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Instability of Factor Strength in Asset Returns

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Abstract

We study the problem of detecting structural instability of factor strength in asset pricing models for financial returns with observable factors. We allow for strong and weaker factors, in which the sum of squared betas grows at a rate equal to and slower than the number of test assets, respectively: this growth rate determines the strength of the corresponding factor. We propose LM and Wald statistics for the null hypothesis of stability and derive their asymptotic distribution when the break fraction is known, as well as when it is unknown and has to be estimated. We corroborate our theoretical results through a comprehensive series of Monte Carlo experiments. An extensive empirical analysis uncovers the dynamics of instability of factor strength in financial returns from equity portfolios.

JEL Classification: C12, C33, C58, G10, G12.

Keywords: Factor strength, structural break, hypothesis testing, stock portfolios.

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1 Introduction

Financial asset returns exhibit a factor structure, as a handful of common factors drives their cross-sectional dependence.[1] This empirical evidence has generated a large number of contributions on factor models in asset pricing: see Giglio et al. (2021) for an overview of the literature. In estimating asset pricing models, it has been common to assume that all factors are strong, meaning that they are pervasive and influence almost all securities: see Fama and MacBeth (1973), and Shanken (1992). The assumption of strong factor structure may be restrictive in practice, as some of the factors may not be strong and do not actually drive the cross-section of all securities: Kan and Zhang (1999), Kleibergen (2009), Bryzgalova (2016), Burnside (2016), Gospodinov et al. (2017), and Anatolyev and Mikusheva (2021), study this scenario when factors are known and observable; in the spirit of Connor and Korajczyk (1986), Lettau and Pelger (2020), Bai and Ng (2021), Freyaldenhoven (2021), Giglio et al. (2021), and Uematsu and Yamagata (2023a,b) consider specifications in which all factors are latent and estimated.

We focus on observable factors. We follow Chudik et al. (2011) and define the strength of a factor based on how the sum of squared betas grows with the number of test assets N. We classify a factor as being strong, semi-strong or weak, depending on whether the sum of squared betas grows at a rate equal to N, between $N^{1/2}$ (excluded) and N (excluded), or less than or equal to $N^{1/2}$, respectively. Bailey et al. (2021), Connor and Korajczyk (2022), and Pesaran and Smith (2021a,b), employ the same classification scheme. Bailey et al. (2021) develops an estimator for the factor strength that is based on the fraction of statistically significant betas and takes into account the associated multiple testing problem. Pesaran and Smith (2021b) show that the convergence rate of the Fama and MacBeth (1973) two-pass estimator depends on pricing errors and factors strength, and thus an estimation of the latter is required.

To the very best of our knowledge, existing studies that allow for semi-strong and weak factors assume that the factor strength is stable over the estimation period. This assumption may not be supported by the data. Bailey et al. (2021), and Pesaran and Smith (2021b), document time-variation in factor strength in large cross-sections of equity returns over rolling estimation windows. Based on the results in Pesaran and Smith (2021b), detecting breaks in factor strength

[1]For example, see Litterman and Scheinkman (1991), Fama and French (1993), and Lustig et al. (2011), in relation to U.S. government bonds, equity returns, and exchange rates, respectively

is important as these breaks may affect the convergence rate of two-pass estimators.

This paper fills a gap in the literature by addressing the problem of instability of factor strength in asset pricing models. It introduces a general testing strategy for the null hypothesis of strength stability. We build LM and Wald-type test statistics based on the difference between the estimator for the factor strength before and after the break. They differ in their variance being estimated under the null and under the alternative, respectively. We derive their asymptotic distribution under the null and show that it is normal. Under the alternative, both statistics asymptotically diverge. Our results are corroborated by an extensive set of Monte Carlo simulations, which shows the good performance of our tests in finite samples.

We stress that we focus on instability in the factor *strength* and not in the factor *betas*. Strength instability can only occur if the corresponding betas experience a break, and betas instability is a *necessary* condition for strength instability. This has implications for deriving the asymptotic distribution of our test statistics under the null. In particular, our proposed test statistics do not suffer from the problem of a nuisance parameter being identified only under the alternative: see Davies (1977, 1987). Stability of factor strength is tested after a break in the betas is detected and the break fraction is identified both under the null and the alternative.

Finally, we illustrate the usefulness of our procedure for empirical work through an analysis of equity portfolios.[2] We consider a large set of 739 portfolios from Chen and Zimmermann (2021). We set up a factor model for the cross-sectional variation of returns and apply our testing procedure using rolling estimation windows of suitable length. Our results shed light on the dynamics of local instability of factor strength over time for the set of test assets and the factor model specification we consider. From an asset pricing perspective, they imply that stability of factor strength may not be a realistic assumption for empirical purposes. Strength instability should be accounted for when running inference on risk premia in order to avoid potentially misleading inferential results.

The rest of the paper is organized as follows. Section 2 sets up the problem. Section 3 introduces the tests. Section 4 runs a set of Monte Carlo experiments. Section 5 performs the empirical analysis. Section 6 concludes. Mathematical proofs are provided in Appendix A.

Notation: $\mathbb{I}(\cdot)$ denotes the indicator function; $\lfloor \cdot \rfloor$ is the integer part of the argument; given

[2]The data used in the empirical analysis are described in details in Section 5.1

a positive integer A, $\boldsymbol{\iota}_A$ is the $A \times 1$ vector of ones; $|\cdot|$ is the absolute value of the argument; $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, and $\Phi^{-1}(\cdot)$ is its inverse; $\stackrel{d}{\to}$ denotes convergence in distribution; $\text{vec}(\mathbf{A})$ denotes the vectorization of the matrix $\mathbf{A}$; the norm of a generic matrix $\mathbf{A}$ is $\|\mathbf{A}\| = [\text{tr}(\mathbf{A}'\mathbf{A})]^{1/2}$, where $\text{tr}(\mathbf{B})$ denotes the trace of a square matrix $\mathbf{B}$; $\stackrel{a.s.}{\to}$ denotes almost sure convergence.

2 Set up

2.1 Econometric model

We assume that asset (excess) returns are generated according to

$$R_{it} = \mathbb{I}(t/T \leq \tau)(\alpha_{1i} + \boldsymbol{\beta}_{1i}'\mathbf{f}_t) + \mathbb{I}(t/T > \tau)(\alpha_{2i} + \boldsymbol{\beta}_{2i}'\mathbf{f}_t) + e_{it}, \tag{1}$$

for $i = 1, \ldots, N$, and $t = 1, \ldots, T$, where N is the total number of assets, and T is the time series dimension: R_{it} is the return on asset i at time t; $0 < \tau < 1$ is the break fraction, which can be either known or unknown; α_{ji} is the asset-specific intercept, for $j = 1, 2$; $\boldsymbol{\beta}_{ji} = (\beta_{ji1}, \ldots, \beta_{jiK})'$ is the $K \times 1$ vector of regression betas, for $j = 1, 2$; $\mathbf{f}_t = (f_{1t}, \ldots, f_{Kt})'$ is the $K \times 1$ vector of observable traded factors; e_{it} is the idiosyncratic component for return i at time t.[3] We further assume that the cross-sectional dispersion of regression betas evolves according to

$$\begin{array}{ll} \beta_{jik} \neq 0, & i = 1, \ldots, \lfloor N^{\lambda_{jk}} \rfloor, \\ \beta_{jik} = 0, & i = \lfloor N^{\lambda_{jk}} \rfloor + 1, \ldots, N, \end{array} \quad , \quad 0 \leq \lambda_{jk} \leq 1, \quad j = 1, 2, \quad k = 1, \ldots, K, \tag{2}$$

where the ordering of the betas is for ease of exposition only and it is not required for the validity of our results, as it becomes clear in the condition in (4) below.

We are interested in the null hypothesis $\mathcal{H}_{0k}$ against the alternative $\mathcal{H}_{1k}$ defined as

$$\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}, \qquad \mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}, \qquad k \in \{1, \ldots, K\} : \tag{3}$$

[3] We focus on the case in which the factors in $\mathbf{f}_t$ are all traded. If some of the factors in $\mathbf{f}_t$ are not returns themselves, following Breeden (1979) we conjecture that our results can be extended using a "mimicking-portfolio" approach. A similar idea underlies the model comparison tests of Barillas et al. (2020). We aim at formalising this interesting extension in future work.

for any k such that $k \in \{1, \ldots, K\}$, λ_{1k} is equal to λ_{2k} under the null hypothesis, whereas λ_{1k} and λ_{2k} are different from each other under the alternative hypothesis. From (3), we can see that our framework is analogous to Bai and Perron (1998), and Qu and Perron (2007), in that we model a break as a discrete change in the parameters of interest.

From an econometric perspective, (1) describes a factor model subject to structural instability occurring at the break fraction τ. The evolution of regression betas in (2) determines the strength of the factors before and after the break. In particular, the strength of the $k-th$ factor within regime j is determined by the parameter λ_{jk}, for $j = 1, 2$, and $k = 1, \ldots, K$. Following Chudik et al. (2011), and Pesaran and Smith (2021a,b), we classify the $k-th$ factor within regime j as strong, semi-strong, and weak, depending on whether $\lambda_{jk} = 1$, $0.5 < \lambda_{jk} < 1$, and $0 \leq \lambda_{jk} \leq 0.5$, respectively. Connor and Korajczyk (2022) use a similar classification. The role played by the factor strength within our testing procedure is discussed in details in Section 3.2.1. Finally, the condition on the cross-sectional dispersion of the betas in (2) may be written more generally as

$$N^{-\lambda_{jk}} \sum_{i=1}^{N} \beta_{jk}^2 \rightarrow C_{jk}, \quad 0 < C_{jk} < \infty, \quad j = 1, 2, \quad k = 1, \ldots, K, \tag{4}$$

as $N \rightarrow \infty$, which states that the sum of squared betas for factor k within regime j grows at rate $N^{\lambda_{jk}}$: this extends the analogous condition given for linear asset pricing models in Pesaran and Smith (2021a,b) and employed in Connor and Korajczyk (2022).

2.2 Interpretation of instability in factor strength

The null and alternative hypotheses $\mathcal{H}_{0k}$ and $\mathcal{H}_{1k}$, respectively, in (3) deserve further considerations. In particular, they do not refer to the regression betas in (1), but to the parameters λ_{jk} that govern the strength of the factors within each regime. In other words, the null and the alternative hypotheses in (3) relate to the stability of the factor strength and not to the stability of the regression betas. The two concepts are distinct although related. In a system of equations with observable factors such as (1), the stability of the regression betas may be assessed through the procedure developed in Qu and Perron (2007) for systems of equations, which suitably extends the seminal work by Bai and Perron (1998) for single equation models. Clearly, instability in the betas is a *necessary* condition for instability in the factor strength. Therefore, a break in the factor strength can occur only conditional upon a break in the factor betas: we explore this intuition in Section 3.3, where we let the break fraction τ be unknown. On the other hand, instability in the factor strength is a *sufficient* condition for instability in the factor betas.

Structural instability in the betas in (1) relates our set up to latent factor models with structural breaks and, more generally, with discrete shifts in the loadings: see Barigozzi and Massacci (2022), and Massacci (2017, 2023), and references therein. To the very best of our knowledge, this literature has mainly worked under the maintained assumption that all latent factors are strong. We explicitly focus on the model in (1) with observable factors.

Finally, the model in (1) has one break fraction. We handle the case of multiple breaks in two ways. We let T be the whole time series dimension and estimate the multiple break fractions using a sequential algorithm, as outlined in Section 3.4 below. Alternatively, (1) may be seen as a local model that applies to a window of length T strictly shorter than the whole available time series. This second approach allows to test the null hypothesis of *local stability*. The notion of local (as opposed to global) stability is not new. For example, in an out-of-sample framework, Giacomini and Rossi (2010) develop a measure of local relative forecasting performance between two competing predictive models, and assess the stability of this measure through a suitable inferential procedure. As further discussed in Section 5.2, inference on local stability is consistent with existing studies, which document a high degree of time-variation in the factor strength by using rolling window estimation strategies: see Bailey et al. (2021), and Pesaran and Smith (2021a).

2.3 Asset pricing implications

Define $\mathbf{R}_t = (R_{1t}, \ldots, R_{Nt})'$, $\boldsymbol{\alpha}_j = (\alpha_{j1}, \ldots, \alpha_{jN})'$, $\mathbf{B}_j = (\boldsymbol{\beta}_{j1}, \ldots, \boldsymbol{\beta}_{jN})'$, and $\mathbf{e}_t = (e_{1t}, \ldots, e_{Nt})'$, for $j = 1, 2$. The model in (1) can then be written as

$$\mathbf{R}_t = \mathbb{I}\left(t/T \leq \tau\right)\left(\boldsymbol{\alpha}_1 + \mathbf{B}_1 \mathbf{f}_t\right) + \mathbb{I}\left(t/T > \tau\right)\left(\boldsymbol{\alpha}_2 + \mathbf{B}_2 \mathbf{f}_t\right) + \mathbf{e}_t. \quad (5)$$

Let $\boldsymbol{\Gamma}_j = \left(\gamma_{j0}, \boldsymbol{\gamma}_{j1}'\right)'$, where γ_{j0} is the zero-beta rate, and $\boldsymbol{\gamma}_{j1}$ is the $K \times 1$ vector of factor risk premia, for $j = 1, 2$. Define as $\mathbf{X}_j = (\boldsymbol{\iota}_N, \mathbf{B}_j)$ the $N \times (K+1)$ beta matrix augmented by the $N \times 1$ vector of ones $\boldsymbol{\iota}_N$.

