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Decentralization in Non-Convex Economies with Externalities

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Abstract
We consider a pure exchange economy with externalities. Individual preferences are affected by the consumption of all other agents in the economy, and to each agent *i* is exogenously associated a nonempty set *Ai*, representing the individuals agent *i* cares about.We adopt a cooperative approach to equilibrium analysis, allowing each individual to cooperate with others and to form coalitions. Following Vasil'ev (2016), Husseinov (1994) and Graziano (2001), we study a notion of generalized fuzzy core and show that, in the case of non-convex preferences, the set of coalitions can be enlarged in such a way that a core allocation can be supported as an *A*-equilibrium by some price system. In the second part of the paper, we consider an economy with Arrowian markets for consumption externalities. For an appropriate definition of generalized fuzzy core, we show that a core allocation can be decentralized as an Information equilibrium in terms of personalized and market prices.

JEL Classification: C71, D51, D62.

Keywords: Exchange economy, interdependent preferences, markets for externalities, generalized fuzzy core.

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1 Introduction

We consider pure exchange economies with a finite number of commodities and individuals. In our framework, preferences may be affected by consumption externalities with each individual i influenced by a group of agents exogenously given. We allow each agent to cooperate with others and to form coalitions and for this model we analyze core solutions. The purpose of the paper is to show that, under suitable notions of cooperation and core, it is still possible to restore equivalence theorems and decentralize core allocations in terms of prices. Furthermore the equivalence theorems are valid even considering, as is natural in this context, the possibility that preferences are not convex.

The importance of consumption externalities has been widely recognized in the recent literature on other-regarding preferences, fairness and altruism, that documents how often individual agents fail to maximize their self interest [1]. On the other hand, collective models of household consumption incorporate a theory of human caring into a general equilibrium setting [2], with real-life examples where an individual cares for others who are unable to make their own decisions [3]. Agents may neglect their own self interest in favor of other agents, who influence their decisions. These situations are very common and deserve to be analyzed in a more general theoretical framework due to their potential impact on the economic environment. In the models of pure exchange economies analyzed in the paper, externalities, human caring and external influence on individual decisions are taken into account. Moreover, the analysis accommodates non-convex preferences. Indeed, it is well known that the assumption of convex individual preferences is especially problematic in the presence of consumption externalities [4].

The decentralization of core allocations in pure exchange economies with selfish traders is formulated in terms of classical Walrasian competitive solutions. For economies with externalities, one may define several notions of competitive equilibria. Some of them (e.g. Walrasian-Nash, Berge and family equilibria) are supported by market prices, others (as for the distributive Lindahl equilibrium) require the additional notion of personalized prices. In this paper we investigate the possibility to restore the equivalence theorems in both of these two situations.

[1] See for instance Levine (1998), Fehr and Schmidt (1999) and Dufwenberg et al. (2011).
[2] See for example, Haller (2000) and Gersbach and Haller (2001).
[3] This is for example the case of elderly persons or children relying on outside help when buying goods or services.
[4] See for example the general discussion on non-convexities and externalities in Starrett (1972)

We consider the general model of an A-economy, which extends the classical general equilibrium model with externalities assuming that the set A_i of agents affecting individual i is not necessarily the same for all individuals and is exogenously given. In the first part, we fix our attention on the so-called *A-equilibrium*, a notion introduced by Vasil'ev (2016) which can be considered as an adaptation of the Berge equilibrium to the pure exchange models with externalities (see Berge (1957)) and naturally generalizes the classical competitive equilibrium à la Nash given in Arrow and Hahn (1971) and Laffont (1988). Under such equilibrium, each individual chooses the best consumption bundle for each member of the associated group A_i, taking as given the commodity prices, the aggregate wealth of the members of the group, and the choices of every other agent in the economy [5] . In the notion of A-equilibrium, agents propose a consumption bundle for each of the traders of their associated set A_i, and exchange the resources of these agents under a unique fixed price. In the second part of the paper, we focus on A-economies in which for each trader i the associated set A_i coincides with the society. We follow Arrow (1969), Laffont (1976) and the model of a pure exchange economy with Arrowian markets for consumption externalities, where agents still choose profiles of consumption bundles for the whole society, but face personalized (Lindahl) prices and a budget constraint defined by their own resources.

On the competitive side, we consider the notion of equilibrium characterized by personalized and market prices which is referred to as *Information equilibrium* under the interpretation of markets for externalities in terms of coordination given by Makarov (1982) and Vasil'ev (1996) [6] .

Our main results show that for two suitable notions of blocking and the corresponding core, each core allocation can be decentralized as an A-equilibrium allocation with market prices in the first model and as an Information equilibrium with personalized prices in the second case. Conversely, price supported allocations belong to the core. Hence a version of the equivalence theorem can still be obtained even without the convexity assumption on preferences. To deal with decentralization theorems in markets with externalities, the notion of blocking and that of core must necessarily be reformulated. It is well known in fact that, in the presence of externalities, a competitive allocation is not necessarily a Pareto optimal allocation. Therefore, a fortiori one cannot expect that the more general notions of A-equilibrium and Information equilibrium allocations belong to the core defined in a standard way. Consequently, for the

[5] As usual, the resulting allocation must be feasible with respect to the initial resources of the economy.

[6] We follow the terminology of Vasil'ev (1996) for competitive equilibria of the exchange economy with Arrowian markets for consumption externalities. Existence of equilibria in this framework are proved by Vasil'ev (1996) and the recent paper by Bonnisseau et al. (2023), among others.

two models under consideration, we adopt an appropriate *fuzzy core* notion relying on fuzzy coalitions [7] and defined in the spirit of Florenzano (1989, 1990).

To decentralize an A-equilibrium allocation in the A-economy in which each agent i is influenced by the members of an exogenously given set A_i, we consider the blocking mechanism and the fuzzy A-core introduced in Vasil'ev (2016). We assume that agents in a blocking coalition S are myopic in the sense that they ignore the choices of the other coalition members. Furthermore, the blocking mechanism is based on an optimistic behavior of each member i of the blocking coalition with respect to the reaction of agents she is not concerned about [8], and the rate of participation in the coalition of an individual may take any nonnegative value [9]. In the case of exchange economies with Arrowian markets for consumption externalities, our approach is instead closely related to the work of Vasil'ev (1996), which assumes convexity and builds on the assumption that the agents are not spiteful, i.e., preference are nondecreasing in their domain. Moreover, in this case, the blocking mechanism defining the core assumes that the utility of coalition members over the alternative allocation depends on the rate of participation of traders in the coalition itself and members outside the coalition do not consume. The basic idea behind the fuzzy coalitions approach, is to broaden the range of blocking mechanism by increasing the number of proposed plans. In doing so, the core shrinks and coincides with the A-equilibrium. As for the decentralization of Information Equilibrium, a similar principle applies, but the number of proposed plans increase in proportional manner.

In both models, we are able to relax the convexity assumption on preferences thanks to the notion of coalition introduced in the seminal work by Husseinov (1994) and studied in the related contributions by Husseinov and Páscoa (1997) and Graziano (2001). We allow agents to participate in more that one fuzzy coalition, by introducing the notion of fuzzy coalition matrix interpreted as a bundle of fuzzy coalitions, or part-time coalitions, in which agents may cooperate simultaneously employing different shares of their own resources. In the case of A-economies with an arbitrary set A_i of agents in-

[7] The idea of fuzzy coalitions has been introduced in Aubin (1979) for pure exchange economies and leads to a fuzzy core that coincides with the competitive equilibria. In the paper, we use the terminology *fuzzy coalitions* and *fuzzy core* to denote the coalitions and the core defined in the Aubin sense, which are also known in the literature as *Aubin coalitions* and *Aubin core*.

[8] That is, the agents outside her reference group A_i. We refer to Graziano et al. (2017), Di Pietro et al. (2022), Hervés-Beloso and Moreno-García (2022) for the difference between optimistic and pessimistic attitude of coalition agents with respect to the behavior of outsiders.

[9] Following Aubin (1979), the rate of participation may be normalized to take all values in the real interval $[0, 1]$.

fluencing i, we introduce the concept of *generalized fuzzy A-core*, by naturally adapting the fuzzy A-core to our setting and by showing that, under convexity, these two notions coincide. We prove that an A-equilibrium allocation belongs to the core, and we show conversely that a core allocation can be supported as an equilibrium allocation for some price system. Our result, as well as the one of Vasil'ev (2016) in the convex case, is based on the crucial assumption that the exogenous family of sets A_i associated to agents must be balanced in the sense of Bondareva with full support. This condition is not too demanding, and it is trivially satisfied by all particular examples analyzed in Section 5. The same idea of simultaneous participation of agents in more than one fuzzy coalition is adopted in the presence of Arrowian markets for consumption externalities, a case where we obtain a version of the core equivalence theorem decentralizing Information equilibria. Here again our notion of *generalized fuzzy Information core* coincide with the fuzzy core introduced in Vasil'ev (1996) when the other-regarding preferences are convex.

It is worth to point out that the models of pure exchange economies analyzed in the paper and the related equilibrium notion are general enough to cover important benchmark cases presented in the literature. Indeed, in the case of A-equilibrium, if for any agent i the associate group A_i reduces to the singleton represented by the agent herself, we recover the classical Walrasian economy with externalities, and the notion of A-equilibrium coincides with the competitive equilibrium à la Nash. If the group is made up of all the agents in the economy other than the individual it is associated with, we end-up with a Berge economy, and the corresponding equilibrium is a natural transposition of the Berge equilibrium for non-cooperative games[10]. In addition to these polar cases, other models are covered by the general framework of an A-economy. A relevant example is the family economy analyzed in Section 5.3, which is in the spirit of Haller (2000) and Gersbach and Haller (2001). In a family economy, the collection of the groups A_i forms a partition of the set of agents, and each A_i is interpreted as a family. Furthermore, under this specification, any individual must belong to her related set (her family), and all the members of the same family should be associated with the same set. Similarly, in the final part of the paper, we compare Information equilibria with some classical equilibrium solutions studied in the literature and with equilibrium notions involving personalized prices. In particular, we show that a Walrasian equilibrium allocation for a pure exchange economy without externalities, and a distributive Lindahl equilibrium as defined by Bergstrom (1970), are particular cases of Information equilibrium.

[10] See Zhukovskii and Chikrii (1994) for more details. This solution notion may be interpreted in some cases as a fully altruistic criteria, since each individual maximizes the goals of all the other agents in the economy. For equilibrium notions in exchange economies involving altruism in the form of donations to social causes see for example Faias and Moreno-García (2025).

The paper is organized as follows. Section 2 presents the model and the basic assumptions; Section 3 introduces the notions of A-equilibrium, fuzzy A-core and the generalized fuzzy A-core, and proves the equivalence of the two core notions under convexity; Section 4 is devoted to our main result, that is the equivalence theorem when preferences are not necessarily convex. Section 5 compares some economic models and the related equilibrium notions studied in literature with the market structure considered in the paper and present examples: Subsections 5.1, 5.2 and 5.3 deal respectively with a classical pure exchange economy with externalities, a Berge economy and family economy, while in Subsection 5.4 we provide examples in which, according to the structure of A, the set of A-equilibria may be empty, or a singleton or may also contain infinitely many allocations. Section 6 discusses the Information equilibrium, the fuzzy information core and the generalized fuzzy information core and also emphasizes that the two core notions coincide under convexity. Section 7 deals with the core equivalence theorem. Section 8 discusses some classical solutions and their relation with the Information equilibrium: Subsection 8.1 deals with a competitive equilibrium of a pure exchange economy without externalities, and Subsection 8.2 with the distributive Lindahl equilibrium. Section 9 summarizes our results and discusses some hints of future research. An Appendix presents the technical proofs.

2 A-Economy: The model and the basic assumptions

There is a finite number l of commodities and the commodity space is $\mathbb{R}^l$. There is a finite number of individuals (agents or traders) denoted by the subscript $i \in N := \{1, \dots, n\}$. To each agent i is exogenously associated a nonempty set $A_i \subseteq N$ describing the individuals agent i cares about or, equivalently, the set of agents influencing agent i; $A := (A_i)_{i \in N}$ [11]. In the first part of the paper, we will focus on the A-equilibrium notion, and A_i can be any arbitrary subset of N. In the model of Arrowian markets for externalities, we will set $A_i = N$ for any agent i. The consumption set associated to each agent is the standard positive cone $\mathbb{R}^l_+$, $x_i := (x_i^1, \dots, x_i^l) \in \mathbb{R}^l_+$ denotes the consumption bundle of individual i, and $x := (x_i)_{i \in N}$ is a vector of consumption bundles. A price vector p is an element of $\mathbb{R}^l_{++}$, where p^c is the price of one unit of the commodity c. Each agent i proposes the consumption for all the members of her reference set A_i. In this respect, we use the following additional notation with obvious meaning: $X_{A_i} := \mathbb{R}_+^{l \cdot |A_i|}$, $x_{A_i} := (x_h)_{h \in A_i}$, $X_{N \setminus A_i} := \mathbb{R}_+^{l \cdot |N \setminus A_i|}$, $x_{N \setminus A_i} := (x_h)_{h \in N \setminus A_i}$ [12]. The set $D_i := \{h \in N : i \in A_h\}$ denotes the agents that are influenced by i. Given x_{A_i} and $x_{N \setminus A_i}$, without loss of generality, we denote x by $(x_{A_i}, x_{N \setminus A_i})$. We also denote x by (x_i, x_{-i}), where $x_{-i} := (x_k)_{k \neq i}$, when we

[11] We are not necessarily requiring that agent i belongs to A_i.
[12] Given a set B, we denote by $|B|$ its cardinality.

compare our economy with classical models treated in the literature. Furthermore, given an agent i, we adopt the notation $x_{(i)} \coloneqq (x_{ih})_{h \in A_i} \in X_{A_i}$, where x_{ih} represents the commodity bundle chosen (or proposed) by agent i for the consumption of an agent $h \in A_i$. When $A_i = N$, the element $x_{(i)} = (x_{ih})_{h \in N}$ belongs to $\mathbb{R}_+^{l \cdot n}$ [13]. Such notation will be useful in the blocking mechanism behind the core notions used in the paper.

The individual preferences $\succsim_i$ of an agent i are affected by the consumption of all the agents. Then, for any agent i and any vector of consumption bundles $x \in \mathbb{R}_+^{l \cdot n}$, $P_i(x) \coloneqq \left\{ x' \in \mathbb{R}_+^{l \cdot n} : x' \succ_i x \right\}$ denotes the set of consumption bundles which are strictly preferred by agent i to x. The initial endowment of individual i is $\omega_i \coloneqq (\omega_i^1, \ldots, \omega_i^l) > 0$, and $\omega \coloneqq (\omega_i)_{i \in N} \in \mathbb{R}_+^{l \cdot n}$.

