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Preliminary Version

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Abstract

We show that a symmetric information Rational Expectations Equilibrium (REE) exists universally (and not generically), it is Pareto efficient and obviously incentive compatible. Agents, in a repeated economy framework, can reach a symmetric information REE (i.e., an efficient and incentive compatible equilibrium outcome) by observing the past asymmetric REE and also by updating their private information. We also prove the converse result, i.e., given a symmetric information REE, we can construct a sequence of approximate asymmetric REE allocations that converges to the symmetric information REE. The approximate REE can be interpreted as the mistakes that agents make due to bounded rationality, nonetheless, in the limit an exact symmetric information REE is reached. In view of the above results, the symmetric information REE provides a rationalization for the asymmetric one.

Keywords: Learning, Rational expectations equilibrium, Asymmetric information, Stability.

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1 Introduction

Debreu introduced uncertainty in the standard Walrasian general equilibrium model (see for example Chapter 7 of the classical treatise, "Theory of Value"). This is the so called "state contingent"model, where agents' preferences and initial endowments depend on the states of nature of the world and agents make contracts ex-ante (before the state of nature is realized) contingent on the exogenously given states of nature of the world. Once the state of nature is realized the previously agreed contract is executed and consumption takes place. For this model to make sense there must be an exogenous enforcer, the government or a court which makes sure that the contract made ex-ante is fulfilled ex-post, otherwise agents may find it beneficial to renege.

Radner (1968) in a seminal paper introduced asymmetric information into the "state contingent"model by allowing each agent to have in addition to his random initial endowment and random utility function, a private information set, which is a partition of the exogenously given state space. By assuming that the net trades are measurable with respect to the private information of each individual the asymmetric information was explicitly introduced in the model of uncertainty.

Kreps (1977), Radner (1979) and Allen (1981) introduced one more notion, called the Rational Expectation Equilibrium (REE), which is also an extension of the deterministic Arrow-Debreu-McKenzie model that allows for asymmetric information. According to the REE, each individual maximizes interim expected utility conditioned on his own private information as well as information that the equilibrium prices have generated.

By now it is well-known that in a finite agent economy with asymmetric information, a rational expectations equilibrium (REE) may not exist (Kreps (1977)), may not be incentive compatible, may not be fully or ex-post Pareto optimal and may not be implementable as a perfect Bayesian equilibrium (Glycopantis and Yannelis (2005) p. 31 and also Example 9.1.1 p. 43). Thus, if the intent of the REE notion is to capture contracts among agents under asymmetric information, then such contracts not only they don't exist universally in well behaved economies (i.e., economies with concave, continuous, monotone utility functions and strictly positive initial endowments), but even if they exist, they fail to have any normative properties, such as incentive compatibility, Pareto optimality and Bayesian rationality. The main conceptual difficulty that one encounters with the REE which creates all the above problems is the fact that individuals are supposed to maximize their interim expected utility conditioned not only on their own private information, but also on the information that the equilibrium prices generate. Since prices are computed on the basis of agents' characteristics, then agents must act as knowing all the characteristics in the economy, which is rather difficult to justify. Perhaps a possible interpretation of the REE concept may be as follows: agents reports all their characteristics to a central planning authority (CPA), i.e., an auctioneer or government. The CPA has all the information needed to compute the equilibrium prices and therefore announces them to all the agents once they are computed. Agents now proceed by maximizing their interim expected utilities based on their own private information and the information the announced equilibrium prices have generated. This optimization of interim utilities by each agent, results in optimal consumption bundles which clear the market for every state of nature, i.e., the sum of the optimal consumption of each agent is equal to their aggregate initial endowment for each state of nature. One may conjecture that if we repeat this process from period to period the asymmetric information may disappear after a large repetition, and all agents will have the same information.

One of our main objectives is to provide a rationalization of the REE which is based on a repeated interim decision making providing the validity of the above conjecture. Indeed, we will show that agents by observing in each period the realized REE outcome, they refine their private information and as time goes on, they reach the symmetric information REE. This is the best outcome that agents can reach and may coincide with the state-contingent Walrasian equilibrium which exists, it is Pareto optimal and clearly incentive compatible.

Furthermore we provide a stability result. We show that any limit symmetric information REE can be approximated by a sequence of approximate REE outcomes. In other words, we can always construct a route indicating how agents reached the symmetric REE. One may view the one shot limit symmetric information REE as a result of the limit of infinitely many repetitions (plays) of asymmetric REE outcomes.

The above results enable us to conclude that the REE does make sense in a repeated framework where agents by observing the realized REE outcome and refining their information, learn how to achieve the limit symmetric information REE. Thus, for all practical purposes we could use the symmetric REE instead of the asymmetric information one, as the symmetric REE, provides a foundation or rationalization for the asymmetric one. The advantage of adopting the symmetric REE is that, it exists universally (and not generically), and obviously it is incentive compatible and interim Pareto optimal, properties that the standard asymmetric information REE fails to have.

The paper proceeds as follows: in Section 2 we describe the model. In Section 3 we give examples of how asymmetrically informed agents may (or may not) learn from the REE prices and allocations. In Section 4 we consider a sequence of repeated economies and describe the corresponding limit economy, then we show in Section 5 that the sequence of REE that emerge in the repetitions approximates a REE in the limit economy. In Section 6 we introduce the *non trivial learning condition* which grants that in the limit economy there exists a REE which is efficient, incentive compatible and implementable as a perfect Bayesian equilibrium of an extensive form game. Under the same condition we show in Section 7 that, in the limit economy, every a REE compatible with the information accumulated in the repetitions is the limit of some sequence of approximated REE that emerge int he repetitions. Finally, we collect in the Appendix some results useful to the discussion.

2 The model

The commodity space is an ordered, separable Banach space Y whose positive cone Y_+ has a nonempty interior. There is a finite or countable set Ω of states of nature, whose realization is uncertain.

An *asymmetric information economy* with commodity space Y and states of nature in Ω is a family $\mathcal{E} = \{(\mathcal{F}_i, X_i, u_i, e_i, q_i) : i \in I\}$ where I is a finite set of agents. For every i it assumes that:

1. $\mathcal{F}_i$ is a σ-algebra on Ω representing i's private information;

2. $X_i \colon \Omega \to 2^{Y_+}$ is an $\mathcal{F}_i$-measurable correspondence that indicates agent i's consumption set in each state;

3. for each $\omega \in \Omega$, $u_i(\omega, \cdot) \colon X_i(\omega) \to \mathbb{R}_+$ is i's *utility function*, which depends on the states;

4. $e_i \colon \Omega \to Y_+$ is an $\mathcal{F}_i$-measurable function specifying for each state $\omega \in \Omega$ the initial endowment vector $e_i(\omega) \in X_i(\omega)$ of agent i;

5. $q_i \colon \Omega \to \mathbb{R}_{++}$ is the prior of agent i, normalized to $\sum_\omega q_i(\omega) = 1$.

An *allocation for agent i* is a summable function $x \colon \Omega \to Y_+$ with the property that $x(\omega) \in X_i(\omega)$ for every $\omega \in \Omega$. We write ℓ_{X_i} for the set of allocations for agent i. Recall that x is *summable* if:

$$\|x\|_1 = \sum_{\omega \in \Omega} \|x(\omega)\| < \infty$$

and that $\ell_1(\Omega, Y)$ denotes the set of all summable functions from Ω to Y. We refer to Appendix 9 for more on summable functions and related concepts. With this notation, the set ℓ_{X_i} of allocations for agent i is:

$$\ell_{X_i} = \{x \in \ell_1(\Omega, Y) : x(\omega) \in X_i(\omega) \text{ for every } \omega \in \Omega\}.$$

Let $\ell_X = \prod_{i \in I} \ell_{X_i}$. We refer to any element of ℓ_X as an *allocation (for the society)* and represent it as a list $x = (x_i)_i$ of allocations, one for each agent.

A *random price* specifies a system of prices for every state of nature. We represent it as a function $p \colon \Omega \to Y^*$ with values in the symplex $\Delta = \left\{q \in Y^*_+ : q \cdot u = 1\right\}$, where u is a vector in the interior of Y_+. The interpretation is that $p(\omega) \cdot y$ gives the worth of the bundle $y \in Y_+$ at the price p, when the state is ω. We write ℓ_P for the set of random prices, in formulas:

$$\ell_P = \left\{p \colon \Omega \to Y^*_+ : p(\omega) \in \Delta \text{ for every } \omega \in \Omega\right\}.$$

2.1 Interim expected utility

Let $\mathcal{G}$ be a σ-algebra on Ω representing the information of agent i in the interim, i.e. after the publication of the prices and before consumption takes place. For every ω, we write $\mathcal{G}(\omega)$ for the smallest element of $\mathcal{G}$ that contains ω.[1] We assume that, when the state ω realizes, agent i cannot

[1] If with an abuse of notation $\mathcal{G}$ also denotes the partition that generates the σ-algebra $\mathcal{G}$, then $\mathcal{G}(\omega)$ is the unique element of the partition that contains ω.

observe ω but only $\mathcal{G}(\omega)$. In this case, her conditional probability on the state of nature being any ω' is:

$$q_i(\omega'|\mathcal{G}(\omega)) = \begin{cases} 0 & if\ \omega' \notin \mathcal{G}(\omega) \\ \dfrac{q_i(\omega')}{\sum_{\bar{\omega}\in\mathcal{G}(\omega)} q_i(\bar{\omega})} & if\ \omega' \in \mathcal{G}(\omega). \end{cases}$$

Therefore, the *conditional interim expected utility* of agent i relative to any $x\colon \Omega \to Y_+$ is the function $v_i(x|\mathcal{G})(\cdot)\colon \Omega \to \mathbb{R}$ given by:

$$v_i(x|\mathcal{G})(\omega) = \sum_{\omega'\in\Omega} u_i(\omega', x(\omega'))\, q_i(\omega'|\mathcal{G}(\omega))$$

whenever this is well-defined[2].

2.2 Rational expectations equilibrium

A rational expectations equilibrium describes a situation in which agents observe the prices to update their information and expectations, they maximize their updated expected utility subject to their budget constraints, and the market clears in every state.

Formally, let $\sigma(p)$ denote the smallest σ-algebra for which the random price $p\colon \Omega \to \Delta$ is measurable. For every $i \in I$ let $\mathcal{G}_i = \sigma(p) \vee \mathcal{F}_i$ be the *join* of the σ-algebras $\sigma(p)$ and $\mathcal{F}_i$, i.e. the smallest σ-algebra on Ω that contains both $\sigma(p)$ and $\mathcal{F}_i$[3]. The following definition is that of Kreps (1977) and Allen (1981).