Consider first the case when the number of test assets N is finite. Since $\mathbf{f}_t$ is a vector of traded factors, then $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are vectors of pricing errors. Therefore, under the assumption of exact pricing (correct model specification), it holds that $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = \mathbf{0}$. In this case, the vector of asset expected returns $\boldsymbol{\mu}_t$ is state-dependent and defined as

$$\boldsymbol{\mu}_t = \mathrm{E}\left(\mathbf{R}_t\right) = \mathbb{I}\left(t/T \leq \tau\right)\mathbf{X}_1\boldsymbol{\Gamma}_1 + \mathbb{I}\left(t/T > \tau\right)\mathbf{X}_2\boldsymbol{\Gamma}_2.$$

Under correct model specification, the model in (1) allows for structural instability in the quantity and in the price of risk, as measured by $\mathbf{X}_j$ and $\boldsymbol{\Gamma}_j$, respectively, for $j = 1, 2$.

In linear asset pricing models, Pesaran and Smith (2021b) show that it is still possible to estimate the risk premia in the presence of non-zero pricing errors as $N \to \infty$. For a given factor, Pesaran and Smith (2021b) show that the rate of convergence as $N \to \infty$ of the Fama and MacBeth (1973) two-pass estimator for the risk premium monotonically increases and decreases in the strength of the factor and of the pricing errors, respectively. The estimator for the risk premia is consistent if the strength of the factor is greater than the strength of the pricing errors, and the convergence rate is slower the smaller the difference between the two.

The result shown for linear asset pricing models in Pesaran and Smith (2021b) holds within each regime of the piecewise linear model in (1) in relation to the risk premia $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$. This is true regardless of whether the break fraction τ is known, or it is unknown and has to be estimated. Following Qu and Perron (2007), and as also discussed in details in Section 3.3, this is because the convergence rate of the least squares estimator for τ is faster than that of the remaining set of parameters in (1). Following Pesaran and Smith (2021b), this implies that the convergence rate of the Fama and MacBeth (1973) two-pass estimator for the risk premium of the $k-th$ factor within regime j is $N^{-(\lambda_{jk}-\lambda_{\alpha j})/2}$, where $\lambda_{\alpha j}$ regulates the strength of the pricing errors in regime j. Formally, $\lambda_{\alpha j}$ satisfies

$$N^{-\lambda_{\alpha j}} \sum_{i=1}^{N} \alpha_{ji}^2 \to C_{\alpha j}, \quad 0 < C_{\alpha j} < \infty, \quad j = 1, 2,$$

which means that, within regime j, the sum of squared pricing errors grows at rate $N^{\lambda_{\alpha j}}$ as $N \to \infty$. Therefore, provided that the strength of the pricing errors is stable over time, testing for stability in factor strength gives valuable information about the stability of the convergence rate of the Fama and MacBeth (1973) two-pass estimator. More generally, inference on the stability of factor strengths is informative to conduct inference on the stability of risk premia. This is an important question in asset pricing.

Existing contributions have studied whether cross-sectional risk premia are stable over time. Fama and MacBeth (2021) asses the stability of the value premium by splitting the sample between the period July 1963 – June 1991 and the period July 1991 – June 2019. This is analogous to considering a model like (1) with a known value of the break fraction τ, which corresponds to a break occurring in June 1991. In a Bayesian setting, Smith and Timmermann (2021) study the more general problem of stability in risk premia by allowing for multiple unknown breaks in the data generating process of asset returns. This setting in analogous to the one discussed in Section 3.4 below. To the best of our knowledge, no existing contribution accounts for the role of factor strength in assessing the stability of risk premia. We make a contribution on this respect by formally studying whether factor strength is constant over time. We do so in the empirical analysis in Section 5 by testing for local stability as discussed in Section 2.2.

3 Detecting instability in factor strength

3.1 Estimation of factor strength under structural instability

In order to estimate the factor strength before and after the break, we extend the estimator developed in Bailey et al. (2021) to allow for the piecewise linear setting of our framework. For ease of exposition, we start by assuming that the break fraction τ in (1) is known. Section 3.3 deals with the case in which τ is unknown and has to be estimated.

We consider the multi-factor model in (1), and the null and the alternative hypothesis in (3). Let $\mathbb{I}_{1t}(\tau) = \mathbb{I}(t/T \leq \tau)$, $\mathbb{I}_{2t}(\tau) = \mathbb{I}(t/T > \tau)$, and the matrix $\mathbf{I}_{jT}(\tau)$ be

$$\mathbf{I}_{jT}(\tau) = \begin{bmatrix} \mathbb{I}_{j1}(\tau) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \mathbb{I}_{jT}(\tau) \end{bmatrix}, \quad j = 1, 2,$$

which is the $T \times T$ diagonal matrix with $t-th$ diagonal element equal to $\mathbb{I}_{jt}(\tau)$. For $k = 1, \ldots, K$, define the $T \times K$ matrix $\mathbf{F}_{j,-k}(\tau)$ as

$$\mathbf{F}_{j,-k}(\tau) = \mathbf{I}_{jT}(\tau)\left(\boldsymbol{\iota}_T, \underline{\mathbf{f}}_1, \ldots, \underline{\mathbf{f}}_{k-1}, \underline{\mathbf{f}}_{k+1}, \underline{\mathbf{f}}_K\right),$$

where $\underline{\mathbf{f}}_k = (f_{k1}, \ldots, f_{kT})'$: the matrix $\mathbf{F}_{j,-k}(\tau)$ collects all but the $k-th$ factor and it is augmented by the $T \times 1$ vector of ones $\boldsymbol{\iota}_T$. Let the $T \times T$ matrix $\mathbf{M}_{jT,-k}(\tau)$ be

$$\mathbf{M}_{jT,-k}(\tau) = \mathbf{I}_{jT}(\tau) - \mathbf{F}_{j,-k}(\tau)\left[\mathbf{F}_{j,-k}(\tau)'\mathbf{F}_{j,-k}(\tau)\right]^{-1}\mathbf{F}_{j,-k}(\tau)',$$

and the $T \times 1$ vector $\underline{\mathbf{f}}_{jkT}(\tau)$ as

$$\underline{\mathbf{f}}_{jkT}(\tau) = \mathbf{M}_{jT,-k}(\tau)\,\underline{\mathbf{f}}_k = \left[f_{jk1}(\tau), \ldots, f_{jkT}(\tau)\right]'.$$

Given the estimator $\hat{\beta}_{jikT}(\tau)$ for β_{jik} defined as

$$\begin{aligned}\hat{\beta}_{jikT}(\tau) &= \left[\underline{\mathbf{f}}_k'\mathbf{M}_{jT,-k}(\tau)\,\underline{\mathbf{f}}_k\right]^{-1}\left[\underline{\mathbf{f}}_k'\mathbf{M}_{jT,-k}(\tau)\,\underline{\mathbf{R}}_i\right] \\ &= \left[\underline{\mathbf{f}}_{jkT}(\tau)'\underline{\mathbf{f}}_{jkT}(\tau)\right]^{-1}\left[\underline{\mathbf{f}}_{jkT}(\tau)'\underline{\mathbf{R}}_i\right],\end{aligned} \tag{6}$$

with $\underline{\mathbf{R}}_i = (R_{i1}, \ldots, R_{iT})'$, the relevant test statistic for the significance of β_{jik} is

$$\hat{t}_{jikT}(\tau) = \frac{\hat{\beta}_{jikT}(\tau)}{\sqrt{\hat{\omega}_{jiT}(\tau)}} = \frac{\left[\underline{\mathbf{f}}_{jkT}(\tau)'\underline{\mathbf{f}}_{jkT}(\tau)\right]^{-1}\left[\underline{\mathbf{f}}_{jkT}(\tau)'\underline{\mathbf{R}}_i\right]}{\sqrt{\hat{\omega}_{jiT}(\tau)}},$$

where

$$\hat{\omega}_{jiT}(\tau) = \hat{\gamma}_{ji0}(\tau) + 2\sum_{l=1}^{L}\left(1 - \frac{l}{L+1}\right)\hat{\gamma}_{jil}(\tau) \tag{7}$$

with

$$\hat{\gamma}_{jil}(\tau) = \frac{\sum_{t=l+1}^{T_j(\tau)} f_{jkt}(\tau)\,\hat{e}_{jit}(\tau)\,f_{jk,t-l}(\tau)\,\hat{e}_{ji,t-l}(\tau)}{T_j(\tau)},$$

$T_j(\tau) = \sum_{t=1}^{T}\mathbb{I}_{jt}(\tau)$ is the number of time series observations within regime j, L is the length of the bandwidth of the Bartlett kernel, and

$$\hat{\mathbf{e}}_{jiT}(\tau) = \left[\hat{e}_{ji1}(\tau), \ldots, \hat{e}_{jiT}(\tau)\right]' = \mathbf{M}_{jT,-k}(\tau)\left[\underline{\mathbf{R}}_i - \underline{\mathbf{f}}_k\hat{\beta}_{jikT}(\tau)\right].$$

Therefore, $\hat{\omega}_{jiT}(\tau)$ is the Newey and West (1987) estimator, which allows for time series dependence and heteroskedasticity in the error terms e_{it} in line with Assumption 1 below.

For given nominal size of the individual tests p and critical value exponent $\delta > 0$, from Chudik et al. (2018) define the critical value function $c_p(N)$ as

$$c_p(N) = \Phi^{-1}\left(1 - \frac{p}{2N^\delta}\right). \quad (8)$$

Following Bailey et al. (2021), the factor strength λ_{jk} is estimated as

$$\hat{\lambda}_{jkNT}(\tau) = \mathbb{I}\left[\hat{\pi}_{jkNT}(\tau) > 0\right] \tilde{\lambda}_{jkNT}(\tau), \quad (9)$$

where

$$\hat{d}_{jikT}(\tau) = \mathbb{I}\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\right], \quad \hat{\pi}_{jkNT}(\tau) = \frac{1}{N}\sum_{i=1}^{N} \hat{d}_{jikT}(\tau), \quad (10)$$

with

$$\tilde{\lambda}_{jkNT}(\tau) = 1 + \frac{\ln \hat{\pi}_{jkNT}(\tau)}{\ln N}, \quad \hat{\pi}_{jkNT}(\tau) > 0. \quad (11)$$

From (9), $\hat{\lambda}_{jkNT}(\tau) = 0$ if $\hat{\pi}_{jkNT}(\tau) = 0$, and $\hat{\lambda}_{jkNT}(\tau) = \tilde{\lambda}_{jkNT}(\tau)$ if $\hat{\pi}_{jkNT}(\tau) > 0$, with $\hat{\pi}_{jkNT}(\tau)$ and $\tilde{\lambda}_{jkNT}(\tau)$ defined in (10) and (11), respectively. By construction, $0 \leq \hat{\pi}_{jkNT}(\tau) \leq 1$, since $\hat{\pi}_{jkNT}(\tau)$ is the proportion of cross-sectional units with non-zero beta on the factor within regime j. Also, $\hat{\lambda}_{jkNT}(\tau)$ and $\tilde{\lambda}_{jkNT}(\tau)$ are asymptotically equivalent since the probability of the event $\hat{\pi}_{jkNT}(\tau) = 0$ is equal to zero as $N \to \infty$.

3.2 Testing for strength instability

3.2.1 Test statistics

Our inferential procedure tests for stability of the strength using the estimators obtained before and after the break as described in Section 3.1. In doing so, we assume the break fraction τ in (1) is known. We relax this assumption in Section 3.3, in which we let the break fraction τ be unknown so that it has to be estimated.

Given $\hat{d}_{jikT}(\tau)$ and $\hat{\pi}_{jkNT}(\tau)$ as in (10), define

$$\hat{D}_{jkNT}(\tau) = \sum_{i=1}^{N} \hat{d}_{jikT}(\tau) = N^{\hat{\lambda}_{jkNT}(\tau)}, \quad D_{jkN} = \sum_{i=1}^{N} d_{jik} = N^{\lambda_{jk}}, \quad d_{jik} = \mathbb{I}(\beta_{jik} \neq 0),$$

so that

$$\frac{\hat{D}_{jkNT}(\tau)}{D_{jkN}} = \frac{N^{\hat{\lambda}_{jkNT}(\tau)}}{N^{\lambda_{jk}}} = N^{\hat{\lambda}_{jkNT}(\tau)-\lambda_{jk}}.$$

Given

$$\hat{A}_{jkNT}(\tau) = \frac{\sum_{i=1}^{N}\left\{\hat{d}_{jikT}(\tau) - \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right]\right\}}{N^{\lambda_{jk}}}, \qquad B_{jkNT}(\tau) = \frac{\sum_{i=1}^{N}\mathrm{E}\left[\hat{d}_{jikT}(\tau)\right] - N^{\lambda_{jk}}}{N^{\lambda_{jk}}},$$

the approximate equality

$$[\ln(N)]\left[\hat{\lambda}_{jkNT}(\tau) - \lambda_{jk}\right] = \hat{A}_{jkNT}(\tau) + B_{jkNT}(\tau), \tag{12}$$

holds.[4] Given (12), interest lies in the difference

$$\begin{aligned} &[\ln(N)]\left\{\left[\hat{\lambda}_{1kNT}(\tau) - \lambda_{1k}\right] - \left[\hat{\lambda}_{2kNT}(\tau) - \lambda_{2k}\right]\right\} \\ = &\left[\hat{A}_{1kNT}(\tau) + B_{1kNT}(\tau)\right] - \left[\hat{A}_{2kNT}(\tau) + B_{2kNT}(\tau)\right]. \end{aligned} \tag{13}$$

Under $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$, (13) simplifies to

$$[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau) - \hat{\lambda}_{2kNT}(\tau)\right] = \left[\hat{A}_{1kNT}(\tau) + B_{1kNT}(\tau)\right] - \left[\hat{A}_{2kNT}(\tau) + B_{2kNT}(\tau)\right].$$

Under Assumptions 1 - 3 in Section 3.2.2 below, for $0 < C_1, C_2, C_3, C_4 < \infty$,

$$\mathrm{Var}\left[\hat{A}_{jkNT}(\tau)\right] = \frac{N - N^{\lambda_{jk}}}{N^{2\lambda_{jk}}} C_T \frac{p}{N^{\delta}}\left(1 - C_T \frac{p}{N^{\delta}}\right) + O\left[\frac{\exp\left(-C_1 T^{C_2}\right)}{N^{\lambda_{jk}}}\right] \tag{14}$$

and

$$B_{jkNT}(\tau) = \frac{N - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} C_T \frac{p}{N^{\delta}} + O\left[\exp\left(-C_3 T^{C_4}\right)\right], \tag{15}$$

for some $0 < C_T < \infty$ such that $C_T \to 1$ as $T \to \infty$.[5]

[4] See equation (A.1) in Appendix A.
[5] See the proof of Theorem 3.1 in Appendix A.