A vector $x = (x_i)_{i \in N} \in \mathbb{R}_+^{l \cdot n}$ is an *allocation* if it satisfies the physical feasibility condition

$$\sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i.$$

We denote by $\mathscr{F}$ the set of allocations and summarize the economy under consideration by the following list of elements:

$$E \coloneqq \langle N, \mathbb{R}_+^l, (A_i, \succsim_i, \omega_i)_{i \in N} \rangle.$$

In order to adopt a cooperative approach to equilibrium analysis, we introduce below the notion of fuzzy coalition matrix [14]. We remind that an ordinary (or crisp) coalition is any nonempty subset S of the set of agents N. A *fuzzy coalition* $\alpha = (\alpha_i) \in \mathbb{R}_+^n$ is a non-zero vector of dimension n, where for each agent $i \in N$, $\alpha^i / \max_{h \in N} \alpha_h$ is interpreted as the share of resources that agent i employs in the coalition (compare Aubin (1979), Husseinov (1994)). Clearly, fuzzy coalitions generalize ordinary coalitions when $\alpha_i \in \{0, 1\}$, for each i. In non-convex economies, a further generalization of the notion of coalition is due to Husseinov (1994).

Definition 1 *A fuzzy coalition matrix* $\alpha = (\alpha_i^j)$ *is a matrix of dimension* $r \times n$, *where* $r \in \mathbb{N}$, *and* $\alpha^j \coloneqq (\alpha_i^j)_{i \in N} \in \mathbb{R}_+^n$ *with* $\alpha^j \neq 0$ *for any* $j = 1, \ldots, r$.

A coalition matrix is interpreted as a finite collection of coalitions ($j = 1, \ldots, r$). Agent i may participate in any of such coalitions employing the share $\alpha_i^j / \max_{h \in N} \sum_{j=1}^r \alpha_h^j$ of her resources for any j. In the case in which $r = 1$, we simply denote α^1 by α and obtain a fuzzy coalition. If $r = 1$ and α_i (the share of participation of agent i in the coalition) takes only $\{0, 1\}$-values for any i, we have usual (crisp) coalitions. For a given coalition matrix $\alpha = (\alpha_i^j)$,

[13] The bundle $(x_{(i)})_{i \in N}$ can be interpreted as a vector of personalized quantities, and it is in a similar spirit of the vector of personalized prices π introduced in Section 6.

[14] *Generalized fuzzy coalition* according to Husseinov (1994).

we denote by $\mathrm{supp}(\alpha^j)$ the support of α^j, i.e., $\mathrm{supp}(\alpha^j) \coloneqq \left\{i \in N \colon \alpha_i^j > 0\right\}$, and by $\mathrm{supp}(\alpha) \coloneqq \bigcup_{j=1}^r \mathrm{supp}(\alpha^j)$ the set of agents who participate actively in the coalition matrix α.

We make the following survival assumption for the aggregate endowment.

Assumption 2 *The aggregate endowment $\sum_{i\in N} \omega_i$ belongs to $\mathbb{R}^l_{++}$.*

The previous assumption has an important role in the proof of equivalence theorems in order to show that the vector p resulting by the application of the Separation Theorem is strictly positive, and consequently, can be interpreted as a supporting price vector [15] .

In the rest of the paper, the exogenous family $A = (A_i)_{i\in N}$ satisfies the following set of assumptions.

Assumption 3 *(1) $N = \bigcup_{i\in N} A_i$;*

(2) $A = (A_i)_{i\in N}$ is balanced in the sense of Bondareva with full support, i.e., there exist weights $\beta = (\beta_i)_{i\in N} \in \mathbb{R}^n_{++}$, such that $\sum_{h\in D_i} \beta_h = 1$, for any $i \in N$.

Point 1 of Assumption 3 assures that each agent is able to influence at least one agent in the economy, i.e., $D_i \neq \emptyset$ for any i. As a consequence, at the equilibrium solution, any agent may potentially consume. Point 2 of Assumption 3 is a stronger version of the condition used by Vasil'ev (2016). It assures that, given an allocation x, if for any agent i, the consumption bundles x_{A_i} lies on the budget hyperplane of i, then the market clearing condition is satisfied (see the proof of Theorem 10 for details). Point 2 of Assumption 3 is equivalent to assume that the following set $\mathscr{B}_A$ is nonempty, i.e.,

$$\mathscr{B}_A \coloneqq \left\{\beta \in \mathbb{R}^n_{++} \colon \sum_{i\in N} \beta_i \chi_{A_i} = \chi_N\right\} \neq \emptyset$$

where, for a given set $S \subseteq N$, we denote by $\chi_S \coloneqq (\chi_S^h)_{h\in N} \in \mathbb{R}^n$ the characteristic vector of $S \subseteq N$, i.e.,

$$\chi_S^h \coloneqq \begin{cases} 1 & \text{if } h \in S \\ 0 & \text{if } h \notin S. \end{cases}$$

Point 2 of Assumption 3 may seem to depend on the agent. However, this is not actually the case when one refers to the equivalent definition formulated in terms of the non-emptiness of the set $\mathscr{B}_A$, which is in line with the classical Bondareva condition used in the literature. We notice that Assumption 3 is

[15] See the definition of an A-equilibrium and the proof of Theorem 10, Claim 3, for details.

automatically satisfied in some cases which are relevant in the rest of the paper. When $A_i = N$ for each agent i, then $D_i = N$ and to have a balanced family it is enough to consider $\beta_i = 1/n$ for each agent i. Similarly, in the case in which $A_i = N \setminus \{i\}$ for each agent i, then $D_i = N \setminus \{i\}$ and point 2 of Assumption 3 holds with $\beta_i = 1/(n-1)$ for each i. Finally, let $\mathscr{H} \coloneqq \{H_j : j = 1, \ldots, k\}$, with $k \leq n$ be a partition of the set of agents, and for every $i \in N$, define A_i as the element of $\mathscr{H}$ containing i. Notice that, members of the same family are associated with the same A_i. Moreover, $i \in A_i$ and $D_i = A_i$, for each $i \in N$. Hence condition 2 is satisfied by choosing $\beta_i = 1/|A_i|$ for each $i \in N$. The latter case also includes the situation in which for each $i \in N$, $A_i = \{i\}$.

3 A-Equilibrium and Generalized Fuzzy A-Core

In this Section we analyze the model of a pure exchange economy with externalities in which the set A_i of agents affecting individual i is not necessarily the same for all individuals and is exogenously given. We start the Section by introducing a first set of assumptions on the preference relations. Then we define the A-equilibrium and the core. The A-equilibrium[16] is a natural generalization of the classical competitive equilibrium in the presence of externalities. The corresponding fuzzy A-core extends the fuzzy core introduced by Aubin (1979) to our context. In the spirit of Husseinov (1994), we further extend the notion of fuzzy A-core introducing the Generalized fuzzy A-Core, and we show that, under convexity, the two notions coincide.

The additional assumptions on preference relations $\succsim_i \subseteq \mathbb{R}_+^{l \cdot n} \times \mathbb{R}_+^{l \cdot n}$ are listed below.

Assumption 4 *For any agent i,*

(1) $\succsim_i$ are complete, transitive and continuous over $\mathbb{R}_+^{l \cdot n}$.
(2) For any vector $x \in X$, $\succsim_i$ are strongly monotone over $X_{A_i} \times \{x_{N \setminus A_i}\}$[17].

Notice that we do not require any convexity assumption on preferences. Moreover, since agent i might not belong to A_i, her individual preferences are not necessary monotone with respect to her own consumption. This implies that our environment might not be altruistic. In this respect, when an individual influencing agent i proposes for her a higher level of consumption compared to a status quo, the influencing individual will feel better, but this does not necessarily imply that agent i will also be better off. Finally we remark that,

[16] See Vasil'ev (2016).
[17] For any $x \in X$, the preference relations $\succsim_i$ are strongly monotone over $X_{A_i} \times \{x_{N \setminus A_i}\}$, if for any vector $v \in \mathbb{R}_+^{l \cdot |A_i|}$ with $v \neq 0$, one has $(x_{A_i} + v, x_{N \setminus A_i}) \succ_i x$.

although we do not make use of utilities, the assumptions stated for preferences ensure that the preference relation $\succsim_i$ can be represented by a continuous utility function u_i defined over the commodity space.

Given the individual preferences $\succsim_i$ of an agent i, we denote by $P_{A_i}(x) \coloneqq \left\{x'_{A_i} \in X_{A_i} \colon (x'_{A_i}, x_{N\setminus A_i}) \in P_i(x)\right\}$ the set of bundles which are strictly preferred by i to x, when the consumption of any agent $h \in N \setminus A_i$ is fixed at x_h [18]. Notice that $P_{A_i}(x)$ is nonempty by Point 2 of Assumption 4. We introduce below the notion of A-equilibrium.

Definition 5 (A-equilibrium) $(x,p) \in \mathbb{R}_+^{l\cdot n} \times \mathbb{R}_{++}^{l}$ *is an A-equilibrium for the economy E if*

1. $x_{A_i} \in B_{A_i}(p,\omega)$ *for all* $i \in N$;
2. $P_{A_i}(x) \bigcap B_{A_i}(p,\omega) = \emptyset$ *for all* $i \in N$;
3. $\sum_{i\in N} x_i = \sum_{i\in N} \omega_i$

where $B_{A_i}(p,\omega) \coloneqq \left\{x_{(i)} \in X_{A_i} \colon p\cdot\left(\sum_{h\in A_i} x_{ih}\right) \leq p\cdot\left(\sum_{h\in A_i}\omega_h\right)\right\}$ *denotes the budget set of agent i.*

In the budget constraint, agent i considers the endowments of all the agents belonging to A_i. Notice that, if two or more agents are influenced by the same group of individuals, they face the same budget constraint. Conditions 1 and 2 state that, for every agent i, the affordable bundle x_{A_i} maximizes the preference of agent i over the budget set, and point 3 is the classical market clearing condition. In equilibrium, by conditions 1 and 2, each agent i chooses the consumption of all agents in A_i by imposing a control of their own endowments when facing the budget constraint. Given an economy E, we denote by $\Omega(E)$ the set of A-equilibria, and by $\mathscr{W}(E)$ the set of A-equilibrium allocations, i.e., $\mathscr{W}(E) \coloneqq \left\{x \in \mathbb{R}_+^{l\cdot n} \mid \exists\, p \gg 0 \colon (x,p) \in \Omega(E)\right\}$ [19]. It should be noted that, in equilibrium, all proposals made by the agents influencing the same individual must be consistent.

[18] The strict preference relation $\succ_i \subseteq \mathbb{R}_+^{l\cdot n} \times \mathbb{R}_+^{l\cdot n}$ is defined in the usual way, i.e., $x \succ_i y$ if and only if $x \succsim_i y$ and not $y \succsim_i x$.

[19] The equilibrium introduced with Definition 5 can be considered close in its spirit to the Berge equilibrium concept for non-cooperative games proposed in Berge (1957). The equilibrium notion studied in Subsection 5.2 is instead similar to the Berge equilibrium in the sense of Zhukovskii and Chikrii (1994). The relevance of Berge equilibrium is due to the fact that it allows to reach cooperative agreements within a non-cooperative framework (see Abalo and Kostreva (2005) and Nessah et al. (2007) for further details).

We introduce now the notion of (fuzzy) A-core and generalized fuzzy A-core. For a given fuzzy coalition α, we define $D_i(\alpha) := D_i \bigcap \operatorname{supp}(\alpha)$ and $\alpha_i^A := \sum_{h \in D_i(\alpha)} \alpha_h$. The set $D_i(\alpha)$ can be interpreted as the set of all agents which are influenced by i and simultaneously participate in the coalition α. The element α_i^A represents the total resources employed by these agents.

Definition 6 (Fuzzy A-Core) *Given an allocation $x \in \mathscr{F}$ and a coalition $\alpha \in \mathbb{R}_+^n$ with $\alpha \neq 0$, we say that α A-improves upon x whenever, for every agent $i \in N$, there exists a vector $x'_{(i)} \in X_{A_i}$ such that*

1. $x'_{(i)} \in P_{A_i}(x)$ *for any* $i \in \operatorname{supp}(\alpha)$;
2. $\displaystyle\sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h x'_{hi} \leq \sum_{i \in N} \alpha_i^A \omega_i.$

The set of allocations which cannot be A-improved upon by any coalition is called (fuzzy) A-Core, and it is denoted by $\mathscr{C}_f(E)$.

In the blocking mechanism just defined, each agent i in the blocking coalition α proposes a new consumption plan for the members of her reference group A_i and uses resources of these agents weighted by her share of participation in the coalition itself. We point out that an agent in the blocking coalition is myopic with respect to the decision made by the other members of the coalition. In this respect, the blocking mechanism follows the one of Florenzano (1990). Furthermore, each member i of the coalition evaluates a new consumption bundle for the members of her reference group A_i assuming that members outside A_i stick to their status-quo consumption. Hence, the blocking mechanism is based on an optimistic attitude of each coalition member i with respect to the reaction of the agents outside A_i. It is proved by Vasil'ev (2016) that, under convexity, the fuzzy A-core introduced with Definition 6 is equivalent to the set $\mathscr{W}(E)$ of A-equilibria.

To investigate the validity of the equivalence theorem in our framework where the preferences are not necessarily convex, we adapt below the (fuzzy) core à la Husseinov (1994) to our economy. In this case, for a coalition matrix α, we define $D_i(\alpha^j) := D_i \bigcap \operatorname{supp}(\alpha^j)$, $\alpha_i^{jA} := \sum_{h \in D_i(\alpha^j)} \alpha_h^j$ and $\alpha_i(r) := \sum_{j=1}^r \alpha_i^{jA}$ [20]. As in the case of fuzzy coalitions, the set $D_i(\alpha^j)$ can be interpreted as the set of all agents which are influenced by i and simultaneously participate in the coalition α^j, while the element α_i^{jA} represents the total resources employed by these agents. The quantity $\alpha_i(r)$ has a similar interpretation referring to the entire coalition matrix α.

Definition 7 (Generalized Fuzzy A-Core) *Given an allocation $x \in \mathscr{F}$*

[20] In the paper, we follow the convention that the empty sum is defined to be equal to the additive identity, i.e., zero. For example, $\alpha_i^{jA} = 0$ if $D_i(\alpha^j) = \emptyset$.

and a coalition matrix $\alpha = (\alpha_i^j)$, we say that α A-improves x whenever, for every agent $i \in N$, there exist vectors $z_{(i)}^j \in X_{A_i}$ with $j = 1, \ldots, r$ such that

1. $z_{(i)}^j \in P_{A_i}(x)$, *for any* $i \in \operatorname{supp}(\alpha^j)$;

2. $\displaystyle\sum_{i\in N}\sum_{j=1}^{r}\sum_{h\in D_i(\alpha^j)} \alpha_h^j z_{hi}^j \leq \sum_{i\in N} \alpha_i(r)\omega_i.$

The set of allocations which cannot be A-improved upon by any coalition matrix is called generalized (fuzzy) A-core (or fuzzy core à la Husseinov), and it is denoted by $\mathscr{C}_h(E)$.

