \in C(\mathcal{E}_B)$.

Suppose B_2 consists of exactly one atom. Being y a core allocation, Shitovitz (1973, Theorem A) implies that agents in B_1, which are all negligible traders, value y at most as much as what they would get in equilibrium, i.e. $u_1(y_1) \leqslant \hat{u}_1(\alpha_B)$. This, with Property $(P3)$, gives that $\alpha_B \leqslant 2$ and so $\lambda(B_1) \leqslant \frac{2}{5}$. Let $C \subseteq T_0 \setminus B$ be such that $\frac{3}{2} < \alpha_C < 2$ and let $z \in \mathcal{W}(\mathcal{E}_C)$ (such a C exists because part of agents of type 1 are excluded from B and there is a portion of negligible agents of type 2). By Lemma B.6, $(C, z) \in Ob(x)$ and so, being C and B disjoint, $(C, z) \in Cob^x(B, y)$. This contradicts the fact that (B, y) is justified.

If B_2 is not one atom then the economy $\mathcal{E}_B$ meets the requirements for the core-Walras equivalence. The allocation y is then competitive and Lemma B.6 gives $\frac{3}{2} \leqslant \alpha_B \leqslant 2$. This implies, in particular, that B does not include both atoms, for otherwise $\lambda(B_2) \geqslant \frac{2}{5}$ and $\alpha_B \leqslant \frac{5}{4}$. We can then assume that $A_1 \notin B$. We divide the analysis by cases.

- *$\alpha_B = \frac{3}{2}$. This is possible only if $\lambda(B_1) \leqslant \frac{9}{20}$ and so there is a coalition $C \subseteq \left[0, \frac{3}{5}\right]$ such that $C_1 \cap B = \emptyset$ and $\frac{3}{2} < \alpha_C < 2$. Such a C exists because B does not include all agents of type 1 and there is a non-null group of negligible agents of type 2. Let $z \in \mathcal{W}(\mathcal{E}_C)$. Then agents in C strictly prefer z to x (by $(P3)$) while $\alpha_B < \alpha_C$ implies that agents in $B \cap C$ (which are all of type 2) strictly prefer z to y.*

- *$\alpha_B > \frac{3}{2}$. In this case there is a coalition C such that $C_2 = \{A_1\}$ and $\frac{3}{2} < \alpha_C < \alpha_B$. Let $z \in \mathcal{W}(\mathcal{E}_C)$. Then agents in C strictly prefer z to x (by $(P3)$) while $\alpha_C < \alpha_B$ implies that every $t \in B \cap C$ (which is of type 1) strictly prefers z to y.*

In both scenarios, we obtained a counterobjection (C, z) to (B, y), contradicting the fact that (B, y) is justified.

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