From (15), the bias terms $B_{1kNT}(\tau)$ and $B_{2kNT}(\tau)$ are *both* asymptotically negligible provided that $\delta > 1 - \min\{\lambda_{1k}, \lambda_{2k}\}$. More importantly, $B_{1kNT}(\tau) + B_{2kNT}(\tau)$ converges to zero exponentially fast as $T \to \infty$ under $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$.

From (14), both $\hat{A}_{1kNT}(\tau) = o_p(1)$ and $\hat{A}_{2kNT}(\tau) = o_p(1)$ if $\delta > 1 - 2\min\{\lambda_{1k}, \lambda_{2k}\}$. This general condition links the factor strength to the critical value exponent δ: in particular, for $\delta > 0$, it is satisfied for $0.5 < \min\{\lambda_{1k}, \lambda_{2k}\} \leq 1$, and the $k-th$ factor is at least semi-strong before and after the break. This is consistent with the empirical findings in Section 5.2, which show that the factors are always either strong of semi-strong given the empirical model we employ. Also, $\hat{A}_{jkNT}(\tau) = O_p\left(N^{1/2-\delta/2-\lambda_{jk}}\right)$ if $0 \leq \lambda_{jk} < 1$, for $j = 1, 2$: as noted in Bailey et al. (2021), when $\lambda_{jk} = 1$ the distribution of $\hat{A}_{jkNT}(\tau)$ is degenerate as the convergence rate is exponential. This implies that a test for the null hypothesis $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$ against the alternative $\mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}$ can be implemented only if $\delta > 1 - 2\min\{\lambda_{1k}, \lambda_{2k}\}$ and $0 \leq \min\{\lambda_{1k}, \lambda_{2k}\} < 1$: in particular, either $\hat{A}_{1kNT}(\tau)$ or $\hat{A}_{2kNT}(\tau)$ (or both) need to have a non-degenerate asymptotic distribution under $\mathcal{H}_{1k}$. It is important to note that the test can be implemented if either $0 \leq \lambda_{1k} < 1$ *or* $0 \leq \lambda_{2k} < 1$: therefore, the test can be implemented if the factor strength were to change from unity to a lower value. From the empirical results in Section 5.2, the more restrictive case $\delta > 1 - 2\min\{\lambda_{1k}, \lambda_{2k}\}$ and $0.5 < \min\{\lambda_{1k}, \lambda_{2k}\} < 1$ is relevant in practice.

Consider the quantity

$$\varphi_N(\lambda_{jk}) = \frac{N - N^{\lambda_{j1k}}}{N^{2\lambda_{jk}}} \frac{p}{N^\delta}\left(1 - \frac{p}{N^\delta}\right), \tag{16}$$

defined in Bailey et al. (2021): $\varphi_N(\lambda_{jk}) = O\left(N^{1-\delta-2\lambda_{jk}}\right)$ for $0 \leq \lambda_{jk} < 1$ and $\varphi_N(\lambda_{jk}) = 0$ for $\lambda_{jk} = 1$. Therefore, $\varphi_N(\lambda_{jk})$ is a consistent estimator for $\text{Var}\left[\hat{A}_{jkNT}(\tau)\right]$ in (14) as $N, T \to \infty$. In order to test the null hypothesis $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$ against the alternative $\mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}$, we propose the test statistics $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ respectively defined as

$$\widehat{\mathcal{LM}}_{kNT}(\tau) = \frac{[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau) - \hat{\lambda}_{2kNT}(\tau)\right]}{\left[2\max\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\tau)\right], \varphi_N\left[\hat{\lambda}_{2kNT}(\tau)\right]\right\}\right]^{1/2}}. \tag{17}$$

and

$$\widehat{\mathcal{W}}_{kNT}(\tau) = \frac{[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau) - \hat{\lambda}_{2kNT}(\tau)\right]}{\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\tau)\right] + \varphi_N\left[\hat{\lambda}_{2kNT}(\tau)\right]\right\}^{1/2}}. \tag{18}$$

The statistics $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ in (17) and (18), respectively, differ in the estimator for the asymptotic variance of $[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau) - \hat{\lambda}_{2kNT}(\tau)\right]$. The Wald statistic $\widehat{\mathcal{W}}_{kNT}(\tau)$ employs the *unrestricted* estimator $\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\tau)\right] + \varphi_N\left[\hat{\lambda}_{2kNT}(\tau)\right]\right\}$. The LM statistic $\widehat{\mathcal{LM}}_{kNT}(\tau)$ uses the *restricted* estimator $\left\{2\max\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\tau)\right], \varphi_N\left[\hat{\lambda}_{2kNT}(\tau)\right]\right\}\right\}$, which deserves some attention. The null hypothesis $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$ does not rule out a break in the factor loadings, as this may occur even if the factor strength is constant over time. Therefore, the factor strength cannot be estimated over the full sample period under the null hypothesis, as the loadings may still experience a break. The denominator of $\widehat{\mathcal{LM}}_{kNT}(\tau)$ accounts for this by taking the maximum between $\varphi_N\left[\hat{\lambda}_{1kNT}(\tau)\right]$ and $\varphi_N\left[\hat{\lambda}_{2kNT}(\tau)\right]$: $\hat{\lambda}_{1kNT}(\tau)$ and $\hat{\lambda}_{2kNT}(\tau)$ converge to the same probability limit under $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$; $\max\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\tau)\right], \varphi_N\left[\hat{\lambda}_{2kNT}(\tau)\right]\right\}$ accounts for the small sample discrepancy between $\hat{\lambda}_{1kNT}(\tau)$ and $\hat{\lambda}_{2kNT}(\tau)$ by making $\widehat{\mathcal{LM}}_{kNT}(\tau)$ more conservative in finite samples.

3.2.2 Asymptotic properties of test statistics

In order to study the asymptotic distribution of the test statistics $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ in (17) and (18), respectively, we consider the following set of assumptions.

Assumption 1 *The error terms e_{it}, and the demeaned factors $\mathbf{f}_t - \mathrm{E}(\mathbf{f}_t)$, are martingale difference processes with respect to $\mathcal{F}_t^{u_i} = \sigma(u_{it-s}, s \leq t)$ and $\mathcal{F}_t^{f} = \sigma(\mathbf{f}_{t-s}, s \leq t)$, respectively. The error terms e_{it} are independent over i and of $\mathbf{f}_t$.*

Assumption 2 $\mathrm{E}\left\{[\mathbf{f}_t - \mathrm{E}(\mathbf{f}_t)][\mathbf{f}_t - \mathrm{E}(\mathbf{f}_t)]'\right\} = \mathbf{\Sigma_f}$*, where $\mathbf{\Sigma_f}$ is a positive definite matrix.*

Assumption 3 *There exist sufficiently large positive constants C_1, $C_2 > 0$, and $q > 0$ such that*

$$\sup_{i,t} \Pr(|e_{it}| > \nu) \leq C_1 \exp(-C_2 \nu^q), \quad \forall \nu > 0,$$

and

$$\sup_{k,t} \Pr(|f_{kt}| > \nu) \leq C_1 \exp(-C_2 \nu^q), \quad \forall \nu > 0.$$

Assumption 4 *The breaks in the regressions betas satisfy* $\mathbf{B}_2 - \mathbf{B}_1 = \mathbf{\Delta}$, *where* $\mathbf{\Delta} \neq \mathbf{0}$ *is independent of the time series dimension* T.

Assumption 5 *The break fraction* τ *satisfies* $0 < \tau < 1$.

Assumption 6 *The size of the bandwidth* L *in* (7) *is such that* $L = O\left(T^{1/3}\right)$.

Assumptions 1 - 3 are the same as the homologous Assumptions 1 - 3 in Bailey et al. (2021) and allow to use results in Lemma A.10 in Chudik et al. (2018). According to Assumption 1, the error terms e_{it} are cross-sectionally independent, which ensures that the central limit theorem that underlies Theorem 3.1 below still holds. On this respect, Assumption 1 could be weakened by assuming some suitable spatial mixing condition, as discussed in Bailey et al. (2021). Assumption 1 also restricts the demeaned factors $\mathbf{f}_t$ to be a martingale difference sequence, as in Chudik et al. (2018): weaker mixing conditions could be employed at the expense of higher mathematical complexity, as discussed in Bailey et al. (2021). Assumption 2 imposes a standard regularity condition on the covariance matrix of the factors, which ensures that the estimators in (6) is well defined. Note that Assumption 2 accommodates a break in the covariance matrix of the factors $\mathbf{f}_t$, as it does not rule out regime-specific covariance matrices: this is important in modelling financial returns, as discussed in Baele et al. (2010). Assumption 3 imposes thin probability tail conditions used for the asymptotic distribution of the test statistics in (17) and (18) stated in Theorem 3.1 below. Assumption 4 is analogous to Assumption A6 in Qu and Perron (2007) and captures the feature of a large shift in the regression betas: this is required because a break in the factor strength can occur only if a break in the betas takes place, as discussed in Section 2.2. Assumption 5 is standard in the literature and allows to identify the model before and after the break: see Assumption A8 in Qu and Perron (2007). Assumption 6 restricts the growth rate of the bandwidth L in (7) (see Hansen (1992)). The following Theorem 3.1 states the properties of the test statistics in (17) and (18).

Theorem 3.1 *Consider the model in* (1), *and let Assumptions 1 - 6 hold. Further, assume that the break fraction* τ *is known. For* $k \in \{1, \ldots, K\}$, *if* $0 \leq \lambda_{1k} < 1$ *or* $0 \leq \lambda_{2k} < 1$ *(or both), with* $\delta > 1 - 2\min\{\lambda_{1k}, \lambda_{2k}\}$, *then the test statistics* $\widehat{\mathcal{LM}}_{kNT}(\tau)$ *and* $\widehat{\mathcal{W}}_{kNT}(\tau)$ *defined in* (17) *and* (18), *respectively, are such that for* $N, T \to \infty$: *(a)* $\widehat{\mathcal{LM}}_{kNT}(\tau) \xrightarrow{d} \mathcal{N}(0,1)$ *and*

$\widehat{\mathcal{W}}_{kNT}(\tau) \xrightarrow{d} \mathcal{N}(0,1)$ *under the null* $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$*; (b)* $\Pr\left(\left|\widehat{\mathcal{LM}}_{kNT}(\tau)\right| > C_1\right) \to 1$ *and* $\Pr\left(\left|\widehat{\mathcal{W}}_{kNT}(\tau)\right| > C_2\right) \to 1$ *for any positive constants* C_1 *and* C_2 *under the alternative* $\mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}$.

For $k \in \{1, \ldots, K\}$, Theorem 3.1 formally shows the validity of the test statistics defined in (17) and (18) for the null hypothesis $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$ against the alternative $\mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}$. The results in the theorem are valid provided that either $0 \leq \lambda_{1k} < 1$ or $0 \leq \lambda_{2k} < 1$ (or both): if $\lambda_{jk} = 1$, from (14) it follows that $\hat{A}_{jkNT}(\tau) \xrightarrow{p} 0$ exponentially fast as $T \to \infty$; therefore, the asymptotic distribution of the test statistics no longer holds under the null hypothesis $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k} = 1$. This implies that we can still test the null hypothesis $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$ even if $\lambda_{jk^\star} = 1$, for $j = 1$ or $j = 2$ (or both), $k^\star \in \{1, \ldots, K\}$ and $k^\star \neq k$. It also implies that we can test if the factor strength changes from unity to a lower value. The results in Theorem 3.1 hold when the break fraction τ is known: Section 3.3 deals with the scenario in which τ is treated as unknown and has to be estimated.

3.3 Unknown change point

Theorem 3.1 holds if the break fraction τ is known. We now relax this assumption and consider the case in which the break fraction τ is unknown and needs to be estimated. The multi-factor model in (1) can be cast within the general framework considered in equation (1) in Qu and Perron (2007). We thus employ relevant findings obtained therein to show that the results stated in Theorem 3.1 apply also when τ no longer is known and needs to be estimated.

