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Preliminary Version

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Abstract

We consider a pure exchange economy with externalities. Individual preferences are affected by the consumption of all other agents in the economy, and to each agent *i* is exogenously associated a nonempty set *Ai*, representing the individuals agent *i* cares about.We adopt a cooperative approach to equilibrium analysis, allowing each individual to cooperate with others and to form coalitions. Following Vasil'ev (2016), Husseinov (1994) and Graziano (2001), we study a notion of generalized fuzzy core and show that, in the case of non-convex preferences, the set of coalitions can be enlarged in such a way that a core allocation can be supported as an *A*-equilibrium by some price system. In the second part of the paper, we consider an economy with Arrowian markets for consumption externalities. For an appropriate definition of generalized fuzzy core, we show that a core allocation can be decentralized as an Information equilibrium in terms of personalized and market prices.

JEL Classification: C71, D51, D62.

Keywords: Exchange economy, interdependent preferences, markets for externalities, generalized fuzzy core.

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1 Introduction

We consider pure exchange economies with a finite number of commodities and individuals. In our framework, preferences may be affected by consumption externalities with each individual i taking care of a group of agents exogenously given. We allow each agent to cooperate with others and to form coalitions and for this model we analyze core solutions. The purpose of the paper is to show that, under suitable definitions of generalized fuzzy core, it is still possible to restore equivalence theorems and decentralize core allocations in terms of prices.

The importance of consumption externalities has been widely recognized in the recent literature on other-regarding preferences, fairness and altruism [1]. On the other hand, collective models of household consumption incorporate a theory of human caring into a general equilibrium setting [2]. Real-life examples where an individual cares for others who are unable to make their own decisions are very common [3]. These situations deserve to be analyzed in a more general theoretical framework since can have a relevant impact on the economic environment. In the models of pure exchange economies analyzed in the paper, externalities and human caring are both taken into account. Moreover, the analysis accommodates non-convex preferences. Indeed, it is well known that the assumption of convex individual preferences is especially problematic in the presence of consumption externalities (see for example the discussion in Starrett (1972)).

In the first part, we fix our attention on the model of A-economy introduced by Vasil'ev (2016), which extends the classical general equilibrium model with externalities assuming that the set A_i of agents affecting individual i is not necessarily the same for all individuals and is exogenously given. The associated *A-equilibrium* notion is a natural generalization of the classical competitive equilibrium à la Nash given in Arrow and Hahn (1971) and Laffont (1988). Under such equilibrium, each individual chooses the best consumption bundle for each member of the associated group A_i, taking as given the commodity prices, the aggregate wealth of the members of the group, and the choices of every other agent in the economy. As usual, the resulting allocation must be feasible with respect to the initial resources of the economy. The economy and the related equilibrium notion are general enough to cover important benchmark cases presented in the literature. Indeed, if for any agent the associate group reduces to the singleton represented by the agent him-

[1] See for instance Levine (1998), Fehr and Schmidt (1999) and Dufwenberg et al. (2011).

[2] See for example, Haller (2000) and Gersbach and Haller (2001).

[3] This is for example the case of elderly persons or children relying on outside help when buying goods or services.

self, we recover the classical Walrasian economy with externalities, and the notion of A-equilibrium coincides with the competitive equilibrium à la Nash. If the group is made up of all the agents in the economy other than the individual it is associated with, we end-up with a Berge economy, and the corresponding equilibrium is a natural transposition of the Berge equilibrium for non-cooperative games [4]. This solution notion may be interpreted in some cases as a fully altruistic criteria, since each individual maximizes the goals of all the other agents in the economy. It is worth noting that in addition to these two polar cases, other models are covered by the general framework of an A-economy. A relevant example is the family economy analyzed in Section 5.3, which is in the spirit of Haller (2000) and Gersbach and Haller (2001). In a family economy, the collection of the groups A_i forms a partition of the set of agents, and each A_i is interpreted as a family. Furthermore, under this specification, any individual must belong to her related set (her family), and all the members of the same family should be associated with the same set.

It is well known that, in the presence of externalities, a competitive allocation is not necessarily a Pareto optimal allocation. As a consequence, one should not expect that an A-equilibrium allocation belongs to the core defined in a standard way. In order to restore a version of the equivalence theorem when the preferences are convex, an appropriate *fuzzy A-core* notion has been introduced by Vasil'ev (2016) relying on fuzzy coalitions. This definition is in the spirit of Florenzano (1989, 1990), and has the feature that agents in a blocking coalition are myopic in the sense that they ignore the choices of the other coalition members. Furthermore, this notion is based on an optimistic behavior of the blocking coalition with respect to the reactions of the outsiders [5], and the rate of participation in the coalition of an individual may take any nonnegative value. [6]

In this paper, we relax the convexity assumption on preferences and provide a version of the Core Equivalence Theorem. We prove our result following the seminal work by Husseinov (1994) and the related contributions by Husseinov and Páscoa (1997) and Graziano (2001). We allow agents to participate in more that one fuzzy coalition, by introducing the notion of fuzzy coalition matrix. Moreover, we introduce the concept of *generalized fuzzy A-core*, by naturally adapting the fuzzy A-core to our setting and by showing that, under convexity, these two notions coincide. Finally, we prove that an A-equilibrium allocation belongs to the core, and by using standard techniques we show that

[4] See Berge (1957) for more details.

[5] We refer to Graziano et al. (2017) and Di Pietro et al. (2022) for difference between optimistic and pessimistic attitude of coalition agents with respect to the behavior of outsiders.

[6] Following Aubin (1979), the rate of participation may be normalized to take all values in the real interval $[0, 1]$.

a core allocation can be supported as an equilibrium allocation for some price system. Our result, as well as the one of Vasil'ev (2016), is based on the crucial assumption that the exogenous family of sets A_i associated to agents must be balanced in the sense of Bondareva with full support. This condition is not too demanding, and it is trivially satisfied by all particular examples analyzed in Section 5.

In the second part of the paper, following Arrow (1969), Laffont (1976), Makarov (1982), Vasil'ev (1996) and the recent paper by Bonnisseau et al. (2023), among others, we consider a pure exchange economy with Arrowian markets for consumption externalities[7]. We show that a core allocation can be decentralized as an equilibrium allocation, and a version of the equivalence theorem can still be obtained even without the convexity assumption on preferences. The focus is on an equilibrium notion named *Information equilibrium* characterized by personalized and market prices. Our result is closely related to the work of Vasil'ev (1996), which assumes convexity and builds on the hypothesis that the agents are not spiteful[8]. In the last part of the paper, we compare some classical equilibrium solutions with the Information equilibrium. In particular, we show that a Walrasian equilibrium allocation for a pure exchange economy without externalities, and a distributive Lindahl equilibrium as defined by Bergstrom (1970), are particular cases of Information equilibrium.

The paper is organized as follows. Section 2 presents the model and the basic assumptions; Section 3 introduces the notions of A-equilibrium, fuzzy A-core and the generalized fuzzy A-core, and proves the equivalence of the two core notions under convexity; Section 4 is devoted to our main result, that is the equivalence theorem. Section 5 compares some economic models and the related equilibrium notions studied in literature with the market structure considered in the paper: Subsections 5.1, 5.2 and 5.3 deal respectively with a a classical pure exchange economy with externalities, a Berge economy and family economy. Section 6 discusses the Information equilibrium, the fuzzy information core and the generalized fuzzy information core. We also emphasize that the two core notions coincide under convexity. Section 7 deals with the proof of the equivalence theorem. Section 8 discusses some classical solutions and their relation with the Information equilibrium: Subsection 8.1 deals with a competitive equilibrium of a pure exchange economy without external-

[7] Bonnisseau et al. (2023) deals with the existence of quasi-equilibria and equilibria under suitable irreducibility and survival conditions. The idea under the paper is based on the intuition of Arrow (1969), that the equilibrium allocations of a pure exchange economy with Arrowian markets for consumption externalities coincide with the equilibrium allocations of an appropriate constant returns to scale production economy without externalities. In contrast with other related works, see for example Vasil'ev (1987), the authors do not assume monotonicity of preferences.

[8] The preference are nondecreasing in their domain.

ities, and Subsection 8.2 with the distributive Lindahl equilibrium. Section 9 summarizes our results and discusses some hints of future research.

2 A-Economy: The model and the basic assumptions

There is a finite number l of commodities. The commodity space is $\mathbb{R}^l$. There is a finite number of individuals (agents or traders) denoted by the subscript $i \in N := \{1, \dots, n\}$. To each agent is exogenously associated a nonempty set $A_i \subseteq N$ describing the individuals agent i cares about; $A \coloneqq (A_i)_{i \in N}$ [9]. The consumption set associated to each agent is the standard positive cone $\mathbb{R}^l_+$, $x_i \coloneqq (x_i^1, \dots, x_i^l)$ denotes the consumption of individual i, and $x \coloneqq (x_i)_{i \in N}$ is a vector of consumption bundles. A price vector p is an element of $\mathbb{R}^l$, where p^c is the price of one unit of the commodity c. Each agent chooses the consumption of all the individuals he cares about. In this respect, we use the following additional notation: $X_{A_i} \coloneqq \mathbb{R}_+^{l \cdot |A_i|}$, $x_{A_i} \coloneqq (x_{ij})_{j \in A_i}$, $X_{N \setminus A_i} \coloneqq \mathbb{R}_+^{l \cdot |N \setminus A_i|}$, $x_{N \setminus A_i} \coloneqq (x_h)_{h \in N \setminus A_i}$. The set $D_i \coloneqq \{h \in N \colon i \in A_h\}$ denotes the agents that takes care of i [10]. Given x_{A_i} and $x_{N \setminus A_i}$, without loss of generality, we denote x by $(x_{A_i}, x_{N \setminus A_i})$. We also denote x by (x_i, x_{-i}), where $x_{-i} \coloneqq (x_k)_{k \neq i}$, when we compare our economy with classical models treated in the literature.

The individual preferences $\succsim_i$ of an agent i are affected by the consumption of all the agents, $P_i(x) \coloneqq \left\{x' \in \mathbb{R}_+^{l \cdot n} \colon x' \succ_i x\right\}$ denotes the set of consumption bundles which are strictly preferred by i to x. $P_{A_i}(x) \coloneqq \left\{x'_{A_i} \in X_{A_i} \colon (x'_{A_i}, x_{N \setminus A_i}) \in P_i(x)\right\}$ is the set of bundles strictly preferred by i to x, when the consumption of any agent $h \in N \setminus A_i$ is fixed at x_h [11]. The initial endowment of individual i is $\omega_i \coloneqq (\omega_i^1, \dots, \omega_i^l) > 0$, and $\omega \coloneqq (\omega_i)_{i \in N} \in \mathbb{R}_+^{l \cdot n}$.

A vector $x = (x_i)_{i \in N} \in \mathbb{R}^{l \cdot n}$ is an *allocation* if it satisfies the physical feasibility condition

$$\sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i$$

and $\mathscr{F}$ denotes the set of allocations.

The economy under consideration is thus formalized by the list of elements summarized below:

$$E \coloneqq \langle N, \mathbb{R}^l_+, (A_i, \succsim_i, \omega_i)_{i \in N} \rangle.$$

In order to adopt a cooperative approach to equilibrium analysis, we introduce the notion of (fuzzy) coalition matrix.

[9] Notice that we are not necessarily requiring that agent i belongs to A_i.

[10] Given a set B, we denote by $|B|$ its cardinality.

[11] The strict preference relation $\succ_i \subseteq \mathbb{R}_+^{l \cdot n} \times \mathbb{R}_+^{l \cdot n}$ is defined in the usual way, i.e., $x \succ_i y$ if and only if $x \succsim_i y$ and not $y \succsim_i x$.