Definition 2.1 *A* rational expectations equilibrium *(REE) consists of an allocation $x = (x_i)_i$ and a random price function p that satisfy the following conditions for every $i \in I$.*

1. *The function x_i is $\mathcal{G}_i$-measurable;*
2. *$x_i(\omega)$ satisfies the budget constraint $p(\omega)\cdot x_i(\omega) \le p(\omega)\cdot e_i(\omega)$ for every $\omega \in \Omega$;*
3. *for every $\mathcal{G}_i$-measurable $y\colon \Omega \to Y_+$, if $v_i(y|\mathcal{G}_i)(\omega) > v_i(x_i|\mathcal{G}_i)(\omega)$ for some $\omega \in \Omega$ then $p(\omega)\cdot y(\omega) > p(\omega)\cdot e_i(\omega)$;*
4. *$\sum_{j\in I} x_j(\omega) = \sum_{j\in I} e_j(\omega)$ for every $\omega \in \Omega$.*

The set of rational expectations equilibria in the economy $\mathcal{E}$ is $R(\mathcal{E})$.

The REE is an interim concept, since agents maximize their conditional expected utility based on their own private information, as well as to the information disclosed by the equilibrium random price. A REE is: (i) *full revealing* if $\sigma(p) = 2^{\Omega}$, (ii) *non-revealing* if $\sigma(p) = \{\emptyset, \Omega\}$, and (iii) *partially revealing* if $\{\emptyset, \Omega\} \subset \sigma(p) \subset 2^{\Omega}$.

[2] In order for $v_i(x|\mathcal{G})(\omega)$ to be defined, it must be that the function $\omega' \mapsto u_i(\omega', x(\omega'))$ is summable when ω' ranges in $\mathcal{G}(\omega)$. In the next sections we will introduce additional assumptions under which this summability condition is always met for every $x \in \ell_{X_i}$ and $\omega \in \Omega$. See also Lemma 9.4.

[3] The σ-algebra of events discernable by every agent is the *coarse* σ-algebra $\bigwedge_{i\in I}\mathcal{F}_i$, which is the largest σ-algebra contained in each $\mathcal{F}_i$. While, agents by pooling their information discern the events in the *fine* σ-algebra $\bigvee_{i\in I}\mathcal{F}_i$, which denotes the smallest σ-algebra containing all $\mathcal{F}_i$.

It is by now well known that a REE may only exist in a generic sense and not universal. Moreover, an REE may fail to be fully Pareto optimal and incentive compatible, and it may not be implementable as a perfect Bayesian equilibrium; see Glycopantis and Yannelis (2005) and Glycopantis, Muir, and Yannelis (2009). The most problematic aspect of the notion of REE is that it requires that agents maximize their interim expected utility conditioned also on the information that the equilibrium prices generate, and the resulting equilibrium allocations are measurable with respect to the private information of each individual and with respect to the information generated by the equilibrium prices. As Kreps (1977)'s example demonstrates, the measurability condition on allocations is what creates the non-existence of the REE equilibrium (see De Castro, Pesce, and Yannelis (2020) for an elaboration of this point)[4].

3 Examples

This section presents some examples that explain how agents can learn from a rational expectations equilibrium if they are involved in a dynamic learning setting. In each case we assume that agents reach a specific equilibrium, and then we ask the following question: if agents could repeat the trades taking in consideration the new information they acquired, how would they behave? Basically, after the realization of a rational expectations equilibrium (REE) we allow agents to refine their private information by observing the REE price and allocation. In a subsequent period, agents repeat the trades with their refined information and reach another (possibly different) REE equilibrium. The same trading situation keeps repeating, but the information that agents have in each period keeps track of the past REE equilibria.

The first example shows a REE in which prices are full revealing, meaning that agents become fully informed in the interim stage. The equilibrium allocation is risk-sharing and represents the best outcome possible. If agents could learn from this equilibrium and had the chance to trade again in the same situation, they would reach the same equilibrium. This is because the learning process stops already in the second period when agents become fully informed and nothing else can be learnt.

Example 3.1 Consider an asymmetric information economy with two agents $i = 1, 2$, three states $\Omega = \{a, b, c\}$ and two goods. For every $i \in I$ and $\omega \in \Omega$, we set $X_i(\omega) = \mathbb{R}^2_+$ and $q_i(\omega) = \frac{1}{3}$. The private information of each agent in period t is:

$$\mathcal{F}_1^t = \sigma\left(\Pi_1^t\right) \text{ with } \Pi_1^t = \{\{a, b\}, \{c\}\} \quad \text{and} \quad \mathcal{F}_2^t = \sigma\left(\Pi_2^t\right) \text{ with } \Pi_2^t = \{\{a, c\}, \{b\}\}.$$

The endowments of agents $i = 1, 2$ in period t are the functions $e_i^t(\omega)$ defined as follows:

$$e_1^t = \left(e_1^t(a), e_1^t(b), e_1^t(c)\right) = ((1, 3), (1, 3), (2, 2)),$$
$$e_2^t = \left(e_2^t(a), e_2^t(b), e_2^t(c)\right) = ((3, 1), (2, 2), (3, 1)).$$

[4]Recently, De Castro, Pesce, and Yannelis (2020) introduced a new notion of REE by allowing for ambiguity in agents' consumption choices and by not imposing that optimal allocations fulfill the measurability condition.

Both agents have the same utility function $u(\omega, x, y) = \sqrt{xy}$ for each $\omega \in \Omega$, where x, y denote the amounts of the two goods assigned to the agent in the state ω.

In this example the information disclosed by the price is the algebra generated by one of the partitions $\{\Omega\}$, Π_1^t, Π_2^t or Ω. Computations show that the only possible REE corresponds to price p that is full revealing, i.e. such that $\sigma(p) = 2^\Omega$, given by:

$$p^t = \left(p^t(a), p^t(b), p^t(c)\right) = \left(\frac{p_y^t(a)}{p_x^t(a)}, \frac{p_y^t(b)}{p_x^t(b)}, \frac{p_y^t(c)}{p_x^t(c)}\right) = \left(1, \frac{3}{5}, \frac{5}{3}\right),$$

where $p_x^t(\omega)$ (resp. $p_y^t(\omega)$) is the price of the first (resp. the second) good in state ω, and $p^t(\omega)$ is the relative price of the second good with respect to first one in state ω. At these prices, the REE allocation $x^t = \left(x_1^t, x_2^t\right)$ is:

$$x_1^t = \left(x_1^t(a), x_1^t(b), x_1^t(c)\right) = \left((2,2), \left(\frac{7}{5}, \frac{7}{3}\right), \left(\frac{8}{3}, \frac{8}{5}\right)\right),$$
$$x_2^t = \left(x_2^t(a), x_2^t(b), x_2^t(c)\right) = \left((2,2), \left(\frac{8}{5}, \frac{8}{3}\right), \left(\frac{7}{3}, \frac{7}{5}\right)\right),$$

where $x_i^t(\omega)$ is the allocation for agent i in state ω.

Suppose now that agents were to trade again in the same economy, only that now they have observed the REE $\left(p^t, x^t\right)$ and have learned from it. Being p^t fully revealing, agents are now fully informed and their updated private information algebra is the whole power set 2^Ω. This new situation is described as a repeated asymmetric information economy $\mathcal{E}^{t+1} = \left\{\left(\mathcal{F}_i^{t+1}, X_i, u_i, e_i^{t+1}, q_i\right) : i \in I\right\}$, where superscript $t+1$ refers to the subsequent period.

$$\mathcal{F}_i^{t+1} = \mathcal{F}_i^t \vee \sigma\left(p^t, x^t\right) = 2^\Omega$$

for every $i \in I$. Here, $\sigma\left(p^t, x^t\right)$ denotes the smallest σ-algebra on Ω making each function p^t and x^t measurable. We ask what REE emerges in this second economy.

If we assume that agents' initial endowment has not changed, i.e. that $e_i^{t+1} = e_i^t$, then the only equilibrium in the repeated economy is exactly the one they obtained in the original one, i.e. $\left(p^{t+1}, x^{t+1}\right) = \left(p^t, x^t\right)$, which is Pareto-optimal. With the same argument, in any further repetition of the economy, if agents endowments do not change then the only possible REE is $\left(p^t, x^t\right)$.

Suppose, on the contrary, that the endowments of agents changes in each repetition, and that it evolves as a martingale. For example, assume that the endowments in the repeated economy are:

$$e_1^{t+1} = ((0,4), (2,2), (2,2)), \quad e_2^{t+1} = ((4,0), (2,2), (2,2)).$$

The interpretation is that the finer information that agents have in period $t+1$ allows them to learn more about their true endowment. In this case, there is only one REE $\left(p^{t+1}, x^{t+1}\right)$ which is given by:

$$\frac{p_y^{t+1}(\omega)}{p_x^{t+1}(\omega)} = 1, \; x_1^{t+1}(\omega) = (2,2), \; x_2^{t+1}(\omega) = (2,2)$$

for every state $\omega \in \Omega$. In this REE, agents receive a higher ex-ante utility than in that of the previous period.

Notice that this second REE is non-revealing in the sense that the algebra it generates is the trivial one. In symbols: $\sigma(p^{t+1}, x^{t+1}) = \{\emptyset, \Omega\} \subset 2^{\Omega} = \sigma(p^t, x^t)$. The equilibrium at time t is therefore more informative than the one in the subsequent period $t+1$. We conclude that repeating the interaction with more information does not imply that agents can learn more from the new REE than from the old ones. ♦

The second example below is similar to the first one, in that agents become fully informed after observing the rational expectations equilibrium. However, while in the first example agents acquire all information in the interim stage by looking at the prices, in this example prices are non-revealing and agents learn how to discern the states only by looking at the equilibrium allocation. This is in contrast with Fama's efficient market hypothesis, according to which it is the prices alone that reflect all available information about future values. See Malkiel (2016).