Recall the formulation in (5), which we repeat for ease of exposition,

$$\mathbf{R}_t = \mathbb{I}_{1t}(\tau)(\boldsymbol{\alpha}_1 + \mathbf{B}_1\mathbf{f}_t) + \mathbb{I}_{2t}(\tau)(\boldsymbol{\alpha}_2 + \mathbf{B}_2\mathbf{f}_t) + \mathbf{e}_t.$$

Let $\hat{\tau}$, $\hat{\boldsymbol{\alpha}}_j$ and $\hat{\mathbf{B}}_j$ be the least squares estimators for τ, $\boldsymbol{\alpha}_j$ and $\mathbf{B}_j$, respectively, for $j = 1, 2$. Denote by $\hat{\boldsymbol{\theta}} = \left[\hat{\tau}, \hat{\boldsymbol{\alpha}}_1', \text{vec}\left(\hat{\mathbf{B}}_1'\right)', \hat{\boldsymbol{\alpha}}_2', \text{vec}\left(\hat{\mathbf{B}}_2'\right)'\right]'$ the estimator for $\boldsymbol{\theta} = \left[\tau, \boldsymbol{\alpha}_1', \text{vec}(\mathbf{B}_1')', \boldsymbol{\alpha}_2', \text{vec}(\mathbf{B}_2')'\right]'$: $\hat{\boldsymbol{\theta}}$ solves

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \frac{1}{NT} \sum_{t=1}^{T} \left\|\mathbf{R}_t - \mathbb{I}_{1t}(\tau)(\boldsymbol{\alpha}_1 + \mathbf{B}_1\mathbf{f}_t) - \mathbb{I}_{2t}(\tau)(\boldsymbol{\alpha}_2 + \mathbf{B}_2\mathbf{f}_t)\right\|^2.$$

For given τ, the estimators $\hat{\boldsymbol{\alpha}}_j(\tau)$ and $\hat{\mathbf{B}}_j(\tau)$ for $\boldsymbol{\alpha}_j$ and $\mathbf{B}_j$, respectively, are obtained by concentrating out τ as

$$\left[\hat{\boldsymbol{\alpha}}_j(\tau), \hat{\mathbf{B}}_j(\tau)\right] = \left[\sum_{t=1}^{T} \mathbb{I}_{jt}(\tau) \mathbf{R}_t \mathbf{g}_t'\right] \left[\sum_{t=1}^{T} \mathbb{I}_{jt}(\tau) \mathbf{g}_t \mathbf{g}_t'\right],$$

for $j = 1, 2$, where $\mathbf{g}_t = (1, \mathbf{f}_t')'$: the estimator $\hat{\tau}$ for τ is then obtained as

$$\hat{\tau} = \arg\min_{\tau} \frac{1}{NT} \sum_{t=1}^{T} \left\| \mathbf{R}_t - \mathbb{I}_{1t}(\tau) \left[\hat{\boldsymbol{\alpha}}_1(\tau) + \hat{\mathbf{B}}_1(\tau) \mathbf{f}_t\right] - \mathbb{I}_{2t}(\tau) \left[\hat{\boldsymbol{\alpha}}_2(\tau) + \hat{\mathbf{B}}_2(\tau) \mathbf{f}_t\right] \right\|^2.$$

Given $\hat{\tau}$, the test statistics $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ in (17) and (18) can be modified as

$$\widehat{\mathcal{LM}}_{kNT}(\hat{\tau}) = \frac{[\ln(N)]\left[\hat{\lambda}_{1kNT}(\hat{\tau}) - \hat{\lambda}_{2kNT}(\hat{\tau})\right]}{\left[2\max\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\hat{\tau})\right], \varphi_N\left[\hat{\lambda}_{2kNT}(\hat{\tau})\right]\right\}\right]^{1/2}}, \tag{19}$$

and

$$\widehat{\mathcal{W}}_{kNT}(\hat{\tau}) = \frac{[\ln(N)]\left[\hat{\lambda}_{1kNT}(\hat{\tau}) - \hat{\lambda}_{2kNT}(\hat{\tau})\right]}{\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\hat{\tau})\right] + \varphi_N\left[\hat{\lambda}_{2kNT}(\hat{\tau})\right]\right\}^{1/2}}, \tag{20}$$

respectively. In order to derive the asymptotic properties of $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$, we consider the following additional set of assumptions.

Assumption 7 *For $l_1 \leq \lfloor \tau T \rfloor$ and $l_2 \leq T - \lfloor \tau T \rfloor$, $(1/l_1)\sum_{t=1}^{l_1} \mathbf{f}_t \mathbf{f}_t' \stackrel{a.s.}{\to} \mathbf{Q}_1$ as $l_1 \to \infty$, and $(1/l_2)\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \tau T \rfloor+l_2} \mathbf{f}_t \mathbf{f}_t' \stackrel{a.s.}{\to} \mathbf{Q}_2$ as $l_2 \to \infty$, where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are nonrandom positive definite matrices non necessarily equal to each other.*

Assumption 8 *There exists a $l_0 > 0$ such that for all $l > l_0$ the minimum eigenvalues of $(1/l)\sum_{t=\lfloor \tau T \rfloor-l}^{\lfloor \tau T \rfloor} \mathbf{f}_t \mathbf{f}_t'$ and of $(1/l)\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \tau T \rfloor+l} \mathbf{f}_t \mathbf{f}_t'$ are bound away from zero.*

Assumption 9 *$\sum_{t=q}^{l} \mathbf{f}_t \mathbf{f}_t'$ is invertible for $l - q \geq q_0$ for some $0 < q_0 < \infty$.*

Assumptions 7, 8 and 9 are analogous to Assumptions A.1, A.2 and A.3, respectively, in Qu and Perron (2007), and impose restrictions on a local neighbourhood of the break fraction τ, which allow for consistent estimation of τ itself. Assumption 7 is stronger than Assumption 2 and still allows the factors to have different distributions before and after the break. Assumption 8 rules out local collinearity. Assumption 9 is an invertibility requirement. The remaining relevant conditions in Assumptions A.4 through A.8 in Qu and Perron (2007) are implied by Assumptions 1, 3, 4, and 5. The asymptotic properties of $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ defined in (19) and (20), respectively, are stated in Theorem 3.2 below.

Theorem 3.2 *Consider the model in* (1). *Let Assumptions 1, and 3 - 9 hold. For* $k \in \{1, \ldots, K\}$, *if* $0 \leq \lambda_{1k} < 1$ *or* $0 \leq \lambda_{2k} < 1$ *(or both), with* $\delta > 1 - 2\min\{\lambda_{1k}, \lambda_{2k}\}$, *then the results in* (*a*) *and* (*b*) *of Theorem* (3.1), *and stated for* $\widehat{\mathcal{LM}}_{kNT}(\tau)$ *and* $\widehat{\mathcal{W}}_{kNT}(\tau)$, *remain valid for* $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ *and* $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$, *respectively, as defined in* (19) *and* (20).

Theorem 3.2 shows that the asymptotic distribution of $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ under the null hypothesis is the same as it would be if τ was known and did not have to be estimated by $\hat{\tau}$: following from Corollary 1 in Qu and Perron (2007), the limiting distribution of the estimator for the betas is the same as it would be if τ was known and the result in Theorem 3.2 naturally follows from Theorem 3.1. Also, neither $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ nor $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ suffer from the problem of having one parameter being identified only under the alternative originally addressed in Davies (1977, 1987), since both statistics are constructed under the maintained assumption that τ is identified also under the null hypothesis: this is because a *necessary* condition for a break in factor strength is the occurrence of a break in the betas, as stated in Assumption 4; this allows to identify τ regardless of whether the factor strength remains stable over time.

3.4 Multiple change points

So far, we have worked under the maintained assumption of a single structural break. In the case of multiple breaks, the specification in (1) generalizes to the following model with J break fractions τ_j such that $0 < \tau_j < 1$, for $j = 1, \ldots, J$, and $J + 1$ regimes

$$R_{it} = \begin{cases} \alpha_{1i} + \boldsymbol{\beta}'_{1i}\mathbf{f}_t + e_{it}, & t/T \leq \tau_1, \\ \alpha_{2i} + \boldsymbol{\beta}'_{2i}\mathbf{f}_t + e_{it}, & \tau_1 < t/T \leq \tau_2, \\ \vdots & \vdots \\ \alpha_{J+1,i} + \boldsymbol{\beta}'_{J+1,i}\mathbf{f}_t + e_{it}, & t/T > \tau_J, \end{cases}, \tag{21}$$

where α_{ji} is the asset-specific intercept, and $\boldsymbol{\beta}_{ji} = (\beta_{ji1}, \ldots, \beta_{jiK})'$, for $j = 1, \ldots, J+1$. In this case, the cross-sectional dispersion of betas in (2) becomes

$$\begin{array}{l} \beta_{jik} \neq 0, \quad i = 1, \ldots, \left\lfloor N^{\lambda_{jk}} \right\rfloor, \\ \beta_{jik} = 0, \quad i = \left\lfloor N^{\lambda_{jk}} \right\rfloor + 1, \ldots, N, \end{array}, \quad 0 \leq \lambda_{jk} \leq 1, \quad j = 1, \ldots, J+1, \quad k = 1, \ldots, K, \tag{22}$$

where the ordering of the betas is for ease of exposition only. We then consider the following null and alternative hypotheses $\mathcal{H}_{0j_1j_2k}$ and $\mathcal{H}_{1j_1j_2k}$, respectively,

$$\mathcal{H}_{0j_1j_2k} : \lambda_{j_1k} = \lambda_{j_2k}, \qquad \mathcal{H}_{1j_1j_2k} : \lambda_{j_1k} \neq \lambda_{j_2k}, \qquad j_1, j_2 = 1, \ldots, J+1, \qquad j_1 \neq j_2, \qquad k \in \{1, \ldots, K\} :$$

we can then test for factor strength equality over any two regimes even if they are not consecutive. Let $\hat{\tau}_j$ be the estimator for τ_j, for $j = 0, \ldots, J+1$, where $\hat{\tau}_0 = \tau_0 = 0$ and $\hat{\tau}_{J+1} = \tau_{J+1} = 1$: $\hat{\tau}_j$ can be estimated using the procedure in Qu and Perron (2007), for $j = 1, \ldots, J$. From (21) and (22), λ_{jk} is the strength of factor k in regime $j = 1, \ldots, J+1$, which occurs for $\tau_{j-1} < t/T \leq \tau_j$. Given the estimators $\hat{\tau}_{j-1}$ and $\hat{\tau}_j$ for τ_{j-1} and τ_j, respectively, we can estimate λ_{jk} following steps analogous to those detailed in Section 3.1. Let $\hat{\lambda}_{jkNT}(\hat{\tau}_{j-1}, \hat{\tau}_j)$ denote the estimator for λ_{jk} obtained within the interval $\hat{\tau}_{j-1} < t/T \leq \hat{\tau}_j$, for $j = 1, \ldots, J+1$. For $j_1, j_2 = 1, \ldots, J+1$, with $j_1 \neq j_2$, and $k \in \{1, \ldots, K\}$, the test statistics $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ defined in (19) and (20), respectively, generalize to

$$\widehat{\mathcal{LM}}_{kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1}, \hat{\tau}_{j_2-1}, \hat{\tau}_{j_2}) = \frac{[\ln(N)] \left[\hat{\lambda}_{j_1kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1}) - \hat{\lambda}_{j_2kNT}(\hat{\tau}_{j_2-1}, \hat{\tau}_{j_2})\right]}{\left[2 \max\left\{\varphi_N\left[\hat{\lambda}_{j_1kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1})\right], \varphi_N\left[\hat{\lambda}_{j_2kNT}(\hat{\tau}_{j_2-1}, \hat{\tau}_{j_2})\right]\right\}\right]^{1/2}},$$

and

$$\widehat{\mathcal{W}}_{kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1}, \hat{\tau}_{j_2-1}, \hat{\tau}_{j_2}) = \frac{[\ln(N)] \left[\hat{\lambda}_{j_1kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1}) - \hat{\lambda}_{j_2kNT}(\hat{\tau}_{j_2-1}, \hat{\tau}_{j_2})\right]}{\left\{\varphi_N\left[\hat{\lambda}_{j_1kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1})\right] + \varphi_N\left[\hat{\lambda}_{j_2kNT}(\hat{\tau}_{j_2-1}, \hat{\tau}_{j_2})\right]\right\}^{1/2}},$$

respectively. Under conditions analogous to those in Theorem 3.2, $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1}, \hat{\tau}_{j_2-1}, \hat{\tau}_{j_2})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1}, \hat{\tau}_{j_2-1}, \hat{\tau}_{j_2})$ inherit the properties of the asymptotic distribution of $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$, respectively, as stated in Theorem 3.2.

4 Monte Carlo study

4.1 Data generating process

For $s = 1, \ldots, S$, $i = 1, \ldots, N$, and $t = 1, \ldots, T$, we consider the DGP

$$R_{it}^{s} = \mathbb{I}\left(t/T \leq \tau\right)\left(\alpha_{1i} + \beta_{1i1} f_{1t}^{s} + \beta_{1i2} f_{2t}^{s}\right) + \mathbb{I}\left(t/T > \tau\right)\left(\alpha_{2i} + \beta_{2i1} f_{1t}^{s} + \beta_{2i2} f_{2t}^{s}\right) + e_{it},$$

where s is the replication index and S is the total number of replications, with $S = 2000$. We consider $N, T \in \{100, 200, 500, 1000\}$. We look at two values for the break fraction τ, namely $\tau = 1/2$ and $\tau = 1/3$. We generate the intercept α_{1i} as $\alpha_{1i} \sim \text{IID}\mathcal{N}\left(0, 1\right)$ fixed in repeated samples and we set $\alpha_{2i} = \alpha_{1i}$, for $i = 1, \ldots, N$.

The factors f_{1t}^{s} and f_{2t}^{s} are generated as

$$f_{kt}^{s} = \rho_{f_k} f_{k,t-1}^{s} + \sqrt{1 - \rho_{f_k}^{2}} \varepsilon_{kt}^{s}, \quad k = 1, 2, \quad t = -99, \ldots, T, \quad f_{k,-100}^{s} = 0,$$

with $\rho_{f1} = \rho_{f2} = 0.5$ and $\varepsilon_{kt}^{s} \sim \text{IID}\mathcal{N}\left(0, 1\right)$, so that $\text{Var}\left(f_{kt}^{s}\right) = \text{Var}\left(\varepsilon_{kt}^{s}\right) = 1$. We minimize the effect of the starting value $f_{k,-100}^{s} = 0$ by discarding the first 100 observations in the DGPs for f_{kt}^{s}, for $k = 1, 2$.