Definition 1 *A (fuzzy) coalition matrix* $\alpha = (\alpha_i^j)$ *is a matrix of dimension* $r \times n$*, where* $r \in \mathbb{N}$*, and* $\alpha^j \coloneqq (\alpha_i^j)_{i \in N} \in \mathbb{R}_+^n$ *with* $\alpha^j \neq 0$ *for any* $j = 1, \ldots, r$*.*

A coalition matrix is interpreted as a finite collection of coalitions ($j = 1, \ldots, r$). Agent i may participate to any of such coalition employing the share α_i^j of her resources for any j. If $r = 1$ and α_i^1 takes only $\{0, 1\}$-values for any i, we have usual (crisp) coalitions. For a given coalition matrix $\alpha = (\alpha_i^j)$, we denote by $\operatorname{supp}(\alpha^j)$ the support of α^j, i.e., $\operatorname{supp}(\alpha^j) \coloneqq \left\{i \in N \colon \alpha_i^j > 0\right\}$, and $\operatorname{supp}(\alpha) \coloneqq \bigcup_{j=1}^r \operatorname{supp}(\alpha^j)$. Furthermore, we also define $D_i(\alpha^j) \coloneqq D_i \bigcap \operatorname{supp}(\alpha^j)$, $\alpha_i^{jA} \coloneqq \sum_{h \in D_i(\alpha^j)} \alpha_h^j$ and $\alpha_i(r) \coloneqq \sum_{j=1}^r \alpha_i^{jA}$ [12]. In the case in which $r = 1$, we simply denote α^1 by α.

We make the following survival assumption for the aggregate endowments.

Assumption 2 *The aggregate endowment* $\sum_{i \in N} \omega_i$ *belongs to* $\mathbb{R}_{++}^l$*.*

The previous assumption has an important role in order to show that the vector p resulting by the application of the Separation Theorem is strictly positive, and consequently, can be interpreted as a supporting prices vector (see the proof of Lemma 14 for details).

The basic assumptions on preference relations $\succsim_i \subseteq \mathbb{R}_+^{l \cdot n} \times \mathbb{R}_+^{l \cdot n}$ are listed below.

Assumption 3 *For any agent* i*,*

(1) $\succsim_i$ *are complete, transitive and continuous over* $\mathbb{R}_+^{l \cdot n}$*.*
(2) For any vector $x_{N \setminus A_i} \in X_{N \setminus A_i}$*,* $\succsim_i$ *are strongly monotone over* $X_{A_i} \times \{x_{N \setminus A_i}\}$*.*

Notice that we do not require any convexity assumption on preferences. Moreover, although in this paper we do not make use of utilities, the assumptions stated for preferences ensure that the preference relation $\succsim_i$ can be represented by a continuous utility function u_i defined over the commodity space.

In the rest of the paper, the exogenous family $A = (A_i)_{i \in N}$ satisfies the following set of assumptions.

Assumption 4 *(1)* $N \subseteq \bigcup_{i \in N} A_i$*;*
(2) $A = (A_i)_{i \in N}$ *is balanced in the sense of Bondareva with full support, i.e., there exists weights* $\beta = (\beta_i)_{i \in N} \in \mathbb{R}_{++}^n$*, such that* $\sum_{h \in D_i} \beta_h = 1$*, for any* $i \in N$*.*

Point 1 of Assumption 4 assures that each agent is taken into consideration by

[12] In the paper, we follow the convention that the empty sum is defined to be equal to the additive identity, i.e. zero. For example, $\alpha_i^{jA} = 0$ if $D_i(\alpha^j) = \emptyset$.

at least one agent in the economy, i.e. $D_i \neq \emptyset$ for any i, and so, at the equilibrium solution, any agent may potentially consume. Point 2 of Assumption 4 assures that, given an allocation x, if for any agent i, the consumption bundles x_{A_i} lay on the budget hyperplane of i, then the market clearing condition is satisfied (see the proof of Theorem 16 for details). Point 2 of Assumption 4 is equivalent to assume that the following set $\mathscr{B}_A$ is nonempty, i.e.,

$$\mathscr{B}_A := \left\{ \beta \in \mathbb{R}^n_{++} \colon \sum_{i \in N} \beta_i \chi_{A_i} = \chi_N \right\} \neq \emptyset$$

where, for a given set $S \subseteq N$, we denote by $\chi_S := (\chi_S^h)_{h \in N} \in \mathbb{R}^n$ the characteristic vector of $S \subseteq N$, i.e.,

$$\chi_S^h := \begin{cases} 1 & \text{if } h \in S \\ 0 & \text{if } h \notin S \end{cases}$$

3 A-Equilibrium and Generalized Fuzzy A-Core

In this section, we introduce the equilibrium and the core notions of our economy. The A-equilibrium is a natural generalization of the classical competitive equilibrium in the presence of externalities (see Vasil'ev (2016)). The (fuzzy) A-core extends the fuzzy core introduced by Aubin (1979) to our contest. In the spirit of Husseinov (1994), we further extend the notion of core introducing the Generalized (fuzzy) A-Core, and we show that, under convexity, the two notions coincide.

Definition 5 (A-equilibrium) *$(x, p) \in \mathbb{R}^{l \cdot n}_{+} \times \mathbb{R}^{l}_{++}$ is an A-equilibrium for the economy E if*

1. *$x_{A_i} \in B_{A_i}(p, \omega)$ for all $i \in N$;*
2. *$P_{A_i}(x) \bigcap B_{A_i}(p, \omega) = \emptyset$ for all $i \in N$;*
3. *$\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$*

where $B_{A_i}(p, \omega) := \left\{ x_{A_i} \in X_{A_i} : p \cdot \left(\sum_{h \in A_i} x_{ih} \right) \leq p \cdot \left(\sum_{h \in A_i} \omega_h \right) \right\}$ denotes the budget set of agent i.

Notice that, agent i considers in the budget constraint the endowments of all the agents belonging to A_i. Conditions 1 and 2 state that, for every agent i, x_{A_i} maximizes the preference of agent i over the budget set, and point 3 is the classical market clearing condition. Given an economy E, we denote by $\Omega(E)$

the set of A-equilibria, and by $\mathscr{W}(E)$ the set of A-equilibrium allocations, i.e., $\mathscr{W}(E) := \left\{x \in \mathbb{R}^{l\cdot n}_+ \mid \exists\, p \gg 0 \colon (x,p) \in \Omega(E)\right\}$.

We introduce now the notion of (fuzzy) A-core.

Definition 6 (Fuzzy A-Core) *Given an allocation $x \in \mathscr{F}$ and a coalition $\alpha \in \mathbb{R}^n_+$ with $\alpha \neq 0$, we say that α A-improves upon x whenever, for every agent $i \in N$, there exists a vector $x'_{A_i} \in X_{A_i}$ such that*

1. $x'_{A_i} \in P_{A_i}(x)$ *for any* $i \in \operatorname{supp}(\alpha)$*;*

2. $\displaystyle\sum_{i\in N}\sum_{h\in D_i(\alpha)} \alpha_h x'_{hi} \leq \sum_{i\in N} \alpha_i^A \omega_i.$

The set of allocations which cannot be A-improved upon by any coalition is called (fuzzy) A-Core, and it is denoted by $\mathscr{C}_f(E)$.

We point out that an agent in the blocking coalition is myopic with respect the decision taken by the others coalition members. Furthermore, this notions is based on an optimistic behavior of the blocking coalition with respect to the reactions of the outsiders. It is proved by Vasil'ev (2016) that the A-core introduced with Definition 6 is equivalent to the set $\mathscr{W}(E)$ of A-equilibria. To investigate the validity of the equivalence theorem in our framework where the preferences are not necessarily convex, we adapt below the fuzzy core à la Husseinov (1994) to our economy.

Definition 7 (Generalized Fuzzy A-Core) *Given an allocation $x \in \mathscr{F}$ and a coalition matrix $\alpha = (\alpha_i^j)$, we say that α A-improves x whenever, for every agent $i \in N$, there exist vectors $z^j_{A_i} \in X_{A_i}$ with $j = 1, \ldots, r$ such that*

1. $z^j_{A_i} \in P_{A_i}(x)$*, for any* $i \in \operatorname{supp}(\alpha^j)$*;*

2. $\displaystyle\sum_{i\in N}\sum_{j=1}^{r}\sum_{h\in D_i(\alpha^j)} \alpha_h^j z_{hi}^j \leq \sum_{i\in N} \alpha_i(r)\omega_i.$

The set of allocations which cannot be A-improved upon by any coalition matrix is called generalized (fuzzy) A-core (or (fuzzy) core à la Husseinov), and it is denoted by $\mathscr{C}_h(E)$.

Remark 8 *Notice that $\mathscr{C}_h(E) \subseteq \mathscr{C}_f(E)$, since the fuzzy coalitions are obtained for $r = 1$.*

In our framework, $P_{A_i}(x)$ is not required to be convex. However, just for completeness, we conclude this section, by showing that, if $P_{A_i}(x)$ is convex for any $x \in \mathbb{R}^{l\cdot n}$, then the fuzzy A-core and the generalized fuzzy A-core coincide.

Theorem 9 *Under convexity of preference relations, $\mathscr{C}_h(E) = \mathscr{C}_f(E)$.*

Proof. By Remark 8, it remains to show that $\mathscr{C}_f(E) \subseteq \mathscr{C}_h(E)$. Let $x \in \mathscr{C}_f(E)$ and suppose by contradiction that $x \notin \mathscr{C}_h(E)$. So, there exist an $r \times n$ coalition matrix $\alpha = (\alpha_i^j)$ with $\alpha^j = (\alpha_i^j)_{i\in N} \in \mathbb{R}_+^n \setminus \{0\}$, and vectors $z_{A_i}^j \in X_{A_i}$, with $i \in N$ and $j = 1, \dots r$, such that $z_{A_i}^j \in P_{A_i}(x)$ ($i \in \operatorname{supp}(\alpha^j)$, with $j = 1, \dots, r$), and $\sum_{i\in N} \sum_{j=1}^r \sum_{h\in D_i(\alpha^j)} \alpha_h^j z_{hi}^j \leq \sum_{i\in N} \alpha_i(r)\omega_i$. For any agent i such that there exists $j(i)$ with $D_i(\alpha^{j(i)}) \neq \emptyset$, define a vector $y_i := \sum_{j=1}^r \frac{\sum_{h\in D_i(\alpha^j)} \alpha_h^j}{\alpha_i(r)} z_{hi}^j$ [13]. Notice that y_i belongs to $P_{A_i}(x)$ since this set is convex and y_i is a convex combination of elements of $P_{A_i}(x)$ [14]. Finally notice that $\sum_{i\in N} \alpha_i(r) y_i = \sum_{i\in N} \sum_{j=1}^r \sum_{h\in D_i(\alpha^j)} \alpha_h^j z_{hi}^j \leq \sum_{i\in N} \alpha_i(r)\omega_i$. This contradicts the fact that x belongs to the core $\mathscr{C}_f(E)$, since $(\alpha_i(r))_{i\in N}$ is a blocking coalition. ■

4 An equivalence theorem for A-Equilibria

The next theorem shows that A-equilibrium allocations belong to the generalized fuzzy A-core.

Theorem 10 $\mathscr{W}(E) \subseteq \mathscr{C}_h(E)$.