Example 3.2 Consider an asymmetric information economy with three agents $i = 1, 2, 3$, three states of nature $\Omega = \{a, b, c\}$ and two goods. For every i and ω, we set $X_i(\omega) = \mathbb{R}^2_+$ and $q_i(\omega) = \frac{1}{3}$. The initial endowment and the private information of each agent are given by:

$$e_1 = (e_1(a), e_1(b), e_1(c)) = ((2,1),(2,1),(3,1)), \quad \text{and} \quad \mathcal{F}_1 = \sigma(\{\{a,b\},\{c\}\}),$$
$$e_2 = (e_2(a), e_2(b), e_2(c)) = ((1,2),(2,2),(1,2)), \quad \text{and} \quad \mathcal{F}_2 = \sigma(\{\{a,c\},\{b\}\}),$$
$$e_3 = (e_3(a), e_3(b), e_3(c)) = ((3,1),(2,1),(2,1)), \quad \text{and} \quad \mathcal{F}_3 = \sigma(\{\{a\},\{b,c\}\}).$$

The utility that agent i receives in sate ω when consuming an amount x of the first commodity and an amount y of the second commodity is given by the function $u_i(\omega, x, y)$, defined as follows:

$$u_1(a,x,y) = \sqrt{xy}, \quad u_1(b,x,y) = \log(xy), \quad u_1(c,x,y) = \sqrt{xy},$$
$$u_2(a,x,y) = \log(xy), \quad u_2(b,x,y) = \sqrt{xy}, \quad u_2(c,x,y) = \sqrt{xy},$$
$$u_3(a,x,y) = \sqrt{xy} \quad u_3(b,x,y) = \sqrt{xy} \quad u_3(c,x,y) = \log(xy).$$

A REE in this economy is given by:

$$(p_x(a), p_y(a)) = \left(1, \tfrac{3}{2}\right) \quad x_1(a) = \left(\tfrac{7}{4}, \tfrac{7}{6}\right) \quad x_2(a) = \left(2, \tfrac{4}{3}\right) \quad x_3(a) = \left(\tfrac{9}{4}, \tfrac{3}{2}\right)$$
$$(p_x(b), p_y(b)) = \left(1, \tfrac{3}{2}\right) \quad x_1(b) = \left(\tfrac{7}{4}, \tfrac{7}{6}\right) \quad x_2(b) = \left(\tfrac{5}{2}, \tfrac{5}{3}\right) \quad x_3(b) = \left(\tfrac{7}{4}, \tfrac{7}{6}\right)$$
$$(p_x(c), p_y(c)) = \left(1, \tfrac{3}{2}\right) \quad x_1(c) = \left(\tfrac{9}{4}, \tfrac{3}{2}\right) \quad x_2(c) = \left(2, \tfrac{4}{3}\right) \quad x_3(c) = \left(\tfrac{7}{4}, \tfrac{7}{6}\right),$$

where $p_x(\omega), p_y(\omega)$ are respectively the prices of the first and second commodity in state ω, and $x_i(\omega)$ denotes the allocation of agent i in state ω.

The equilibrium price $p = (p_x, p_y)$ is constant across the states, and so it is non-revealing (in symbols, $\sigma(p) = \{\emptyset, \Omega\}$). This implies that in the interim stage agents do not acquire any new information and $\mathcal{G}_i = \mathcal{F}_i \vee \sigma(p) = \mathcal{F}_i$. At the same time, the algebra $\sigma(x)$ on Ω generated by the allocation $x = (x_i)_i$ is the power set 2^{Ω}, meaning that x reveals the finest information possible. We conclude that, after having observed the equilibrium (p, x), agents become immediately fully informed in any repetition of the economy. Thus, in this example agents learned nothing by observing the equilibrium prices, but they became fully informed by observing the equilibrium allocation. ♦

The last example describes a situation in which the only REE is constant across the states of nature, meaning that neither the price nor the allocation reveal any new information. In this case agents don't learn and remain partially and asymmetrically informed in every repetition of the economy.

Example 3.3 Consider an asymmetric information economy with three agents $i = 1, 2, 3$, three states of nature $\Omega = \{a, b, c\}$ and two commodities. For every i and ω, we set $X_i(\omega) = \mathbb{R}^2_+$ and $q_i(\omega) = \frac{1}{3}$. Agents have the same utility function $u(\omega, x, y) = \sqrt{xy}$ for any $\omega \in \Omega$ and $x, y \geq 0$. Their endowments and information algebras are given by:

$$(e_1(a), e_1(b), e_1(c)) = ((1,3), (2,2), (1,3)), \quad \text{and} \quad \mathcal{F}_1 = \sigma(\{\{a,c\}, \{b\}\}),$$
$$(e_2(a), e_2(b), e_2(c)) = ((3,1), (2,2), (3,1)), \quad \text{and} \quad \mathcal{F}_2 = \sigma(\{\{a,c\}, \{b\}\}),$$
$$(e_3(a), e_3(b), e_3(c)) = ((2,2), (2,2), (2,2)), \quad \text{and} \quad \mathcal{F}_3 = \sigma(\{\{a,b,c\}\}).$$

Notice that for each $i \in I$, $\mathcal{F}_i = \sigma(e_i, X_i)$. The allocation $x_i(\omega) = (2, 2)$ for all $i \in I$ and all $\omega \in \Omega$ is a REE allocation with respect to the price system $(p_x(\omega), p_y(\omega)) = (1, 1)$ for all $\omega \in \Omega$. Then, $\sigma(p, x) = \{\emptyset, \Omega\}$.

The equilibrium (p, x) does not reveal any new information to the agents, whose private information algebras strictly contain all the events disclosed by the equilibrium. Any repetition of the economy would then generate the same REE, since agents don't acquire any new information from the previous equilibria and so they remain asymmetrically informed. ♦

The next sections consider a condition, called *non trivial learning*, which implies that in at least one of the repetitions there is an agent who learns something from the REE. Clearly, this example violates the non trivial learning condition.

4 Infinitely repeated rational expectations equilibria

This section considers an asymmetric information economy in a dynamic setting. Agents engage repeatedly in the same trading situation, and each time they reach a REE they learn from it and update their private information. This process generates a sequence of repeated economies, one per period, and a corresponding sequence of REE's.

Time is discrete and indexed by the set T of positive integers. Let $\mathcal{E}^1 = \left\{\left(\mathcal{F}_i^1, X_i, u_i, e_i, q_i\right) : i \in I\right\}$ denote the initial asymmetric information economy, and let $\left(p^1, x^1\right)$ be a REE in $\mathcal{E}^1$. We define recursively the sequence of economies and REE's generated from $\mathcal{E}^1$. Precisely, suppose you have defined the economy $\mathcal{E}^t$ at time t and that $\left(p^t, x^t\right)$ is an REE in $\mathcal{E}^t$. In the next period, the economy is $\mathcal{E}^{t+1} = \left\{\left(\mathcal{F}_i^{t+1}, X_i, u_i, e_i, q_i\right) : i \in I\right\}$, where $\mathcal{F}_i^{t+1}$ is defined recursively as:

$$\mathcal{F}_i^{t+1} = \mathcal{F}_i^t \vee \sigma\left(p^t, x^t\right)$$

and $\sigma\left(p^t, x^t\right)$ is the σ-algebra generated by the REE in the previous period. $\mathcal{F}_i^t \vee \sigma\left(p^t, x^t\right)$ is the join (i.e. the coarsest σ-algebra containing both $\mathcal{F}_i^t$ and $\sigma\left(p^t, x^t\right)$) and represents the information that i held in the previous step, updated with that revealed by the random price p^t and the allocation x^t. Once the economy $\mathcal{E}^{t+1}$ is defined, we take a REE $\left(p^{t+1}, x^{t+1}\right)$ in $\mathcal{E}^{t+1}$.

The *limit full information economy* is $\mathcal{E}^* = \left\{\left(\mathcal{F}_i^*, X_i, u_i, e_i, q_i\right) : i \in I\right\}$, where agent i's limit information algebra is:

$$\mathcal{F}_i^* = \bigvee_{k=1}^{\infty} \mathcal{F}_i^k.$$

We refer to any REE (p^*, x^*) in $\mathcal{E}^*$ as a *limit rational expectations equilibrium.*

Definition 4.1 *Let $\left\{\mathcal{E}^t : t \in T\right\}$ be a sequence of repeated economies. A* limit rational expectations equilibrium *consists of an allocation x^* and a random price p^* that satisfy the following conditions for every $i \in I$.*

1. *The consumption bundle x_i^* is $\mathcal{G}_i^*$-measurable, where $\mathcal{G}_i^*$ denotes the interim information algebra $\mathcal{F}_i^* \vee \sigma\left(p^*\right)$ of agent i;*
2. *$x_i^*(\omega)$ satisfies the budget constraint $p^*(\omega) \cdot x_i^*(\omega) \leq p^*(\omega) \cdot e_i(\omega)$ for every $\omega \in \Omega$;*
3. *for every $\mathcal{G}_i^*$-measurable $y \colon \Omega \to Y_+$, if $v_i\left(y|\mathcal{G}_i^*\right)(\omega) > v_i\left(x_i^*|\mathcal{G}_i^*\right)(\omega)$ for some ω then $p^*(\omega) \cdot y(\omega) > p^*(\omega) \cdot e_i(\omega)$;*
4. *$\sum_{j \in I} x_j^*(\omega) = \sum_{j \in I} e_j(\omega)$ for every $\omega \in \Omega$.*

We make a few considerations. The first one is that each repetition $\mathcal{E}^t$ differs from the initial economy only in the private information of agents, and hence in their interim expected utility functions. In particular, for every i and period t we have:

$$\mathcal{F}_i^t \subseteq \mathcal{F}_i^{t+1} \subseteq \mathcal{F}_i^{t+2} \subseteq \cdots \subseteq \mathcal{F}_i^*$$

which we interpret as a learning process for agent i. In particular, if $\mathcal{G}_i^t$ denotes the interim information algebra of agent i in the period t (i.e. the algebra $\mathcal{F}_i^t \vee \sigma\left(p^t\right)$) then it must be that $\mathcal{F}_i^t \subseteq \mathcal{G}_i^t \subseteq \mathcal{F}_i^{t+1}$ for each t. This does not mean that equilibria become more and more informative, for it is possible that $\sigma\left(p^t, x^t\right) \supset \sigma\left(p^{t+1}, x^{t+1}\right)$ in some period t. This eventuality is described in Example 3.1.

Our second observation is that we can write the information of an agent i in period t in the form:

$$\mathcal{F}_i^t = \mathcal{F}_i^1 \vee \left(\bigvee_{k=1}^{t-1} \sigma\left(p^k, x^k\right)\right)$$

where each $\left(p^k, x^k\right)$ is the equilibrium realized in the k-th repetition of the economy. This means that the private information of agent i has two components: the first is her initial information $\mathcal{F}_i^1$, which is private and contributes to the information asymmetry in $\mathcal{E}^t$; the second one is generated by all the REE's obtained in the previous steps and it is common to all agents (because they all observe and remember every past equilibria).

Last, we stress that the expression "full information" does not mean "complete information". In the limit full information economy, in fact, agents may still have partial and differential information. This happens, for instance, when the sequence of REE's does not reveal any new information to the agents, i.e. when $\sigma\left(p^t, x^t\right) \subset \mathcal{F}_i^1$ for every i and every t. In this situation agents don't learn from the process and that the limit economy $\mathcal{E}^*$ coincides with the initial one $\mathcal{E}^1$.[5] This is precisely the case in Example 3.3. In this paper, we refer to $\mathcal{E}^*$ as the limit "full" information economy simply because the $\mathcal{F}_i^*$'s represent everything that agents can learn in the specific process $\left\{\mathcal{E}^t : t \in T\right\}$ by looking at the corresponding sequence of REE's.