The blocking mechanism defined by a fuzzy coalition matrix maintains the same characteristic of the one introduced with Definition 6, but now each agent i may cooperate to more than one coalition employing in each coalition α^j a share α_i^j of resources of the members of her group A_i. The enlargement of the set of coalitions produces a stronger corresponding core. In particular, since the fuzzy coalitions are obtained for $r = 1$, it is clear that $\mathscr{C}_h(E) \subseteq \mathscr{C}_f(E)$. Without convexity of preference relations, one can also show that the previous inclusion is strict (see Husseinov (1994)).

In our framework, preferences $P_{A_i}(x)$ are not required to be convex. However, for the sake of completeness, we conclude this Section, by showing that, if $P_{A_i}(x)$ is convex for any $x \in \mathbb{R}^{l \cdot n}$, then the fuzzy A-core and the generalized fuzzy A-core coincide [21] .

Theorem 8 *Under convexity of preference relations, $\mathscr{C}_h(E) = \mathscr{C}_f(E)$.*

Proof. We have only to show that $\mathscr{C}_f(E) \subseteq \mathscr{C}_h(E)$. Let $x \in \mathscr{C}_f(E)$ and suppose by contradiction that $x \notin \mathscr{C}_h(E)$. So, there exist an $r \times n$ coalition matrix $\alpha = (\alpha_i^j)$ with $\alpha^j = (\alpha_i^j)_{i\in N} \in \mathbb{R}_+^n \setminus \{0\}$, and vectors $z_{(i)}^j \in X_{A_i}$, with $i \in N$ and $j = 1, \ldots r$, such that $z_{(i)}^j \in P_{A_i}(x)$ ($i \in \operatorname{supp}(\alpha^j)$, with $j = 1, \ldots, r$), and $\sum_{i\in N}\sum_{j=1}^{r}\sum_{h\in D_i(\alpha^j)} \alpha_h^j z_{hi}^j \leq \sum_{i\in N}\sum_{j=1}^{r}\sum_{h\in D_i(\alpha^j)} \alpha_h^j \omega_i$. The previous inequality can be written as $\sum_{i\in N}\sum_{h\in N} \gamma_h \sum_{j=1}^{r} \frac{\alpha_h^j}{\gamma_h} \tilde{z}_{hi}^j \leq \sum_{i\in N}\sum_{h\in N} \gamma_h \sum_{j=1}^{r} \frac{\alpha_h^j}{\gamma_h}\omega_i = \sum_{i\in N}\sum_{h\in N} \gamma_h \omega_i$, with $\gamma_h := \sum_{j=1}^{r} \alpha_h^j$, $\tilde{z}_{hi}^j := z_{hi}^j$ if $h \in D_i(\alpha^j)$ and $\tilde{z}_{hi}^j := 0$ otherwise. Define now for any $i \in N$ the vectors $t_{(i)} := (\sum_{j=1}^{r} \frac{\alpha_i^j}{\gamma_i} z_{ih}^j)_{h\in A_i} \in X_{A_i}$. Then each $t_{(i)}$ belongs to $P_{A_i}(x)$ since this set is convex and $t_{(i)}$ is a convex combination of elements of $P_{A_i}(x)$. Moreover, from $\sum_{i\in N}\sum_{h\in D_i(\gamma)} \gamma_h t_{hi} \leq \sum_{i\in N}\sum_{h\in D_i(\gamma)} \gamma_h \omega_i$ it follows that $\gamma = (\gamma_i)_{i\in N} > 0$ is a blocking coalition and this contradicts the fact that x belongs to the core $\mathscr{C}_f(E)$. ■

[21] For any $x \in X$, the preference relations $\succsim_i$ are convex over $X_{A_i} \times \{x_{N\setminus A_i}\}$, if for any $x'_{(i)}, x''_{(i)} \in X_{A_i}$ such that $(x'_{(i)}, x_{N\setminus A_i}) \succsim_i x$ and $(x''_{(i)}, x_{N\setminus A_i}) \succsim_i x$, and for any $0 \leq \tau \leq 1$, one has $(\tau x'_{(i)} + (i - \tau)x''_{(i)}, x_{N\setminus A_i}) \succsim_i x$.

4 An equivalence theorem for A-Equilibria

The next theorem shows that A-equilibrium allocations belong to the generalized fuzzy A-core.

Theorem 9 $\mathscr{W}(E) \subseteq \mathscr{C}_h(E)$.

Proof. Let $x \in \mathscr{W}(E)$. Suppose by contradiction that $x \notin \mathscr{C}_h(E)$. So there exist an $r \times n$ coalition matrix α and vectors $z^j_{(i)} \in X_{A_i}$, with $j = 1, \ldots, r$ and $i \in N$, such that $z^j_{(i)} \in P_{A_i}(x)$ for any $i \in \operatorname{supp}(\alpha^j)$ [22]. Since $x \in \mathscr{W}(E)$, it must be the case that $z^j_{(i)} \notin B_{A_i}(p, \omega)$, where $p \gg 0$ is the associated equilibrium price. Thus, for any $j = 1, \ldots, r$, we have $p \cdot \sum_{h \in A_i} \alpha^j_i z^j_{ih} > p \cdot \sum_{h \in A_i} \alpha^j_i \omega_h$ for each $i \in \operatorname{supp}(\alpha^j)$. The previous inequalities can be written as $p \cdot \sum_{h \in N} \alpha^j_i \tilde{z}^j_{ih} > p \cdot \sum_{h \in N} \alpha^j_i \tilde{\omega}_h$ for each $i \in \operatorname{supp}(\alpha^j)$, where the vectors $\tilde{z}^j_i$ and $\tilde{\omega}_i$ belong to $\mathbb{R}^{l \cdot n}_+$, and they are defined by $\tilde{z}^j_{ih} := z^j_{ih}$ and $\tilde{\omega}_{ih} := \omega_h$ if $h \in A_i$, and they are equal to zero otherwise. Summing over $i \in \operatorname{supp}(\alpha^j)$ and using the associative and commutative properties of the sum operators, one gets $p \cdot \left(\sum_{h \in N} \sum_{i \in \operatorname{supp}(\alpha^j)} \alpha^j_i \tilde{z}^j_{ih}\right) > p \cdot \left(\sum_{h \in N} \sum_{i \in \operatorname{supp}(\alpha^j)} \alpha^j_i \tilde{\omega}_{ih}\right)$, which is equivalent to $p \cdot \left(\sum_{h \in N} \sum_{i \in D_h(\alpha^j)} \alpha^j_i z^j_{ih}\right) > p \cdot \left(\sum_{h \in N} \alpha^{jA}_h \omega_h\right)$. Finally, summing over j and rearranging, one obtains $p \cdot \left(\sum_{h \in N} \sum_{j=1}^r \sum_{i \in D_h(\alpha^j)} \alpha^j_i z^j_{ih} - \sum_{h \in N} \alpha_h(r) \omega_h\right) > 0$, which contradicts the fact that the coalition matrix α blocks the allocation x via vectors $z^j_{(i)} \in X_{A_i}$, with $j = 1, \ldots, r$. ■

As consequence of Theorem 9 and $\mathscr{C}_h(E) \subseteq \mathscr{C}_f(E)$, one deduces Theorem 2.1 in Vasil'ev (2016).

To show the converse inclusion and decentralize core allocations as A-equilibrium allocations, define for any $x \in \mathbb{R}^{l \cdot n}_+$ and for each $i \in N$, the set

$$F_i(x, \omega) := \left\{ \sum_{h \in A_i} z_{ih} \in \mathbb{R}^l_+ \colon z_{(i)} = (z_{ih})_{h \in A_i} \in P_{A_i}(x) \right\} - \left\{ \sum_{h \in A_i} \omega_h \right\}$$

Notice that the set $F_i(x, \omega)$ is nonempty by Point 2 of Assumption 4. Denote by $\operatorname{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right)$ the convex hull of $\bigcup_{i \in N} F_i(x, \omega)$. The proof of the equivalence theorem is based on separation arguments applied to the convex set $\operatorname{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right)$.

Theorem 10 (Equivalence Theorem) *The set of A-equilibrium allocations coincides with the set of the generalized fuzzy core allocations: $\mathscr{W}(E) = \mathscr{C}_h(E)$.*

Proof. Take $x \in \mathscr{C}_h(E)$. We have to show that (x, p) belongs to $\Omega(E)$ for a strictly positive price vector p. We prove this result through several claims

[22] Notice that, $z^j_{(i)} \in P_{A_i}(x)$ and Point 2 of Assumption 4 implies $z^j_{(i)} \neq 0$.

whose proof is presented in the Appendix.

Claim 1. For the allocation $x \in \mathscr{C}_h(E)$ we have that $0 \notin \text{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right)$.

As consequence of Claim 1, we have that $\text{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right) \cap \{0\} = \emptyset$. So, applying the Separating Hyperplane Theorem, there exists a vector $p \in \mathbb{R}^l$ with $p \neq 0$ such that $p \cdot \zeta \geq 0$ for any $\zeta \in \text{co}\left(\bigcup_{i \in N} F_i(x_i, \omega)\right)$. The price p is nonnegative and the allocation x of the generalized fuzzy A-core lies on the budget hyperplane associated to p as stated in the next claim.

Claim 2. The vector p, with $p \neq 0$, is nonnegative and $p \cdot \sum_{h \in A_i} x_h = p \cdot \sum_{h \in A_i} \omega_h$ for any $i \in N$.

We claim now that the vector p is strictly positive and moreover for any agent i and for any $x'_{(i)} \in P_{A_i}(x)$, the inequality $p \cdot \sum_{h \in A_i} x'_{ih} > p \cdot \sum_{h \in A_i} \omega_h$ holds true.

Claim 3. The vector p is strictly positive and $p \cdot \zeta > 0$ for any $\zeta \in \text{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right)$.

As consequence of Claim 2, one gets $x_{A_i} \in B_{A_i}(p, \omega)$, and by Claim 3, $P_{A_i}(x) \cap B_{A_i}(p, \omega) = \emptyset$ holds true for any agent $i \in N$. It remains to show that market clearing condition is satisfied. Indeed, by Claim 2, $x \in \mathscr{C}_h(E)$ implies $p \cdot \sum_{h \in A_i} x_h = p \cdot \sum_{h \in A_i} \omega_h$ for any $i \in N$. By Point 2 of Assumption 3, the family A is balanced in the sense of Bondareva with full support. So there exists $\beta_i > 0$ with $i \in N$ such that $\sum_{i \in N} \beta_i \chi_{A_i} = \chi_N$. Consequently, $p \cdot \beta_i \sum_{h \in A_i} x_h = p \cdot \beta_i \sum_{h \in A_i} \omega_h$ for any agent i, and $p \cdot \sum_{i \in N} \beta_i \sum_{h \in A_i} x_h = p \cdot \sum_{i \in N} \beta_i \sum_{h \in A_i} \omega_h$. As in the proof of Theorem 9 [23], the previous equality can be written in the form $p \cdot \sum_{h \in N} \sum_{i \in D_h(\beta)} \beta_i x_h = p \cdot \sum_{h \in N} \sum_{i \in D_h(\beta)} \beta_i \omega_h$. Since $\text{supp}(\beta) = N$, we get $p \cdot \sum_{h \in N} \sum_{i \in D_h} \beta_i (x_h - \omega_h) = 0$. Finally, by $\beta \in \mathscr{B}_A$, one gets $\sum_{i \in D_h} \beta_i = 1$, and thus $p \cdot \sum_{h \in N} (x_h - \omega_h) = 0$. As a consequence, market clearing condition in the definition of A-equilibrium is satisfies by $p \gg 0$ and the fact that x is an allocation i.e., $\sum_{h \in N} (x_h - \omega_h) \leq 0$. Therefore, all the conditions in Definition 5 are verified, and $x \in \mathscr{W}(E)$. ■

If preferences are convex, as a consequence of Theorem 8 and Theorem 10, one trivially obtains the equivalence $\mathscr{W}(E) = \mathscr{C}_f(E)$. Furthermore, notice that, as highlighted in Remark 15, the balanced condition stated in Point 2 of Assumption 3 is necessary in order for the equivalence theorem to hold.

Remark 11 Theorem 10 extends at several instances (Husseinov, 1994, Theorem 6) [24]. In particular, it proves the core equivalence Theorem for models with externalities and non convex preferences presented in Section 5. The in-

[23] see also the proof of Claim 1 in the Appendix.
[24] See also Theorem 4.5 in Graziano (2001) and Theorem 1 in Husseinov and Páscoa (1997) for the production case.

terdependency effects due to the set A_i for each trader i in a blocking coalition, forces us to provide a direct proof of the equivalence result which does not use any continuum associated model (compare with the proof of (Husseinov, 1994, Theorem 6)).

Remark 12 Although our result shows that A-equilibrium allocations are core allocations, one should not expect that they are also Pareto optimal in presence of externalities. However, the inclusion $\mathscr{W}(E) \subseteq \mathscr{C}_h(E)$ implies that an A-equilibrium allocation x is not blocked by the society N in the following sense: it is not possible to find the vectors $x'_{(i)} \in X_{A_i}$ for $i \in N$ such that

1. $x'_{(i)} \in P_{A_i}(x)$ for each $i \in N$;
2. $\sum_{i \in N} \sum_{h \in D_i} x'_{hi} \leq \sum_{i \in N} |D_i| \omega_i$,

where D_i represents the set of agents affected by i and $|D_i|$ denotes its cardinality. Notice that even when each A_i reduces to the set $\{i\}$, an allocation x which is not dominated in the previous sense is not necessarily Pareto optimal, because, according to our notion of preferences, the consumption of the others is fixed.

5 Economic models analyzed in literature - Part I

This Section focuses on some models of exchange economies with externalities studied in literature and compares them with an A-economy. For each benchmark case analyzed below, our equivalence theorem provides a core notion such that allocations in the core can be decentralized as equilibrium allocations. A relevant motivation to analyze the decentralization results lies in the development of methods for establishing the existence of competitive equilibria by studying the non-emptiness of the core. For simplicity, we suppose that preferences are represented by utility functions [25] .

5.1 Walrasian Economy

Suppose that the family A is the following partition of the set of agents: $A \equiv \{A_i = \{i\} : i \in N\}$, i.e., the only agent influencing the stability of i is the agent i herself. In this framework, (x, p) is an equilibrium if

[25] As already mentioned in the paper, under Assumption 4 the preference relations can be represented by continuous utility functions.

1. $x_i \in \arg\max_{x'_i \in \mathbb{R}^l_+} \{u_i(x'_i, x_{-i}) \mid p \cdot x'_i \leq p \cdot \omega_i\}$, for all $i \in N$;

2. $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$.

The reader may notice that, under this specification of the family A, an A-economy coincides with the standard pure exchange economy with externalities, and the definition of A-equilibrium coincides with the one of competitive equilibrium à la Nash [26]. The existence of competitive equilibria à la Nash is proved, among other authors, by Dufwenberg et al. (2011) and Florenzano (1990).