Proof. Let $x \in \mathscr{W}(E)$. Suppose by contradiction that $x \notin \mathscr{C}_h(E)$. So there exist an $r \times n$ coalition matrix α and vectors $z_{A_i}^j \in X_{A_i}$, with $j = 1, \dots, r$ and $i \in N$, such that $z_{A_i}^j \in P_{A_i}(x)$ for any $i \in \operatorname{supp}(\alpha^j)$ [15]. Since $x \in \mathscr{W}(E)$, it must be the case that $z_{A_i}^j \notin B_{A_i}(p, \omega)$, where $p \gg 0$ is the associated equilibrium price, i.e., $(x, p) \in \Omega(E)$. Thus, for any $j = 1, \dots, r$, we have $p \cdot \sum_{h\in A_i} \alpha_i^j z_{ih}^j > p \cdot \sum_{h\in A_i} \alpha_i^j \omega_h$ for each $i \in \operatorname{supp}(\alpha^j)$. The previous inequalities can be written as $p \cdot \sum_{h\in N} \alpha_i^j \tilde{z}_{ih}^j > p \cdot \sum_{h\in N} \alpha_i^j \tilde{\omega}_h$ for each $i \in \operatorname{supp}(\alpha^j)$, where the vectors $\tilde{z}_i^j$ and $\tilde{\omega}_i$ belong to $\mathbb{R}_+^{l\cdot n}$, and they are defined by $\tilde{z}_{ih}^j := z_{ih}^j$ and $\tilde{\omega}_{ih} := \omega_h$ if $h \in A_i$, and they are equal to zero otherwise. Summing over $i \in \operatorname{supp}(\alpha^j)$ and using the associative and commutative properties of the sum operators, one gets $p \cdot \left(\sum_{h\in N} \sum_{i\in\operatorname{supp}(\alpha^j)} \alpha_i^j \tilde{z}_{ih}^j\right) > p \cdot \left(\sum_{h\in N} \sum_{i\in\operatorname{supp}(\alpha^j)} \alpha_i^j \tilde{\omega}_{ih}\right)$, which is equivalent to $p \cdot \left(\sum_{h\in N} \sum_{i\in D_h(\alpha^j)} \alpha_i^j z_{ih}^j\right) > p \cdot \left(\sum_{h\in N} \alpha^{jA} \omega_h\right)$. Finally, summing over j and rearranging, one obtains $p \cdot \left(\sum_{h\in N} \sum_{j=1}^r \sum_{i\in D_h(\alpha^j)} \alpha_i^j z_{ih}^j - \sum_{h\in N} \alpha(r)\omega_h\right) > 0$,

---

[13] By the nonemptiness of A_i, $i \in N$ and $N \subseteq \bigcup_{i\in N} A_i$, there exists at least one agent i such that $D_i(\alpha^{j(i)}) \neq \emptyset$ for some $j(i)$.

[14] By $\alpha_h(r) > 0$ and $\bigcup_{j=1}^r D_i(\alpha^j) \neq \emptyset$, one gets $\frac{\sum_{h\in D_i(\alpha^j)} \alpha_h^j}{\alpha_h(r)} \geq 0$ for any $j = 1, \dots, r$ and $\sum_{j=1}^r \frac{\sum_{h\in D_i(\alpha^j)} \alpha_h^j}{\alpha_i(r)} = 1$. Thus, the bundle y_i is a linear convex combination of elements of $P_{A_i}(x)$.

[15] Notice that, $z_{A_i}^j \in P_{A_i}(x)$ and Point 2 of Assumption 3 implies $z_{A_i}^j \neq 0$.

which contradicts the fact that the coalition matrix α blocks the allocation x via vectors $z^j_{A_i} \in X_{A_i}$, with $j = 1, \dots, r$. ■

For any $x \in \mathbb{R}^{l \cdot n}_+$ and for each $i \in N$, define the set

$$F_i(x, \omega) := \left\{ \sum_{h \in A_i} z_{ih} \in \mathbb{R}^l_+ : z_{A_i} = (z_{ih})_{h \in A_i} \in P_{A_i}(x) \right\} - \left\{ \sum_{h \in A_i} \omega_h \right\}$$

Notice that the set $F_i(x, \omega)$ is nonempty by Point 2 of Assumption 3. Denote by $\operatorname{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right)$ its convex hull.

Lemma 11 *If $x \in \mathscr{C}_h(E)$ then $0 \notin \operatorname{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right)$.*

Proof. By contradiction, suppose $0 \in \operatorname{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right)$. Therefore, for any $i \in N$ there exist vectors $z^{j(i)}_{A_i} \in X_{A_i}$ and scalars $\alpha^{j(i)}_i \geq 0$ with $j(i) = 1, \dots, r_i$ such that $\sum_{i \in N} \sum_{j(i)=1}^{r_i} \alpha^{j(i)}_i = 1$ which meet $\sum_{i \in N} \sum_{j(i)=1}^{r_i} \alpha^{j(i)}_i \sum_{h \in A_i} (z^{j(i)}_{ih} - \omega_h) = 0$. By the Caratheodory theorem, we have $\sum_{i \in N} r_i \leq l + 1$. Consider a $(l + 1) \times n$ matrix M with element $m^j_i = \alpha^{j(i)}_i$ for $j \leq r_i$ and $m^j_i = 0$ for any $j > r_i$, and for any $i \in N$. Define for any $i \in N$ and for any $j = 1, \dots, l + 1$, vectors $\tilde{z}^j_i$ and $\tilde{\omega}_i$ of $\mathbb{R}^{l \cdot n}_+$, given by $\tilde{z}^j_{ih} := z^{j(i)}_{ih}$ and $\tilde{\omega}_{ih} := \omega_h$ if $h \in A_i$ and $j \leq r_i$, and equal to zero otherwise. Thus, $0 = \sum_{i \in N} \sum_{j(i)=1}^{r_i} \sum_{h \in A_i} \alpha^{j(i)}_i (z^{j(i)}_{ih} - \omega_h) = \sum_{h \in N} \sum_{j=1}^{l+1} \sum_{i \in N} m^j_i (\tilde{z}^j_{ih} - \tilde{\omega}_{ih}) = \sum_{h \in N} \sum_{j=1}^{l+1} \sum_{i \in D_h(m^j)} m^j_i (z^j_{ih} - \omega_h)$ implying that the $l + 1$ coalitions $(m^j)_{j=1}^{l+1} = ((m^j_i)_{i \in N})_{j=1}^{l+1}$ block allocation x in sense of Husseinov. ■

By Lemma 11, if $x \in \mathscr{C}_h(E)$ then $\operatorname{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right) \cap \{0\} = \emptyset$. So, applying the Separating Hyperplane Theorem, there exists a vector $p \in \mathbb{R}^l$ with $p \neq 0$ such that $p \cdot \zeta \geq 0$ for any $\zeta \in \operatorname{co}\left(\bigcup_{i \in N} F_i(x_i, \omega)\right)$.

Lemma 12 *The vector p, with $p \neq 0$, is nonnegative.*

Proof. By strong monotonicity of the preferences over $X_{A_i} \times \{x_{N \setminus A_i}\}$, one gets $P_{A_i}(x) + X_{A_i} \subseteq P_{A_i}(x)$ for any agent i, which implies $F_i(x, \omega) + \mathbb{R}^l_+ \subseteq F_i(x, \omega)$[16]. From $F_i(x, \omega) \subseteq \operatorname{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right)$, it follows that $\operatorname{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right)$ contains a shifted nonnegative orthant. Therefore, we must have $p > 0$. Indeed, suppose $p^c < 0$ for some commodity c. Then, one may choose an element ζ of $\operatorname{co}\left(\bigcup_{i \in N} F_i(x_i, \omega)\right)$ with ζ^c large enough, such that $p \cdot \zeta < 0$, obtaining a contradiction. ■

The next lemma shows that allocations of the generalized fuzzy A-core, lay on the budget hyperplane associated to the vector p.

[16] Indeed, let $y_i \in F_i(x, \omega) + \mathbb{R}^l_+$. So, there exists $v \in \mathbb{R}^l_+$ such that $y_i = \eta_i + v$ for some $\eta_i \in F_i(x, \omega)$. Since $\eta_i \in F_i(x, \omega)$, there exists $z_{A_i} = (z_{ih})_{h \in A_i} \in P_{A_i}(x)$ such that $\eta_i = \sum_{h \in A_i} (z_{ih} - \omega_h)$. Finally, notice that $y_i = \left(\sum_{h \in A_i} z_{ih} + v\right) - \sum_{h \in A_i} \omega_h$ belongs to $F_i(x, \omega)$ since $z_{A_i} + \tilde{v} \in P_{A_i}(x) + X_{A_i} \subseteq P_{A_i}(x)$, with $\tilde{v} = (v, 0 \dots, 0) \in X_{A_i}$.

Lemma 13 *If $x \in \mathscr{C}_h(E)$ then $p \cdot \sum_{h \in A_i} x_{ih} = p \cdot \sum_{h \in A_i} \omega_h$ for any $i \in N$.*

Proof. Since x_{A_i} belongs to $\operatorname{cl} P_{A_i}(x)$, there exists a sequence $(x^{\nu}_{A_i})_{\nu \in \mathbb{N}} = ((x^{\nu}_{ih})_{i \in A_i})_{\nu \in \mathbb{N}} \subseteq P_{A_i}(x)$ such that $x^{\nu}_{A_i}$ converges to x_{A_i}. Notice that, by construction, for any agent i and any $\nu \in \mathbb{N}$, $\sum_{h \in A_i}(x^{\nu}_{ih} - \omega_h) \in F_i(x_i, \omega)$ which is included in $\operatorname{co}\left(\bigcup_{i \in N} F_i(x_i, \omega)\right)$. So, by $x \in \mathscr{C}_h(E)$, it must be true that $p \cdot \sum_{h \in A_i} x^{\nu}_{ih} \geq p \cdot \sum_{h \in A_i} \omega_h$ for any agent i and any $\nu \in \mathbb{N}$. Taking the limit, we get $p \cdot \sum_{h \in A_i} x_h \geq p \cdot \sum_{h \in A_i} \omega_h$ for each agent $i \in N$. Suppose by contradiction that there exists k such that $p \cdot \sum_{h \in A_k} x_h > p \cdot \sum_{h \in A_k} \omega_h$. By Point 2 of Assumption 4, the set $\mathscr{B}_A$ is non empty. So, take $\beta = (\beta_i)_{i \in N} \in \mathscr{B}_A$ and multiply each of the previous inequality by the corresponding weigh $\beta_i > 0$. Summing over $i \in N$ one gets $p \cdot \sum_{i \in N} \beta_i \sum_{h \in A_i} x_h > p \cdot \sum_{i \in N} \beta_i \sum_{h \in A_i} \omega_h$. By an argument similar to the one used in the proof of Theorem 10 or Lemma 11, the previous inequality can be written as $p \cdot \sum_{h \in N} \sum_{i \in D_h(\beta)} \beta_i x_h > p \cdot \sum_{h \in N} \sum_{i \in D_h(\beta)} \beta_i \omega_h$ [17]. Since $\operatorname{supp}(\beta) = N$, then we get $p \cdot \sum_{h \in N} \sum_{i \in D_h} \beta_i (x_h - \omega_h) > 0$. Finally, $\sum_{i \in D_h} \beta_i = 1$, implies $p \cdot \sum_{h \in N}(x_h - \omega_h) > 0$, which contradicts the fact that x is an allocation, since $p > 0$. ∎

We show now that the vector p is strictly positive.