In the following we fix a sequence of economies $\left\{\mathcal{E}^t : t \in T\right\}$ generated through the infinite repetition process described here. We refer to this as a *sequence of repeated economies* and write $\mathcal{E}^*$ for the corresponding limit full information economy. For every agent i, we let $\mathcal{F}_i^t$ denote her information at time t and $\mathcal{F}_i^*$ her information in the limit full information economy. A sequence $\left\{\left(p^t, x^t\right) : t \in T\right\}$ of price-allocation pairs *generates* the sequence of repeated economies if, for every t, the pair $\left(p^t, x^t\right)$ is the REE in $\mathcal{E}^t$ that generates $\mathcal{E}^{t+1}$, i.e. if $\mathcal{F}_i^{t+1} = \mathcal{F}_i^t \vee \sigma\left(p^t, x^t\right)$ for every i.

5 The convergence of the rational expectations equilibria

This section studies the asymptotic behavior of the REE's obtained in a sequence of repeated economies. Its main result provides conditions under which a subsequence of the REE's converges to an equilibrium in the limit full information economy.

We require the following assumptions on the initial economy.

Assumption 5.1 *For each $i \in I$, the correspondence $X_i \colon \Omega \to 2_+^Y$ is such that:*

(i) *$X_i(\omega)$ is a nonempty, convex, norm compact set for every $\omega \in \Omega$;*

(ii) *it is summably bounded in the sense that there exists a $f \in \ell_1(\Omega)$ such that $\|x\| \leq f(\omega)$ for every $\omega \in \Omega$ and $x \in X_i(\omega)$.*

Assumption 5.2 *For each $i \in I$, the utility function u_i is such that*

(i) *for each $\omega \in \Omega$, $u_i(\omega, \cdot) \colon X_i(\omega) \to \mathbb{R}$ is continuous;*

(ii) *u_i is uniformly summably bounded on allocations, in the sense that there exists a $g \in \ell_1(\Omega)$ such that $|u_i(\omega, x)| \leq g(\omega)$ for every $\omega \in \Omega$ and $x \in X_i(\omega)$;*

[5] When there are infinitely many states, it is also possible that every REE reveals new information to the agents, and still the asymmetry of information does not vanish in the limit. This is because one can define a sequence $\left(\mathcal{F}^t\right)$ of σ-algebras on Ω, each strictly larger than the former, with the property that $\bigvee_t \mathcal{F}^t \neq 2^\Omega$.

(*iii*) *for each* $\omega \in \Omega$, $u_i(\omega, \cdot): X_i(\omega) \to \mathbb{R}$ *is monotone in the sense that* $x \gg y \Rightarrow u_i(\omega, x) > u_i(\omega, y)$*;*

(*iv*) *for each* $\omega \in \Omega$, $u_i(\omega, \cdot): X_i(\omega) \to \mathbb{R}$ *is concave.*

Assumption 5.3 *For each* $i \in I$*, the endowment* e_i *is such that the set* $\{z \in X_i(\omega) : q \cdot z < q \cdot e_i(\omega)\}$ *is nonempty for every* $\omega \in \Omega$ *and* $q \in \Delta$*.*

Theorem 1 *Suppose that the sequence* $\{\mathcal{E}^t : t \in T\}$ *of repeated economies satisfies Assumptions 5.1, 5.2(i)-(ii) and 5.3. Let* $\{(p^t, x^t) : t \in T\}$ *be the REE in each economy* $\mathcal{E}^t$*. Then we can extract a subsequence* $\{(p^{t_n}, x^{t_n}) : n = 1, 2, \dots\}$ *of REE with the following properties:*

1. $x_i^{t_n}$ *converges to some* $x_i^* \in \ell_{X_i}$ *in norm, for every* i*;*
2. p^{t_n} *converges to some* $p^* \in \ell_P$ *in the weak***-topology;*
3. (p^*, x^*) *is a limit REE in the limit economy* $\mathcal{E}^*$*.*

In addition, x_i^* *is measurable with respect to* $\mathcal{F}_i^*$ *for every* $i \in I$*.*

The convergence of the REE's suggests that, after sufficiently many repetitions, acquiring additional information does not change drastically the equilibrium outcome. The failure of this result would have significant implications on the robustness of the equilibrium concept, for it would imply that small perturbations of the information structure would have profound effects on the REE outcome.

The last claim of the Theorem states that the limit equilibrium x^* is measurable with respect to all the information accumulated in the repetitions. This ensures that the price p^* does not disclose any new information that is relevant to the realization of x^* in the limit full information economy.

5.1 Proof of Theorem 1

The proof consists of several steps. First we show that the set $\ell_p \times \ell_X$ of all price-allocations pairs is compact, and use this result to find a subsequence of the REE's that converges to some (p^*, x^*). Second, we show that x_i^* is $\mathcal{F}_i^*$-measurable for every i. Last we prove that each x_i^* maximizes the interim expected utility of agent i subject to the measurability and budget constraints imposed by p^*. This will show that (p^*, x^*) is a REE and conclude the proof.

We split the proof in lemmata.

Lemma 5.4 *There exist a subsequence* $\{(p^{t_n}, x^{t_n}) : n = 1, 2, \dots\}$ *and a* $(p^*, x^*) \in \ell_p \times \ell_X$ *such that* $p^{t_n} \to p^*$ *in the weak***-topology and* $x^{t_n} \to x^*$ *in the norm topology.*

Proof. First we show that the set $\ell_p \times \ell_X$ is a compact set when ℓ_p is considered with the weak*-topology and ℓ_{X_i} with the norm one. The set ℓ_p, seen as a subset of $[\ell_1(\Omega, Y)]^*$, is weakly* closed and bounded, and so it is weakly* compact by Alaoglu's Theorem. We show that ℓ_{X_i} is norm-compact for every i. This, in fact, will imply that ℓ_X is compact too.

Every function $x: \Omega \to Y_+$ such that $x(\omega) \in X_i(\omega)$ is dominated by a $f \in \ell_1(\Omega)$ (Assumption 5.1, (ii)) and so it is summable and belongs to ℓ_{X_i}. It follows that the set ℓ_{X_i} is closed, it is bounded and has equismall tails (because it is summably bounded, see Appendix 9), and it is such that $\{x(\omega) : x \in \ell_{X_i}\}$ coincides with the compact set $X_i(\omega)$. An application of Ascoli-Arzelà Theorem for summable functions gives that ℓ_{X_i} is a compact set (see Fact 9.2, or Leonard (1976), Theorem 5.1).

We conclude that $\{(p^t, x^t) : t \in T\}$ is a sequence in the compact space $\ell_p \times \ell_X$ and so it has a subsequence converging to a (p^*, x^*). ■

The next lemma proves that (p^*, x^*) satisfies condition (1) in Definition 2.1.

Lemma 5.5 *For every $i \in I$, the allocation x_i^* is measurable with respect to $\mathcal{F}_i^*$.*

Proof. Fix an agent i and consider the sequence $\{x_i^t : t \in T\}$. By assumption, x_i^t is the allocation that i receives in equilibrium in period t, and so it is measurable with respect to the interim information algebra $\mathcal{G}_i^t = \mathcal{F}_i^t \vee \sigma(p^t)$. This, in turn, is a subset of $\mathcal{F}_i^*$. It follows that every x_i^t is an element in the set:

$$\ell_{X_i}^* = \{x \in \ell_1(\Omega, Y) : x \text{ is } \mathcal{F}_i^*\text{-measurable and } x(\omega) \in X_i(\omega) \text{ for every } \omega \in \Omega\}$$

which is closed in the norm topology. But x_i^* a limit point of the sequence $\{x_i^t : t \in T\}$, and so it belongs to $\ell_{X_i}^*$ as well. ■

We now prove that (p^*, x^*) satisfies conditions (2) and (4) in Definition 2.1.

Lemma 5.6 *Let $\omega \in \Omega$. Then $\sum_{j \in I} x_j^*(\omega) = \sum_{j \in I} e_j(\omega)$, and $p^*(\omega) \cdot x_i^*(\omega) \leq p^*(\omega) \cdot e_i(\omega)$ for every $i \in I$.*

Proof. Since $p^{t_n} \to p^*$ in the weak* topology and $x^{t_n} \to x^*$ in the norm topology, it must be that $\sum_j x_j^{t_n}(\omega) \to \sum_j x_j^*(\omega)$, and that $p^{t_n}(\omega) \cdot x_i^{t_n}(\omega) \to p^*(\omega) \cdot x_i^*(\omega)$ and $p^{t_n}(\omega) \cdot e_i(\omega) \to p^*(\omega) \cdot e_i(\omega)$ for every $i \in I$ (Aliprantis and Border (2005), Theorem 6.40). The claim follows from the fact that, for every n, the pair (p^{t_n}, x^{t_n}) is a REE in the economy $\mathcal{E}^{t_n}$ and so it satisfies $\sum_j x_j^{t_n}(\omega) = \sum_j e_j(\omega)$ and $p^{t_n}(\omega) \cdot x_i^{t_n}(\omega) \leq p^{t_n}(\omega) \cdot e_i(\omega)$ for every $i \in I$. ■

Our last Lemma proves that (p^*, x^*) satisfies condition (3) in Definition 2.1, from which we conclude that it is a REE. To this end, recall that, for every $i \in I$, $\mathcal{G}_i^t = \mathcal{F}_i^t \vee \sigma(p^t)$ for every $t \in T$, and $\mathcal{G}_i^* = \mathcal{F}_i^* \vee \sigma(p^*)$.