We now look at the core. Notice that, when $A_i = \{i\}$ for each $i \in N$, the set of agents whose consumption is affected by i is just the singleton $D_i = \{i\}$ for any agent i, and thus $\alpha_i^A = \alpha_i$. Under this specification, a coalition $\alpha = (\alpha_i)_{i \in N}$ blocks a status quo x if there exist assignments $x'_{(i)} = (x'_{ih})_{h \in \{i\}} = x'_{ii}$, for each $i \in \text{supp}(\alpha)$ such that

1. $u_i(x'_{ii}, x_{-i}) > u_i(x)$ for any $i \in \text{supp}(\alpha)$;

2. $\sum_{i \in N} \alpha_i x'_{ii} \leq \sum_{i \in N} \alpha_i \omega_i$.

Each agent improves her utility when the preferences are evaluated over (x'_{ii}, x_{-i}). Thus, the assignment changes over agent i, but it is fixed for all the others. If there are no externalities at all, we have the usual notion of fuzzy core. If, in the presence of externalities, α is an ordinary coalition, that is, $\alpha = \chi_S$, with $S \subseteq N$, then a coalition S blocks a status quo x if for any agent which belongs to the blocking coalition, there exists an assignment $x'_{(i)}$, such that $u_i(x'_{ii}, x_{-i}) > u_i(x)$ for any $i \in S$ and $\sum_{i \in S} x'_{ii} \leq \sum_{i \in S} \omega_i$. This is the cooperative solution concept analyzed by Florenzano (1989, 1990).

5.2 Berge Economy

Suppose that the structure A is now given by $A \equiv \{A_i = N \setminus \{i\} \colon i \in N\}$. In this situation, each agent is influenced by all the other agents, and A is not a partition. Equivalently, one may think that agents are supposed to act using an altruistic behavior, regardless to their own consumption. Thus, (x, p) is an equilibrium if

1. $x_{-i} \in \arg\max_{x'_{-i} \in \mathbb{R}^{l \cdot (n-1)}_+} \{u_i(x'_{-i}, x_i) \mid p \cdot \sum_{h \in N \setminus \{i\}} x'_h \leq p \cdot \sum_{N \setminus \{i\}} \omega_h\}$, for all $i \in N$;

[26] See for example Borglin (1973).

2. $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$.

This definition can be considered as an adaptation of the classical Berge equilibrium in the sense of Zhukovskii and Chikrii (1994) for non-cooperative games [27].

We now look at the cooperative solution. When $A_i = N \setminus \{i\}$, for each $i \in N$, then the set of agents whose consumption is affected by i is the set $D_i = N \setminus \{i\}$. Thus, given a fuzzy coalition $\alpha = (\alpha_i)_{i \in N}$ one has $\alpha_i^A = \sum_{j \in N \setminus \{i\}} \alpha_j = \sum_{j \neq i} \alpha_j$ and α blocks an allocation x if there exist vectors $x'_{(i)} = (x'_{ih})_{h \neq i}$ with $i \in N$ such that

1. $u_i((x'_{ih})_{h \neq i}, x_i) > u_i(x)$ for any $i \in \text{supp}(\alpha)$;

2. $\sum_{i \in N} \sum_{h \in \text{supp}(\alpha) \setminus \{i\}} \alpha_h x'_{hi} \leq \sum_{i \in N} \sum_{h \in \text{supp}(\alpha) \setminus \{i\}} \alpha_h \omega_i$.

The consumption of agent i is fixed for each member in the coalition, and each member proposes a different consumption plan. When α is a crisp coalition, $S \subseteq N$ blocks the status quo x if $u_i((x'_{ih})_{h \neq i}, x_i) > u_i(x)$ for any $i \in S$, and $\sum_{i \in N} \sum_{h \in (S \setminus \{i\})} x'_{ih} \leq \sum_{i \in N} |S \setminus \{i\}| \cdot \omega_i$.

In a game theoretical setting, the issue related to the existence of a Berge equilibrium is a difficult task that have been addressed for example in Abalo and Kostreva (2005) and Nessah et al. (2007) [28]. For the pure exchange economy studied in this paper, the fact that different agents may share the same A-relevant arguments in their utility functions, makes standard approaches to the existence problem not applicable (see the discussion in Vasil'ev (2016)). Hence the existence of altruistic Berge equilibria as well as the more general problem of existence of A-equilibria are open questions that deserve to be analyzed.

5.3 Family Economy

Assume that $\mathscr{H} := \{H_j : j = 1, \ldots, k\}$, with $k \leq n$ is a partition of the set of agents, and interpret each $H_j \in \mathscr{H}$ as a family. For every $i \in N$, define A_i as

[27] See for instance, Vasil'ev (2016).

[28] More recently, Riedel and Torrente (2025) study algorithms to determine the existence and computation of Berge equilibria in finite games, Haller (2024) and Haller (2025) investigate respectively welfare properties of Berge equilibria and their existence. With a different approach to the definition of core, the problem of the existence of core allocations in the presence of externalities has recently been addressed, among others, in Martins-da-Rocha and Yannelis (2026).

the element of $\mathscr{H}$ containing i. Notice that, in this case, the sets A_i are not necessarily different, and $i \in A_i$ for each $i \in N$. In particular, observe that members of the same family are associated with the same A_i, and as consequence the budget set in the definition of an A-equilibrium is the same for all members of the same family (see Section 3). In particular, the budget set of the family A_i is given by $B_{A_i}(p,\omega) := \left\{x_{(i)} \in X_{A_i} : p \cdot \left(\sum_{h\in A_i} x_{ih}\right) \leq p \cdot \left(\sum_{h \in A_i} \omega_h\right)\right\}$, where $p \cdot \left(\sum_{h\in A_i} \omega_h\right)$ is the income of the family. Therefore, (x,p) is an equilibrium if

1. $x_{A_i} \in \underset{x_{(i)} \in \mathbb{R}_+^{l\cdot|A_i|}}{\arg\max} \{u_i(x_{(i)}, x_{N\setminus A_i}) \mid p \cdot \sum_{h\in A_i} x_{ih} \leq p \cdot \sum_{h\in A_i} \omega_h\}$, for all $i \in N$;
2. $\sum_{i\in N} x_i = \sum_{i \in N} \omega_i$.

In equilibrium, the consumption bundle of the family A_i maximizes the utility of all the members $h \in A_i$ under the same budget constraint. Notice also that the utility may be different for different members of the same family. We call this equilibrium *family equilibrium* for the following reason. Suppose that the family structure $\mathscr{H}$ is given, and the utility of each member of the family A_i only depends on the members of the family, i.e., $u_h(x) = u_h(x_{A_i})$, for all $h \in A_i$. This form of externalities is known in the literature as intra-household externalities [29]. Thus, we end up with the notion of equilibrium defined by Haller (2000) and Gersbach and Haller (2001) for their collective consumption models [30]. The non-emptiness of the set of competitive equilibrium among household is provided by Gersbach and Haller (1999). We now look at the core of a family economy. A coalition α blocks an allocation x if there exists a plan for the family $x'_{(i)} = (x'_{ih})_{h\in A_i}$, with $i \in N$, such that

1. $u_i(x'_{(i)}, x_{N\setminus A_i}) > u_i(x)$ for each $i \in \text{supp}(\alpha)$;
2. $\sum_{i\in N} \sum_{h \in A_i \cap \text{supp}(\alpha)} \alpha_h x'_{hi} \leq \sum_{i\in N} \sum_{h\in A_i\cap \text{supp}(\alpha)} \alpha_h \omega_i$.

In this setting, $D_i(\alpha) = A_i \cap \text{supp}(\alpha)$ for each agent i. If $\alpha = \chi_S$, then S blocks x if $u_i(x'_{(i)}, x_{N\setminus A_i}) > u_i(x)$ for each $i \in S$ and $\sum_{i\in N} \sum_{h\in A_i \cap S} x'_{hi} \leq \sum_{i \in N} |A_i \cap S| \cdot \omega_i$. Assume the following restriction on coalition formation: α is an admissible coalition if $\alpha_i \neq 0$ for a member i of a family A_i implies $\alpha_h \neq 0$

[29] See for instance, Haller (2000) and Gersbach and Haller (2001).

[30] Haller (2000) and Gersbach and Haller (2001) define a competitive equilibrium among households as a pair (x,p) such that, $\sum_{i\in N} x_i = \sum_{i\in N} \omega_i$, and for any $i \in N$: (1) $x_{A_i} \in B_{A_i}(p,\omega)$; (2) there is no $z_{(i)} \in B_{A_i}(p,\omega)$ which meets $(u_h(z_{(i)}))_{h\in A_i} > (u_h(x_{A_i}))_{h\in A_i}$. Notice that, if x is a family equilibrium allocation, then $u_h(z_{(i)}) \leq u_h(x_{A_i})$ for any $z_{(i)}$ belongs to $B_{A_i}(p,\omega)$ and for any $h \in A_i$. Consequently, x is also an equilibrium allocation in the sense of Haller (2000) and Gersbach and Haller (2001).

for any other member h of A_i. For a crisp coalition, this condition implies that a coalition is formed by the union of families. Furthermore, if the agents in a family have the same utility function, and if it is affected only by the family consumption, under quasi-concavity, the sum of what an agent i receives from the family members, under a blocking allocation i.e., $\sum_{h \in A_i \cap S} x'_{hi}$, can be replaced by a convex combination, and we obtain the Household core notion introduced by Haller (2000). In this case, each family proposes an alternative plan for the family and globally for the blocking coalition S.

5.4 Two examples

In this Section, we provide two examples. Example 13 considers an economy in which there are two agents with the same utility function and only one commodity. Under this simple setting, according to the structure of A, the set of A-equilibria may be empty or a singleton. Example 14 generalizes the first example, by considering two commodities and showing that we may also have infinitely many A-equilibria.

Example 13 *Consider an A-economy with $N = \{1, 2\}$, one commodity and an initial endowment of $\omega_i = 1$ for any $i = 1, 2$. Suppose that the preferences of agent i are represented by the utility function $u_i(x_1, x_2) = x_1 + 2x_2$ with $i = 1, 2$. We may consider four cases.*

<u>Case 1:</u> Nash-Walrasian Economy, i.e., $A_i = \{i\}$ for any agent i. Under this framework, at the equilibrium, each agent i solves the problem $\max\{ix_i \mid p \cdot x_i \leq p\}$, which leads to the no-trade A-equilibrium given by $(x_1, x_2) = (\omega_1, \omega_2)$.

<u>Case 2:</u> Berge Economy, i.e, $A_i = \{h\}$ with $h \neq i$, for any agent i. By a similar argument used for case 1, one easily gets the no-trade A-equilibrium $(x_1, x_2) = (\omega_1, \omega_2)$.

<u>Case 3:</u> $A_i = \{1, 2\}$ and $A_h = \{k\}$ with $k \in \{1, 2\}$ and $i \neq h$. Agent i solves $\max\{x_{i1} + 2x_{i2} \mid p \cdot (x_{i1} + x_{i2}) \leq 2p\}$ which implies $x_{(i)} = (x_{i1}, x_{i2}) = (0, 2)$. Agent h solves $\max\{kx_{hk} \mid p \cdot x_{hk} \leq p\}$, and the optimal bundle is $x_{(h)} = x_{hk} = 1$. Notice that, independently by the identity of the agent k, the A-equilibrium does not exist, since $x_{ik} \neq x_{hk}$.

<u>Case 4:</u> The family economy with $A_i = \{1, 2\}$ for any i. Each agent i solves $\max\{x_{i1} + 2x_{i2} \mid p \cdot (x_{i1} + x_{i2}) \leq 2p\}$, and the corresponding A-equilibrium allocation is $(x_1, x_2) = (0, 2)$.

Example 14 *Consider an A-economy with $N = \{1, 2\}$, two commodities, and initial endowments $\omega_i = (1, 1)$ for any $i = 1, 2$. Suppose that the preferences of agent i are represented by the utility function $u_i((x_1, y_1), (x_2, y_2)) = (x_1 +$*

$y_1) + 2(x_2 + y_2)$ *with* $i = 1, 2$. *Again, we may consider four cases.*

<u>*Case 1:*</u> *Nash-Walrasian Economy, i.e.,* $A_i = \{i\}$ *for any agent* i. *At the equilibrium, agent* i *solves* $\max\{i(x_i + y_i) \mid px_i + qy_i \leq p + q\}$. *If* $p = q$ *there are infinite many A-equilibria allocations, given by* $(x_1, y_1) = (x, 2 - x)$ *and* $(x_2, y_2) = (2 - x, x)$, *with* $x \geq 0$. *Notice that the no-trade allocation is an A-equilibrium allocation for* $x = 1$. *If* $p < q$ *the A-equilibrium does not exist. Indeed, the agent maximization problem leads to* $(x_i, y_i) = ((p + q)/p, 0)$ *for any* $i = 1, 2$. *However, using market clearing condition for commodity* 1, *we get* $q = 0$ *which is a contradiction. In a similar way one may show that if* $p > q$ *there are no A-equilibria.*

<u>*Case 2:*</u> *Berge Economy, i.e,* $A_i = \{h\}$ *with* $h \neq i$, *for any agent* i. *By a similar argument used for case* 1, *one may show that there are infinite many A-equilibria allocations given by* $(x_1, y_1) = (x, 2 - x)$ *and* $(x_2, y_2) = (2 - x, x)$, *with* $x \geq 0$, *and* $(p, q) \gg 0$ *with* $p = q$ *are the associated equilibrium prices.*

<u>*Case 3:*</u> $A_i = \{1, 2\}$ *and* $A_h = \{k\}$ *with* $k \in \{1, 2\}$ *and* $i \neq h$. *At the equilibrium, agent* h *solves* $\max\{k(x_{hk} + y_{hk}) \mid px_{hk} + qy_{hk} \leq p + q\}$. *Notice that:* (1) *if* $p = q$ *the optimal solution satisfies* $x_{hk} + y_{hk} = 2$ *with* $x_{hk} \geq 0$*;* (2) *if* $p < q$ *we obtain* $x_{hk} = (p+q)/p$ *and* $y_{hk} = 0$*;* (3) *if* $p > q$ *we have* $x_{hk} = 0$ *and* $y_{hk} = (p+q)/q$. *Agent* i *solves* $\max\{(x_{i1}+y_{i2})+2(x_{i2}+y_{i2}) \mid p(x_{i1}+x_{i2})+q(y_{i1}+y_{i2}) \leq 2p + 2q\}$. *Suppose that* $k = 1$, *we show that the set of A-equilibria is empty. If* $p = q$, *since at the equilibrium we must have* $(x_{h1}, y_{h1}) = (x_{i1}, y_{i1}) = (x_1, y_1)$, *using the fact that* $x_1 + y_1 = 2$, *the optimization problem of agent* i *can be rewritten as* $\max\{2 + 2(x_{i2} + y_{i2}) \mid x_{i2} + y_{i2} \leq 2\}$ *which implies* $x_{i2} + y_{i2} = 2$. *Finally, by market clearing conditions we obtain* $(x_1, y_1) = (x, 2 - x)$ *and* $(x_2, y_2) = (2 - x, x)$, *with* $x \geq 0$. *However, this is not an A-equilibrium, since the allocation* $z = (z_1, z_2)$ *with* $z_1 = (0, 0)$, *and* $z_2 = (2, 2)$ *ensures to agent* i *an higher utility, indeed* $u_i(z) = 8 > x + (2 - x) + 2(2 - x + x) = 6 = u_i((x_1, y_1), (x_2, y_2))$. *If* $p < q$, *by* $(x_{h1}, y_{h1}) = (x_{i1}, y_{i1}) = (x_1, y_1)$, *and* $(x_1, y_1) = ((p + q)/p, 0)$, *the optimization problem of agent* i *can be written as* $\max\{(p + q)/p + 2(x_{i2} + y_{i2}) \mid px_{i2} + py_{i2} \leq 2(p + q)\}$, *and consequently,* $(x_{i2}, y_{i2}) = ((p + q)/p, 0)$. *By the market clearing condition for commodity* 1, *we get* $q = 0$ *which is a contradiction. One may get a similar contradiction when* $p > q$. *By similar arguments it is possible to show that there are no A-equilibria when* $k = 2$.