We consider two cases for the idiosyncratic components e_{it}: (a) $e_{it} \sim \text{IID}\mathcal{N}\left(0, \sigma_i^2\right)$, with $\sigma_i^2 \sim \chi^2\left(1\right)$ fixed in repeated samples; (b) $e_{it} = \sigma_i\left[\left(u_{it} - 2\right)/2\right]$, with $u_{it} \sim \text{IID}\chi^2\left(2\right)$. In both cases, we have $\text{E}\left(e_{it}\right) = 0$, $\text{Var}\left(e_{it}\right) = \sigma_i^2$ and $\lim_{N \to \infty} \sum_{i=1}^{N} \text{Var}\left(e_{it}\right) = 1$. In (a) the idiosyncratic components are normally distributed. In the set up in (b), which is analogous to the one used in Chudik et al. (2018), they have a non-Gaussian distribution.

As for the factor loadings, we first consider those on f_{1t}^{s}. We begin by generating $v_i \sim \text{IID}\mathcal{U}\left(\mu_v - d_v, \mu_v + d_v\right)$ fixed in repeated samples, with $\mu_v = 1.00$ and $d_v = 0.2$. We then randomly assign $\lfloor N^{\lambda_{11}} \rfloor$ elements of v_i to $\lfloor N^{\lambda_{11}} \rfloor$ elements of the sequence $\{\beta_{1i1}\}_{i=1}^{N}$ and set to zero the remaining elements of $\{\beta_{1i1}\}_{i=1}^{N}$. In a similar way, we randomly assign $\lfloor N^{\lambda_{21}} \rfloor$ elements of v_i to $\lfloor N^{\lambda_{21}} \rfloor$ elements of the sequence $\{\beta_{2i1}\}_{i=1}^{N}$ and set to zero the remaining elements of $\{\beta_{2i1}\}_{i=1}^{N}$. In this way, under the null hypothesis $\mathcal{H}_{01} : \lambda_{11} = \lambda_{21} = \lambda_1$, the sequences $\{\beta_{1i1}\}_{i=1}^{N}$ and $\{\beta_{2i1}\}_{i=1}^{N}$ have the same number of non-zero elements, although those elements may be different since they are obtained from independent draws from $\{v_i\}_{i=1}^{N}$. Under the alternative hypothesis $\mathcal{H}_{11} : \lambda_{11} \neq \lambda_{21}$, the sequences $\{\beta_{1i1}\}_{i=1}^{N}$ and $\{\beta_{2i1}\}_{i=1}^{N}$ have a different number of non-zero elements: in this case, we define $\kappa_1 = \lambda_{21} - \lambda_{11}$, so that if $\kappa_1 < 0$ the factor strength decreases, whereas f_{1t}^{s} becomes stronger if $\kappa_1 > 0$.

As for the loadings of f_{2t}^{s}, we consider two cases: (a) the one-factor model with $\beta_{1i2} = \beta_{2i2} = 0$, for $i = 1, \ldots, N$; (b) the two-factor model such that, for $j = 1, 2$, we randomly assign $\lfloor N^{\lambda_{j2}} \rfloor$ elements of v_i generated as described above to as many elements of the sequence $\{\beta_{ji2}\}_{i=1}^{N}$ and set to zero the remaining elements of $\{\beta_{ji2}\}_{i=1}^{N}$, with $\lambda_{12} = \lambda_{22} = 0.85$. Therefore, the strength of f_{2t}^{s} is kept fixed, although its betas may experience a break as in the latter case.

4.2 Results

We group our results in relation to the underlying Monte Carlo experiment and consider five scenarios given by Experiments 1 through 5 as discussed below: Experiments 1 through 4 treat the break fraction τ as known and study the test statistics $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ defined in (17) and (18), respectively; Experiment 5 assumes that τ is unknown and looks at the test statistics $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ in (19) and (20), respectively. In all experiments we run, we follow Bailey et al. (2021) and implement the critical value function in (8) by setting $p = 0.10$ and $\delta = 1/4$. We consider the size of $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ as being equal to 0.05.

4.2.1 Experiment 1

The focus is on the size of the test statistics. We set $\tau = 0.50$. We also choose $\lambda_1 = 0.75, 0.80, 0.85, 0.90, 0.95, 0.99$. Finally, we consider two scenarios, namely Experiments 1A and 1B, depending on whether the DGP is a one-factor or a two-factor model, and whose results are displayed in Tables 1A and 1B, respectively.

Tables 1A and 1B about here

The results in Table 1A show that the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ statistic has good size properties for $T = 500, 1000$ irrespective of the values of N and λ_1, and of the distribution of the idiosyncratic components (see Panel A). On the other hand, the $\widehat{\mathcal{W}}_{kNT}(\tau)$ statistic tends to overreject the null hypothesis more often than the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ statistic does (see Panel B): intuitively, this is due to the different estimator for the asymptotic variance of $[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau)-\hat{\lambda}_{2kNT}(\tau)\right]$, which we discuss extensively in Section 3.2.1. In particular, the $\widehat{\mathcal{W}}_{kNT}(\tau)$ statistic overrejects when $\lambda_1 = 0.99$ for $N = 500, 1000$ and $T = 1000$, when instead the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ statistic performs particularly well: this result is relevant within the set up of the empirical analysis in Section 5, to which we refer to for further comments. The results from Experiment 1B shown in Table 1B confirm for the two-factor model the findings for the one-factor model shown in Table 1A. We conclude that the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ statistic has the right empirical coverage for large enough T irrespective of N and λ_1, and it also has a hedge over the $\widehat{\mathcal{W}}_{kNT}(\tau)$ statistic.

4.2.2 Experiment 2

We study the power of the test statistics. We fix $\tau = 0.5$ and consider the two-factor model only. We fix $T = 500$ and consider $\kappa_1 = -0.02, -0.01, 0.01, 0.02$ with the exception of $\lambda_{11} = 0.99$, in which case we consider $\kappa_1 = -0.02, -0.01, 0.01$ only.

Table 2 about here

The results from Experiment 2 are collected in Table 2. The $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ statistics have similar power properties: it increases in the cross-sectional dimension N, in the factor strength λ_{11}, and in the magnitude of the break as measured by κ_1. These findings hold true irrespective of the distribution of the idiosyncratic components. We can thus conclude that both $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ have good empirical power properties.

4.2.3 Experiment 3

The aim is to verify the robustness of the size of the test statistics with respect to the location of the break fraction, which we now set to $\tau = 1/3$. As in Experiment 1, we have $\lambda_1 = 0.75, 0.80, 0.85, 0.90, 0.95, 0.99$. For ease of exposition, we only show results for the two-factor model with non-Gaussian idiosyncratic components.

Table 3 about here

The results collected in Table 3 show that the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ statistics are relatively unaffected by the location of the break fraction τ, as the results are similar to the homologous findings shown in Table 1B.

4.2.4 Experiment 4

This complements Experiment 3 by looking at the power of the test statistics when $\tau = 1/3$. As in Experiment 2, we show results for $T = 500$ and consider $\kappa_1 = -0.02, -0.01, 0.01, 0.02$, with the exception of $\lambda_{11} = 0.99$, in which case we consider $\kappa_1 = -0.02, -0.01, 0.01$ only.

Table 4 about here

The results in Table 4 are aligned with the homologous findings in Table 2 and show that, even when $\tau = 1/3$, the two test statistics have good empirical power properties: in particular, the power increases in N, λ_{11} and κ_1.

4.2.5 Experiment 5

Experiment 5 studies how the two test statistics under consideration perform when the break fraction τ is estimated and no longer assumed to be known. In line with Experiments 1 and 2, we consider $T = 500$, $\tau = 1/2$, $\lambda_{12} = \lambda_{22} = 0.85$, $\lambda_{21} = \lambda_{11} + \kappa_1$, with $\lambda_{11} = 0.75, 0.80, 0.85, 0.90, 0.95, 0.99$ and $\kappa_1 = -0.02, -0.01, 0.00, 0.01, 0.02$: when $\kappa_1 = 0.00$, we study the size of the test, whereas the power is computed for the remaining values of κ_1. We estimate τ using the algorithm detailed in Section 3.3 through a grid of values made of the set $\{0.05, 0.10, 0.15, \ldots, 0.85, 0.90, 0.95\}$.

Table 5 about here

The findings shown in Table 5 support the theoretical results stated in Theorem 3.2: the asymptotic behaviour of the test statistics under consideration is unaffected by the estimation noise induced by the estimator $\hat{\tau}$ for τ. In particular, the statistics $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ is correctly sized (with size computed for $\kappa_1 = 0.00$), with power that increases in N, λ_{11} and in the magnitude of

κ_1. The statistics $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ is more often oversized, especially when $N = 1000$ and $\lambda_{11} = 0.99$.

4.3 Discussion

The Monte Carlo results presented in Section 4.2 support the validity of the theoretical findings of this paper. In particular, the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ statistic has an edge over the $\widehat{\mathcal{W}}_{kNT}(\tau)$ statistic in terms of superior empirical size properties, especially when N and T are large and the factor strength is very close to unity. For this reason, in the empirical analysis in Section 5 we will focus upon the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ statistic.

5 Empirical analysis

5.1 Data and empirical specification

We study the Chen and Zimmermann (2021) large dataset of equity portfolios and use the April 2021 version of it. Given 205 characteristics, Chen and Zimmermann (2021) build a number of portfolios whose returns are then provided; we then obtain the excess returns of those portfolios by subtracting the risk-free rate measured as the one-month Treasury bill rate.[6] The sample period of interest runs from July 1967 through December 2020, a total of $T = 690$ time series observations. To ensure that inference on the factor strength is not affected by the time-varying dimension and nature of the cross-section, we balance the dataset and retain only those portfolios that are available over the entire sample period. This results in $N = 739$ portfolios.

We consider the six factor model proposed in Fama and French (2016).[7] This is made of the following factors: the market return in excess of the risk-free rate as measured by the one-month Treasury bill rate ($RmRf$), size (SMB), value (HML), operating profitability (RMW), investment (CMA), and momentum (MOM).

We estimate our empirical model using rolling windows of length equal to 240 months. Given the discussion in Section 4.3, we then test for the stability of the factor strength over two consecutive non-overlapping windows using the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ test defined in (17) and discussed in

[6] The Chen and Zimmermann (2021) dataset is available at `https://www.openassetpricing.com/` .

[7] The data for the pricing factors are available from Kenneth French website at `https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`.

Section 3.2. From a methodological standpoint, this is equivalent to estimating the model over $T = 480$ time series observations and testing for a break in factor strength at a known break fraction $\tau = 0.50$. This set up is consistent with the Monte Carlo results in Section 4, which show the good finite sample properties of the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ statistic for T approximately equal to 500, as stressed in Section 4.3. This strategy therefore is informative about *local stability* of factor strength, which we further motivate in Section 5.2 below. Note also that pre-break and post-break estimation windows of 240 month are aligned with the set up in Fama and MacBeth (2021), who consider the first and the second half of the July 1963–June 2019 period to test for the stability of the value premium. As in the Monte Carlo experiments in Section 4, we set $p = 0.10$ and $\delta = 1/4$ in (8). We consider the size of $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ equal to 0.05.

5.2 Results

We first empirically motivate the detection of local instability as discussed in Section 5.1. Following the strategy adopted in Bailey et al. (2021), and Pesaran and Smith (2021a), we document substantial time-variation in the strength of the six factors included in our specification: this is a first empirical contribution of our paper. As discussed in Section 5.1, we estimate the model using rolling windows of length equal to 240 months.

Figure 1 and Figure 2 about here

The sequences of estimated strength for the six factors are displayed in Figure 1. The factor $RmRf$ is strong over the whole sample period, since its estimated strength is always equal to unity: as such, given Theorem 3.1, in this case we cannot run inference on the strength stability. Turning to SMB, it is a semi-strong factor, although its estimated strength always lies in the proximity of unity: in particular, the estimated values fall between 0.991 and 0.993. The remaining factors displays a higher degree of strength variation over time: HML is characterized by a cyclical behaviour around an average value of 0.921; RMW displays a clear upward trend, starting from 0.808 at the beginning of the sample, and reaching an average value of approximately 0.941 from early 2000s onwards; CMA has a very pronounced cyclical behaviour, with a peak of 0.945 in January 2000 and a trough of 0.748 in October 1990; MOM reaches an average value approximately equal to 0.986 from January 2000 onwards.

We conduct inference on the local stability as discussed in Section 5.1. Figure 2 displays the evolution over time of the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ test statistic together with the 95% confidence band. The SMB factor is stable over the whole sample period. To a different degree, the remaining factors display evidence of strength instability: HML is locally unstable at the beginning of the sample and during a short spell between January 1996 and September 1998; RMW exhibit significant local increases until January 2000, whereas this behaviour is somehow reverted after June of the same year; CMA is unstable from April 1995 onwards; MOM has dynamics similar to those of RMW, in that local increases in factor strength take place almost until the end of the sample.

6 Conclusions

This paper studies the detection of structural instability in factor strength in asset pricing models for financial returns. We distinguish between strong and weaker factors. We construct LM and Wald statistics and show that they are asymptotically normally distributed under the null hypothesis of factor strength stability. The empirical analysis conducted over a rolling estimation window uncovers the dynamics of factor strength instability in empirical models for equity portfolio returns. Given the tools we have developed, future work will focus upon the consequences of structural instability in factor strength for asset pricing and portfolio choice.