Lemma 5.7 *Suppose that, for $i \in I$, $y \in \ell_{X_i}$ is a $\mathcal{G}_i^*$-measurable function such that $v_i(y|\mathcal{G}_i^*)(\omega) > v_i(x_i^*|\mathcal{G}_i^*)(\omega)$ for some $\omega \in \Omega$. Then $p^*(\omega) \cdot y(\omega) > p^*(\omega) \cdot e_i(\omega)$.*

Proof. For every $t \in T$, define $y^t = E[y|\mathcal{G}_i^t]$. Since y is measurable with respect to $\mathcal{G}_i^*$, Lemma 9.2 gives that $y^t \to y$ in norm. We can therefore apply Lemma 9.5 to the sequences of the y^t's and of the $\mathcal{G}^t$'s and obtain that:

$$\lim_t v_i(y^t|\mathcal{G}_i^t)(\omega) = v_i(y|\mathcal{G}_i^*)(\omega). \tag{1}$$

Similarly, applying Lemma 9.5 to the $x_i^{t_n}$'s gives:

$$\lim_t v_i(x_i^{t_n}|\mathcal{G}_i^{t_n})(\omega) = v_i(x_i^*|\mathcal{G}_i^*)(\omega). \tag{2}$$

Equations (1) and (2), combined with the fact that $v_i\left(y|\mathcal{G}_i^*\right)(\omega) > v_i\left(x_i^*|\mathcal{G}_i^*\right)(\omega)$ by assumption, imply that $v_i\left(y^{t_n}|\mathcal{G}_i^{t_n}\right)(\omega) > v_i\left(x_i^{t_n}|\mathcal{G}_i^{t_n}\right)(\omega)$ for n sufficiently large. But then y^{t_n} is an allocation $\mathcal{G}_i^{t_n}$-measurable, that gives an interim expected utility higher than the equilibrium allocation $x_i^{t_n}$, and so it must be that $p^{t_n}(\omega)\cdot y^{t_n}(\omega) > p^{t_n}(\omega)\cdot e_i(\omega)$. Taking it to the limit, we must have that:

$$p^*(\omega)\cdot y(\omega) \geq p^*(\omega)\cdot e_i(\omega).$$

We show that it cannot be that $p^*(\omega)\cdot y(\omega) = p^*(\omega)\cdot e_i(\omega)$. Let $z \in \ell_{X_i}$ be such that $p^*(\omega')\cdot z < p^*(\omega')\cdot e_i(\omega')$ for every $\omega' \in \Omega$ (such z exists because of Assumption 5.3). For every n, set $z^n = 2^{-n}y + (1-2^{-n})z$ and observe that: $z^n \in \ell_{X_i}$ (because ℓ_{X_i} is convex by Assumption 5.1(i)); $p^*(\omega)\cdot z^n(\omega) < p^*(\omega)\cdot y(\omega)$; and z^n converges to y in norm. Therefore, $v_i\left(z^n|\mathcal{G}_i^*\right)(\omega)$ converges to $v_i\left(y|\mathcal{G}_i^*\right)(\omega)$. For n sufficiently large it must be that:

$$v_i\left(z^n|\mathcal{G}_i^*\right)(\omega) > v_i\left(x_i^*|\mathcal{G}_i^*\right)(\omega).$$

Apply the same argument above, replacing y with z^n. We obtain that $p^*(\omega)\cdot z^n(\omega) \geq p^*(\omega)\cdot e_i(\omega)$. But since $p^*(\omega)\cdot y(\omega) > p^*(\omega)\cdot z^n(\omega)$, we conclude that $p^*(\omega)\cdot y(\omega) > p^*(\omega)\cdot e_i(\omega)$. ■

6 The limit symmetric information REE

In Section 4 it was noted that in the sequence of repeated economies it is possible that the progression of equilibrium prices and allocations may not reveal all the information privately held by agents. In these situations, the learning process fails and agents remain incompletely and asymmetrically informed even in the limit full information economy. This is the case of Example 3.3, in which agents learn nothing from the REE's and so they maintain the same initial private information in every repetition, as well as in the limit economy.

This section focuses on those situations in which the learning process is effective and resolves the asymmetry of information in the limit economy. This requires that at least an agent learns something in at least one period, and that in the limit the public information revealed by the equilibria prevails over individuals' private information. We refer to this condition as *non trivial learning* and formalize it as follows:

$$(NTL) \quad \mathcal{F}_i^1 \subseteq G^\infty = \bigvee_{k=1}^{\infty} \sigma(p^k, x^k) \quad \text{for all } i \in I.$$

The NTL condition states that, in the limit economy, the pooled information generated by all the past equilibria is at least as fine as the initial private information of any agent. The sequence of REE's gradually reveals the information held privately by the agents to the point that, in the limit full information economy, no agent knows something that is not disclosed in some repetition. The NTL condition is violated in Example 3.3.

Notice that the NTL is a condition on the whole sequence of repetitions, and not on the single economies. The condition, in fact, depends on agents characteristics as well as on the specific sequence of equilibria that emerge in each repetition, which is typically unpredictable. Knowing the state of the economy after t repetitions does not allow to anticipate what information the endogenous equilibria are going to generate, and hence to verify if the NTL condition will hold. It is only in the limit full information economy, when all past equilibria are observable, that we can certainly tell whether the NTL condition is met or not.

Under the NTL condition, in the limit full information economy every agent has the same information algebra G^{∞}, which corresponds to what one can learn by looking at the REE's that emerged in each repetition. The asymmetry in the information disappears, and so any limit symmetric information REE is immediately incentive compatible and implementable as a perfect Bayesian equilibrium of an extensive form game. In addition to that, our next theorem shows that the limit symmetric information REE exists (universally and not generically) and is efficient.

Theorem 2 *Let $\{\mathcal{E}^t : t \in T\}$ be a sequence of repeated economies that satisfies the NTL condition, and Assumptions 5.1, 5.2 and 5.3. Then there exists a limit REE (p^*, x^*) in $\mathcal{E}^*$ such that:*

1. *p^* is measurable with respect to $\mathcal{F}_i^*$ for every $i \in I$;*
2. *there exists no $y = (y_i)_i$ and $\omega \in \Omega$ such that $\sum_{i\in I} y_i(\omega) = \sum_{i\in I} e_i(\omega)$ and, for every $i \in I$, y_i is $\mathcal{F}_i^*$-measurable and $v_i\left(y_i|\mathcal{F}_i^*\right)(\omega) \geq v_i\left(x_i^*|\mathcal{F}_i^*\right)(\omega)$, with a strict inequality for at least a $i \in I$.*

Condition (1) ensures that the equilibrium price p^* does not reveal any new information to agents, who maintain in the interim stage the same private information they had ex-ante. This implies that each equilibrium allocation x_i^* is itself measurable with respect to $\mathcal{F}_i^*$, and so it is compatible with the information that agents accumulate through the repetitions. This condition is consistent with Theorem 1, which shows that the limit REE's have the same property. Condition (2) corresponds to a form of state-wise efficiency of the REE allocation x^* in the interim stage: if in a state ω there is an allocation y at which every agent receives at least the same interim conditional expected utility of x and someone a strictly higher one, then either y is not compatible with agents' information or y is not feasible.

6.1 Proof of Theorem 2

By the NTL condition, in the limit full information economy every agent has the same private information, which coincides with the σ-algebra G^{∞}. Let $\{A^n : n = 1, 2, \dots\}$ be the family of atoms that generate the algebra G^{∞}. The idea of the proof is to define for every n an auxiliary exchange economy $\mathcal{E}_n^*$ that captures agents' behaviour when they learn that a state in A^n has realized, but they still don't know which one. Fix a n and a generic $\omega^n \in A^n$. The economy $\mathcal{E}_n^*$ is given by:

$$\mathcal{E}_n^* = \left\{Y, I, \left(X_i^n, e_i^n, U_i^n\right)_i\right\}$$

where Y is the commodity space and I the set of agents. For every $i \in I$, $X_i^n = X_i(\omega^n)$ is agent's consumption set, $e_i^n = e_i(\omega^n)$ is her initial endowment. The utility that i receives from consuming a $x \in X_i^n$ is $U_i^n(x) = v_i\left(\hat{x}|G^{\infty}\right)(\omega^n)$, where $\hat{x} \colon \Omega \to Y$ is the constant function equal to x. Notice that $\mathcal{E}_n^*$ is indifferent on how one chooses ω^n in A^n.

Each $\mathcal{E}_n^*$ is an economy that satisfies the assumptions of the Auxiliary Theorem in Khan and Yannelis (1991), pg. 239, and so there exists a Walrasian equilibrium[6] (p^n, x^n) in $\mathcal{E}_n^*$. For every $\omega \in \Omega$, define:

$$p^*(\omega) = p^n, \quad x_i^*(\omega) = x_i^n$$

where n is the only number such that $\omega \in A^n$. We claim that (p^*, x^*) is the desired REE in $\mathcal{E}^*$.

First observe that p^* and x^* are constant on each atom in the algebra G^∞, and so they are measurable with respect to it. This implies that $\mathcal{F}_i^* \vee \sigma(p^*) = G^\infty$ for every i and so the measurability of the equilibrium allocations is satisfied. Second, notice that in every state ω the allocation x_i^* maximizes the interim expected utility of agent i conditional on G^∞ subject to the budget constraints imposed by $p^*(\omega)$, and the market clears. We conclude that (p^*, x^*) is a REE in $\mathcal{E}^*$.

We show that the REE (p^*, x^*) satisfies the conditions (1) and (2) in the claim. The measurability condition (1) follows immediately from the way p^* was constructed. We focus only on condition (2). Fix a ω and let $y \in \ell_X$ be a G^∞-measurable allocation such that $\sum_{i\in I} y_i(\omega) = \sum_{i\in I} e_i(\omega)$. If A^n is the atom that contains ω, then $y_i(\omega')$ is constantly equal to some $\tilde{y}_i \in Y$ on A^n, and $\tilde{y} = (\tilde{y}_i)_i$ is a feasible allocation in the auxiliary economy $\mathcal{E}_n^*$. By contradiction, assume that $v_i(y_i|G^\infty)(\omega) \geq v_i(x_i|G^\infty)(\omega)$ for every $i \in I$, with a strict inequality for at least one i. It follows that:

$$U_i^n(\tilde{y}_i) = v_i(y_i|G^\infty)(\omega) \geq v_i(x_i|G^\infty)(\omega) = U_i^n\left(x_i^n\right)$$

for every $i \in I$, with a strict inequality for at least one i. But this implies that $\tilde{y}$, seen as an allocation in the auxiliary economy $\mathcal{E}_n^*$, Pareto dominates the Walrasian allocation x^n, violating the first welfare Theorem.

7 The stability of limit rational expectations equilibria

This section considers a sequence of repeated economies and studies the stability of the REE's in the corresponding limit economy. Precisely, it asks when an equilibrium in the limit can be approximated with the REE's that emerge in the repetitions. We provide an answer to this question in terms of approximated REE outcomes.

An approximate (or ε-) REE describes a situation similar to that of a standard REE, if not that agents maximize their interim conditional expected utility within a small error $\varepsilon > 0$ in every state, with few exceptions. The interpretation is that agents have bounded rationality.

Definition 7.1 *Given $\varepsilon > 0$, an ε-*rational expectations equilibrium *(ε-REE) consists of an allocation $x = (x_i)$ and a random price function p that satisfy the following conditions for every $i \in I$.*

1. *The consumption bundle x_i is $\mathcal{G}_i$-measurable, where $\mathcal{G}_i = \mathcal{F}_i \vee \sigma(p)$.*

2. *There exists a $B_i \subseteq \Omega$ with $\sum_{\omega\in B_i} q_i(\omega) \geq 1 - \varepsilon$ such that, for every $\omega \in B_i$:*

[6] A Walrasian equilibrium in $\mathcal{E}_n^*$ consists of a $p \in \Delta$ and a list $x = (x_i)$ with $x_i \in X_i^n$ for every i, with the property that, for every $i \in I$: (i) $p \cdot x_i \leq p \cdot e_i^n$, (ii) if $U_i^n(y) > U_i^n(x_i)$ for some $y \in X_i^n$ then $p \cdot y > p \cdot e_i^n$, (iii) $\sum_{j\in I} x_j = \sum_{j\in I} e_j^n$.