<u>*Case 4:*</u> *The family equilibrium economy with* $A_i = \{1, 2\}$ *for any* i. *Each agent* i *solves* $\max\{(x_{i1} + y_{i1}) + 2(x_{i2} + y_{i2}) \mid p(x_{i1} + x_{i2}) + q(y_{i1} + y_{i2}) \leq 2(p + q)\}$. *We show now that* $(x_1, y_1) = (0, 0)$ *and* $(x_2, y_2) = (2, 2)$ *is an A-equilibrium allocation associated to the equilibrium prices* (p, q) *with* $p = q$. *First of all, notice that, it is feasible and it belongs to the budget set of agent* i. *By contradiction, suppose that there exists a vector* $z = ((x_1, y_1), (x_2, y_2))$ *such that* $u_i(z) = (x_1 + y_1) + 2(x_2 + y_2) > 8 = u_i((0, 0), (2, 2))$ *and* z *satisfies*

the budget set at $p = q$, i.e., $(x_1+x_2)+(y_1+y_2) \leq 4$. The last two inequalities imply $x_2+y_2 > 4$, and by market clearing conditions we get $x_1+y_1 < 0$ which is a contradiction.

Remark 15 We come back to Case 3 of Example 13 with $A_1 = \{1,2\}$ and $A_2 = \{2\}$ in order to show that $\mathscr{C}_f(E) \neq \emptyset$. We already know that the set of A-equilibria is empty. We verify that $(x_1, x_2) = (0,2)$ belong to the fuzzy A-core. Suppose by contradiction that there exist a fuzzy coalition $\alpha = (\alpha_1, \alpha_2) > 0$ and $y_{(1)} = (y_{11}, y_{12})$ and $y_{(2)} = (y_{22})$ such that: (1) $y_{11} + 2y_{12} > 4$ if $\alpha_1 > 0$; (2) $2y_{22} > 4$ if $\alpha_2 > 0$; (3) $\alpha_1 y_{11} + \alpha_1 y_{12} + \alpha_2 y_{22} \leq 2\alpha_1 + \alpha_2$. We must consider three cases. Case 1: $\alpha_1 = 0$ and $\alpha_2 > 0$. The relevant inequalities (2) and (3) become $y_{22} > 2$ and $y_{22} \leq 1$ which give arise to a contradiction. Case 2: $\alpha_1 > 0$ and $\alpha_2 = 0$. The inequalities (1) and (3) become $y_{11} + 2y_{12} > 4$ and $y_{11} + y_{12} \leq 2$. Therefore, $4 - 2y_{12} < y_{11} \leq 2 - y_{12}$ which implies $y_{12} > 2$. Using inequality (3), one obtains $y_{11} < 0$ which leads to a contradiction. Case 3: $\alpha_1 > 0$ and $\alpha_2 > 0$. Multiplying inequalities (1) and (2) respectively by α_1 and α_2 one obtains $\alpha_1 y_{11} > 4\alpha_1 - 2\alpha_1 y_{12}$ and $\alpha_2 y_{22} > 2\alpha_2$, which, when added to (3), yield $\alpha_1 y_{12} > 2\alpha_1 + \alpha_2$. This is a contradiction, since by (3) one gets $\alpha_1 y_{11} + \alpha_2 y_{22} < 0$. We have showed that the allocation $(0,2)$ cannot be blocked by a fuzzy coalition, and consequently, it belongs to the fuzzy A-core. Therefore, $\mathscr{W}(E) \subseteq \mathscr{C}_f(E)$, but the two sets are not equivalent. We conclude the remark by underlying that the structure $A_1 = \{1,2\}$ and $A_2 = \{2\}$ is not balanced in the sense of Bondareva with full support.[31] As a consequence, the example shows that Point 2 of Assumption 3 is a necessary condition for our equivalence theorem.

6 Information equilibrium and Generalized Fuzzy Information-Core

One particular case of the A-economy analyzed in Section 3 is the one in which every agent takes care of all individuals in the economy, including himself, i.e., $A_i = N$ for each $i \in N$. Under this specification for the structure A, the notion of A-equilibrium introduced with Definition 5 requires that agents maximize their utility functions on the same budget set. The common budget set is formed by allocations whose value does not exceed the value at the fixed price system of the total initial endowment. Hence agents propose a consumption bundle for each of the traders of their reference set, but exchange under a unique fixed price.

In the present Section, we follow a different approach and assume that in the

[31] With similar arguments, one may verify that in the Example 14, the fuzzy A-core is nonempty when the structure is $A_1 = \{1,2\}$ and $A_2 = \{2\}$, and in particular, $((0,0),(2,2)) \in \mathscr{C}_f(E)$, whereas there do not exist A-equilibria.

exchange economy with externalities, agents still choose allocations, that is profiles of consumption bundles for the whole society, i.e., $A_i = N$ for each i, but they face personalized (Lindahl) prices and a budget constraint defined by their own resources[32]. In particular, following the interpretation of markets for externalities in terms of coordination given by Makarov (1982) and Vasil'ev (1996), we consider the notion of competitive equilibrium called *Information equilibrium* in which the demand of agents for additional commodities permit to exchange the information on the consumption structure of the economic system as a whole. Under this interpretation, a personalized price p_{ih} can be interpreted as the price of commodities which present to agent i the information on consumption of agent h[33] and the budget set is defined accordingly. We introduce a notion of blocking in terms of the fuzzy coalition matrix and provide a complete characterization of Information equilibrium allocations as allocations in the Generalized Fuzzy Information-Core.

We consider an A-economy in which $A_i = N$ for any agent i. To simplify the notation, we will denote the economy by E, that is, $E := \langle N, \mathbb{R}^l_+, (\succsim_i, \omega_i)_{i \in N} \rangle$. In this setting, for a given bundle x, the set $P_i(x) = \left\{ x' \in \mathbb{R}^{l \cdot n}_+ \colon x' \succ_i x \right\}$ coincides with the set $P_{A_i}(x)$ defined in Section 3.

The Basic Assumptions 4 on preference relations $\succsim_i \subseteq \mathbb{R}^{l \cdot n}_+ \times \mathbb{R}^{l \cdot n}_+$ stated in Section 3, are replaced by the following

Assumption 16 *For any agent i,*

(1) $\succsim_i$ are complete, transitive and continuous over $\mathbb{R}^{l \cdot n}_+$.
(2) For any vector $x \in \mathbb{R}^{l \cdot n}_+$, $\succsim_i$ are strongly monotone over $\mathbb{R}^l_+ \times \{x_{-i}\}$, and non-decreasing over $\mathbb{R}^{l \cdot n}_+$.[34]

Following Makarov (1982) and Vasil'ev (1996), we introduce below the Information equilibrium[35]. In contrast with the notion of A-equilibrium which is characterized by a unique price system, in the Information equilibrium there are a personalized price system $\pi := (p_{(i)})_{i \in N}$, with $p_{(i)} := (p_{ih})_{h \in N}$, and a market price p. The personalized price p_{ih} is interpreted as the price vector which makes up the information for agent i about the consumption of agent

[32] See the seminal papers by Arrow (1969) and Laffont (1976).
[33] Under the classical definitions in Arrow (1969) and Laffont (1976), the extended consumption vector of individual i incorporates her effective consumption and the external effect of other agents consumption as perceived by individual i. Hence the personalized price p_{ih} is interpreted as the price paid by individual i for the consumption externalities created by h on i (compare Bonnisseau et al. (2023)).
[34] A preference relation $\succsim_i \subseteq \mathbb{R}^{l \cdot n}_+ \times \mathbb{R}^{l \cdot n}_+$ is non-decreasing if for any $x \in \mathbb{R}^{l \cdot n}_+$ and for any vector $v \in \mathbb{R}^{l \cdot n}_+$, one has $x + v \succsim_i x$.
[35] We follow the terminology of Vasil'ev (1996) to denote an equilibrium for markets with externalities.

h.

Definition 17 (Information equilibrium) $(x, \pi, p) = ((x_i, p_{(i)})_{i \in N}, p) \in \mathbb{R}_+^{l \cdot n} \times \mathbb{R}_+^{l \cdot n^2} \times \mathbb{R}_{++}^l$ *is an Information equilibrium for the economy* E *if*

1. $x \in B_i(\pi, p, \omega)$ *for all* $i \in N$*;*
2. $P_i(x) \bigcap B_i(\pi, p, \omega) = \emptyset$ *for all* $i \in N$*;*
3. $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$*;*
4. $\sum_{i \in N} p_{ih} = p$*, for all* $h \in N$

where the set $B_i(\pi, p, \omega) \coloneqq \left\{ x_{(i)} \in \mathbb{R}_+^{l \cdot n} \colon p_{(i)} \cdot x_{(i)} \leq p \cdot \omega_i \right\}$ *denotes the budget set of agent* i*.*

Notice that the budget set is defined in terms of the personalized and market prices. In contrast with the budget set in the definition of an $A-$equilibrium, in an Information equilibrium, agent i considers her own endowment. Conditions 1 and 2 state that, for every agent i, x maximizes preference under the budget constraint, point 3 is the classical market clearing condition, and point 4 is a compatibility condition for prices. We denote by $\Omega_I(E)$ the set of information equilibria, and by

$$\mathscr{W}_I(E) \coloneqq \left\{ x \in \mathbb{R}_+^{l \cdot n} \mid \exists\, (\pi, p) \in \mathbb{R}_+^{l \cdot n^2} \times \mathbb{R}_{++}^l \colon (x, \pi, p) \in \Omega_I(E) \right\}$$

the set of all information equilibrium allocations. Conditions that guarantee the existence of equilibria with personalized prices are studied by Vasil'ev (1983) and Bonnisseau et al. (2023), among other authors.

We report below the notion of fuzzy Information-Core, introduced by Vasil'ev (1996).

Definition 18 (Fuzzy Information-Core) *Given an allocation* $x \in \mathscr{F}$ *and a coalition* $\alpha \in \mathbb{R}_+^n$ *with* $\alpha \neq 0$*, we say that* α *(information)-improves upon* x *whenever, for every agent* $i \in N$*, there exists a vector* $(z_{(i)}, \xi_i) = ((z_{ih})_{h \in N}, \xi_i) \in \mathbb{R}_+^{l \cdot n} \times \mathbb{R}_+^l$ *such that*

1. $z_{(i)} \in P_i(x)$ *for any* $i \in \operatorname{supp}(\alpha)$*;*
2. $\sum_{i \in N} \alpha_i \xi_i \leq \sum_{i \in N} \alpha_i \omega_i$*;*
3. $\alpha_i z_{(i)} = (\alpha_h \xi_h)_{h \in N}$ *for any* $i \in \operatorname{supp}(\alpha)$*.*

The set of allocations which cannot be (information)-improved upon by any coalition is called fuzzy Information-Core, and it is denoted by $\mathscr{C}_f^I(E)$*.*

Notice that condition 3. in Definition 18 says that for the vector $z_{(i)}$ containing the information on the consumption of each trader h provided to agent i, it is true that $z_{ih} = 0$ when h does not belong to α. Moreover, since $z_{ih} = \frac{\alpha_h}{\alpha_i}\xi_h$, each agent i joining a blocking coalition receives the vector ξ_i and through condition 1. evaluates the allocation $(\xi_h)_{h\in N}$ taking into account the share of participation of all the agents in the coalition [36]. As in Section 3, we adapt below the generalized fuzzy core to our economy. For a coalition matrix α, with $\alpha^j \in \mathbb{R}^n_+$ and $\alpha^j \neq 0$, denote by $\alpha_i(r)$ the element $\sum_{j=1}^r \alpha_i^j$.

Definition 19 (Generalized Fuzzy Information-Core) *Given an allocation $x \in \mathscr{F}$ and a coalition matrix $\alpha = (\alpha_i^j)$, with $\alpha^j \in \mathbb{R}^n_+$ with $\alpha^j \neq 0$, we say that α (information)-improves upon x whenever, for every agent $i \in N$, there exist vectors $(z^j_{(i)}, \xi^j_i) = ((z^j_{ih})_{h\in N}, \xi^j_i) \in \mathbb{R}^{l\cdot n}_+ \times \mathbb{R}^l_+$ with $j = 1, \dots, r$ such that*

1. $z^j_{(i)} \in P_i(x)$, *for any* $i \in \mathrm{supp}(\alpha^j)$;
2. $\sum_{i\in N}\sum_{j=1}^{r} \alpha_i^j \xi_i^j \leq \sum_{i\in N} \alpha_i(r)\omega_i$;
3. $\sum_{j=1}^{r} \alpha_i^j z^j_{(i)} = \left(\sum_{j=1}^{r} \alpha_h^j \xi_h^j\right)_{h\in N}$ *for any* $i \in \mathrm{supp}(\alpha)$.

The set of allocations which cannot be (information)-improved by any coalition matrix is called generalized fuzzy Information-core and it is denoted by $\mathscr{C}_h^I(E)$.

Notice that $\mathscr{C}_h^I(E) \subseteq \mathscr{C}_f^I(E)$, since the fuzzy coalitions are obtained for $r = 1$. Under convexity, the fuzzy information-core and the generalized fuzzy information-core coincide.