Lemma 14 *The vector p is strictly positive.*

Proof. We first claim that there exists an agent k, such that $p \cdot \sum_{h \in A_k} x'_{kh} > p \cdot \sum_{h \in A_k} \omega_k$ for any $x'_{A_k} \in P_{A_k}(x)$, with $x \in \mathscr{C}_h(E)$. Since $p > 0$ and $\sum_{i \in N} \omega_i \gg 0$, then by Point 1 of Assumption 4, there exists k such that $p \cdot \sum_{h \in A_k} \omega_h > 0$. Consider this agent. By $F_k(x, \omega) \subseteq \operatorname{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right)$, one gets $p \cdot \sum_{h \in A_k} x'_{kh} \geq p \cdot \sum_{h \in A_k} \omega_h$ for any $x'_{A_k} \in P_{A_k}(x)$. Suppose that $p \cdot \sum_{h \in A_k} x''_{A_k} = p \cdot \sum_{h \in A_k} \omega_k > 0$ for some $x''_{A_k} \in P_{A_k}(x)$. By continuity of the preference, there exists $V_\delta(x''_{A_k}) \coloneqq \{\xi_{A_k} \in X_{A_k} : (\xi_{A_k}, x_{N \setminus A_k}) \in N_\delta(x''_{A_k}, x_{N \setminus A_k}) \cap \mathbb{R}^{l \cdot n}_+\}$ included in $P_{A_k}(x)$, where $N_\delta(x''_{A_k}, x_{N \setminus A_k}) \subseteq \mathbb{R}^{l \cdot n}$ is an open ball centered at $(x''_{A_k}, x_{N \setminus A_k})$ with radius $\delta > 0$. Let $\varepsilon > 0$ such that $0 < (1 - \varepsilon)\|x''_{A_k}\| < \delta$ and consider the vector $(\varepsilon x''_{A_k}, x_{N \setminus A_k})$. Thus, $\varepsilon x''_{A_k} \in V_\delta(x''_{A_k})$ and consequently $p \cdot \sum_{h \in A_k} \varepsilon x''_{kh} \geq p \cdot \sum_{h \in A_k} \omega_h$ since $\varepsilon x''_{A_k} \in P_{A_k}(x)$. By $p \cdot \sum_{h \in A_k} x''_{kh} = p \cdot \sum_{h \in A_k} \omega_k > 0$ and $\varepsilon < 1$, we get $\varepsilon p \cdot \sum_{h \in A_k} x''_{kh} < p \cdot \sum_{h \in A_k} x''_{kh}$. So,

$$p \cdot \sum_{h \in A_k} \omega_k \leq \varepsilon p \cdot \sum_{h \in A_k} x''_{kh} < p \cdot \sum_{h \in k} x''_{kh} = p \cdot \sum_{h \in A_k} \omega_h$$

and a contradiction. This completes the proof of the claim.

Fix an agent $\bar{h} \in A_k$ and a commodity $\bar{c}$. By strong monotonicity, $x_{A_k} +$

[17] Indeed, $p \cdot \sum_{i \in N} \beta_i \sum_{h \in A_i} x_h > p \cdot \sum_{i \in N} \beta_i \sum_{h \in A_i} \omega_h$ can be written as $p \cdot \sum_{i \in N} \beta_i \sum_{h \in N} \widetilde{x}_{ih} > p \cdot \sum_{i \in N} \beta_i \sum_{h \in N} \widetilde{\omega}_{ih}$, where for any agent i, $\widetilde{x}_{ih} \coloneqq x_h$ and $\widetilde{\omega}_{ih} \coloneqq \omega_h$ if $h \in A_i$, and they are equal to zero otherwise. Using the associative and commutative properties of the sum operators, one gets $p \cdot \sum_{h \in N} \sum_{i \in N} \beta_i \widetilde{x}_{ih} > p \cdot \sum_{h \in N} \sum_{i \in N} \beta_i \widetilde{\omega}_{ih}$, which is equivalent to $p \cdot \sum_{h \in N} \sum_{i \in D_h(\beta)} \beta_i x_h > p \cdot \sum_{h \in N} \sum_{i \in D_h(\beta)} \omega_h$.

$e_{A_k}(\overline{h}, \overline{c}) \in P_{A_k}(x)$, where $e_{A_k}(\overline{h}, \overline{c}) := (e_h(\overline{h}, \overline{c}))_{h \in A_k} = ((e^c_h(\overline{h}, \overline{c}))^l_{c=1})_{h \in A_k}$ is a vector in $\mathbb{R}^{l \cdot |A_k|}$ with $e^c_h(\overline{h}, \overline{c}) = 0$ for any $c \neq \overline{c}$ and $e^{\overline{c}}_{\overline{h}}(\overline{h}, \overline{c}) = 1$. So $p \cdot \sum_{h \in A_k}(x_h + e_h(\overline{h}, \overline{c})) > p \cdot \sum_{h \in A_k} \omega_h$ by the previous claim. Since $x \in \mathscr{C}_h(E)$, by Lemma 13 and the property of the inner product, one gets $p \cdot \sum_{h \in A_k} e_h(\overline{h}, \overline{c}) > 0$ which implies, by the definition of $e_{A_k}(\overline{h}, \overline{c})$, that $p^{\overline{c}} > 0$. Repeating the same argument for any commodity c, one obtains, $p \gg 0$. ■

Lemma 15 *If $x \in \mathscr{C}_h(E)$ then $p \cdot \zeta > 0$ for any $\zeta \in \operatorname{co}(\bigcup_{i \in N} F_i(x, \omega))$.*

Proof. Suppose otherwise that there exists $\zeta \in \operatorname{co}(\bigcup_{i \in N} F_i(x, \omega))$ such that $p \cdot \zeta = 0$. Furthermore, $\zeta = \sum_{j=1}^r \alpha^j \zeta^j$ for some scalars $\alpha^j > 0$ with $j = 1, \ldots, r$, $r \in \mathbb{N}$ and $r \neq 0$, such that $\sum_{j=1}^r \alpha^j = 1$ and $\zeta^j \in \bigcup_{i \in N} F_i(x, \omega) \subseteq \operatorname{co}(\bigcup_{i \in N} F_i(x, \omega))$ for any $j = 1, \ldots, r$. Therefore, $p \cdot \zeta^j = 0$ for any $j = 1, \ldots, r$, otherwise one gets a contradiction with $0 = p \cdot \zeta = p \cdot \sum_{j=1}^r \alpha^j \zeta^j$, since $\alpha^j > 0$ for any j [18]. Consider the agent $i(j)$ such that $\zeta^j \in F_{i(j)}(x, \omega)$. We claim that $P_{A_{i(j)}}(x) \subseteq X_{A_{i(j)}} \setminus \{0\}$. By Point 2 of Assumption 3, $(x_{A_{i(j)}}, x_{N \setminus A_{i(j)}}) \succsim_{i(j)} (0, x_{N \setminus A_{i(j)}})$, therefore, by transitivity, $P_{A_{i(j)}}(x) \subseteq P_{A_{i(j)}}(0, x_{N \setminus A_{i(j)}})$. Since the binary relation $\succ_{i(j)}$ is irreflexive, then $0 \notin P_{A_{i(j)}}(0, x_{N \setminus A_{i(j)}})$ and consequently, $P_{A_{i(j)}}(x) \subseteq X_{A_{i(j)}} \setminus \{0\}$, which completes the proof of the claim. So, there exists $x'_{A_{i(j)}} > 0$ such that: (i) $(x'_{A_{i(j)}}, x_{N \setminus A_{i(j)}}) \succ_{i(j)} (x_{A_{i(j)}}, x_{N \setminus A_{i(j)}})$, (ii) $\zeta^j = \sum_{h \in A_{i(j)}}(x'_{i(j)h} - \omega_h)$ and (iii) $p \cdot \zeta_{i(j)} = 0$. By continuity of the preference relation and (i), there exists $\widetilde{x}_{A_{i(j)}} < x'_{A_{i(j)}}$ such that $\sum_{h \in A_{i(j)}}(\widetilde{x}_{i(j)h} - \omega_h) \in F_{i(j)}(x, \omega)$. Furthermore, by $\widetilde{x}_{A_{i(j)}} < x'_{A_{i(j)}}$, (ii), (iii) and $p \gg 0$, we get $p \cdot \sum_{h \in A_{i(j)}}(\widetilde{x}_{i(j)h} - \omega_h) < 0$. This is a contradiction, since by $x \in \mathscr{C}_h(E)$ and $\sum_{h \in A_{i(j)}}(\widetilde{x}_{i(j)h} - \omega_h) \in \operatorname{co}(\bigcup_{i \in N} F_i(x_i, \omega))$ one should have $p \cdot \sum_{h \in A_{i(j)}}(\widetilde{x}_{i(j)h} - \omega_h) \geq 0$. ■

As a consequence of Lemma 15, for any agent i and for any $x'_{A_i} \in P_{A_i}(x)$, the inequality $p \cdot \sum_{h \in A_i} x'_{ih} > p \cdot \sum_{h \in A_i} x_h = p \cdot \sum_{h \in A_i} \omega_h$ holds true (by Lemma 13). We are now ready to prove the equivalence theorem.

Theorem 16 (Equivalence Theorem) *The set of A-equilibrium allocations concides with the set of the generalized fuzzy core: $\mathscr{W}(E) = \mathscr{C}_h(E)$.*

Proof. Take $x \in \mathscr{C}_h(E)$. We are going to show that (x, p) belongs to $\Omega(E)$. By Lemma 13, one gets $x_{A_i} \in B_{A_i}(p, \omega)$, and by Lemma 15, $P_{A_i}(x) \cap B_{A_i}(p, \omega) = \emptyset$ holds true for any agent $i \in N$. Finally, it remains to show that when $x \in \mathscr{C}_h(E)$ then market clearing condition is satisfied. Indeed, by Lemma 13, $x \in \mathscr{C}_h(E)$ implies $p \cdot \sum_{h \in A_i} x_h = p \cdot \sum_{h \in A_i} \omega_h$ for any $i \in N$. By Point 2 of Assumption 4, the family A is balanced in the sense of Bondareva with full support. So there exists $\beta_i > 0$ with $i \in N$ such that $\sum_{i \in N} \beta_i \chi_{A_i} = \chi_N$. So, $p \cdot \beta_i \sum_{h \in A_i} x_h = p \cdot \beta_i \sum_{h \in A_i} \omega_h$ for any agent i, and

[18] Since $x \in \mathscr{C}_h(E)$ and $\{\zeta^j : j = 1, \ldots, k\} \subseteq \operatorname{co}(\bigcup_{i \in N} F_i(x, \omega))$, then $p \cdot \zeta^j \geq 0$ for any $j = 1, \ldots, r$.

consequently $p \cdot \sum_{i \in N} \beta_i \sum_{h \in A_i} x_h = p \cdot \sum_{i \in N} \beta_i \sum_{h \in A_i} \omega_h$. As in the proof of Theorem 10 or Lemma 11, the previous equality can be written in the form $p \cdot \sum_{h \in N} \sum_{i \in D_h(\beta)} \beta_i x_h = p \cdot \sum_{h \in N} \sum_{i \in D_h(\beta)} \omega_h$ [19]. Since $\text{supp}(\beta) = N$, we get $p \cdot \sum_{h \in N} \sum_{i \in D_h} \beta_i (x_h - \omega_h) = 0$. Finally, by $\beta \in \mathscr{B}_A$, one gets $\sum_{i \in D_h} \beta_i = 1$, and thus $p \cdot \sum_{h \in N} (x_h - \omega_h) = 0$. As a consequence, market clearing condition is satisfies by $p \gg 0$ and the fact that x is an allocation i.e., $\sum_{h \in N} (x_h - \omega_h) \leq 0$. Therefore, all the conditions in the definition 5 are verified. So, $x \in \mathscr{W}(E)$. ■

In the case in which the preferences are convex, as a consequence of Theorem 9 one trivially obtains that $\mathscr{W}(E) = \mathscr{C}_f(E)$.