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Table 1A, Experiment 1A: $\lambda_{11} = \lambda_{21} = \lambda_1$, $\tau = 0.50$, $\beta_{1i2} = \beta_{2i2} = 0$

Panel A: LM test

(a) $e_{it} \sim \text{IID}\mathcal{N}\left(0, \sigma_i^2\right)$, $i = 1, \ldots, N$

N	100						200						500						1000					
λ_1	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
T																								
100	0.0850	0.1475	0.1380	0.1575	0.2800	0.4480	0.1170	0.1330	0.2125	0.2225	0.3515	0.7010	0.1310	0.2300	0.2730	0.3770	0.5715	0.7665	0.1730	0.1905	0.2700	0.3790	0.5735	0.8145
200	0.0455	0.0965	0.0670	0.0695	0.0865	0.1480	0.0690	0.0630	0.1000	0.0750	0.0725	0.2340	0.0570	0.0860	0.0720	0.0820	0.1680	0.3095	0.0810	0.0655	0.0785	0.0870	0.1320	0.3580
500	0.0375	0.0805	0.0535	0.0505	0.0420	0.0325	0.0530	0.0390	0.0685	0.0395	0.0295	0.0520	0.0430	0.0515	0.0410	0.0420	0.0665	0.0380	0.0600	0.0550	0.0650	0.0545	0.0600	0.0595
1000	0.0385	0.0830	0.0550	0.0550	0.0395	0.0200	0.0590	0.0435	0.0640	0.0380	0.0385	0.0675	0.0420	0.0545	0.0430	0.0350	0.0485	0.0320	0.0540	0.0415	0.0470	0.0475	0.0510	0.0460

(b) $e_{it} = \sigma_i \left[(u_{it} - 2)/2\right]$, $u_{it} \sim \text{IID}\chi^2(2)$, $i = 1, \ldots, N$

N	100						200						500						1000					
λ_1	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
T																								
100	0.0900	0.1315	0.1260	0.1355	0.2830	0.4590	0.1100	0.1215	0.2075	0.2125	0.3515	0.7015	0.1255	0.2325	0.2630	0.3845	0.5885	0.7725	0.1855	0.1985	0.2760	0.3985	0.6100	0.8210
200	0.0410	0.0910	0.0745	0.0725	0.1135	0.1595	0.0680	0.0610	0.0920	0.0675	0.0765	0.2740	0.0495	0.0875	0.0820	0.1040	0.1895	0.3275	0.0695	0.0675	0.0755	0.0850	0.1375	0.3680
500	0.0370	0.0755	0.0580	0.0465	0.0380	0.0310	0.0485	0.0390	0.0705	0.0335	0.0305	0.0590	0.0385	0.0530	0.0485	0.0450	0.0740	0.0460	0.0685	0.0570	0.0520	0.0425	0.0440	0.0590
1000	0.0275	0.0760	0.0545	0.049	0.0345	0.0165	0.0485	0.0350	0.0690	0.0320	0.0240	0.0560	0.0420	0.0585	0.0475	0.0400	0.0590	0.0385	0.0480	0.0440	0.0490	0.0560	0.0465	0.0375

Panel B: Wald test

(a) $e_{it} \sim \text{IID}\mathcal{N}\left(0, \sigma_i^2\right)$, $i = 1, \ldots, N$

N	100						200						500						1000					
λ_1	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
T																								
100	0.1645	0.1715	0.1380	0.2695	0.2800	0.4480	0.1170	0.1390	0.2125	0.2325	0.3545	0.7065	0.1760	0.2300	0.3385	0.3815	0.5715	0.7675	0.1730	0.2340	0.2700	0.3790	0.5735	0.8145
200	0.1140	0.1000	0.0670	0.1510	0.0865	0.1480	0.0690	0.0655	0.1000	0.1115	0.0915	0.2340	0.0890	0.0860	0.1075	0.1030	0.1680	0.3095	0.0810	0.0960	0.0785	0.0870	0.1320	0.3850
500	0.0980	0.0805	0.0535	0.1110	0.0420	0.0325	0.0530	0.0455	0.0685	0.0850	0.0545	0.0520	0.0745	0.0515	0.0700	0.0700	0.0665	0.0380	0.0600	0.0785	0.0650	0.0545	0.0600	0.1250
1000	0.1005	0.0830	0.0550	0.1100	0.0395	0.0200	0.0590	0.0475	0.0640	0.0825	0.0635	0.0675	0.0655	0.0545	0.0725	0.0575	0.0485	0.0320	0.0540	0.0640	0.0470	0.0475	0.0510	0.1070

(b) $e_{it} = \sigma_i \left[(u_{it} - 2)/2\right]$, $u_{it} \sim \text{IID}\chi^2(2)$, $i = 1, \ldots, N$

N	100						200						500						1000					
λ_1	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
T																								
100	0.1610	0.1600	0.1260	0.2400	0.2830	0.4590	0.1100	0.1270	0.2075	0.2250	0.3525	0.7065	0.1705	0.2325	0.3330	0.3855	0.5910	0.7740	0.1855	0.2430	0.2760	0.3985	0.6100	0.8210
200	0.1090	0.0945	0.0745	0.1630	0.1135	0.1595	0.0680	0.0665	0.0920	0.1005	0.0995	0.2740	0.0870	0.0875	0.1210	0.1210	0.1895	0.3275	0.0695	0.0955	0.0755	0.0850	0.1375	0.3835
500	0.0995	0.0755	0.0580	0.1080	0.0380	0.0310	0.0485	0.0420	0.0705	0.0760	0.0515	0.0590	0.0730	0.0530	0.0850	0.0710	0.0740	0.0460	0.0685	0.0810	0.0520	0.0425	0.0440	0.1255
1000	0.0860	0.0760	0.0545	0.1035	0.0345	0.0165	0.0485	0.0365	0.0690	0.0670	0.0540	0.0560	0.0695	0.0585	0.0800	0.0650	0.0590	0.0385	0.0480	0.0715	0.0490	0.0560	0.0465	0.0935

Table 1B, Experiment 1B: $\lambda_{11} = \lambda_{21} = \lambda_1$, $\tau = 0.50$, $\lambda_{12} = \lambda_{22} = 0.85$

Panel A: LM test

(a) $e_{it} \sim \text{IID}\mathcal{N}\left(0, \sigma_i^2\right), i = 1, \ldots, N$

N	100						200						500						1000					
λ_1	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
T																								
100	0.0970	0.1425	0.1425	0.1580	0.3000	0.4700	0.1085	0.1440	0.2330	0.2430	0.3815	0.7095	0.1300	0.2170	0.2635	0.3795	0.5880	0.7970	0.1810	0.2170	0.2935	0.4050	0.6195	0.8390
200	0.0500	0.1020	0.0760	0.0820	0.0965	0.1520	0.0620	0.0590	0.0985	0.0725	0.0825	0.2850	0.0700	0.0885	0.0700	0.0885	0.1565	0.3020	0.0830	0.0775	0.0850	0.0890	0.1465	0.3715
500	0.0405	0.0790	0.0545	0.0485	0.0335	0.0290	0.0550	0.0425	0.0675	0.0405	0.0370	0.0570	0.0430	0.0645	0.0405	0.0450	0.0605	0.0435	0.0590	0.0495	0.0525	0.0485	0.0510	0.0545
1000	0.0340	0.0665	0.0450	0.0440	0.0315	0.0160	0.0490	0.0350	0.0595	0.0300	0.0280	0.0460	0.0445	0.0595	0.0465	0.0415	0.0630	0.0370	0.0655	0.0510	0.0610	0.0465	0.0515	0.0545

(b) $e_{it} = \sigma_i \left[(u_{it} - 2)/2\right]$, $u_{it} \sim \text{IID}\chi^2(2)$, $i = 1, \ldots, N$

N	100						200						500						1000					
λ_1	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
T																								
100	0.0895	0.1310	0.1250	0.1440	0.2925	0.4570	0.1105	0.1355	0.2225	0.2325	0.3870	0.7215	0.1390	0.2415	0.2705	0.3785	0.5840	0.7925	0.1820	0.2045	0.2825	0.4050	0.6005	0.8470
200	0.0415	0.0895	0.0765	0.0710	0.1100	0.1525	0.0670	0.0610	0.0940	0.0695	0.0910	0.3015	0.0605	0.0895	0.0865	0.0900	0.1765	0.3210	0.0750	0.0750	0.0815	0.0930	0.1395	0.4060
500	0.0365	0.0765	0.0580	0.0500	0.0350	0.0255	0.0490	0.0375	0.0675	0.0325	0.0295	0.0550	0.0410	0.0540	0.0470	0.0395	0.0685	0.0390	0.0630	0.0475	0.0500	0.0410	0.0490	0.0595
1000	0.0555	0.0755	0.0555	0.0510	0.0365	0.0170	0.0670	0.0375	0.0670	0.0325	0.0220	0.0535	0.0425	0.0585	0.0390	0.0395	0.0530	0.0305	0.0550	0.0450	0.0505	0.0585	0.0445	0.0350

Panel B: Wald test

(a) $e_{it} \sim \text{IID}\mathcal{N}\left(0, \sigma_i^2\right), i = 1, \ldots, N$

N	100						200						500						1000					
λ_1	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
T																								
100	0.1725	0.1680	0.1425	0.2815	0.3000	0.4700	0.1085	0.1460	0.2330	0.2545	0.3835	0.7170	0.1810	0.2170	0.3295	0.3800	0.5885	0.7975	0.1810	0.2605	0.2935	0.4050	0.6195	0.8390
200	0.1245	0.1035	0.0760	0.1715	0.0965	0.1520	0.0620	0.0640	0.0985	0.1100	0.1040	0.2850	0.1130	0.1090	0.1130	0.1090	0.1565	0.3020	0.0830	0.1120	0.0850	0.0890	0.1465	0.3985
500	0.1025	0.0790	0.0545	0.0935	0.0335	0.0290	0.0550	0.0470	0.0675	0.0780	0.0640	0.0570	0.0770	0.0645	0.0655	0.070	0.0605	0.0435	0.0590	0.0740	0.0525	0.0485	0.0510	0.1100
1000	0.0875	0.0665	0.0450	0.0970	0.0315	0.0160	0.0490	0.0400	0.0595	0.0690	0.0530	0.0460	0.0650	0.0595	0.0770	0.0645	0.0630	0.0370	0.0655	0.0745	0.0610	0.0465	0.0515	0.1065

(b) $e_{it} = \sigma_i \left[(u_{it} - 2)/2\right]$, $u_{it} \sim \text{IID}\chi^2(2)$, $i = 1, \ldots, N$

N	100						200						500						1000					
λ_1	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
T																								
100	0.1580	0.1600	0.1250	0.2530	0.2925	0.4570	0.1105	0.1375	0.2225	0.2430	0.3880	0.7275	0.1855	0.2415	0.3360	0.3795	0.5850	0.7930	0.1820	0.2425	0.2825	0.4050	0.6005	0.8470
200	0.1095	0.0935	0.0765	0.1605	0.1100	0.1525	0.0670	0.0665	0.0940	0.1035	0.1120	0.3015	0.0995	0.0895	0.1290	0.1130	0.1765	0.3210	0.0750	0.1025	0.0815	0.0930	0.1395	0.4195
500	0.1000	0.0765	0.0580	0.1070	0.0350	0.0255	0.0490	0.0415	0.0675	0.0750	0.0525	0.0550	0.0780	0.0540	0.0790	0.0640	0.0685	0.0390	0.0630	0.0715	0.0500	0.0410	0.0490	0.1185
1000	0.0555	0.0755	0.0555	0.1070	0.0365	0.0170	0.0670	0.0405	0.0670	0.0730	0.0500	0.0535	0.0760	0.0585	0.0725	0.0600	0.0530	0.0305	0.0550	0.0740	0.0505	0.0585	0.0445	0.0870

Table 2, Experiment 2: $\lambda_{21} = \lambda_{11} + \kappa_1$, $\tau = 0.50$, $\lambda_{12} = \lambda_{22} = 0.85$, $T = 500$

Panel A: LM test

(a) $e_{it} \sim \text{IID}\mathcal{N}\left(0, \sigma_i^2\right)$, $i = 1, \ldots, N$

N	100						200						500						1000					
λ_{11}	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
κ_1																								
-0.02	0.2145	0.2410	0.7700	0.9170	0.9980	1.0000	0.5465	0.6915	0.9590	0.9985	1.0000	1.0000	0.7785	0.9885	1.0000	1.0000	1.0000	1.0000	0.9835	1.0000	1.0000	1.0000	1.0000	1.0000
-0.01	0.0565	0.0920	0.3615	0.3345	0.8620	0.9960	0.1855	0.2840	0.5540	0.7295	0.9905	1.0000	0.2695	0.6420	0.8825	0.9960	1.0000	1.0000	0.5895	0.8750	0.9940	1.0000	1.0000	1.0000
0.01	0.1105	0.2135	0.1840	0.5310	0.9045	1.0000	0.1770	0.2640	0.5635	0.8660	0.9995	1.0000	0.3415	0.6230	0.9575	1.0000	1.0000	1.0000	0.6595	0.9115	0.9990	1.0000	1.0000	1.0000
0.02	0.2220	0.5855	0.5875	0.9875	1.0000	-	0.5475	0.7960	0.9830	1.0000	1.0000	-	0.8955	0.9990	1.0000	1.0000	1.0000	-	0.9970	1.0000	1.0000	1.0000	1.0000	-

(b) $e_{it} = \sigma_i \left[(u_{it} - 2)/2\right]$, $u_{it} \sim \text{IID}\chi^2(2)$, $i = 1, \ldots, N$