Theorem 20 *Under convexity of preference relations, $\mathscr{C}_h^I(E) = \mathscr{C}_f^I(E)$.*

Proof. We have only to show that $\mathscr{C}_f^I(E) \subseteq \mathscr{C}_h^I(E)$. Let $x \in \mathscr{C}_f^I(E)$ and suppose by contradiction that $x \notin \mathscr{C}_h^I(E)$. So, there exist an $r \times n$ coalition matrix $\alpha = (\alpha_i^j)$ and vectors $(z^j_{(i)}, \xi^j_i) \in \mathbb{R}^{l\cdot n}_+ \times \mathbb{R}_+$, with $i \in N$ and $j = 1, \dots r$, such that $z^j_{(i)} \in P_i(x)$ ($i \in \mathrm{supp}(\alpha^j)$, with $j = 1, \dots, r$), $\sum_{i\in N}\sum_{j=1}^r \alpha_i^j \xi_i^j \leq \sum_{i\in N} \alpha_i(r)\omega_i$ and $\sum_{j=1}^r \alpha_i^j z^j_{(i)} = (\sum_{j=1}^r \alpha_h^j \xi_h^j)_{h\in N}$ for each $i \in \mathrm{supp}(\alpha)$. For any

[36] In this respect, the notion of blocking assumes that the preference of agents in the veto mechanism may change depending on the coalition that traders join. This is in line with Ichiishi (1981). In a different context, see also Allouch and Wooders (2008), where individuals may participate to multiple clubs and preferences depends on the clubs they belong to.

agent i, define a vector

$$(y_{(i)}, \eta_i) := \left(\frac{\sum_{j=1}^{r} \alpha_i^j z_{(i)}^j}{\alpha_i(r)}, \frac{\sum_{j=1}^{r} \alpha_i^j \xi_i^j}{\alpha_i(r)} \right) \in \mathbb{R}_+^{l \cdot n} \times \mathbb{R}_+^l$$

Notice that $y_{(i)}$ belongs to $P_i(x)$ since $P_i(x)$ is a convex set and $y_{(i)}$ is a linear convex combination of elements of $P_i(x)$[37]. Furthermore,

$$\sum_{i \in N} \alpha_i(r) \eta_i = \sum_{i \in N} \sum_{j=1}^{r} \alpha_i^j \xi_i^j \leq \sum_{i \in N} \alpha_i(r) \omega_i$$

and

$$\alpha_i(r) y_{(i)} = \sum_{j=1}^{r} \alpha_i^j z_{(i)}^j = \left(\alpha_h(r) \frac{\sum_{j=1}^{r} \alpha_h^j \xi_h^j}{\alpha_h(r)} \right)_{h \in N} = (\alpha_h(r) \eta_h)_{h \in N}$$

This contradicts the fact that $x \in \mathscr{C}_f^I(E)$, since $(\alpha_i(r))_{i \in N}$ is a blocking coalition. ■

7 An equivalence theorem for Information equilibria

The present Section contains the technical steps necessary to show the fuzzy core equivalence theorem for Information Equilibrium when preferences are not convex. We start the Section by proving that an informational equilibrium allocation belongs to the generalized fuzzy information-core.

Theorem 21 $\mathscr{W}_I(E) \subseteq \mathscr{C}_h^I(E)$.

Proof. Let $x \in \mathscr{W}_I(E)$ and suppose that $x \notin \mathscr{C}_h^I(E)$. So, there exists a coalition matrix $\alpha = (\alpha_i^j)$ such that $z_{(i)}^j \in P_i(x)$ for any $i \in \text{supp}(\alpha^j)$ and for any $j = 1, \ldots, r$. Since $x \in \mathscr{W}_I(E)$, it must be the case that $z_{(i)}^j \notin B_i(\pi, p, \omega)$, where π and p are such that $(x, \pi, p) \in \Omega_I(E)$. Thus, $p_{(i)} \cdot \alpha_i^j z_{(i)}^j > p \cdot \alpha_i^j \omega_i$ for any agent $i \in \text{supp}(\alpha^j)$ and for any $j = 1, \ldots, r$. Summing over $i \in \text{supp}(\alpha^j)$, we get $\sum_{i \in \text{supp}(\alpha^j)} p_{(i)} \cdot \alpha_i^j z_{(i)}^j > p \cdot \sum_{i \in \text{supp}(\alpha^j)} \alpha_i^j \omega_i$, which is equivalent to $\sum_{i \in N} p_{(i)} \cdot \alpha_i^j z_{(i)}^j > p \cdot \sum_{i \in N} \alpha_i^j \omega_i$. Summing with respect to $j = 1, \ldots, r$ and inverting the sum operators, we obtain $\sum_{i \in N} p_{(i)} \cdot \sum_{j=1}^{r} \alpha_i^j z_{(i)}^j > p \cdot \sum_{i \in N} \alpha_i(r) \omega_i$. Using the fact that $\sum_{j=1}^{r} \alpha_i^j z_{(i)}^j = (\sum_{j=1}^{r} \alpha_h^j \xi_h^j)_{h \in N}$ for any $i \in \text{supp}(\alpha)$, for some ξ_h^j, the previous inequalities can be written as $\sum_{i \in N} \sum_{h \in N} p_{ih} \cdot \sum_{j=1}^{r} \alpha_h^j \xi_h^j > p \cdot \sum_{i \in N} \alpha_i(r) \omega_i$. Finally, inverting the sum operators on the left side of the

[37] Notice that $\frac{\alpha_h^j}{\alpha_h(r)} \geq 0$ for any $j = 1, \ldots, r$, and $\sum_{j=1}^{r} \frac{\alpha_h^j}{\alpha_h(r)} = 1$ for any agent h.

inequality, and by $\sum_{i\in N} p_{ih} = p$, for all $h \in N$, we get $\sum_{h\in N} p \cdot \sum_{j=1}^{r} \alpha_h^j \xi_h^j > p \cdot \sum_{i\in N} \alpha_i(r)\omega_i = p \cdot \sum_{h\in N} \alpha_h(r)\omega_h$. Therefore,

$$p \cdot \sum_{h\in N} \left(\sum_{j=1}^{r} \alpha_h^j \xi_h^j - \alpha_h(r)\omega_h \right) > 0$$

contradicting the fact that the coalition matrix α blocks the allocation x. ■

The reader may notice that, Proposition 3.1 in Vasil'ev (1996) can be deduced by Theorem 21 and $\mathscr{C}_h^I(E) \subseteq \mathscr{C}_f^I(E)$.

We now introduce some notations. Define the set Q as follows,

$$Q := \{(i, h)|\ i, h \in N, i \neq h\} \cup \{(0, 0)\}$$

the linear mapping $\varphi_i \colon z_{(i)} \in \mathbb{R}_+^{l\cdot n} \mapsto \varphi_i(z_{(i)}) := (\varphi_i^q(z_{(i)}))_{q\in Q} \in \mathbb{R}_+^{l\cdot|Q|}$, by

$$\varphi_i^q(z_{(i)}) := \begin{cases} z_{ii} & \text{if } q = (0,0) \\ -z_{ii} & \text{if } q = (h,i) \\ z_{ih} & \text{if } q = (i,h) \\ 0 & \text{otherwise} \end{cases}$$

and the vector $\widehat{\omega} \in \mathbb{R}^{l\cdot|Q|}$, by

$$\widehat{\omega}_i^q := \begin{cases} \omega_i & \text{if } q = (0,0) \\ 0 & \text{otherwise} \end{cases}$$

For any $x \in \mathbb{R}_+^{l\cdot n}$ and for any agent $i \in N$, consider the set

$$G_i(x, \omega) := \left\{ \varphi_i(z_{(i)}) \in \operatorname{Im} \varphi_i \colon z_{(i)} \in P_i(x) \right\} - \{\widehat{\omega}_i\}.$$

Notice that the set $G_i(x, \omega) \subseteq \mathbb{R}^{l\cdot|Q|}$ is nonempty by Point 2 of Assumption 16. Denote by $\operatorname{co}(\bigcup_{i\in N} G_i(x, \omega))$ the convex hull of $\bigcup_{i\in N} G_i(x, \omega)$.

We are now ready to prove the equivalence theorem.

Theorem 22 (Equivalence Theorem) *The set of information equilibrium allocations coincides with the generalized fuzzy Information-core.*

Proof. Take $x \in \mathscr{C}_h^I(E)$. We are going to show that $(x, (\pi, p))$ belongs to the set of Information equilibria $\Omega_I(E)$ for a system of prices (π, p). The proof of the claims below is contained in the Appendix.

Claim 1. For the allocation $x \in \mathscr{C}_h^I(E)$ then $0 \notin \operatorname{co}(\bigcup_{i\in N} G_i(x, \omega))$.

As consequence of Claim 1, applying the Separating Hyperplane Theorem, there exists a vector $\widehat{p} = (\widehat{p}^q) \in \mathbb{R}^{l \cdot |Q|}$, with $\widehat{p} \neq 0$, such that $\widehat{p} \cdot \zeta \geq 0$ for any $\zeta \in \mathrm{co}\left(\bigcup_{i \in N} G_i(x_i, \omega)\right)$. Define the vector $(\pi, p) = ((p_{(i)})_{i \in N}, p) \in \mathbb{R}^{l \cdot n^2} \times \mathbb{R}^l$ as follows,

$$p := \widehat{p}^{(0,0)} \text{ and } p_{(i)} = (p_{ih})_{h \in N} \quad \text{with} \quad p_{ih} := \begin{cases} \widehat{p}^{(i,h)} & \text{if } i \neq h \\ \widehat{p}^{(0,0)} - \sum_{k \neq i} \widehat{p}^{(k,h)} & \text{if } i = h \end{cases}$$

Notice that $\sum_{i \in N} p_{ih} = p$, for all $h \in N$. Indeed,

$$\sum_{i \in N} p_{ih} = p_{hh} + \sum_{i \neq h} p_{ih} = \widehat{p}^{(0,0)} - \sum_{i \neq h} \widehat{p}^{(i,h)} + \sum_{i \neq h} \widehat{p}^{(i,h)} = \widehat{p}^{(0,0)} = p.$$

We claim that the hyperplane associated to (π, p) satisfies the following important properties.

Claim 2. For any $i \in N$ and for any $z_{(i)} \in P_i(x)$, $p_{(i)} \cdot z_{(i)} \geq p \cdot \omega_i$ holds true and the vector (π, p), with $(\pi, p) \neq (0, 0)$, is nonnegative.

In particular, Claim 2 implies that $\pi > 0$ and $p > 0$. Indeed, we have already proved that (π, p) is non negative with $p \neq 0$. The fact that π is different from the null vector follows from $\sum_{i \in N} p_{ih} = p > 0$ for any $h \in N$. In particular, for any agent h there is an agent i such that $p_{ih} > 0$.

Claim 3. For the allocation $x \in \mathscr{C}_h^I(E)$ it is true that $p_{(i)} \cdot x = p \cdot \omega_i$ for any $i \in N$.

As consequence of Claim 3, we obtain that $x \in B_i(\pi, p, \omega)$. Moreover, we claim that

Claim 4. The vector p is strictly positive and $p_{(i)} \cdot z_{(i)} > p \cdot \omega_i$ for any $i \in N$ and for any $z_{(i)} \in P_i(x)$.

Hence, by Claim 4, $P_i(x) \cap B_i(\pi, p, \omega) = \emptyset$ holds true for any agent $i \in N$, and by the definition of (π, p), one has $\sum_{i \in N} p_{ih} = p$ for all $h \in N$. Finally, it remains to show that if $x \in \mathscr{C}_h^I(E)$ then market clearing condition is satisfied. Indeed, by Claim 3, $x \in \mathscr{C}_h^I(E)$ implies $\sum_{h \in N} p_{ih} \cdot x_h = p \cdot \omega_i$ for any $i \in N$. So summing with respect to i, inverting the sum operator, using the bilinearity property of the inner product and the feasibility condition for prices, we get

$$\sum_{i \in N} \sum_{h \in N} p_{ih} \cdot x_h = \sum_{h \in N} \sum_{i \in N} p_{ih} \cdot x_h = p \cdot \sum_{h \in N} x_h = p \cdot \sum_{h \in N} \omega_h$$

Since $p \gg 0$, the feasibility of x implies $\sum_{h \in N} x_h = \sum_{h \in N} \omega_h$. ∎

Remark 23 If preferences are convex, as a consequence of Theorem 8, one trivially obtains Theorem 3.2 in Vasil'ev (1996), that is $\mathscr{W}_I(E) = \mathscr{C}_f^I(E)$.

8 Economic models analyzed in literature - Part II

In this Section, we underline that a Walrasian equilibrium allocation for a pure exchange economy without externalities and a distributive Lindahl equilibrium for an economy with externalities are both particular cases of information equilibrium.

8.1 Pure exchange economy without externalities

If E is a pure exchange economy without externalities at all, then a competitive equilibrium allocation x is also an Information equilibrium allocation. Indeed, if p is the equilibrium price, then the corresponding personalized price of agent i is given by $p_{ih} := 0$ for any agent $h \neq i$, and $p_{ii} := p$. Under this setting, the budget set of agent i is $B_i(\pi, p, \omega) = \{x \in \mathbb{R}^{l \cdot n}_+ \colon p \cdot x_i \leq p \cdot \omega_i\}$ and the feasibility condition for the personalized prices is trivially satisfied. So, we end up in the model of a pure exchange economies without externalities, in which Information equilibrium allocations exist, and the validity of the equivalence theorem holds true.

8.2 Distributive Lindahl equilibrium

We consider a distributive Lindahl equilibrium as defined by Bergstrom (1970), and we show that it is a special case of an Information equilibrium [38]. Define the set of admissible Lindahl shares as the set Γ of $n \times n$ matrices given by $\Gamma := \{\gamma = (\gamma^i_h) \colon \gamma^i_h \geq 0, \sum_{i \in N} \gamma^i_h = 1, \forall\, h \in N\}$. For an agent i, one may interpret the vector $(\gamma^i_h)_{h \in N}$ as the contribution of agent i to the cost of consumption of all the other agents. The total contribution to consumption of one agent is normalized to 1. Given a price system $p \in \mathbb{R}^l_+$ and admissible Lindahl shares $\gamma \in \Gamma$, the budget set of agent i is $B_i(\gamma, p, \omega) := \{x \in \mathbb{R}^{l \cdot n}_+ \colon \sum_{h \in N} \gamma^i_h p \cdot x_h \leq p \cdot \omega_i\}$. Thus, $(x, \gamma, p) \in \mathbb{R}^{l \cdot n}_+ \times \Gamma \times \mathbb{R}^l_{++}$ is a distributive Lindahl equilibrium for the economy E if

1. $x \in \underset{x' \in \mathbb{R}^{l \cdot n}_+}{\arg\max} \{u_i(x') \mid \sum_{h \in N} \gamma^i_h p \cdot x'_h \leq p \cdot \omega_i\}$, for all $i \in N$;
2. $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$.

In a distributive Lindahl equilibrium, individual bundles can be considered as public goods and consequently they are consumed in the same amount by

[38] See Theorem 1 in Bergstrom (1970) for the existence of distributive Lindahl equilibria.

every agent. At the equilibrium, the Lindahl shares are personalized in such a way that the levels of these public goods are optimal choices for the agents. Notice that a distributive Lindahl equilibrium is an Information equilibrium. Indeed, suppose that (x, γ, p) is a distributive Lindahl equilibrium, then for every agent i, one may define a system of personalized price $\pi = (p_{(i)})_{i \in N}$ as $p_{ih} \coloneqq \gamma^i_h p$ for any $h \in N$. Condition 4 in Definition 17 is satisfied, since $\sum_{i \in N} p_{ih} = \sum_{i \in N} \gamma^i_h p = p$. Finally observe that the budget set of an agent in the distributive Lindahl equilibrium coincides with the one of the Information equilibrium with (π, p) system of prices, since $B_i(\gamma, p, \omega) = \{x \in \mathbb{R}^{l \cdot n}_+ \colon \sum_{h \in N} \gamma^i_h p \cdot x_h \leq p \cdot \omega_i\} = \{x \in \mathbb{R}^{l \cdot n}_+ \colon \sum_{h \in N} p_{ih} \cdot x_h \leq p \cdot \omega_i\} = B_i(\pi, p, \omega_i)$. Therefore, a distributive Lindahl equilibrium is an information equilibrium in which the personalized prices of agent i are all on the same direction, given by p.