5 Economic models analyzed in literature - Part I

This section focuses on economies studied in literature and compares them with an A-economy. For simplicity, we suppose that preferences are represented by utility functions.

5.1 Walrasian Economy

Suppose that the family A is the following partition: $A = \{A_i = \{i\} : i \in N\}$, i.e., the only agent contributing to i's wealth is i, or equivalently, each agent cares only about himself. In this framework, (x, p) is an equilibrium if

1. $x_i \in \arg\max_{x'_i \in \mathbb{R}^l_+} \{u_i(x'_i, x_{-i}) \mid p \cdot x'_i \leq p \cdot \omega_i\}$, for all $i \in N$;
2. $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$.

The reader may notice that, under this specification of the family A, an A-economy coincides with the standard pure exchange economy with externalities, and the definition of A-equilibrium coincides with the one of competitive equilibrium à la Nash [20].

We now look at the core. Notice that, when $A = \{A_i = \{i\} : i \in N\}$, the set of

[19] Indeed, $p \cdot \sum_{i \in N} \beta_i \sum_{h \in A_i} x_h = p \cdot \sum_{i \in N} \beta_i \sum_{h \in A_i} \omega_h$ can be written as $p \cdot \sum_{i \in N} \beta_i \sum_{h \in N} \widetilde{x}_{ih} = p \cdot \sum_{i \in N} \beta_i \sum_{h \in N} \widetilde{\omega}_{ih}$, where for any agent i, $\widetilde{x}_{ih} := x_h$ and $\widetilde{\omega}_{ih} := \omega_h$ if $h \in A_i$, and they are equal to zero otherwise. Using the associative and commutative properties of the sum operators, one gets $p \cdot \sum_{h \in N} \sum_{i \in N} \beta_i \widetilde{x}_{ih} = p \cdot \sum_{h \in N} \sum_{i \in N} \beta_i \widetilde{\omega}_{ih}$, which is equivalent to $p \cdot \sum_{h \in N} \sum_{i \in D_h(\beta)} \beta_i x_h = p \cdot \sum_{h \in N} \sum_{i \in D_h(\beta)} \omega_h$.

[20] See for example Borglin (1973).

agents whose consumption is affected by i is just the singleton $D_i = \{i\}$ for any agent i, and thus $\alpha_i^A = \alpha_i$. Under this specification, a coalition $\alpha = (\alpha_i)_{i \in N}$ blocks a status quo x if there exist assignments $x'_i = x'_{A_i} = (x'_{ih})_{h \in \{i\}} = x'_{ii}$, for each $i \in \text{supp}(\alpha)$ such that

1. $u_i(x'_i, x_{-i}) > u_i(x)$ for any $i \in \text{supp}(\alpha)$;

2. $\displaystyle\sum_{i \in N} \alpha_i x'_i \leq \sum_{i \in N} \alpha_i \omega_i$.

Each agent improves her utility when the preferences are evaluated over (x'_i, x_{-i}). Thus, the assignment changes over agent i, but it is fixed for all the others. If there are no externalities at all, we have the usual notion of fuzzy core. If, in the presence of externalities, α is an ordinary coalition, that is, $\alpha = \chi_S$, with $S \subseteq N$, then a coalition S blocks a status quo x if for any agent which belongs to the blocking coalition, there exists an assignment x'_i, such that $u_i(x'_i, x_{-i}) > u_i(x)$ for any $i \in S$ and $\sum_{i \in S} x'_i \leq \sum_{i \in S} \omega_i$. This is the cooperative solution concept analyzed by Florenzano (1989, 1990).

5.2 Berge Economy

Suppose that the structure A is now given by $A = \{A_i = N \setminus \{i\} \colon i \in N\}$. In this situation, each agent is affected by all the other agents, and A is not a partition. Equivalently, one may thing that agents are supposed to act using an altruistic behaviour, regardless to their own consumption. Thus, (x, p) is an equilibrium if

1. $x_{-i} \in \underset{x'_{-i} \in \mathbb{R}_+^{l \cdot (n-1)}}{\arg\max} \{u_i(x'_{-i}, x_i) \mid p \cdot \sum_{h \in N \setminus \{i\}} x'_h \leq p \cdot \sum_{N \setminus \{i\}} \omega_h\}$, for all $i \in N$;
2. $\displaystyle\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$.

This definition can be considered as an adaptation of the classical Berge equilibrium for non-cooperative games [21] .

We now look at the cooperative solution. When $A = \{A_i = N \setminus \{i\} \colon i \in N\}$, then the set of agents whose consumption is affected by i is the set $D_i = \{j \in N \colon i \in N \setminus \{j\}\} = N \setminus \{i\}$. Thus, given a fuzzy coalition $\alpha = (\alpha_i)_{i \in N}$ one has $\alpha_i^A = \sum_{j \in N \setminus \{i\}} \alpha_j = \sum_{j \neq i} \alpha_j$. Under this framework, α blocks an allocation x if there exist vectors $x'_{-i} = x'_{A_i} = (x'_{ih})_{h \neq i}$ with $i \in N$ such that

1. $u_i(x'_{-i}, x_i) > u_i(x)$ for any $i \in \text{supp}(\alpha)$;

[21] See for instance, Berge (1957) and Vasil'ev (2016).

2. $\sum_{i \in N} \sum_{h \in \mathrm{supp}(\alpha) \setminus \{i\}} x'_{ih} \leq \sum_{i \in N} \sum_{h \in \mathrm{supp}(\alpha) \setminus \{i\}} \alpha_h \omega_h$.

The consumption of agent i is fixed for each member in the coalition, and each member proposes a different consumption plan. When α is a crisp coalition, $S \subseteq N$ blocks the status quo x if $u_i(x'_{-i}, x_i) > u_i(x)$ for any $i \in S$, and $\sum_{i \in N} \sum_{h \in (S \setminus \{i\})} x'_{ih} \leq \sum_{i \in N} |S \setminus \{i\}| \cdot \omega_h$.

5.3 Family Economy

Assume that $\mathscr{H} \coloneqq \{H_j : j = 1, \ldots, k\}$, with $k \leq n$ is a partition of the set of agents, and interpret each $H_j \in \mathscr{H}$ as a family. For every $i \in N$, define A_i as the element of $\mathscr{H}$ containing i, i. e., $A_i \coloneqq \{H_j \in \mathscr{H} : i \in H_j\}$. Notice that, in this case, the sets A_i are not necessarily different, and $i \in A_i$ for each $i \in N$. In particular, observe that members of the same family are associated with the same A_i, and as consequence the budget set in the definition of an A-equilibrium is the same for all members of the same family (see Section 3). In particular, the budget set of the family A_i is given by $B_{A_i}(p, \omega) \coloneqq \left\{x_{A_i} \in X_{A_i} : p \cdot \left(\sum_{h \in A_i} x_{ih}\right) \leq p \cdot \left(\sum_{h \in A_i} \omega_h\right)\right\}$, where $p \cdot \left(\sum_{h \in A_i} \omega_h\right)$ is the income of the family. Therefore, (x, p) is an equilibrium if

1. $x_{A_i} \in \arg\max_{x'_{A_i} \in \mathbb{R}_+^{l \cdot |A_i|}} \{u_i(x'_{A_i}, x_{N \setminus A_i}) \mid p \cdot \sum_{h \in A_i} x'_h \leq p \cdot \sum_{h \in A_i} \omega_h\}$, for all $i \in N$;
2. $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$.

In equilibrium, the consumption bundle of the family A_i maximizes the utility of all the members $h \in A_i$ under the same budget constraint. Notice also that the utility may be different for members of the same family. We call this equilibrium *family equilibrium* for the following reason. Suppose that the family structure $\mathscr{H}$ is given, and the utility of each member of the family A_i only depends on the members of the family, i.e., $u_h(x) = u_h(x_{A_i})$, for all $h \in A_i$. This form of externalities is known in the literature as intra-household externalities [22]. Thus, we end up with the notion of equilibrium defined by Haller (2000) and Gersbach and Haller (2001) for they collective consumption models [23].

We now look at the core. A coalition α blocks an allocation x if there exists a

[22] See for instance, Haller (2000) and Gersbach and Haller (2001).

[23] Haller (2000) and Gersbach and Haller (2001) define a competitive equilibrium among households as a pair (x, p) such that, $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$, and for any $i \in N$: (1) $x_{A_i} \in B_{A_i}(p, \omega)$; (2) there is no $z_{A_i} \in B_{A_i}(p, \omega)$ which meets $u_h(z_{A_i}) \geq u_h(x_{A_i})$ for any $h \in A_i$, and $u_h(z_{A_i}) > u_h(x_{A_i})$ for some $h \in A_i$.

plan for the family $x'_{A_i} = (x'_{ih})_{h \in A_i}$, with $i \in N$, such that

1. $u_i(x'_{A_i}, x_{N \setminus A_i}) > u_i(x)$ for each $i \in \text{supp}(\alpha)$;
2. $\sum_{i \in N} \sum_{h \in A_i \cap \text{supp}(\alpha)} \alpha_h x'_{hi} \leq \sum_{i \in N} \sum_{h \in A_i \cap \text{supp}(\alpha)} \alpha_h \omega_i$.

In this setting, $D_i(\alpha) = A_i \cap \text{supp}(\alpha)$ for each agent i. If $\alpha = \chi_S$, then S blocks x if $u_i(x'_{A_i}, x_{N \setminus A_i}) > u_i(x)$ for each $i \in S$ and $\sum_{i \in N} \sum_{h \in A_i \cap S} x'_{hi} \leq \sum_{i \in N} \sum_{h \in A_i \cap S} |A_i \cap S| \cdot \omega_i$. Assume the following restriction on coalition formation: α is an admissible coalition if $\alpha_i \neq 0$ for a member i of a family A_i implies $\alpha_h \neq 0$ for any other member h of A_i. For a crisp coalition, this condition implies that a coalition is formed by the union of families. In this case, each family proposes an alternative plan for the family and globally for the coalition S.

6 Information equilibrium and Generalized Fuzzy Information-Core

In this section, we consider an A-economy in which $A_i = N$ for any agent i. To simplify the notation, we will denote the economy by E, that is, $E := \langle N, \mathbb{R}^l_+, (\succsim_i, \omega_i)_{i \in N} \rangle$. Given a bundle x, the set $P_i(x) = \left\{ x' \in \mathbb{R}^{l \cdot n}_+ : x' \succ_i x \right\}$ coincides with $P_{A_i}(x)$. Following Arrow (1969), Laffont (1976), Makarov (1982) and Vasil'ev (1996), we introduce below the *Information equilibrium*[24]. In contrast with the notion of A-equilibrium which is characterized by a unique price system, in the Information equilibrium there are a personalized price system $\pi := (p_{(i)})_{i \in N}$, with $p_{(i)} := (p_{ih})_{h \in N}$, and a market price p. The personalized price p_{ih} is interpreted as the vector which make up the information for agent i about the consumption of agent h. An element $z_{A_i} = (z_{ih})_{h \in N}$, will be denoted by $z_{(i)}$.

Below we introduce the equilibrium notion for market with externalities.

Definition 17 (Information equilibrium) $(x, \pi, p) = ((x_i, p_{(i)})_{i \in N}, p) \in \mathbb{R}^{l \cdot n}_+ \times \mathbb{R}^{l \cdot n^2} \times \mathbb{R}^l$ *is an information equilibrium for the economy E if*

1. $x \in B_i(\pi, p, \omega)$ for all $i \in N$;

2. $P_i(x) \bigcap B_i(\pi, p, \omega) = \emptyset$ for all $i \in N$;

3. $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$;

[24] We follow the terminology of Vasil'ev (1996).