N	100						200						500						1000					
λ_{11}	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
κ_1																								
-0.02	0.2230	0.2335	0.7655	0.9135	0.9995	1.0000	0.5510	0.6760	0.9530	0.9995	1.0000	1.0000	0.7720	0.9890	1.0000	1.0000	1.0000	1.0000	0.9870	1.0000	1.0000	1.0000	1.0000	1.0000
-0.01	0.0565	0.0865	0.3480	0.3190	0.8540	0.9975	0.1855	0.2850	0.5740	0.7405	0.9920	1.0000	0.2750	0.6200	0.8655	0.9970	1.0000	1.0000	0.5885	0.8870	0.9975	1.0000	1.0000	1.0000
0.01	0.1105	0.2075	0.1785	0.5435	0.9105	1.0000	0.1950	0.2815	0.5470	0.8710	0.9995	1.0000	0.3575	0.6280	0.9575	1.0000	1.0000	1.0000	0.6430	0.9020	0.9990	1.0000	1.0000	1.0000
0.02	0.2060	0.5960	0.6030	0.9870	1.0000	-	0.5425	0.7940	0.9835	1.0000	1.0000	-	0.8845	0.9990	1.0000	1.0000	1.0000	-	0.9970	1.0000	1.0000	1.0000	1.0000	-

Panel B: Wald test

(a) $e_{it} \sim \text{IID}\mathcal{N}\left(0, \sigma_i^2\right)$, $i = 1, \ldots, N$

N	100						200						500						1000					
λ_{11}	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
κ_1																								
-0.02	0.2610	0.3780	0.7700	0.9365	0.9980	1.0000	0.5465	0.6915	0.9590	0.9985	1.0000	1.0000	0.8380	0.9885	1.0000	1.0000	1.0000	1.0000	0.9835	1.0000	1.0000	1.0000	1.0000	1.0000
-0.01	0.1190	0.1155	0.3615	0.5285	0.8620	0.9960	0.1855	0.2840	0.5540	0.7325	0.9905	1.0000	0.3575	0.6420	0.9250	0.9960	1.0000	1.0000	0.5895	0.8990	0.9940	1.0000	1.0000	1.0000
0.01	0.2305	0.2135	0.1840	0.6195	0.9045	1.0000	0.1770	0.3110	0.5635	0.9350	0.9995	1.0000	0.4360	0.6230	0.9575	1.0000	1.0000	1.0000	0.6595	0.9410	0.9990	1.0000	1.0000	1.0000
0.02	0.3880	0.5855	0.5875	0.9875	1.0000	-	0.5475	0.8765	0.9830	1.0000	1.0000	-	0.9330	0.9990	1.0000	1.0000	1.0000	-	0.9970	1.0000	1.0000	1.0000	1.0000	-

(b) $e_{it} = \sigma_i \left[(u_{it} - 2)/2\right]$, $u_{it} \sim \text{IID}\chi^2(2)$, $i = 1, \ldots, N$

N	100						200						500						1000					
λ_{11}	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
κ_1																								
-0.02	0.2645	0.3650	0.7655	0.9135	0.9995	1.0000	0.5510	0.6760	0.9530	0.9995	1.0000	1.0000	0.8330	0.9890	1.0000	1.0000	1.0000	1.0000	0.9870	1.0000	1.0000	1.0000	1.0000	1.0000
-0.01	0.1105	0.1085	0.3480	0.4995	0.8540	0.9975	0.1855	0.2855	0.5740	0.7420	0.9920	1.0000	0.3615	0.6200	0.9180	0.9970	1.0000	1.0000	0.5885	0.9095	0.9975	1.0000	1.0000	1.0000
0.01	0.2145	0.2075	0.1785	0.6395	0.9110	1.0000	0.1950	0.3145	0.5470	0.9450	0.9995	1.0000	0.4395	0.6280	0.9575	1.0000	1.0000	1.0000	0.6430	0.9355	0.9990	1.0000	1.0000	1.0000
0.02	0.3805	0.5960	0.6030	0.9870	1.0000	-	0.5425	0.8840	0.9835	1.0000	1.0000	-	0.9275	0.9990	1.0000	1.0000	1.0000	-	0.9970	1.0000	1.0000	1.0000	1.0000	-

Table 3, Experiment 3: $\tau = 1/3$, $\lambda_{12} = \lambda_{22} = 0.85$, $e_{it} = \sigma_i \left[(u_{it} - 2)/2 \right]$, $u_{it} \sim \text{IID}\chi^2(2)$, $i = 1, \ldots, N$

Panel A: LM test

N	100						200						500						1000					
λ_{11}	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
T																								
100	0.1290	0.1790	0.2105	0.2895	0.5290	0.7270	0.1295	0.1715	0.3455	0.4400	0.6305	0.8895	0.2070	0.4030	0.5170	0.6935	0.8740	0.9585	0.2470	0.3255	0.4825	0.7060	0.8790	0.9670
200	0.0475	0.0935	0.0750	0.0865	0.1840	0.3735	0.0730	0.0705	0.1180	0.0950	0.1680	0.5410	0.0740	0.1220	0.1315	0.2055	0.4665	0.7810	0.0910	0.0920	0.1325	0.2230	0.4805	0.8505
500	0.0310	0.0645	0.0510	0.0525	0.0460	0.0505	0.0510	0.0355	0.0585	0.0315	0.0310	0.0725	0.0560	0.0670	0.0465	0.0465	0.0760	0.0635	0.0640	0.0525	0.0620	0.0575	0.0535	0.0715
1000	0.0375	0.0770	0.0515	0.0475	0.0295	0.0200	0.0425	0.0305	0.0510	0.0240	0.0245	0.0505	0.0455	0.0615	0.0435	0.0425	0.0650	0.0380	0.0575	0.0440	0.0475	0.0365	0.0465	0.0485

Panel B: Wald test

N	100						200						500						1000					
λ_{11}	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
T																								
100	0.2025	0.2260	0.2105	0.3995	0.5290	0.7270	0.1295	0.1745	0.3455	0.4470	0.6330	0.8920	0.2530	0.4030	0.5675	0.6940	0.8740	0.9585	0.2470	0.3745	0.4825	0.7060	0.8790	0.9670
200	0.1130	0.0980	0.0750	0.1900	0.1840	0.3735	0.0730	0.0755	0.1180	0.1295	0.1825	0.5410	0.1090	0.1220	0.1845	0.2225	0.4665	0.7810	0.0910	0.1285	0.1325	0.2230	0.4805	0.8540
500	0.0810	0.0645	0.0510	0.1130	0.0460	0.0505	0.0510	0.0390	0.0585	0.0755	0.0640	0.0725	0.0860	0.0670	0.0770	0.0660	0.0760	0.0635	0.0640	0.0750	0.0620	0.0575	0.0535	0.1425
1000	0.0915	0.0770	0.0515	0.1000	0.0295	0.0200	0.0425	0.0355	0.0510	0.0635	0.0475	0.0505	0.0720	0.0615	0.0800	0.0600	0.0650	0.0380	0.0575	0.0675	0.0475	0.0365	0.0465	0.1035

Table 4, Experiment 4: $\lambda_{21} = \lambda_{11} + \kappa_1$, $\tau = 1/3$, $\lambda_{12} = \lambda_{22} = 0.85$, $T = 500$, $e_{it} = \sigma_i [(u_{it} - 2)/2]$, $u_{it} \sim \text{IID}\chi^2(2)$, $i = 1, \ldots, N$

Panel A: LM test

N	100						200						500						1000					
λ_{11}	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
κ_1																								
-0.02	0.2460	0.2510	0.7875	0.9075	0.9975	1.0000	0.5685	0.6935	0.9645	0.9990	1.0000	1.0000	0.8150	0.9835	1.0000	1.0000	1.0000	1.0000	0.9875	1.0000	1.0000	1.0000	1.0000	1.0000
-0.01	0.0525	0.0890	0.3845	0.3145	0.8275	0.9895	0.2030	0.2955	0.5860	0.7430	0.9955	1.0000	0.3135	0.6560	0.8800	0.9950	1.0000	1.0000	0.6425	0.8945	0.9960	1.0000	1.0000	1.0000
0.01	0.0875	0.1945	0.1720	0.5560	0.9245	1.0000	0.1580	0.2590	0.5255	0.8655	0.9990	1.0000	0.2950	0.5870	0.9540	0.9995	1.0000	1.0000	0.5965	0.8920	0.9985	1.0000	1.0000	1.0000
0.02	0.1935	0.5540	0.5700	0.9870	1.0000	-	0.5350	0.7870	0.9870	1.0000	1.0000	-	0.8680	0.9970	1.0000	1.0000	1.0000	-	0.9965	1.0000	1.0000	1.0000	1.0000	-

Panel B: Wald test

N	100						200						500						1000					
λ_{11}	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
κ_1																								
-0.02	0.2840	0.4020	0.7875	0.9225	0.9975	1.0000	0.5685	0.6935	0.9645	0.9990	1.0000	1.0000	0.8680	0.9835	1.0000	1.0000	1.0000	1.0000	0.9875	1.0000	1.0000	1.0000	1.0000	1.0000
-0.01	0.1075	0.1115	0.3845	0.4915	0.8275	0.9895	0.2030	0.2955	0.5860	0.7430	0.9955	1.0000	0.4110	0.6560	0.9195	0.9950	1.0000	1.0000	0.6425	0.9150	0.9960	1.0000	1.0000	1.0000
0.01	0.2055	0.1945	0.1720	0.6555	0.9245	1.0000	0.1580	0.3000	0.5255	0.9455	0.9990	1.0000	0.3865	0.5870	0.9550	0.9995	1.0000	1.0000	0.5965	0.9270	0.9985	1.0000	1.0000	1.0000
0.02	0.3575	0.5540	0.5700	0.9870	1.0000	-	0.5350	0.8725	0.9870	1.0000	1.0000	-	0.9130	0.9970	1.0000	1.0000	1.0000	-	0.9965	1.0000	1.0000	1.0000	1.0000	-

Table 5, Experiment 5: $\lambda_{21} = \lambda_{11} + \kappa_1$, $\tau = 1/2$, $\lambda_{12} = \lambda_{22} = 0.85$, $T = 500$, $e_{it} = \sigma_i [(u_{it} - 2)/2]$, $u_{it} \sim \text{IID}\chi^2(2)$, $i = 1, \ldots, N$

Panel A: LM test

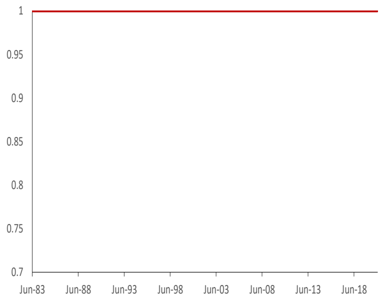
N	100						200						500						1000					
λ_{11}	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
κ_1																								
-0.02	0.2230	0.2325	0.7655	0.9135	0.9995	1.0000	0.5510	0.6755	0.9530	0.9995	1.0000	1.0000	0.7720	0.9890	1.0000	1.0000	1.0000	1.0000	0.9870	1.0000	1.0000	1.0000	1.0000	1.0000
-0.01	0.0590	0.0880	0.3480	0.3190	0.8540	0.9975	0.1860	0.2850	0.5735	0.7405	0.9920	1.0000	0.2750	0.6200	0.8655	0.9970	1.0000	1.0000	0.5885	0.8870	0.9975	1.0000	1.0000	1.0000
0.00	0.0485	0.0800	0.0590	0.0515	0.0360	0.0260	0.0720	0.0495	0.0755	0.0365	0.0295	0.0550	0.0440	0.0560	0.0475	0.0395	0.0685	0.0390	0.0650	0.0490	0.0500	0.0410	0.0490	0.0595
0.01	0.1115	0.2080	0.1790	0.5430	0.9105	1.0000	0.1980	0.2815	0.5470	0.8710	0.9995	1.0000	0.3575	0.6280	0.9575	1.0000	1.0000	1.0000	0.6430	0.9020	0.9990	1.0000	1.0000	1.0000
0.02	0.2060	0.5960	0.6030	0.9870	1.0000	-	0.5420	0.7940	0.9835	1.0000	1.0000	-	0.8845	0.9990	1.0000	1.0000	1.0000	-	0.9970	1.0000	1.0000	1.0000	1.0000	-

Panel B: Wald test

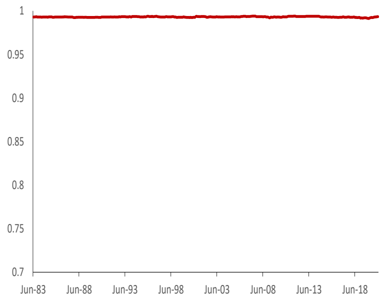
N	100						200						500						1000					
λ_{11}	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99	0.75	0.80	0.85	0.90	0.95	0.99
κ_1																								
-0.02	0.2650	0.3640	0.7655	0.9315	0.9995	1.0000	0.5510	0.6755	0.9530	0.9995	1.0000	1.0000	0.8330	0.9890	1.0000	1.0000	1.0000	1.0000	0.9870	1.0000	1.0000	1.0000	1.0000	1.0000
-0.01	0.1135	0.1100	0.3480	0.4990	0.8540	0.9975	0.1860	0.2855	0.5735	0.7420	0.9920	1.0000	0.3615	0.6200	0.9180	0.9970	1.0000	1.0000	0.5885	0.9095	0.9975	1.0000	1.0000	1.0000
0.00	0.1150	0.0800	0.0590	0.1080	0.0360	0.0260	0.0720	0.0540	0.0755	0.0790	0.0540	0.0550	0.0820	0.0560	0.0800	0.0640	0.0685	0.0390	0.0650	0.0730	0.0500	0.0410	0.0490	0.1185
0.01	0.2160	0.2080	0.1790	0.6390	0.9110	1.0000	0.1980	0.3145	0.5470	0.9450	0.9995	1.0000	0.4395	0.6280	0.9575	1.0000	1.0000	1.0000	0.6430	0.9355	0.9990	1.0000	1.0000	1.0000
0.02	0.3810	0.5960	0.6030	0.9870	1.0000	-	0.5420	0.8840	0.9835	1.0000	1.0000	-	0.9275	0.9990	1.0000	1.0000	1.0000	-	0.9970	1.0000	1.0000	1.0000	1.0000	-