9 Conclusions

We have shown that the equivalence theorem can be restored for two non-standard market models with externalities and non-convex preferences. For an equilibrium allocation to be a core allocation, an appropriate blocking procedure is needed. In particular, (i) an agent in the blocking coalition needs to be myopic with respect to the choices of all the other agents; (ii) an optimistic behavior of the blocking coalition with respect to the reactions of the outsiders is required. Vice-versa, to show that a core allocation belongs to the set of the equilibrium allocations, we use a standard approach based on Separation theorems. In order to overcome the difficulties arising by removing the convexity assumption, we adapt the idea of Husseinov (1994) to our frameworks by allowing agents to participate in more than one fuzzy coalitions simultaneously. In the A-economy, our main result is based on the assumption that the family of the exogenously given sets $A = (A_i)_{i \in N}$ is balanced in the sense of Bondereva with full support. This assumption is trivially satisfied for the model studied in the second part of the paper, that is a pure exchange economy with Arrowian markets for externalities. The proof of the equivalence theorem for this latter case requires the additional assumption of non spiteful agents. We have also shown that the two market models analyzed in the paper are sufficiently general to cover some well-known cases. Several important aspects of these two non-classical market models deserve to be investigated.[39] In the absence of externalities, non-convex preferences are easily accommodated in models of exchange economy with a continuum of agents. In our ongoing research, following the idea of Husseinov (1994) and Husseinov and Páscoa

[39] Under convexity, some new cooperative characterizations of A-equilibria in terms of the blocking power of the whole society have been already analyzed in the companion paper Graziano et al. (2025).

(1997), the fuzzy core will be related to the core of an appropriate economy with a continuum of agents. Although it is natural, this correspondence is not easy to construct due to the presence of externalities. Moreover, due to limited availability of resources and the impact on the economic environment of an inefficient allocation, the concept of resources-core and its characterization in terms of a measure of social loss deserve to be studied [40].

Appendix

We present below the proofs of the technical claims.

Proof of Theorem 10, Claim 1. Suppose $0 \in \operatorname{co}(\bigcup_{i\in N} F_i(x,\omega))$. Thus, for any $i \in N$ there exist vectors $z_{(i)}^{j(i)} \in X_{A_i}$ and scalars $\alpha_i^{j(i)} \geq 0$ with $j(i) = 1,\ldots,r_i$ such that $\sum_{i\in N}\sum_{j(i)=1}^{r_i} \alpha_i^{j(i)} = 1$ which meet $\sum_{i\in N}\sum_{j(i)=1}^{r_i} \alpha_i^{j(i)} \sum_{h\in A_i}(z_{ih}^{j(i)} - \omega_h) = 0$. By the Caratheodory theorem, we have $\sum_{i\in N} r_i \leq l+1$. Consider a $(l+1)\times n$ matrix M with element $m_i^j = \alpha_i^{j(i)}$ for $j \leq r_i$ and $m_i^j = 0$ for any $j > r_i$, and for any $i \in N$. Define for any $i \in N$ and for any $j = 1,\ldots,l+1$, vectors $\widetilde{z}_i^j$ and $\widetilde{\omega}_i$ of $\mathbb{R}_+^{l\cdot n}$, given by $\widetilde{z}_{ih}^j \coloneqq z_{ih}^{j(i)}$ and $\widetilde{\omega}_{ih} \coloneqq \omega_h$ if $h \in A_i$ and $j \leq r_i$, and equal to zero otherwise. Thus, $0 = \sum_{i\in N}\sum_{j(i)=1}^{r_i}\sum_{h\in A_i}\alpha_i^{j(i)}(z_{ih}^{j(i)} - \omega_h) = \sum_{h\in N}\sum_{j=1}^{l+1}\sum_{i\in N} m_i^j(\widetilde{z}_{ih}^j - \widetilde{\omega}_{ih}) = \sum_{h\in N}\sum_{j=1}^{l+1}\sum_{i\in D_h(m^j)} m_i^j(z_{ih}^j - \omega_h)$ implying that the $l+1$ coalitions $(m^j)_{j=1}^{l+1} = ((m_i^j)_{i\in N})_{j=1}^{l+1}$ block the allocation x. ■

Proof of Theorem 10, Claim 2. By strong monotonicity of the preferences over $X_{A_i} \times \{x_{N\setminus A_i}\}$, one gets $P_{A_i}(x) + X_{A_i} \subseteq P_{A_i}(x)$ for any agent i, which implies $F_i(x,\omega) + \mathbb{R}_+^l \subseteq F_i(x,\omega)$ [41]. From $F_i(x,\omega) \subseteq \operatorname{co}(\bigcup_{i\in N} F_i(x,\omega))$, it follows that $\operatorname{co}(\bigcup_{i\in N} F_i(x,\omega))$ contains a shifted nonnegative orthant. Therefore, we must have $p > 0$. Indeed, suppose $p^c < 0$ for some commodity c. Then, we can choose an element ζ of $\operatorname{co}(\bigcup_{i\in N} F_i(x_i,\omega))$ with ζ^c large enough, such that $p\cdot\zeta < 0$, obtaining a contradiction. To show that the allocation x lays on the budget hyperplane associated to p, first observe that since x_{A_i} belongs to $\operatorname{cl} P_{A_i}(x)$, there exists a sequence $(x_{(i)}^\nu)_{\nu\in\mathbb{N}} = ((x_{ih}^\nu)_{h\in A_i})_{\nu\in\mathbb{N}} \subseteq P_{A_i}(x)$ such that $x_{(i)}^\nu$ converges to x_{A_i}. By construction, for any agent i and any $\nu \in \mathbb{N}$, $\sum_{h\in A_i}(x_{ih}^\nu - \omega_h) \in F_i(x_i,\omega)$ which is included in $\operatorname{co}(\bigcup_{i\in N} F_i(x_i,\omega))$. So, by $x \in \mathscr{C}_h(E)$, it must be true that $p\cdot\sum_{h\in A_i} x_{ih}^\nu \geq p\cdot\sum_{h\in A_i}\omega_h$ for any agent

[40] We refer to Di Pietro et al. (2022) and Graziano and Platino (2024) for the notion of resource core and its characterization in terms of a measure of social loss, in presence of externalities.

[41] Indeed, let $y_i \in F_i(x,\omega)+\mathbb{R}_+^l$. So, there exists $v \in \mathbb{R}_+^l$ such that $y_i = \eta_i + v$ for some $\eta_i \in F_i(x,\omega)$. Since $\eta_i \in F_i(x,\omega)$, there exists $z_{(i)} = (z_{ih})_{h\in A_i} \in P_{A_i}(x)$ such that $\eta_i = \sum_{h\in A_i}(z_{ih} - \omega_h)$. Finally, notice that $y_i = (\sum_{h\in A_i} z_{ih} + v) - \sum_{h\in A_i}\omega_h$ belongs to $F_i(x,\omega)$ since $z_{(i)} + \widetilde{v} \in P_{A_i}(x) + X_{A_i} \subseteq P_{A_i}(x)$, with $\widetilde{v} = (v, 0\ldots,0) \in X_{A_i}$.

i and any $\nu \in \mathbb{N}$. Taking the limit, we get $p \cdot \sum_{h\in A_i} x_h \geq p \cdot \sum_{h\in A_i} \omega_h$ for each agent $i \in N$. Suppose by contradiction that there exists k such that $p \cdot \sum_{h\in A_k} x_h > p \cdot \sum_{h\in A_k} \omega_h$. By Point 2 of Assumption 3, the set $\mathscr{B}_A$ is non empty. So, take $\beta = (\beta_i)_{i\in N} \in \mathscr{B}_A$ and multiply each of the previous inequality by the corresponding weigh $\beta_i > 0$. Summing over $i \in N$ one gets $p \cdot \sum_{i\in N} \beta_i \sum_{h\in A_i} x_h > p \cdot \sum_{i\in N} \beta_i \sum_{h\in A_i} \omega_h$. By an argument similar to the one used in the proof of Theorem 9, the previous inequality can be written as $p \cdot \sum_{h\in N} \sum_{i\in D_h(\beta)} \beta_i x_h > p \cdot \sum_{h\in N} \sum_{i\in D_h(\beta)} \beta_i \omega_h$. Since $\text{supp}(\beta) = N$, then we get $p \cdot \sum_{h\in N} \sum_{i\in D_h} \beta_i (x_h - \omega_h) > 0$. Finally, $\sum_{i\in D_h} \beta_i = 1$, implies $p \cdot \sum_{h\in N} (x_h - \omega_h) > 0$, which contradicts the fact that x is an allocation, since $p > 0$. ∎

Proof of Theorem 10, Claim 3. We first show that there exists an agent k, such that $p \cdot \sum_{h\in A_k} x'_{kh} > p \cdot \sum_{h\in A_k} \omega_k$ for any $x'_{(k)} \in P_{A_k}(x)$, with $x \in \mathscr{C}_h(E)$. Since $p > 0$ and $\sum_{i\in N} \omega_i \gg 0$, then by Point 1 of Assumption 3, there exists k such that $p \cdot \sum_{h\in A_k} \omega_h > 0$. Consider this agent. By $F_k(x, \omega) \subseteq \text{co}\left(\bigcup_{i\in N} F_i(x, \omega)\right)$, one gets $p \cdot \sum_{h\in A_k} x'_{kh} \geq p \cdot \sum_{h\in A_k} \omega_h$ for any $x'_{(k)} \in P_{A_k}(x)$. Suppose that $p \cdot \sum_{h\in A_k} x''_{kh} = p \cdot \sum_{h\in A_k} \omega_k > 0$ for some $x''_{(k)} \in P_{A_k}(x)$. By continuity of the preference, there exists $V_\delta(x''_{(k)}) \coloneqq \{\xi_{(k)} \in X_{A_k} : (\xi_{(k)}, x_{N\setminus A_k}) \in N_\delta(x''_{(k)}, x_{N\setminus A_k}) \cap \mathbb{R}^{l\cdot n}_+\}$ included in $P_{A_k}(x)$, where $N_\delta(x''_{(k)}, x_{N\setminus A_k}) \subseteq \mathbb{R}^{l\cdot n}$ is an open ball centered at $(x''_{(k)}, x_{N\setminus A_k})$ with radius $\delta > 0$. Let $\varepsilon > 0$ such that $0 < (1-\varepsilon)\|x''_{(k)}\| < \delta$ and consider the vector $(\varepsilon x''_{(k)}, x_{N\setminus A_k})$. Thus, $\varepsilon x''_{(k)} \in V_\delta(x''_{(k)})$ and consequently $p \cdot \sum_{h\in A_k} \varepsilon x''_{kh} \geq p \cdot \sum_{h\in A_k} \omega_h$ since $\varepsilon x''_{(k)} \in P_{A_k}(x)$. By $p \cdot \sum_{h\in A_k} x''_{kh} = p \cdot \sum_{h\in A_k} \omega_k > 0$ and $\varepsilon < 1$, we get $\varepsilon p \cdot \sum_{h\in A_k} x''_{kh} < p \cdot \sum_{h\in A_k} x''_{kh}$.[42] So,

$$p \cdot \sum_{h\in A_k} \omega_k \leq \varepsilon p \cdot \sum_{h\in A_k} x''_{kh} < p \cdot \sum_{h\in k} x''_{kh} = p \cdot \sum_{h\in A_k} \omega_h$$

which is a contradiction.
Fix now an agent $\overline{h} \in A_k$ and a commodity $\overline{c}$. By strong monotonicity, $x_{A_k} + e_{A_k}(\overline{h}, \overline{c}) \in P_{A_k}(x)$, where $e_{A_k}(\overline{h}, \overline{c}) \coloneqq (e_h(\overline{h}, \overline{c}))_{h\in A_k} = ((e^c_h(\overline{h}, \overline{c}))^l_{c=1})_{h\in A_k}$ is a vector in $\mathbb{R}^{l\cdot|A_k|}$ with $e^c_h(\overline{h}, \overline{c}) = 0$ for any $c \neq \overline{c}$ and for any $h \neq \overline{h}$, and $e^{\overline{c}}_{\overline{h}}(\overline{h}, \overline{c}) = 1$. So $p \cdot \sum_{h\in A_k} (x_h + e_h(\overline{h}, \overline{c})) > p \cdot \sum_{h\in A_k} \omega_h$ by the previous claim. Since $x \in \mathscr{C}_h(E)$, by Claim 2 and the bilinearity property of the inner product, one gets $p \cdot \sum_{h\in A_k} e_h(\overline{h}, \overline{c}) > 0$ which implies, by the definition of $e_{A_k}(\overline{h}, \overline{c})$, that $p^{\overline{c}} > 0$. Repeating the same argument for any commodity c, one obtains, $p \gg 0$.

To show the second part of Claim 3, suppose by contradiction that for $\zeta \in \text{co}\left(\bigcup_{i\in N} F_i(x, \omega)\right)$, $p \cdot \zeta = 0$ holds true. Then $\zeta = \sum_{j=1}^r \alpha^j \zeta^j$ for some scalars $\alpha^j > 0$ with $j = 1, \ldots, r$, $r \in \mathbb{N}$ and $r \neq 0$, such that $\sum_{j=1}^r \alpha^j = 1$ and $\zeta^j \in \bigcup_{i\in N} F_i(x, \omega) \subseteq \text{co}\left(\bigcup_{i\in N} F_i(x, \omega)\right)$ for any $j = 1, \ldots, r$. Therefore, $p \cdot$

[42] Observe that $\varepsilon < 1$, since $\|x''_{(k)}\| \neq 0$ by $x''_{(k)} \in P_{A_k}(x)$ and by Point 2 of Assumption 4.