4. $\sum_{i \in N} p_{ih} = p$*, for all* $h \in N$

where the set $B_i(\pi, p, \omega) \coloneqq \left\{ x_i \in \mathbb{R}^{l \cdot n}_+ \colon p_{(i)} \cdot x \leq p \cdot \omega_i \right\}$ *denotes the budget set of agent* i*.*

Notice that the budget is defined in terms of the personalized and market prices. In contrast with the budget set in the definition of an $A-$equilibrium, in an information equilibrium, agent i considers her own endowment. Conditions 1 and 2 state that, for every agent i, x maximizes preference under the budget constraint, point 3 is the classical market clearing condition, and point 4 is a feasibility condition for prices. Given an economy E, we denote by $\Omega_I(E)$ the set of information equilibria, and $\mathscr{W}_I(E) \coloneqq \left\{ x \in \mathbb{R}^{l \cdot n}_+ | \; \exists \, (\pi, p) \in \mathbb{R}^{l \cdot n^2} \times \mathbb{R}^l \colon (x, \pi, p) \in \Omega_I(E) \right\}$ is the set of all information equilibrium allocations.

The Basic Assumptions 3 on preference relations $\succsim_i \subseteq \mathbb{R}^{l \cdot n}_+ \times \mathbb{R}^{l \cdot n}_+$ stated in Section 2, are replaced by the following,

Assumption 18 *For any agent* i*,*

(1) $\succsim_i$ *are complete, transitive and continuous over* $\mathbb{R}^{l \cdot n}_+$*.*
(2) For any vector $x_i \in \mathbb{R}^l_+$*,* $\succsim_i$ *are strongly monotone over* $\mathbb{R}^l_+ \times \{x_{-i}\}$*, and non-decreasing over* $\mathbb{R}^{l \cdot n}_+$*.*[25]

We remark that Assumptions 4 is trivially satisfies. We report below the notion of fuzzy Information-Core, introduced by Vasil'ev (1996).

Definition 19 (Fuzzy Information-Core) *Given an allocation* $x \in \mathscr{F}$ *and a coalition* $\alpha \in \mathbb{R}^n_+$ *with* $\alpha \neq 0$*, we say that* α *(information)-improves upon* x *whenever, for every agent* $i \in N$*, there exists a vector* $(z_{(i)}, \xi_i) = ((z_{ih})_{h \in N}, \xi_i) \in \mathbb{R}^{l \cdot n}_+ \times \mathbb{R}^l_+$ *such that*

1. $z_{(i)} \in P_i(x)$ *for any* $i \in \operatorname{supp}(\alpha)$*;*

2. $\sum_{i \in N} \alpha_i \xi_i \leq \sum_{i \in N} \alpha_i \omega_i$*;*

3. $\alpha_i z_{(i)} = (\alpha_h \xi_h)_{h \in N}$ *for any* $i \in \operatorname{supp}(\alpha)$*.*

The set of allocations which cannot be (information)-improved upon by any coalition is called (fuzzy) Information-Core, and it is denoted by $\mathscr{C}^I_f(E)$*.*

As in section 3, we adapt below the generalized fuzzy core to our economy. For a coalition matrix α, with $\alpha^j \in \mathbb{R}^n_+$ with $\alpha^j \neq 0$, denote by $\alpha_i(r)$ the element

[25] A preference relation $\succsim_i \subseteq \mathbb{R}^{l \cdot n}_+ \times \mathbb{R}^{l \cdot n}_+$ is non-decreasing if for any $x \in \mathbb{R}^{l \cdot n}_+$ and for any vector $v \in \mathbb{R}^{l \cdot n}_+$, one has $x + v \succsim_i x$.

$\sum_{j=1}^r \alpha_i^j$.

Definition 20 (Generalized Fuzzy Information-Core) *Given an allocation $x \in \mathscr{F}$ and a coalition matrix $\alpha = (\alpha_i^j)$, with $\alpha^j \in \mathbb{R}_+^n$ with $\alpha^j \neq 0$, we say that α (information)-improves upon x whenever, for every agent $i \in N$, there exist vectors $(z_{(i)}^j, \xi_i^j) = ((z_{ih}^j)_{h\in N}, \xi_i^j) \in \mathbb{R}_+^{l\cdot n} \times \mathbb{R}_+^l$ with $j = 1, \ldots, r$ such that*

1. $z_{(i)}^j \in P_i(x)$, *for any* $i \in \operatorname{supp}(\alpha^j)$;
2. $\displaystyle\sum_{i\in N}\sum_{j=1}^r \alpha_i^j \xi_i^j \leq \sum_{i\in N} \alpha_i(r)\omega_i$;
3. $\displaystyle\sum_{j=1}^r \alpha_i^j z_{(i)}^j = \left(\sum_{j=1}^r \alpha_h^j \xi_h^j\right)_{h\in N}$ *for any* $i \in \operatorname{supp}(\alpha)$.

The set of allocations which cannot be (information)-improved by any coalition matrix is called generalized (fuzzy) information-core (or fuzzy core à la Husseinov) and it is denoted by $\mathscr{C}_h^I(E)$.

Remark 21 *Notice that $\mathscr{C}_h^I(E) \subseteq \mathscr{C}_f^I(E)$, since the fuzzy coalitions are obtained for $r = 1$.*

Under convexity, the fuzzy information-core and the generalized fuzzy information-core coincide.

Theorem 22 *Under convexity of preference relations, $\mathscr{C}_h^I(E) = \mathscr{C}_f^I(E)$.*

Proof. By Remark 21, it remains to show that $\mathscr{C}_f^I(E) \subseteq \mathscr{C}_h^I(E)$. Let $x \in \mathscr{C}_f^I(E)$ and suppose by contradiction that $x \notin \mathscr{C}_h^I(E)$. So, there exist an $r \times n$ coalition matrix $\alpha = (\alpha_i^j)$ and vectors $(z_{(i)}^j, \xi_i^j) \in \mathbb{R}_+^{l\cdot n} \times \mathbb{R}_+$, with $i \in N$ and $j = 1, \ldots r$, such that $z_{(i)}^j \in P_i(x)$ ($i \in \operatorname{supp}(\alpha^j)$, with $j = 1, \ldots, r$), $\sum_{i\in N}\sum_{j=1}^r \alpha_i^j \xi_i^j \leq \sum_{i\in N} \alpha_i(r)\omega_i$ and $\sum_{j=1}^r \alpha_i^j z_{(i)}^j = (\sum_{j=1}^r \alpha_h^j \xi_h^j)_{h\in N}$ for each $i \in \operatorname{supp}(\alpha)$. For any agent i, define a vector

$$(y_{(i)}, \eta_i) := \left(\frac{\sum_{j=1}^r \alpha_i^j}{\alpha_i(r)} z_{(i)}^j, \frac{\sum_{j=1}^r \alpha_i^j}{\alpha_i(r)} \xi_i^j\right) \in \mathbb{R}_+^{l\cdot n} \times \mathbb{R}_+^l$$

Notice that $y_{(i)}$ belongs to $P_i(x)$ since $P_i(x)$ is a convex set and $y_{(i)}$ is a linear convex combination of elements of $P_i(x)$ [26] . Furthermore,

$$\sum_{i\in N} \alpha_i(r)\eta_i = \sum_{i\in N}\sum_{j=1}^r \alpha_i^j \xi_i^j \leq \sum_{i\in N} \alpha_i(r)\omega_i$$

[26] Notice that $\frac{\alpha_h^j}{\alpha_i(r)} \geq 0$ for any $j = 1, \ldots, r$, and $\sum_{j=1}^r \frac{\alpha_h^j}{\alpha_h(r)} = 1$ for any agent h.

and

$$\alpha_i(r)y_{(i)} = \sum_{j=1}^{r} \alpha_i^j z_{(i)}^j = \left(\sum_{j=1}^{r} \alpha_h^j \xi_h^j \right)_{h\in N} = \left(\alpha_h(r) \frac{\sum_{j=1}^{r} \alpha_h^j \xi_h^j}{\alpha_h(r)} \right)_{i\in N} = (\alpha_h(r)\eta_h)_{h\in N} .$$

This contradicts the fact that $x \in \mathscr{C}_f^I(E)$, since $(\alpha_i(r))_{i\in N}$ is a blocking coalition. ∎

7 An equivalence theorem for Information equilibria

We start the section by proving that an informational equilibrium allocation belongs to the generalized fuzzy information-core.

Theorem 23 *$\mathscr{W}_I(E) \subseteq \mathscr{C}_h^I(E)$.*

Proof. Let $x \in \mathscr{W}_I(E)$ and suppose that $x \notin \mathscr{C}_h^I(E)$. There exists a coalition matrix $\alpha = (\alpha_i^j)$ such that $z_{(i)}^j \in P_i(x)$ for any $i \in \mathrm{supp}(\alpha^j)$ and for any $j = 1, \ldots, r$. Since $x \in \mathscr{W}_I(E)$, it must be the case that $z_{(i)}^j \notin B_i(\pi, p, \omega)$, where π and p are such that $(x, \pi, p) \in \Omega_I(E)$. Thus, $p_{(i)} \cdot \alpha_i^j z_{(i)}^j > p \cdot \alpha_i^j \omega_i$ for any agent $i \in \mathrm{supp}(\alpha^j)$ and for any $j = 1, \ldots, r$. Summing over $i \in \mathrm{supp}(\alpha^j)$, we get $\sum_{i\in \mathrm{supp}(\alpha^j)} p_{(i)} \cdot \alpha_i^j z_{(i)}^j > p \cdot \sum_{i\in \mathrm{supp}(\alpha^j)} \alpha_i^j \omega_i$, which is equivalent to $\sum_{i\in N} p_{(i)} \cdot \alpha_i^j z_{(i)}^j > p \cdot \sum_{i\in N} \alpha_i^j \omega_i$. Summing with respect to $j = 1, \ldots, r$ and inverting the sum operators, we obtain $\sum_{i\in N} p_{(i)} \cdot \sum_{j=1}^r \alpha_i^j z_{(i)}^j > p \cdot \sum_{i\in N} \alpha_i(r)\omega_i$. Using the fact that $\sum_{j=1}^r \alpha_i^j z_{(i)}^j = (\sum_{j=1}^r \alpha_h^j \xi_h^j)_{h\in N}$ for any $i \in \mathrm{supp}(\alpha)$, for some ξ_h^j, the previous inequalities can be written as $\sum_{i\in N} \sum_{h\in N} p_{ih} \cdot \sum_{j=1}^r \alpha_h^j \xi_h^j > p \cdot \sum_{i\in N} \alpha_i(r)\omega_i$. Finally, inverting the sum operators on the left side of the inequality, and using the fact that $\sum_{i\in N} p_{ih} = p$, for all $h \in N$ we get $\sum_{h\in N} p \cdot \sum_{j=1}^r \alpha_h^j \xi_h^j > p \cdot \sum_{i\in N} \alpha_i(r)\omega_i = p \cdot \sum_{h\in N} \alpha_h(r)\omega_h$. Therefore,

$$p \cdot \sum_{h\in N} \left(\sum_{j=1}^{r} \alpha_h^j \xi_h^j - \alpha_h(r)\omega_h \right) > 0$$

which contradicts the fact that the coalition matrix α blocks the allocation x. ∎

We now introduce some notations. Define the set Q as follows,

$$Q := \{(i, h) |\ i, h \in N, i \neq h\} \cup \{(0, 0)\}$$

the linear mapping $\varphi_i\colon z_{(i)} \in \mathbb{R}_+^{l\cdot n} \mapsto \varphi_i(z_{(i)}) \in \mathbb{R}_+^{l\cdot|Q|}$, by

$$\varphi_i^q(z_{(i)}) \coloneqq \begin{cases} z_{ii} & \text{if } q = (0,0) \\ -z_{ii} & \text{if } q = (h,i) \\ z_{ih} & \text{if } q = (i,h) \\ 0 & \text{otherwise} \end{cases}$$

and the vector $\widehat{\omega} \in \mathbb{R}^{l\cdot|Q|}$, by

$$\widehat{\omega}_i^q \coloneqq \begin{cases} \omega_i & \text{if } q = (0,0) \\ 0 & \text{otherwise} \end{cases}$$

For any $x \in \mathbb{R}_+^{l\cdot n}$ and for any agent $i \in N$, consider the set

$$G_i(x,\omega) \coloneqq \left\{\varphi_i(z_{(i)}) \in \operatorname{Im}\varphi_i\colon z_{(i)} \in P_i(x)\right\} - \{\widehat{\omega}_i\}.$$

Notice that the set $G_i(x,\omega) \subseteq \mathbb{R}^{l\cdot|Q|}$ is nonempty by Point 2 of Assumption 18. Denote by $\operatorname{co}\left(\bigcup_{i\in N} G_i(x,\omega)\right)$ its convex hull.