(i) $x_i(\omega)$ *meets the approximated budget constraints* $p(\omega) \cdot x_i(\omega) \leq p(\omega) \cdot e_i(\omega) + \varepsilon$,

(ii) for every $\mathcal{G}_i$*-measurable* $y\colon \Omega \to Y$, *if* $v_i(y|\mathcal{G}_i)(\omega) > v_i(x_i|\mathcal{G}_i)(\omega) + \varepsilon$ *then* $p(\omega) \cdot y(\omega) > p(\omega) \cdot e_i(\omega)$.

3. $\sum_{j \in I} x_j(\omega) = \sum_{j \in I} e_j(\omega)$ *for every* $\omega \in \Omega$.

The set of ε*-rational expectations equilibria in the economy* $\mathcal{E}$ *is* $R_\varepsilon(\mathcal{E})$.

It is clear that, for $\varepsilon = 0$, the definition of ε-REE coincides with the standard definition of REE.

The main result of this section proves that, under mild conditions, if one takes a sequence of repeated economies and selects a REE (p^*, x^*) in the limit economy, then there exists a sequence of approximated REE outcomes, one for each repetition, that converges to (p^*, x^*). This result, in a way, constitute a partial converse to Theorem 1, which shows that there exists a REE in the limit economy to which the sequence of repeated REE's converges.

The result requires the additional assumption that every agent has the same information about the total endowment vector, i.e. about the sum of everyones' endowments. We write $\bigwedge_{i \in I} \mathcal{F}_i$ to denote their *meet*, which is the largest σ-algebra on Ω contained in each $\mathcal{F}_i$.

Assumption 7.2 *The endowment vector* $e(\omega) = \sum_{i \in I} e_i(\omega)$ *is measurable with respect to* $\bigwedge_{i \in I} \mathcal{F}_i$.[7]

Theorem 3 *Let* $\{\mathcal{E}^t : t \in T\}$ *be a sequence of repeated economies that satisfies the NTL condition, and Assumptions 5.1, 5.2(i)-(ii) and 7.2. Let* (p^*, x^*) *be a limit REE in* $\mathcal{E}^*$ *such that* x^* *is measurable with respect to* $\mathcal{F}_i^*$ *for every* i. *Then for every* $\varepsilon > 0$ *there exists a sequence of allocations* $\{z^t : t \in T\}$ *such that:*

1. $z^t \to x^*$ *in norm,*

2. *for* $t \in T$ *sufficiently large,* (p^*, z^t) *is a* ε*-REE in* $\mathcal{E}^t$.

Theorem 3 states that the sequence of repeated economies allows to describe every REE in the limit economy by means of a converging sequence of approximated equilibria. Therefore, no equilibrium in the limit economy is extraneous to the learning process. The only restriction on the limit REE is on the measurability of the equilibrium allocation x^* with respect to agents limit information algebras. Notice that Theorem 1 proves that all the equilibria that are obtained as the limit of the generating REE's satisfy this measurability requirement, and so do the REE's whose existence is proved in Theorem 2.

7.1 Proof of Theorem 3

Fix a $\varepsilon > 0$. For every $t \in T$ let $\mathcal{F}_I^t = \bigwedge_{i \in I} \mathcal{F}_i^t$ denote the common information at time t and let $z^t = (z_i^t)$ be the allocation defined by:

$$z_i^t = E\left[x_i^* \middle| \mathcal{F}_I^t\right].$$

[7] Notice that if the initial endowment is constant across the states, this assumption is satisfied.

Each z_i^t is the expectation of x_i^* conditional on the algebra $\mathcal{F}_I^t$, i.e. the common information on x_i^* that is available at time t. We claim that the sequence $\{z^t : t \in T\}$ satisfies the two conditions of the Theorem, which we prove separately.

Lemma 7.3 *For every $i \in I$, the sequence z_i^t converges to x_i^* in norm.*

Proof. By the NTL condition, $\mathcal{F}_i^*$ coincides with the common information algebra $\mathcal{F}_I^* = \bigvee_{t\in T} \mathcal{F}_I^t$. Being the allocation x_i^* measurable with respect to $\mathcal{F}_i^*$ by assumption, it must be that $x_i^* = E\left[x_i^* \middle| \mathcal{F}_I^*\right]$. We apply Lemma 9.2 and obtain $\lim_t z^t = \lim_t E\left[x_i^t \middle| \mathcal{F}_I^t\right] = E\left[x_i^* \middle| \mathcal{F}_I^*\right] = x_i^*$. ■

We now prove that (p^*, z^t) is a ε-REE in $\mathcal{E}^t$ for all but a finite number of periods t. We divide this part in steps, the first of which proves that the allocations z^t meet the measurability and feasibility requirements.

Lemma 7.4 *For every $t \in T$ and $i \in I$ the allocation z_i^t is measurable with respect to $\mathcal{H}_i^t = \mathcal{F}_i^t \vee \sigma(p^*)$. Furthermore, $\sum_{j\in I} z_j^t(\omega) = \sum_{j\in I} e_j(\omega)$.*

Proof. The first part of the claim follows directly from the definition of the z_i^t's. For the second part, observe that:

$$\sum_{j\in I} z_j^t = \sum_{j\in I} E\left[x_j^* \middle| \mathcal{F}_I^t\right] = E\left[\sum_{j\in I} x_j^* \middle| \mathcal{F}_I^t\right] = E\left[\sum_{j\in I} e_j \middle| \mathcal{F}_I^t\right] = \sum_{j\in I} e_j$$

where the last equivalence follows from Assumption 7.2. ■

We are only left to show that (p^*, z^t) satisfies condition 2 in Definition 7.1 when t is sufficiently large. This requires that, in all but a "small" set of states, the allocations z^t eventually solve the approximated utility maximization problems subject to the budget constraints imposed by p^*. We do it in three steps.

The first lemma considers the set C_i^t of all states in which z_i^t violates the approximate budget constraints in period t, then it shows that C_i^t is "small" for all but a finite number of periods.

Lemma 7.5 *For every $i \in I$ and $t \in T$, let $C_i^t \subset \Omega$ be the set:*

$$C_i^t = \left\{\omega \in \Omega : p^*(\omega) \cdot z_i^t(\omega) > p^*(\omega) \cdot e^i(\omega) + \varepsilon\right\}.$$

Then $\sum_{\omega\in C_i^t} q_i(\omega) < \varepsilon/2$ for all but a finite number of periods t.

Proof. Suppose that this was not the case, and that $\sum_{\omega\in C_i^t} q_i(\omega) > \varepsilon/2$ for infinitely many t's. Since $\sum_{\omega\in\Omega} q_i(\omega) = 1$, this implies that the set:

$$C_i^* = \bigcap_{t\in T} \bigcup_{s\geq t} C_i^s$$

is nonempty. Notice that C_i^* consists of all the $\omega \in \Omega$ for which there are infinitely many values of t for which $z_i^t(\omega)$ violates the approximate budget constraints.

Let $\omega \in C_i^*$. Without loss of generality, we may assume that $\omega \in C_i^t$ for every $t \in T$ (if this is not the case, just replace the t's with a subsequence for which the condition holds). This means that:

$$p^*(\omega) \cdot z_i^t(\omega) > p^*(\omega) \cdot e_i(\omega) + \varepsilon \quad \text{for every } t \in T.$$

But the z_i^t's converge in norm to x_i^*, and so they converge pointwise. This implies that:

$$\lim_t p^*(\omega) \cdot z_i^t(\omega) = p^*(\omega) \cdot x_i^*(\omega) \le p^*(\omega) \cdot e_i(\omega)$$

which is a contradiction. ■

The second lemma considers the set D_i^t of states in which z_i^t is not approximately maximal in the budget set of period t, then it shows that D_i^t is "small" for all but but a finite number of periods.

Lemma 7.6 *For every $i \in I$ and $t \in T$, let $D_i^t \subset \Omega$ be the set:*

$$D_i^t = \left\{\omega \in \Omega : \exists y \in \ell_{X_i} \text{ with } v_i\left(y|\mathcal{H}_i^t\right)(\omega) > v_i\left(z_i^t|\mathcal{H}_i^t\right)(\omega) + \varepsilon \text{ and } p^*(\omega) \cdot y(\omega) \le p^*(\omega) \cdot e_i(\omega)\right\}$$

where $\mathcal{H}_i^t = \mathcal{F}_i^t \vee \sigma(p^*)$. Then $\sum_{\omega \in D_i^t} q_i(\omega) < \varepsilon/2$ for all but a finite number of periods t.

Proof. Suppose that this was not the case, and that $\sum_{\omega \in D_i^t} q_i(\omega) > \varepsilon/2$ for infinitely many t's. By the same argument used in Lemma 7.5 the set:

$$D_i^* = \bigcap_{t \in T} \bigcup_{s \ge t} D_i^s$$

is nonempty. Take a $\omega \in D_i^*$ and assume, without loss of generality, that $\omega \in D_i^t$ for every t. This means that for every $t \in T$ there is a $y^t \in \ell_{X_i}$ such that $p^*(\omega) \cdot y^t(\omega) \le p^*(\omega) \cdot e_i(\omega)$ and:

$$v_i\left(y^t|\mathcal{H}_i^t\right)(\omega) > v_i\left(z_i^t|\mathcal{H}_i^t\right)(\omega) + \varepsilon. \tag{3}$$

The sequence $\{y^t : t \in T\}$ ranges in the compact set ℓ_{X_i}, and so it has subsequence (which we don't relabel) that converges in norm (and hence pointwise) to a $y^* \in \ell_{X_i}$.