$\zeta^j = 0$ for any $j = 1, \ldots, r$, otherwise one gets a contradiction with $0 = p \cdot \zeta = \sum_{j=1}^{r} \alpha^j p \cdot \zeta^j$, since $\alpha^j > 0$ for any j [43]. Consider the agent $i(j)$ such that $\zeta^j \in F_{i(j)}(x, \omega)$. We claim that $P_{A_{i(j)}}(x) \subseteq X_{A_{i(j)}} \setminus \{0\}$. By Point 2 of Assumption 4, $(x_{A_{i(j)}}, x_{N \setminus A_{i(j)}}) \succsim_{i(j)} (0, x_{N \setminus A_{i(j)}})$, therefore, by transitivity, $P_{A_{i(j)}}(x) \subseteq P_{A_{i(j)}}(0, x_{N \setminus A_{i(j)}})$. Since the binary relation $\succ_{i(j)}$ is irreflexive, then $0 \notin P_{A_{i(j)}}(0, x_{N \setminus A_{i(j)}})$ and consequently, $P_{A_{i(j)}}(x) \subseteq X_{A_{i(j)}} \setminus \{0\}$, which completes the proof of the claim. So, there exists $x'_{(i(j))} > 0$ such that: (i) $(x'_{(i(j))}, x_{N \setminus A_{i(j)}}) \succ_{i(j)} (x_{A_{i(j)}}, x_{N \setminus A_{i(j)}})$, (ii) $\zeta^j = \sum_{h \in A_{i(j)}} (x'_{i(j)h} - \omega_h)$ and (iii) $p \cdot \zeta_{i(j)} = 0$. By continuity of the preference relation and (i), there exists $\widetilde{x}_{(i(j))} < x'_{A_{i(j)}}$ such that $\sum_{h \in A_{i(j)}} (\widetilde{x}_{i(j)h} - \omega_h) \in F_{i(j)}(x, \omega)$. Furthermore, by $\widetilde{x}_{(i(j))} < x'_{(i(j))}$, (ii), (iii) and $p \gg 0$, we get $p \cdot \sum_{h \in A_{i(j)}} (\widetilde{x}_{i(j)h} - \omega_h) < 0$. This is a contradiction, since by $x \in \mathscr{C}_h(E)$ and $\sum_{h \in A_{i(j)}} (\widetilde{x}_{i(j)h} - \omega_h) \in \operatorname{co}\left(\bigcup_{i \in N} F_i(x_i, \omega)\right)$ one should have $p \cdot \sum_{h \in A_{i(j)}} (\widetilde{x}_{i(j)h} - \omega_h) \geq 0$. ■

Proof of Theorem 22, Claim 1. By contradiction, suppose $0 \in \operatorname{co}\left(\bigcup_{i \in N} G_i(x, \omega)\right)$. Therefore, for any $i \in N$, there exist vectors $z_{(i)}^{j(i)} \in P_i(x) \subseteq \mathbb{R}_+^{1 \cdot n}$ and scalars $\alpha_i^{j(i)} \geq 0$ with $j(i) = 1, \ldots, r_i$ such that $\sum_{i \in N} \sum_{j(i)=1}^{r_i} \alpha_i^{j(i)} = 1$ which meets $\sum_{i \in N} \sum_{j(i)=1}^{r_i} \alpha_i^{j(i)} (\varphi_i(z_{(i)}^{j(i)}) - \widehat{\omega}_i) = 0$. By the Caratheodory theorem, we have $\sum_{i \in N} r_i \leq l + 1$. Consider a $(l+1) \times n$ matrix M with element $m_i^j = \alpha_i^{j(i)}$ for $j \leq r_i$ and $m_i^j = 0$ for any $j > r_i$, and for any $i \in N$. Thus $0 = \sum_{i \in N} \sum_{j(i)=1}^{r_i} \alpha_i^{j(i)} (\varphi_i(z_{(i)}^{j(i)}) - \widehat{\omega}_i) = \sum_{i \in N} \sum_{j=1}^{l+1} m_i^j (\varphi_i(z_{(i)}^j) - \widehat{\omega}_i)$. By the definition of φ_i and $\widehat{\omega}_i$, for any agent i and for $q = (0, 0)$, we have

$$0 = \sum_{i \in N} \sum_{j=1}^{l+1} m_i^j (\varphi_i^{(0,0)}(z_{(i)}^j) - \widehat{\omega}_i^{(0,0)}) = \sum_{i \in N} \sum_{j=1}^{l+1} m_i^j (z_{ii}^j - \omega_i)$$

Defining $\xi_i^j \coloneqq z_{ii}^j$ for any agent i and any j, we get

$$\sum_{i \in N} \sum_{j=1}^{l+1} m_i^j \xi_i^j \leq \sum_{i \in N} m_i(l+1) \omega_i$$

where $m_i(l+1) = \sum_{j=1}^{l+1} m_i^j$, according to our notation. Furthermore, for any $q \in Q$, with $q \neq (0, 0)$, we obtain $\sum_{j=1}^{l+1} m_h^j z_{hh}^j = \sum_{j=1}^{l+1} m_i^j z_{ih}^j$ for any $i, h \in$

[43] Since $x \in \mathscr{C}_h(E)$ and $\{\zeta^j : j = 1, \ldots, r\} \subseteq \operatorname{co}\left(\bigcup_{i \in N} F_i(x, \omega)\right)$, then $p \cdot \zeta^j \geq 0$ for any $j = 1, \ldots, r$.

$\text{supp}(m) \subseteq N$.[44] By $\xi_h^j = z_{hh}^j$ for any agent h, one gets

$$\sum_{j=1}^{l+1} m_i^j z_{(i)}^j = \left(\sum_{j=1}^{l+1} m_h^j \xi_h^j\right)_{h\in N} \quad \forall i \in \text{supp}(m)$$

implying that the coalitions $(m^j)_{j=1}^{l+1} = ((m_i^j)_{i\in N})_{j=1}^{l+1}$ block the allocation x. ■

Proof of Theorem 22, Claim 2. By $G_i(x,\omega) \subseteq \text{co}\left(\bigcup_{i\in N} G_i(x,\omega)\right)$, one gets $\widehat{p}\cdot\varphi(z_{(i)}) > \widehat{p}\cdot\widehat{\omega}$ for any $z_{(i)} \in P_i(x)$. By the definition of $\varphi(\cdot)$,

$$\begin{aligned}\sum_{q\in Q} \widehat{p}^q \cdot \varphi^q(z_{(i)}) &= \widehat{p}^{(0,0)} \cdot z_{ii} + \sum_{h\neq i} \widehat{p}^{(h,i)} \cdot (-z_{ii}) + \sum_{h\neq i} \widehat{p}^{(i,h)} \cdot (z_{ih}) + \sum_{h\neq i, k\neq i} \widehat{p}^{(h,k)} \cdot 0 = \\ &= \left(\widehat{p}^{(0,0)} - \sum_{h\neq i} \widehat{p}^{(h,i)}\right) \cdot z_{ii} + \sum_{h\neq i} \widehat{p}^{(i,h)} \cdot (z_{ih}) \\ &\geq \sum_{q\in Q} \widehat{p}^q \cdot \widehat{\omega}_i^q = \widehat{p}^{(0,0)} \cdot \omega_i + \sum_{q\neq 0} \widehat{p}^q \cdot 0 = \widehat{p}^{(0,0)} \cdot \omega_i\end{aligned}$$

Finally, by the definition of (π, p), we get the first part of the Claim.

Observe now that by Point (ii) of Assumption 16, the preference relation $\succsim_i$ is non-decreasing over $\mathbb{R}_+^{l\cdot n}$. Thus, one gets $P_i(x) + \mathbb{R}_+^{l\cdot n} \subseteq P_i(x)$ for any agent i, which implies that $\pi = (p_{(i)})_{i\in N} \geq 0$. Indeed, suppose that there exists an agent i such that $p_{ih}^c < 0$ for some individual h and a commodity c, then, since ω_i^c is exogenously given, one may choose an element $z_{(i)} \in P_i(x)$ with z_{ih}^c large enough, such that $p_{(i)} \cdot z_{(i)} < p \cdot \omega_i$, and obtain a contradiction. Moreover, by $\sum_{i\in N} p_{ih} = p$, we get $p \geq 0$, which implies that (π, p) is nonnegative. It remains to show that (π, p) is different from the null vector. It follows from the fact that p must be different from the null vector. Indeed, if $p = \widehat{p}^{(0,0)} = 0$, then $\sum_{i\in N} p_{ih} = p = 0$ for any $h \in N$, and $p_{(i)} \geq 0$, imply that $p_{(i)} = 0$ for any agent i. This implies $\widehat{p} = 0$ which is a contradiction. ■

Proof of Theorem 22, Claim 3. As a consequence of 22, Claim 2, one obtains $p_{(i)} \cdot x \geq p \cdot \omega_i$ for any agent $i \in N$. Indeed, since $x \in \text{cl}\, P_i(x)$, we may find a sequence $(x_{(i)}^\nu) \subseteq P_i(x)$ converging to x. Again by Claim 2, we obtain $p_{(i)} \cdot x_{(i)}^\nu \geq p \cdot \omega_i$ for any $\nu \in \mathbb{N}$. So, taking the limit we get $p_{(i)} \cdot x \geq p \cdot \omega_i$. Suppose that there exists an agent h such that $p_{(h)} \cdot x > p \cdot \omega_h$. Summing over h, we derive $\sum_{h\in N} p_{(h)} \cdot x > p \cdot \sum_{h\in N} \omega_h$. By $p = \sum_{h\in N} p_{hi}$ for any agent i, the previous

[44] Indeed, for any $k, h \in N$ with $k \neq h$, we have $0 = \sum_{i\in N} \sum_{j=1}^{l+1} m_i^j (\varphi_i^{(k,h)}(z_{(i)}^j) - \widehat{\omega}_i^{(k,h)}) = \sum_{i\in N} \sum_{j=1}^{l+1} m_i^j \varphi_i^{(k,h)}(z_{(i)}^j)$, where the second equality follows by $\widehat{\omega}_i^q = 0$ for $q \neq (0,0)$. Developing the sum, and noting that $\varphi_i^{(k,h)}(z_{(i)}^j) = 0$ for any $i \notin \{k, h\}$, we get $\sum_{j=1}^{l+1} m_h^j \varphi_h^{(k,h)}(z_{(h)}^j + \sum_{j=1}^{l+1} m_k^j \varphi_k^{(k,h)}(z_{(k)}^j) = 0$. Finally, by definition of φ_h and φ_k we obtain $\sum_{j=1}^{l+1} m_h^j(-z_{hh}^j) + \sum_{j=1}^{l+1} m_k^j z_{kh}^j = 0$. So, $\sum_{j=1}^{l+1} m_h^j z_{hh}^j = \sum_{j=1}^{l+1} m_k^j z_{kh}^j$ for any $k, h \in \text{supp}(m)$.

inequality can be written as $\sum_{h\in N}(\sum_{i\in N} p_{hi} \cdot x_i) = \sum_{i\in N}(\sum_{h\in N} p_{hi}) \cdot x_i = p \cdot (\sum_{i\in N} x_i) > p \cdot \sum_{h\in N} \omega_h$, which is equivalent to $p \cdot \sum_{i\in N}(x_i - \omega_i) > 0$. This is a contradiction since $x \in \mathscr{F}$ and $p > 0$. ■

Proof of Theorem 22, Claim 4. By $p > 0$ and $\sum_{i\in N} \omega_i \gg 0$, there exists an agent h such that $p \cdot \omega_h > 0$. We first claim that for this agent h, it must be the case that $p_{(h)} \cdot z_{(h)} > p \cdot \omega_h$ for any $z_{(h)} \in P_h(x)$, with $x \in \mathscr{C}^I_h(E)$. By previous Claim 2 we know that $p_{(h)} \cdot z_{(h)} \geq p \cdot \omega_h$ for any $z_{(h)} \in P_h(x)$. Suppose that $p_{(h)} \cdot z'_{(h)} = p \cdot \omega_h > 0$ for some $z'_{(h)} \in P_{(h)}(x)$. By continuity of preferences, there exists $V_\delta(z'_{(h)}) \coloneqq N_\delta(z'_{(h)}) \cap \mathbb{R}^{l\cdot n}_+$ included in $P_h(x)$, where $N_\delta(z'_{(h)}) \subseteq \mathbb{R}^{l\cdot n}$ is an open ball centered at $z'_{(h)}$ with radius $\delta > 0$. Let $\varepsilon > 0$ such that $0 < (1-\varepsilon)\|z'_{(h)}\| < \delta$ and consider the vector $\varepsilon z'_{(h)}$ [45]. Thus, $\varepsilon z'_{(h)} \in V_\delta(z'_{(h)})$ and consequently $p_{(h)} \cdot \varepsilon z'_{(h)} \geq p \cdot \omega_h$ since $\varepsilon z'_{(h)} \in P_h(x)$. By $p_{(h)} \cdot z'_{(h)} = p \cdot \omega_h > 0$ and $\varepsilon < 1$, we get $\varepsilon p_{(h)} \cdot z'_{(h)} < p_{(h)} \cdot z' = p \cdot \omega_h$. Therefore,

$$p \cdot \omega_h \leq \varepsilon p_{(h)} \cdot z'_{(h)} < p_{(h)} \cdot z'_{(h)} = p \cdot \omega_h$$

which is a contradiction. This completes the proof of the claim. We now claim that the vector p_{hh} is strictly positive. Fix a commodity c. By strong monotonicity of the preference of agent h with respect to her own consumption, $x + e(h,c) \in P_h(x)$, where $e(h,c) \coloneqq (e_i(h,c))_{i\in N} = ((e^s_i(h,c))^l_{s=1})_{i\in N}$ is a vector in $\mathbb{R}^{l\cdot n}$ with $e^s_i(h,c) = 0$ for any $s \neq c$ and for any $i \neq h$, $e^c_h(h,c) = 1$. So $p_{(h)} \cdot (x + e(h,c)) > p \cdot \omega_h$ by the previous claim. Since $x \in \mathscr{C}^I_h(E)$, by the previous Claim 3 and the bilinearity property of the inner product, one gets $p_{(h)} \cdot e(h,c) > 0$ which implies, by definition of $e(h,c)$, that $p^c_{hh} > 0$. Repeating the same argument for any commodity, one obtains, $p_{hh} \gg 0$, which proves the claim. Finally, by $p = \sum_{i\in N} p_{ih}$ one gets $p \gg 0$.

By $p_{(i)} \cdot z_{(i)} \geq p \cdot \omega_i$, $p \gg 0$ and $\omega_i > 0$, we get $p_{(i)} \cdot z_{(i)} > 0$ for any $z_{(i)} \in P_i(x)$. By continuity of preferences, one my find a vector $z'_{(i)} \in P_i(x)$ such that $p_{(i)} \cdot z_i > p_{(i)} \cdot z'_{(i)}$ [46]. Since $z'_{(i)} \in P_i(x)$, by Claim 2, we must have $p_{(i)} \cdot z_{(i)} > p_{(i)} \cdot z'_{(i)} \geq p \cdot \omega_i$, which completely proves the statement. ■

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[45] Notice that, since $\succ_i$ is not reflexive, then $z_{(i)} \neq x$. Thus, $z_{(i)} \in P_i(x)$ and the non-decreasing assumption implies that $z_{(i)}$ is different from the null vector. Therefore, $\|z'_{(h)}\| > 0$, $\varepsilon < 1$ and so, $\varepsilon z'_{(h)} < z'_{(h)}$.

[46] As in the proof of the first part, we can define $z'_{(i)} \coloneqq \varepsilon z_{(i)}$, with $\varepsilon > 0$ such that $0 < (1-\varepsilon)\|z_{(i)}\| < \delta$. Thus, $z'_{(i)} \in V_\delta(z_{(i)}) \coloneqq N_\delta(z_{(i)}) \cap \mathbb{R}^{l\cdot}_+ \subseteq P_i(x)$, where $N_\delta(z_{(i)})$ is an appropriate open ball in $\mathbb{R}^{l\cdot n}$.

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