Lemma 24 *If $x \in \mathscr{C}_h^I(E)$ then $0 \notin \operatorname{co}\left(\bigcup_{i\in N} G_i(x,\omega)\right)$.*

Proof. By contradiction, suppose $0 \in \operatorname{co}\left(\bigcup_{i\in N} G_i(x,\omega)\right)$. Therefore, for any $i \in N$, there exist vectors $z_{(i)}^{j(i)} \in P_i(x) \subseteq \mathbb{R}_+^{1\cdot n}$ and scalars $\alpha_i^{j(i)} \geq 0$ with $j(i) = 1,\ldots,r_i$ such that $\sum_{i\in N}\sum_{j(i)=1}^{r_i} \alpha_i^{j(i)} = 1$ which meet $\sum_{i\in N}\sum_{j(i)=1}^{r_i} \alpha_i^{j(i)}(\varphi_i(z_{(i)}^{j(i)}) - \widehat{\omega}_i) = 0$. By the Caratheodory theorem, we have $\sum_{i\in N} r_i \leq l+1$. Consider a $(l+1)\times n$ matrix M with element $m_i^j = \alpha_i^{j(i)}$ for $j \leq r_i$ and $m_i^j = 0$ for any $j > r_i$, and for any $i \in N$. Thus $0 = \sum_{i\in N}\sum_{j(i)=1}^{r_i} \alpha_i^{j(i)}(\varphi_i(z_{(i)}^{j(i)}) - \widehat{\omega}_i) = \sum_{i\in N}\sum_{j=1}^{l+1} m_i^j(\varphi_i(z_{(i)}^j) - \widehat{\omega}_h)$. By the definition of φ_i and $\widetilde{\omega}_i$, for any agent i, we have

$$0 = \sum_{i\in N}\sum_{j=1}^{l+1} m_i^j(\varphi_i^{(0,0)}(z_{(i)}^j) - \widehat{\omega}_h^{(0,0)}) = \sum_{i\in N}\sum_{j=1}^{l+1} m_i^j(z_{ii}^j - \omega_h)$$

Defining $\xi_i^j \coloneqq z_{ii}^j$ for any agent i and any j, we get

$$\sum_{i\in N}\sum_{j=1}^{l+1} m_i^j \xi_i^j \leq \sum_{i\in N} m_i(l+1)\omega_i$$

Furthermore, for any $q \in Q$, with $q \neq (0,0)$, we obtain $\sum_{j=1}^{l+1} m_h^j z_{hh}^j = \sum_{j=1}^{l+1} m_i^j z_{ih}^j$ for any $i,h \in \operatorname{supp}(m) \subseteq N$. By $\xi_h^j = z_{hh}^j$ for any agent h, one gets

$$\sum_{j=1}^{l+1} m_i^j z_{(i)}^j = \left(\sum_{j=1}^{l+1} m_h^j \xi_h^j\right)_{h\in N} \qquad \forall i \in \operatorname{supp}(m)$$

implying that the $l+1$ coalitions $(m^j)_{j=1}^{l+1} = ((m_i^j)_{i\in N})_{j=1}^{l+1}$ block the allocation x. ■

By Lemma 24, if $x \in \mathscr{C}_h^I(\mathscr{E})$ then $\mathrm{co}\,(\bigcup_{i\in N} G_i(x,\omega)) \cap \{0\} = \emptyset$. So, applying the Separating Hyperplane Theorem, there exists a vector $\widehat{p} = (\widehat{p}^q) \in \mathbb{R}^{l\cdot|Q|}$, with $\widehat{p} \neq 0$, such that $\widehat{p}\cdot\zeta \geq 0$ for any $\zeta \in \mathrm{co}\,(\bigcup_{i\in N} G_i(x_i,\omega))$. Define the vector $(\pi, p) = ((p_i)_{i\in N}, p) \in \mathbb{R}^{l\cdot n^2} \times \mathbb{R}^l$ as follows,

$$p \coloneqq \widehat{p}^{(0,0)} \text{ and } p_{(i)} = (p_{ih})_{h\in N} \quad \text{with} \quad p_{ih} \coloneqq \begin{cases} \widehat{p}^{(i,h)} & \text{if } i \neq h \\ \widehat{p}^{(0,0)} - \sum_{k\neq i} \widehat{p}^{(k,h)} & \text{if } i = h \end{cases}$$

Notice that $\sum_{i\in N} p_{ih} = p$, for all $h \in N$. Indeed,

$$\sum_{i\in N} p_{ih} = p_{hh} + \sum_{i\neq h} p_{ih} = \widehat{p}^{(0,0)} - \sum_{i\neq h} \widehat{p}^{(i,h)} + \sum_{i\neq h} \widehat{p}^{(i,h)} = \widehat{p}^{(0,0)} = p.$$

The following results show important properties of the hyperplane associated to (π, p).

Lemma 25 *For any $i \in N$ and for any $z_{(i)} \in P_i(x)$, $p_{(i)} \cdot z_{(i)} \geq p \cdot \omega_i$ holds true.*

Proof. By $G_i(x,\omega) \subseteq \mathrm{co}\,(\bigcup_{i\in N} G_i(x,\omega))$, one gets $\widehat{p}\cdot\varphi(z_{(i)}) > \widehat{p}\cdot\widehat{\omega}$ for any $z_{(i)} \in P_i(x)$. By the definition of $\varphi(\cdot)$,

$$\begin{aligned}\sum_{q\in Q} \widehat{p}^q \cdot \varphi^q(z_{(i)}) &= \widehat{p}^{(0,0)} \cdot z_{ii} + \sum_{h\neq i} \widehat{p}^{(h,i)} \cdot (-z_{ii}) + \sum_{h\neq i} \widehat{p}^{(i,h)} \cdot (z_{ih}) + \sum_{h\neq i, k\neq i} \widehat{p}^{(h,k)} \cdot 0 = \\ &= \left(\widehat{p}^{(0,0)} - \sum_{h\neq i} \widehat{p}^{(h,i)}\right) \cdot z_{ii} + \sum_{h\neq i} \widehat{p}^{(i,h)} \cdot (z_{ih}) \\ &\geq \sum_{q\in Q} \widehat{p}^q \cdot \widehat{\omega}_i^q = \widehat{p}^{(0,0)} \cdot \omega_i + \sum_{q\neq 0} \widehat{p}^q \cdot 0 = \widehat{p}^{(0,0)} \cdot \omega_i\end{aligned}$$

Finally, by the definition of (π, p), we get the desired result. ■

Lemma 26 *The vector (π, p), with $(\pi, p) \neq (0, 0)$, is nonnegative.*

Proof. By Point (ii) of Assumption 18, the preference relation $\succsim_i$ is nondecreasing over $\mathbb{R}_+^{l\cdot n}$. Thus, one gets $P_i(x) + \mathbb{R}_+^{l\cdot n} \subseteq P_i(x)$ for any agent i, which implies that $\pi = (p_{(i)})_{i\in N} \geq 0$. Indeed, suppose that there exists an agent i such that $p_{ih}^c < 0$ for some individual h and a commodity c, then, since ω_i^c is exogenously given, one may choose an element $z_{(i)} \in P_i(x)$ with z_{ih}^c large enough, such that $p_{(i)} \cdot z_{(i)} < p\cdot\omega_i$, and obtain a contradiction. Moreover, by $\sum_{i\in N} p_{ih} = p$, we get $p \geq 0$, which implies that (π, p) is nonnegative. It remains to show that (π, p) is different from the null vector. It follows from the fact that p must be different from the null vector. Indeed, if $p = \widehat{p}^{(0,0)} = 0$, then $\sum_{i\in N} p_{ih} = p = 0$ for any $h \in N$, and $p_{(i)} \geq 0$, imply that $p_{(i)} = 0$ for any agent i. This implies $\widehat{p} = 0$ which is a contradiction. ■

Remark 27 By Lemma 26, it follows that $\pi > 0$ and $p > 0$. Indeed, we have already proved that (π, p) is non negative with $p \neq 0$. The fact that π is different from the null vector follows from $\sum_{i\in N} p_{ih} = p > 0$ for any $h \in N$. In particular, for any agent h there is an agent i such that $p_{ih} > 0$.

Lemma 28 *If $x \in \mathscr{C}_h^I(E)$ then $p_{(i)} \cdot x = p \cdot \omega_i$ for any $i \in N$.*

Proof. As a consequence of Lemma 25, one obtains $p_{(i)} \cdot x \geq p \cdot \omega_i$ for any agent $i \in N$. Indeed, since $x \in \operatorname{cl} P_i(x)$, we may find a sequence $(x_{(i)}^\nu) \subseteq P_i(x)$ converging to x. By Lemma 25, we obtain $p_{(i)} \cdot x_{(i)}^\nu \geq p \cdot \omega_i$ for any $\nu \in \mathbb{N}$. So, taking the limit we get $p_{(i)} \cdot x \geq p \cdot \omega_i$. Suppose that there exists an agent h such that $p_{(h)} \cdot x > p \cdot \omega_h$. Summing over h, we derive $\sum_{h\in N} p_{(h)} \cdot x > p \cdot \sum_{h\in N} \omega_h$. By $p = \sum_{h\in N} p_{hi}$ for any agent i, the previous inequality can be written as $\sum_{h\in N}(\sum_{i\in N} p_{hi} \cdot x_i) = \sum_{i\in N}(\sum_{h\in N} p_{hi}) \cdot x_i = p \cdot (\sum_{i\in N} x_i) > p \cdot \sum_{h\in N} \omega_h$, which is equivalent to $p \cdot \sum_{i\in N}(x_i - \omega_i) > 0$. This is a contradiction since $x \in \mathscr{F}$ and $p > 0$. ■