The sequence $\left\{y^t : t \in T\right\}$ converges to y^* in ℓ_{X_i}, while $\left\{\mathcal{H}_i^t : t \in T\right\}$ is an increasing sequence of σ-algebras converging (in the order) to $\mathcal{F}_i^* \vee \sigma(p^*)$. By the continuity of the interim expected utility (see Lemma 9.5) we have that:

$$\lim_t v_i\left(y^t|\mathcal{H}_i^t\right)(\omega) = v_i\left(y^*|\mathcal{F}_i^* \vee \sigma(p^*)\right)(\omega).$$

As the z_i^t converge to x_i^* (Lemma 7.3) the same argument gives that:

$$\lim_t v_i\left(z_i^t|\mathcal{H}_i^t\right)(\omega) = v_i\left(x^*|\mathcal{F}_i^* \vee \sigma(p^*)\right)(\omega).$$

Combining these two equations with Equation (3), it must be that:

$$v_i\left(y^*|\mathcal{F}_i^* \vee \sigma(p^*)\right)(\omega) \ge v_i\left(x^*|\mathcal{F}_i^* \vee \sigma(p^*)\right)(\omega) + \varepsilon. \tag{4}$$

We show that this is a contradiction. We know that $y^t(\omega) \to y^*(\omega)$, and so $p^*(\omega) \cdot y^t(\omega) \to p^*(\omega) \cdot y^*(\omega)$. As every $y^t(\omega)$ satisfies the budget constraints imposed by $p^*(\omega)$, even $y^*(\omega)$ must do so. However, being (p^*, x^*) a REE in the limit economy, the fact that $y^*(\omega)$ is in the budget implies that:

$$v_i\left(y^*|\mathcal{F}^* \vee \sigma\left(p^*\right)\right)(\omega) \leq v_i\left(x^*|\mathcal{F}_i^* \vee \sigma\left(p^*\right)\right)(\omega)$$

in contradiction with Equation (4). ■

To conclude the proof define, for every $i \in I$ and $t \in T$, the set:

$$B_i^t = \Omega \setminus \left(C_i^t \cup D_i^t\right).$$

By Lemmas 7.5 and 7.6, for all but a finite number of t the set B_i^t is such that $\sum_{\omega \in B_i^t} q_i(\omega) \geq 1 - \varepsilon$. Furthermore, for every $\omega \in B_i^t$ one has: (i) $p^*(\omega) \cdot z_i^t(\omega) \leq p^*(\omega) \cdot e_i(\omega)$ (because $\omega \notin C_i^t$) and (ii) if $v_i\left(y|\mathcal{H}_i^t\right)(\omega) > v_i\left(z_i^t|\mathcal{H}_i^t\right)(\omega) + \varepsilon$ for some y, then $p^*(\omega) \cdot y(\omega) > p^*(\omega) \cdot e_i(\omega)$ (because $\omega \notin D_i^t$). We conclude that, for t sufficiently large, every z_i^t is a ε-REE in $\mathcal{E}^t$.

8 Conclusions

Our analysis starts from some common arguments against the notion of REE in asymmetric information economies, which include the fact that it may not exist universally and it may not be efficient and incentive compatible. We argue that, despite these non-attractive features, the asymmetric REE can be rationalized by a symmetric one which has nice properties. Thus, for all practical purposes one can focus on the Bayesian symmetric REE that we know it exists and it is efficient.

Our conclusions are driven by the fact that iterated repetitions of the same trading situation may reduce the information asymmetry to the point that it vanishes in the limit. When this is the case, the asymmetric REE that emerge in the iterated repetitions converge to a symmetric REE in the limit (Theorem 1) which exists, it is Pareto efficient and it is obviously incentive compatible (Theorem 2). Therefore, the repeated REE become asymptotically similar to a well-behaving REE in the limit. We also show a partial inverse to this result: given a symmetric well-behaving REE in the limit we can always construct a sequence of approximated REE in the repeated economies that converge to it (Theorem 3).

De Castro, Pesce, and Yannelis (2020) showed that the asymmetric REE under ambiguity exists universally, it is Pareto optimal and it is incentive compatible contrary to the asymmetric Bayesian REE concept of Kreps (1977), Radner (1979) and Allen (1981) examined in this paper. Similar results with the ones obtained here, can also be proved for the asymmetric REE under ambiguity and show that it can be rationalized by a symmetric ambiguous REE. Thus, not only the Bayesian asymmetric REE that can be rationalized by a symmetric one but the same holds true if allow for ambiguity, i.e., we replace the interim Bayesian utility by the Wald interim maxmin utility.

Throughout the paper the set of agents is finite. It is not obvious how one can extend the current results to a continuum of agents (e.g. Sun and Yannelis (2007)) in order to capture the idea of perfect competition. At this stage this seems to be an open problem.

9 Appendix

9.1 Notation and general results

Let Y be a Banach space. We write Y^* for the topological dual of Y, i.e. the space of all continuous, linear functionals on Y. For $x \in Y$ and $p \in Y^*$ we use both the notations $p \cdot x$ or $\langle x, p \rangle$ to denote the value of x at p (the context will make it clear). Seen as a function on $Y \times Y^*$, the evaluation map $\langle \cdot, \cdot \rangle$ is jointly continuous when both Y and Y^* are considered with their norms (Aliprantis and Border (2005) Theorem 6.37). If $B \subset Y^*$ is a norm-bounded set, then $\langle \cdot, \cdot \rangle$ is jointly continuous on $Y \times B$ when Y is considered with its norm and Y^* with the weak* topology (Aliprantis and Border (2005) Theorem 6.40).

The following result is known as Alaoglu's Theorem.

Fact 9.1 (Aliprantis and Border (2005), Theorem 6.10) *A subset K of the dual space Y^* is compact in the weak* topology if and only if it is weak*-closed and norm-bounded.*

A *random price* is a function $p \colon \Omega \to Y^*$ with values in the symplex $\Delta = \left\{ q \in Y^*_+ : q \cdot u = 1 \right\}$, where u is a vector in the interior of Y_+. We write ℓ_P for the set of random prices, i.e.

$$\ell_P = \left\{ p \colon \Omega \to Y^*_+ : p(\omega) \in \Delta \text{ for every } \omega \in \Omega \right\}.$$

Notice that, being Δ a weakly*-compact set by Alaoglu's Theorem (see Fact 9.1, or Jameson (1970) Theorem 3.8.6), every $p \in \ell_P$ belongs to the space $\ell_\infty(\Omega, Y^*)$ of bounded functions from Ω to Y^*, and can be seen as an element in the dual of $\ell_1(\Omega, Y)$ (see Leonard (1976), pg. 246). The set ℓ_P is then a closed and bounded subset of a dual space, and so it is weakly*-compact by Alaoglu's Theorem.

Let $\Omega = \{\omega_n\}_n$ be a finite or countable set and Y a Banach space. We write $\ell_1(\Omega, Y)$ for the set of all functions $x \colon \Omega \to Y$ that are *summable* in the sense that

$$\|x\|_1 = \sum_{\omega \in \Omega} \|x(\omega)\| < \infty$$

Endowed with the norm $\| \cdot \|_1$, $\ell_1(\Omega, Y)$ is a Banach space. Notice that $\ell_1(\Omega, Y)$ coincides with the space of $L_1(\mu, Y)$ of μ-Bochner integrable functions with values in Y when μ denotes the counting measure on Ω. When $Y = \mathbb{R}$ we also write $\ell_1(\Omega)$ instead of $\ell_1(\Omega, \mathbb{R})$.

The set $\ell_\infty(\Omega, Y)$ denotes the collection of all functions $x \colon \Omega \to Y$ that are bounded. Endowed with the norm $\|x\|_\infty = \sup_\omega \|x(\omega)\|$, the set $\ell_\infty(\Omega, Y)$ is a Banach space. In particular, if Y^* is the topological dual of Y then $\ell_\infty(\Omega, Y^*)$ is the dual of $\ell_1(\Omega, Y)$, see (Leonard, 1976, pg. 246). The corresponding duality evaluation map is given by:

$$\langle x, y \rangle = \sum_{\omega \in \Omega} x(\omega) \cdot y(\omega), \quad \text{for } x \in \ell_1(\Omega, Y) \text{ and } y \in \ell_\infty(\Omega, Y).$$

A subset K of $\ell_1(\Omega, Y)$ is *summably dominated* if there exists a $g \in \ell_1(\Omega)$ such that $\|x(\omega)\| \le g(\omega)$ for every $x \in K$ and $\omega \in \Omega$. The following version of the dominated convergence Theorem holds.

Lemma 9.1 *Let* $\{x^t : t \in T\}$ *be a sequence in* $\ell_1(\Omega, Y)$ *and* $x^*: \Omega \to Y$ *a function such that:* (i) $x^t(\omega) \to x^*(\omega)$ *for every* $\omega \in \Omega$*; and* (ii) $\{x^t : t \in T\}$ *is summably dominated. Then* $x^* \in \ell_1(\Omega, Y)$ *and* $\lim_t x^t = x^*$ *in the* ℓ_1*-norm.*

Proof. The sequence of summable, scalar functions $\{\|x^t(\cdot)\| : t \in T\}$ converges pointwise to the function $\|x^*(\cdot)\|$ and it is dominated by a $g \in \ell_1(\Omega)$. The scalar version of the dominated convergence Theorem applies (see Aliprantis and Border (2005) Theorem 11.21), proving that $\|x^*(\cdot)\|$ (and hence x^*) is summable and that $\|x^t\|_1 \to \|x^*\|_1$. Then the claim follows from Theorem 2.1 in Leonard (1976). ■

Notice that a summably bounded set K is automatically bounded, and has *equismall tails* in the following sense: for every $\varepsilon > 0$ there is a finite $J_\varepsilon \subseteq \Omega$ (depending only on ε) such that $\sum_{\omega \notin J_\omega} \|x(\omega)\| < \varepsilon$ for every $x \in K$. The following result is a version of Ascoli-Arzelà's Theorem for summable functions.

Fact 9.2 (Leonard (1976), Theorem 5.1) *A set* $K \subseteq \ell_1(\Omega, Y)$ *is compact if and only if:* (i) *it is closed and bounded,* (ii) *it has equismall tails, and* (iii) *it is such that* $K(\omega) = \{x(\omega) : x \in K\}$ *is compact for every* $\omega \in \Omega$.

If $\{\mathcal{G}^t : t \in T\}$ is a sequence of σ-algebras on Ω, the join $\bigvee_{t \in T} \mathcal{G}^t$ is the smallest σ-algebra on Ω that contains all the $\mathcal{G}^t$'s. The meet $\bigwedge_{t \in T} \mathcal{G}^t$ is the intersection of the $\mathcal{G}^t$'s. For a σ-algebra, for every $\omega \in \Omega$ we write $\mathcal{G}(\omega)$ for the smallest element of $\mathcal{G}$ that contains ω. The expectation of a summable function x conditional on $\mathcal{G}$ is the function $E[x|\mathcal{G}]$ defined by:

$$E[x|\mathcal{G}](\omega) = \begin{cases} \frac{1}{|\mathcal{G}(\omega)|} \sum_{\bar{\omega} \in \mathcal{G}(\omega)} x(\bar{\omega}) & \text{if } \mathcal{G}(\omega) \text{ is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

where $|\mathcal{G}(\omega)|$ denotes the cardinality of $\mathcal{G}(\omega)$.

Lemma 9.2 *Let* $\{\mathcal{G}^t : t \in T\}$ *be an increasing sequence of* σ*-algebras on* Ω*, and let* $\mathcal{G}^* = \bigvee_t \mathcal{G}^t$*. Then* $\lim_t E[x|\mathcal{G}^t] = E[x|\mathcal{G}^*]$ *for every* $x \in \ell_1(\Omega, Y)$.

Proof. By construction, $\{E[x|\mathcal{G}^t] : t \in T\}$ is a sequence that converges to $E[x|\mathcal{G}^*]$ pointwise and that is summably dominated by $g(\omega) = \|x(\omega)\|$. The claim follows the Theorem of dominated convergence (Lemma 9.1). ■

9.2 Joint continuity of the interim expected utility

This appendix shows that the conditional interim expected utility function $v_i(x|\mathcal{G})(\omega)$ is, in a sense, jointly continuous with respect to x and $\mathcal{G}$. Precisely, it shows that: if ω is fixed, if x^t converges (topologically) to a x^* and if $\mathcal{G}^t$ converges (in the order sense) to a $\mathcal{G}^*$, then $v_i(x^t|\mathcal{G}^t)(\omega)$ converges to $v_i(x^*|\mathcal{G}^*)(\omega)$.

Some preliminary lemmas are needed.

Lemma 9.3 *Let $\{\mathcal{G}^t : t \in T\}$ be an increasing sequence of σ-algebras on Ω, and let $\mathcal{G}^* = \bigvee_t \mathcal{G}^t$. Then, for every $i \in I$ and $\omega \in \Omega$, one has that $q_i\left(\cdot\, |\mathcal{G}^t(\omega)\right)$ is a function in $\ell_\infty(\Omega)$, and:*

$$\lim_t q_i\left(\cdot\, |\mathcal{G}^t(\omega)\right) = q_i\left(\cdot\, |\mathcal{G}^*(\omega)\right)$$

in the ℓ_∞-norm.

Proof. The relation $q_i(F) = \sum_{\omega\in F} q_i(\omega)$ defines a σ-additive probability measure on 2^Ω, the power set of Ω. As $q_i\left(\mathcal{G}^*(\omega)\right)$ is strictly positive and finite for every $\omega \in \Omega$, and q_i is summable (and hence bounded), $q_i\left(\cdot\, |\mathcal{G}^t(\omega)\right)$ is itself a bounded function.

Let us fix a $\hat{\omega} \in \Omega$. Since $\mathcal{G}^* = \bigvee_t \mathcal{G}^t$, $\mathcal{G}^t(\hat{\omega})$ is a decreasing sequence of sets with $\mathcal{G}^*(\hat{\omega}) = \bigcap_t \mathcal{G}^t(\hat{\omega})$, and so $q_i\left(\mathcal{G}^*(\hat{\omega})\right) = \inf_t q_i\left(\mathcal{G}^t(\hat{\omega})\right)$. Therefore, for every $\varepsilon > 0$ there is a $t \in T$ such that:

$$|q_i\left(\mathcal{G}^s(\hat{\omega})\right) - q_i\left(\mathcal{G}^*(\hat{\omega})\right)| = q_i\left[\mathcal{G}^s(\hat{\omega}) \setminus \mathcal{G}^*(\hat{\omega})\right] < \varepsilon, \quad \text{for every } s > t.$$

Take a $s > t$. To conclude the proof it is enough to show that $\left|q_i\left(\omega|\mathcal{G}^t(\hat{\omega})\right) - q_i\left(\omega|\mathcal{G}^*(\hat{\omega})\right)\right| < \frac{\varepsilon}{q_i(\mathcal{G}^*(\hat{\omega}))^2}$ for every $\omega \in \Omega$. We prove this by cases.

If $\omega \notin \mathcal{G}^s(\hat{\omega})$, then $q_i\left(\omega|\mathcal{G}^s(\hat{\omega})\right) = q_i\left(\omega|\mathcal{G}^*(\hat{\omega})\right) = 0$ and the condition is satisfied. If $\omega \in \mathcal{G}^s(\hat{\omega}) \setminus \mathcal{G}^*(\hat{\omega})$, then $q_i(\omega) \leq q_i\left[\mathcal{G}^s(\hat{\omega}) \setminus \mathcal{G}^*(\hat{\omega})\right] < \varepsilon$ and $q_i\left(\omega|\mathcal{G}^*(\hat{\omega})\right) = 0$. But then:

$$\left|q_i\left(\omega|\mathcal{G}^t(\hat{\omega})\right) - q_i\left(\omega|\mathcal{G}^*(\hat{\omega})\right)\right| = \frac{q_i(\omega)}{q_i\left(\mathcal{G}^*(\hat{\omega})\right)} < \frac{\varepsilon}{q_i\left(\mathcal{G}^*(\hat{\omega})\right)} \leq \frac{\varepsilon}{q_i\left(\mathcal{G}^*(\hat{\omega})\right)^2}$$

where the last inequality follows the fact that $q_i(F) \leq 1$ for every $F \subseteq \Omega$. Last, if $\omega \in \mathcal{G}^*(\hat{\omega})$ then:

$$\left|q_i\left(\omega|\mathcal{G}^t(\hat{\omega})\right) - q_i\left(\omega|\mathcal{G}^*(\hat{\omega})\right)\right| = \left(\frac{q_i(\omega)}{q_i\left(\mathcal{G}^*(\hat{\omega})\right)} - \frac{q_i(\omega)}{q_i\left(\mathcal{G}^s(\hat{\omega})\right)}\right) < \frac{\varepsilon}{q_i\left(\mathcal{G}^*(\hat{\omega})\right)^2}.$$

■

Lemma 9.4 *Under Assumptions 5.1(ii) and 5.2(ii), let $x\colon \Omega \to Y$ be such that $x(\omega) \in X_i(\omega)$ for every $\omega \in \Omega$. Then:*

1. *the function x is summable;*
2. *the function $u_i(\cdot, x(\cdot))$ is summable;*
3. *$v_i(x|\mathcal{G})(\hat{\omega})$ is well defined for every σ-algebra $\mathcal{G}$ on Ω and every $\hat{\omega} \in \Omega$. Furthermore:*

$$v_i(x|\mathcal{G})(\hat{\omega}) = \langle u_i(\cdot, x(\cdot)), q_i(\cdot\, |\mathcal{G})(\hat{\omega})\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the dual evaluation map between $\ell_1(\Omega)$ and $\ell_\infty(\Omega)$.

Proof. To prove point (1) recall that, by Assumption 5.1(ii), there exists a $f \in \ell_1(\Omega)$ such that $\|x(\omega)\| \leq f(\omega)$ for every $\omega \in \Omega$. Therefore, x is summable by the dominated convergence Theorem (Lemma 9.1).

Consider now the function $u_i(\cdot, x(\cdot))$. Assumption 5.2(ii) gives that there is a $g \in \ell_1(\Omega)$ such that $|u_i(\omega, x(\omega))| \leq g(\omega)$ for every $\omega \in \Omega$. But then, the dominated convergence Theorem (Lemma 9.1) gives that $u(\cdot, x(\cdot))$ is summable.

For the last point, fix a $\hat{\omega} \in \Omega$ and observe that $v_i(x|\mathcal{G})(\hat{\omega})$ corresponds to the result of evaluating the function $u_i(\cdot, x(\cdot))$ (as a function in $\ell_1(\Omega)$) at $q_i(\cdot|\mathcal{G}(\hat{\omega}))$ (as a function in $\ell_1(\Omega)$) via the standard duality map. In fact:

$$v_i(x|\mathcal{G})(\hat{\omega}) = \sum_{\omega} u_i(\omega, x(\omega))\, q_i(\omega|\mathcal{G}(\hat{\omega})) = \langle u_i(\cdot, x(\cdot)), q_i(\cdot|\mathcal{G}(\hat{\omega}))\rangle .$$

where $\langle \cdot, \cdot \rangle$ is the standard duality map between $\ell_1(\Omega)$ and $\ell_\infty(\Omega)$. ■

Lemma 9.5 *Under Assumptions 5.1(ii) and 5.2(i)-(ii), suppose that:*

- $\{x^t : t \in T\}$ *is a sequence in* ℓ_{X_i} *that converges to a* $x^* \in \ell_{X_i}$ *in norm,*
- $\{\mathcal{G}^t : t \in T\}$ *is an increasing sequence of* σ*-algebras on* Ω*, and* $\mathcal{G}^* = \bigvee_{t \in T} \mathcal{G}^t$*.*

Then $\{v_i(x^t|\mathcal{G}^t)(\cdot) : t \in T\}$ *converges to* $v_i(x^*|\mathcal{G}^*)(\cdot)$ *pointwise.*

Proof. Fix a $\hat{\omega} \in \Omega$. By point (3) in Lemma 9.4, we may write $v_i(x^t|\mathcal{G}^t)(\hat{\omega})$ and $v_i(x^*|\mathcal{G}^*)(\hat{\omega})$ in the form:

$$v_i(x^t|\mathcal{G}^t)(\hat{\omega}) = \langle u_i(\cdot, x^t(\cdot)), q_i(\cdot|\mathcal{G}^t)(\hat{\omega})\rangle, \quad v_i(x^*|\mathcal{G}^*)(\hat{\omega}) = \langle u_i(\cdot, x^*(\cdot)), q_i(\cdot|\mathcal{G}^*)(\hat{\omega})\rangle \tag{5}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual evaluation map between $\ell_1(\Omega)$ and $\ell_\infty(\Omega)$. We already know that $q_i(\cdot|\mathcal{G}^t(\omega)) \to q_i(\cdot|\mathcal{G}^*(\omega))$ in $\ell_\infty(\Omega)$ (Lemma 9.4). If we knew that $u_i(\cdot, x^t(\cdot)) \to u^*(\cdot, x^*(\cdot))$ in $\ell_1(\Omega)$, then Equation (5) would give:

$$\lim_t v_i(x^t|\mathcal{G}^t)(\hat{\omega}) = \lim_t \langle u(\cdot, x^t(\cdot)), q_i(\cdot|G^t)\rangle = \langle u(\cdot, x^*(\cdot)), q_i(\cdot|G^*)\rangle = v_i(x^*|\mathcal{G}^*)(\hat{\omega})$$

by the joint continuity of the map $\langle \cdot, \cdot \rangle$.

So we only hage to prove that $\{u_i(\cdot, x^t(\cdot)) : t \in T\}$ converges to $u^*(\cdot, x^*(\cdot))$ in $\ell_1(\Omega)$. By assumption, $x^t \to x^*$ in norm, and hence pointwise. Being $u_i(\omega, \cdot)$ continuous for every $\omega \in \Omega$ (Assumption 5.2(i)), it must be that $u_i(\omega, x^t(\omega)) \to u_i(\omega, x^*(\omega))$. The sequence $\{u(\cdot, x^t(\cdot)) : t \in T\}$ converges pointwise to $u_i(\cdot, x^*(\cdot))$, and it is dominated by a summable $g \in \ell_1(\Omega)$ by Assumption 5.2(iii). An application of the Theorem of dominated convergence (Lemma 9.1) gives that $u_i(\cdot, x^*(\cdot))$ is summable and that it is the limit of $\{u(\cdot, x^t(\cdot)) : t \in T\}$ in the ℓ_1-norm. This concludes the proof.
■

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