Lemma 29 *The vector p is strictly positive.*

Proof. By $p > 0$ and $\sum_{i\in N} \omega_i \gg 0$, there exists an agent h such that $p \cdot \omega_h > 0$. We first claim that for this agent h, it must be the case that $p_{(h)} \cdot z_{(h)} > p \cdot \omega_h$ for any $z_{(h)} \in P_h(x)$, with $x \in \mathscr{C}_h^I(E)$. By Lemma 25 we know that $p_{(h)} \cdot z_{(h)} \geq p \cdot \omega_h$ for any $z_{(h)} \in P_h(x)$. Suppose that $p_{(h)} \cdot z'_{(h)} = p \cdot \omega_h > 0$ for some $z'_{(h)} \in P_{(h)}(x)$. By continuity of preferences, there exists $V_\delta(z'_{(h)}) \coloneqq N_\delta(z'_{(h)}) \cap \mathbb{R}_+^{l\cdot n}$ included in $P_h(x)$, where $N_\delta(z'_{(h)}) \subseteq \mathbb{R}^{l\cdot n}$ is an open ball centered at $z'_{(h)}$ with radius $\delta > 0$. Let $\varepsilon > 0$ such that $0 < (1-\varepsilon)\|z'_{(h)}\| < \delta$ and consider the vector $\varepsilon z'_{(h)}$ [27]. Thus, $\varepsilon z'_{(h)} \in V_\delta(z'_{(h)})$ and consequently $p_{(h)} \cdot \varepsilon z'_{(h)} \geq p \cdot \omega_h$ since $\varepsilon z'_{(h)} \in P_h(x)$. By $p_{(h)} \cdot z'_{(h)} = p \cdot \omega_h > 0$ and $\varepsilon < 1$, we get $\varepsilon p_{(h)} \cdot z'_{(h)} < p_{(h)} \cdot z' = p \cdot \omega_h$. Therefore,

$$p \cdot \omega_h \leq \varepsilon p_{(h)} \cdot z'_{(h)} < p_{(h)} \cdot z'_{(h)} = p \cdot \omega_h$$

which is a contradiction. This completes the proof of the claim. We now claim that the vector p_{hh} is strictly positive. Fix a commodity c. By strong monotonicity of the preference of agent h with respect to her own consumption, $x + e(h,c) \in P_h(x)$, where $e(h,c) \coloneqq (e_i(h,c))_{i\in N} = ((e_i^s(h,c))_{s=1}^l)_{i\in N}$ is a vector in $\mathbb{R}^{l\cdot n}$ with $e_i^s(h,c) = 0$ for any $s \neq c$ and $e_h^c(h,c) = 1$. So $p_{(h)} \cdot (x + e(h,c)) > p \cdot \omega_h$ by the previous claim. Since $x \in \mathscr{C}_h^I(\mathscr{E})$, by Lemma 28 and the bilinearity property of the inner product, one gets $p_{(h)} \cdot e(h,c) > 0$ which implies, by definition of $e(h,c)$, that $p_{hh}^c > 0$. Repeating the same argument for any commodity s, one obtains, $p_{hh} \gg 0$, which proves the claim. Finally, by $p = \sum_{i\in N} p_{ih}$ one gets $p \gg 0$. ■

[27] Notice that, since $\succ_i$ is not reflexive, then $z_{(i)} \neq x$. Thus, $z_{(i)} \in P_i(x)$ and the non-decreasing assumption implies that $z_{(i)}$ is different from the null vector. Therefore, $\varepsilon z'_{(h)} < z'_{(h)}$.

Lemma 30 *If* $x \in \mathscr{C}_h^I(E)$ *then* $p_{(i)} \cdot z_{(i)} > p \cdot \omega_i$ *for any* $i \in N$ *and for any* $z_{(i)} \in P_i(x)$.

Proof. By $p_{(i)} \cdot z_{(i)} \geq p \cdot \omega_i$, $p \gg 0$ and $\omega_i > 0$, we get $p_{(i)} \cdot z_{(i)} > 0$ for any $z_{(i)} \in P_i(x)$. By continuity of preferences, one my find a vector $z'_{(i)} \in P_i(x)$ such that $p_{(i)} \cdot z_i > p_{(i)} \cdot z'_{(i)}$ [28] . Since $z'_{(i)} \in P_i(x)$, by Lemma 25, we must have $p_{(i)} \cdot z_{(i)} > p_{(i)} \cdot z'_{(i)} \geq p \cdot \omega_i$, which completely proves the statement. ■

We are now ready to prove the equivalence theorem.

Theorem 31 (Equivalence Theorem) *The set of information equilibrium allocations concides with the generalized fuzzy information-core.*

Proof. Take $x \in \mathscr{C}_h^I(E)$. We are going to show that $(x, (\pi, p))$ belongs to $\Omega_I(E)$. By Lemma 28, one gets $x \in B_i(\pi, p, \omega)$, by Lemma 30, $P_i(x) \cap B_i(\pi, p, \omega) = \emptyset$ holds true for any agent $i \in N$, and by the definition of (π, p), one has $\sum_{i \in N} p_{ih} = p$ for all $h \in N$. Finally, it remains to show that if $x \in \mathscr{C}_h^I(E)$ then market clearing condition is satisfied. Indeed, by Lemma 28, $x \in \mathscr{C}_h^I(E)$ implies $\sum_{h \in N} p_{ih} \cdot x_h = p \cdot \omega_i$ for any $i \in N$. So summing with respect to i, inverting the sum operator and using the bilinearity property of the inner product, we get

$$\sum_{i \in N} \sum_{h \in N} p_{ih} \cdot x_h = \sum_{h \in N} \sum_{i \in N} p_{ih} \cdot x_h = p \cdot \sum_{i \in N} \omega_i = p \cdot \sum_{h \in N} \omega_h$$

By $\sum_{i \in N} p_{ih} = p$, the previous equality is equivalent to $p \cdot \sum_{h \in N}(x_h - \omega_h) = 0$. Since $p \gg 0$, the feasibility of x implies $\sum_{h \in N} x_h = \sum_{h \in N} \omega_h$. ■

8 Economic models analyzed in literature - Part II

In this section, we underline that a Walrasian equilibrium allocation for a pure exchange economy without externalities and a distributive Lindahl equilibrium for an economy with externalities are both particular cases of information equilibrium.

8.1 Pure exchange economy without externalities

If $\mathscr{E}$ is a pure exchange economy without externalities at all, then a competitive equilibrium allocation x is also an Information equilibrium allocation. Indeed,

[28] As in the proof of Lemma 29, we can define $z'_{(i)} := \varepsilon z_{(i)}$, with $\varepsilon > 0$ such that $0 < (1 - \varepsilon)\|z_{(i)}\| < \delta$. Thus, $z'_{(i)} \in V_\delta(z_{(i)}) := N_\delta(z_{(i)}) \cap \mathbb{R}^{l \cdot}_+ \subseteq P_i(x)$, where $N_\delta(z_{(i)})$ is an appropriate open ball in $\mathbb{R}^{l \cdot n}$.

if p is the equilibrium price, then the corresponding personalized price of agent i is given by $p_{ih} \coloneqq 0$ for any agent $h \neq i$, and $p_{ii} \coloneqq p$. Under this setting, the budget set of agent i is $B_i(\pi, p, \omega) = \{x \in \mathbb{R}_+^{l \cdot n} : p \cdot x_i \leq p \cdot \omega\}$ and the feasibility condition for the personalized prices is trivially satisfied.

8.2 Distributive Lindahl equilibrium

Define the set of admissible Lindahl shares as the set Γ of $n \times n$ matrices given by $\Gamma \coloneqq \{\gamma = (\gamma_h^i) \colon \gamma_h^i \geq 0, \sum_{i \in N} \gamma_h^i = 1, \forall\ h \in N\}$. For an agent i, one may interpret the vector $(\gamma_h^i)_{h \in N}$ as the contribution of agent i to the cost of consumption of all the other agents. The total contribution to consumption of one agent is normalized to 1. Given a price system $p \in \mathbb{R}_+^l$ and admissible Lindahl shares $\gamma \in \Gamma$, the budget set of agent i is $B_i(\gamma, p, \omega) \coloneqq \{x \in \mathbb{R}_+^{l \cdot n} \colon \sum_{h \in N} \gamma_h^i p \cdot x_h \leq p \cdot \omega_i\}$. Thus, $(x, \gamma, p) \in \mathbb{R}_+^{l \cdot n} \times \Gamma \times \mathbb{R}_+^l$ is a distributive Lindahl equilibrium for the economy E if

1. $x \in \underset{x' \in \mathbb{R}_+^{l \cdot n}}{\arg\max} \{u_i(x') \mid \sum_{h \in N} \gamma_h^i p \cdot x_h \leq p \cdot \omega_i\}$, for all $i \in N$;
2. $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$.

In a distributive Lindahl equilibrium, individual bundles are the public goods of the economy, and as such, they are enjoyed by every agent in the same amount. Equilibrium shares are personalized in such a way that these bundles are an optimal choice for each agent. Given a price system p, in a distributional Lindahl equilibrium, there is a procedure which assigns to each agent a share in the consumption of every other agent, so that at equilibrium agents agree on the share. Notice that a distributive Lindahl equilibrium is an Information equilibrium. Indeed, suppose that (x, γ, p) is a distributive Lindahl equilibrium, then for every agent i, one may define a system of personalized price $\pi = (p_{(i)})_{i \in N}$ as $p_{ih} \coloneqq \gamma_h^i p$ for any $h \in N$. Notice that, condition 4 in Definition 17 is satisfied, since $\sum_{i \in N} p_{ih} = \sum_{i \in N} \gamma_h^i p = p$. Finally observe that the budget set of an agent in the distributive Lindahl equilibrium coincides with the one of the Information equilibrium with (π, p) sistem of prices, since $B_i(\gamma, p, \omega) = \{x \in \mathbb{R}_+^{l \cdot n} \colon \sum_{h \in N} \gamma_h^i p \cdot x_h \leq p \cdot \omega_i\} = \{x \in \mathbb{R}_+^{l \cdot n} \colon \sum_{h \in N} p_{ih} \cdot x_h \leq p \cdot \omega_i\} = B_i(\pi, p, \omega_i)$. Therefore, a distributive Lindahl equilibrium is an information equilibrium in which prices of agent i are all on the same direction, given by p.

9 Conclusions

We have shown that the equivalence theorem can be restored for two non-standard market models with externalities and non-convex preferences. For an equilibrium allocation to be a core allocation, an appropriate blocking procedure is needed. In particular, (i) an agent in the blocking coalition needs to be myopic with respect to the choices of all the other agents; (ii) an optimistic behavior of the blocking coalition with respect to the reactions of the outsiders is required. Vice-versa, to show that a core allocation belongs to the set of the equilibrium allocations, we use a standard approach based on Separation theorems. In order to overcome the difficulties arising by removing the convexity assumption, we adapt the idea of Husseinov (1994) to our frameworks by allowing agents to participate in more than one fuzzy coalitions simultaneously. In the A-economy, our main result is based on the assumption that the family of the exogenously given sets $A = (A_i)_{i \in N}$ is balanced in the sense of Bondereva with full support. This assumption is trivially satisfied for the model studied in the second part of the paper, that is a pure exchange economy with Arrowian markets for externalities. The proof of the equivalence theorem for this latter case requires the additional assumption of non spiteful agents. We have also shown that the two market models analyzed in the paper are sufficiently general to cover some well-known cases. Several important aspects of these two non-classical market models deserve to be investigated. In the absence of externalities, non-convex preferences are easily accommodated in models of exchange economy with a continuum of agents. In our ongoing research, following the idea of Husseinov (1994) and Husseinov and Páscoa (1997), the fuzzy core will be related to the core of an appropriate economy with a continuum of agents. Although it is natural, this correspondence is not easy to construct due to the presence of externalities. Moreover, due to limited availability of resources and the impact on the economic environment of an inefficient allocation, the concept of resources-core and its characterization in terms of a measure of social loss deserve to be studied [29] .

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[29] We refer to Di Pietro et al. (2022) and Graziano and Platino (2023) for the notion of resource core and its characterization in terms of a measure of social loss, in presence of externalities.

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