Figure 1: Factor strength, equity portfolios, six-factor model

(a) $RmRf$



(b) SMB



(c) HML

(d) RMW

(e) CMA

(f) MOM

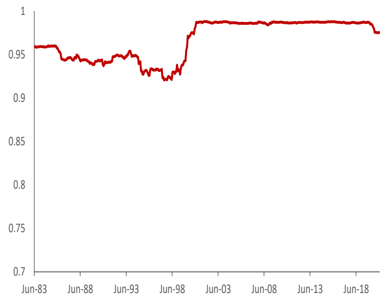
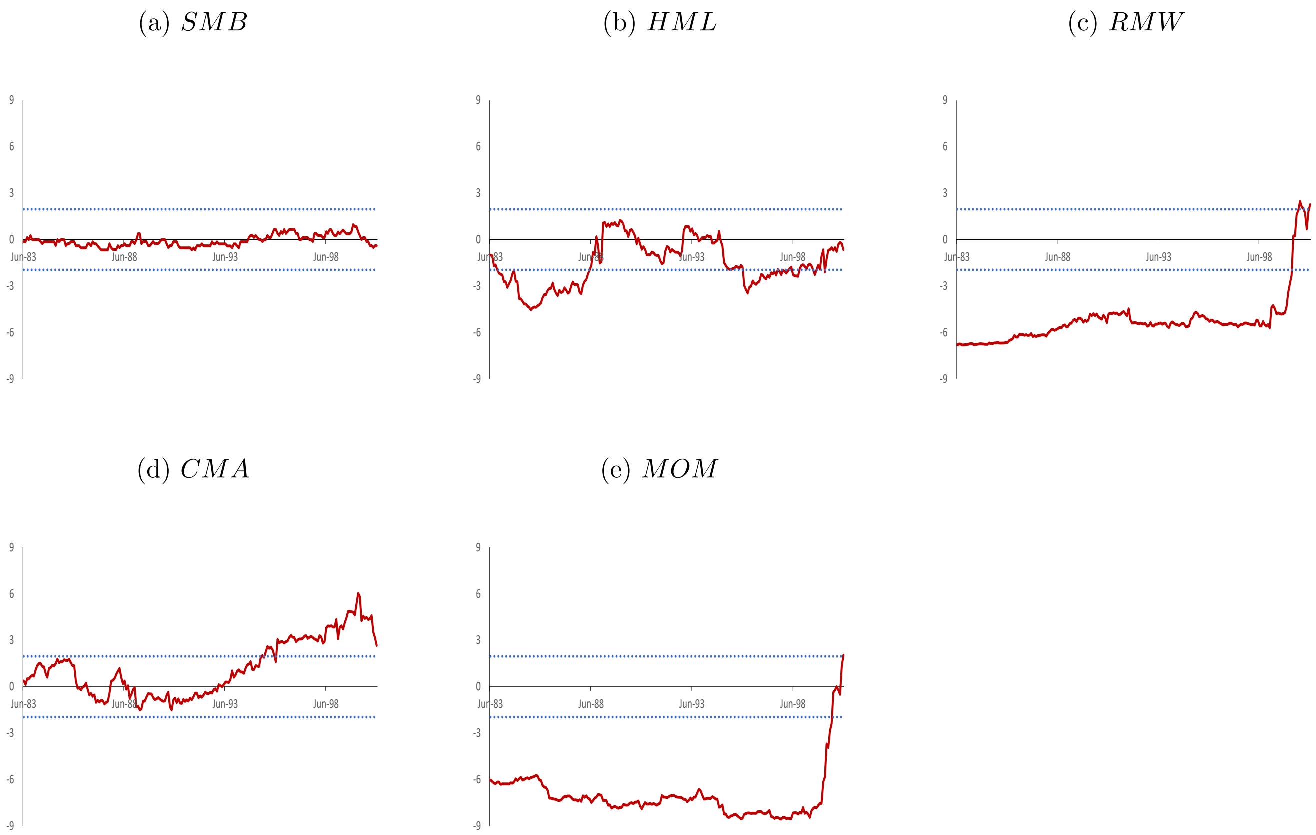


Figure 2: LM statistic, equity portfolios, six-factor model



Supplementary material to "Instability of Factors Strength in Asset Returns"

A Appendix: proofs of theorems

Proof of Theorem 3.1. For $k \in \{1, \ldots, K\}$, consider $\hat{d}_{jikT}(\tau)$ defined in (10), and let

$$\hat{D}_{jkNT}(\tau) = \sum_{i=1}^{N} \hat{d}_{jikT}(\tau) = N^{\hat{\lambda}_{jkNT}(\tau)}, \quad D_{jkN} = \sum_{i=1}^{N} d_{jik} = N^{\lambda_{jk}}, \quad d_{jik} = \mathbb{I}(\beta_{jik} \neq 0).$$

We have

$$\begin{aligned}
[\ln(N)]\left[\hat{\lambda}_{jkNT}(\tau) - \lambda_{jk}\right] &= \ln\left[\frac{\hat{D}_{jkNT}(\tau)}{D_{jkN}}\right] \\
&= \ln\left[\frac{\hat{D}_{jkNT}(\tau) + N^{\lambda_{jk}} - N^{\lambda_{jk}}}{N^{\lambda_{jk}}}\right] \\
&= \ln\left[1 + \frac{\hat{D}_{jkNT}(\tau) - N^{\lambda_{jk}}}{N^{\lambda_{jk}}}\right] \\
&\simeq \frac{\hat{D}_{jkNT}(\tau) - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} \\
&= \frac{\sum_{i=1}^{N} \hat{d}_{jikT}(\tau) - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} \\
&= \frac{\sum_{i=1}^{N}\left\{\hat{d}_{jikT}(\tau) - \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right]\right\}}{N^{\lambda_{jk}}} + \frac{\sum_{i=1}^{N} \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right] - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} \\
&= \hat{A}_{jkNT}(\tau) + B_{jkNT}(\tau),
\end{aligned} \tag{A.1}$$

where

$$\hat{A}_{jkNT}(\tau) = \frac{\sum_{i=1}^{N}\left\{\hat{d}_{jikT}(\tau) - \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right]\right\}}{N^{\lambda_{jk}}}, \quad B_{jkNT}(\tau) = \frac{\sum_{i=1}^{N} \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right] - N^{\lambda_{jk}}}{N^{\lambda_{jk}}}.$$

Since $\mathrm{E}\left[\hat{d}_{jikT}(\tau)\right] = \pi_{jik} = \Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\right]$, then

$$\begin{aligned}
B_{jkNT}(\tau) &= \frac{\sum_{i=1}^{N} \Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\right] - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} \\
&= \frac{\sum_{i=1}^{\lfloor N^{\lambda_{jk}} \rfloor} \Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N) \,|\beta_{jik} \neq 0\right] - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} \\
&\quad + \frac{\sum_{i=\lfloor N^{\lambda_{jk}} \rfloor + 1}^{N} \Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N) \,|\beta_{jik} = 0\right]}{N^{\lambda_{jk}}}.
\end{aligned} \tag{A.2}$$

Following steps analogous to those in the proof of Theorem 1 in Bailey et al. (2021),

$$\Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\,|\beta_{jik} \neq 0\right] = 1 - \exp\left(-C_1 T^{C_2}\right), \tag{A.3}$$

for some $0 < C_1, C_2 < \infty$, so that

$$\frac{\sum_{i=1}^{\lfloor N^{\lambda_{jk}} \rfloor} \Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\,|\beta_{jik} \neq 0\right] - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} = \exp\left(-C_1 T^{C_2}\right). \tag{A.4}$$

Further,

$$\Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\,|\beta_{jik} = 0\right] \leq C_T \frac{p}{N^{\delta}}, \tag{A.5}$$

for some $0 < C_T < \infty$ such that $C_T \to 1$ as $T \to \infty$, since the distribution of $\hat{t}_{jikT}(\tau)$ converges to a standard normal for $T \to \infty$. This implies that

$$\frac{\sum_{i=\lfloor N^{\lambda_{jk}} \rfloor + 1}^{N} \Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\,|\beta_{jik} = 0\right]}{N^{\lambda_{jk}}} = C_T \frac{p\left(N - N^{\lambda_{jk}}\right)}{N^{\delta + \lambda_{jk}}}. \tag{A.6}$$

Therefore, taking into account (A.2), (A.4) and (A.6), it follows that

$$B_{jkNT}(\tau) = C_T \frac{p\left(N - N^{\lambda_{jk}}\right)}{N^{\delta + \lambda_{jk}}} + O\left[\exp\left(-C_1 T^{C_2}\right)\right].$$

Under Assumption (1), the error terms e_{it} are cross-sectionally independent and

$$\begin{aligned}
\operatorname{Var}\left[\hat{A}_{jkNT}(\tau)\right] &= \operatorname{Var}\left\{ \frac{\left\{\sum_{i=1}^{N} \hat{d}_{jikT}(\tau) - \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right]\right\}}{N^{\lambda_{jk}}} \right\} \\
&= \frac{1}{N^{2\lambda_{jk}}} \sum_{i=1}^{N} \operatorname{Var}\left\{\hat{d}_{jikT}(\tau) - \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right]\right\} \\
&= \frac{1}{N^{2\lambda_{jk}}} \sum_{i=1}^{N} \pi_{jikT}(\tau)\left[1 - \pi_{jikT}(\tau)\right] \\
&= \frac{1}{N^{2\lambda_{jk}}} \sum_{i=1}^{N} \pi_{jikT}(\tau) - \frac{1}{N^{2\lambda_{jk}}} \sum_{i=1}^{N} \left[\pi_{jikT}(\tau)\right]^2 \\
&= \frac{1}{N^{2\lambda_{jk}}} \sum_{i=1}^{\lfloor N^{\lambda_{jk}} \rfloor} \pi_{jikT}(\tau) + \frac{1}{N^{2\lambda_{jk}}} \sum_{i=\lfloor N^{\lambda_{jk}} \rfloor + 1}^{N} \pi_{jikT}(\tau) \\
&\quad - \frac{1}{N^{2\lambda_{jk}}} \sum_{i=1}^{\lfloor N^{\lambda_{jk}} \rfloor} \left[\pi_{jikT}(\tau)\right]^2 - \frac{1}{N^{2\lambda_{jk}}} \sum_{i=\lfloor N^{\lambda_{jk}} \rfloor + 1}^{N} \left[\pi_{jikT}(\tau)\right]^2.
\end{aligned}$$

Therefore, taking into account (A.3) and (A.5), we have

$$\begin{aligned}
\operatorname{Var}\left[\hat{A}_{jkNT}(\tau)\right] &= \frac{1}{N^{2\lambda_{jk}}}\sum_{i=1}^{\lfloor N^{\lambda_{jk}}\rfloor}\left[1-\exp\left(-C_1T^{C_2}\right)\right]+\frac{1}{N^{2\lambda_{jk}}}\sum_{i=\lfloor N^{\lambda_{jk}}\rfloor+1}^{N}C_T\frac{p}{N^{\delta}} \\
&\quad -\frac{1}{N^{2\lambda_{jk}}}\sum_{i=1}^{\lfloor N^{\lambda_{jk}}\rfloor}\left[1-\exp\left(-C_1T^{C_2}\right)\right]^2-\frac{1}{N^{2\lambda_{jk}}}\sum_{i=\lfloor N^{\lambda_{jk}}\rfloor+1}^{N}\left(C_T\frac{p}{N^{\delta}}\right)^2 \\
&= \frac{1-\exp\left(-C_1T^{C_2}\right)}{N^{\lambda_{jk}}}+C_T\frac{p\left(N-N^{\lambda_{jk}}\right)}{N^{\delta+2\lambda_{jk}}} \\
&\quad -\frac{\left[1-\exp\left(-C_1T^{C_2}\right)\right]^2}{N^{\lambda_{jk}}}+\frac{N-N^{\lambda_{jk}}}{N^{2\lambda_{jk}}}\left(C_T\frac{p}{N^{\delta}}\right)^2 \\
&= \frac{1}{N^{\lambda_{jk}}}\left[1-\exp\left(-C_1T^{C_2}\right)\right]\left\{1-\left[1-\exp\left(-C_1T^{C_2}\right)\right]\right\} \\
&\quad +\frac{N-N^{\lambda_{jk}}}{N^{2\lambda_{jk}}}C_T\frac{p}{N^{\delta}}\left(1-C_T\frac{p}{N^{\delta}}\right) \\
&= \frac{N-N^{\lambda_{jk}}}{N^{2\lambda_{jk}}}C_T\frac{p}{N^{\delta}}\left(1-C_T\frac{p}{N^{\delta}}\right)+O\left[\frac{\exp\left(-C_1T^{C_2}\right)}{N^{\lambda_{jk}}}\right].
\end{aligned}$$

This implies that, for $0\le\lambda_{jk}<1$ we have

$$\hat{A}_{jkNT}(\tau)=O_p\left(N^{1/2-\delta/2-\lambda_{jk}}\right).$$

whereas for $\lambda_{jk}=1$

$$\hat{A}_{jkNT}(\tau)=O_p\left[\exp\left(-C_1T^{C_2}\right)/N^{0.5\lambda_{jk}}\right].$$

Recall $\varphi_N(\lambda_{jk})$ defined in (16) and define $\zeta_N(\lambda_{jk})$ as

$$\zeta_N(\lambda_{jk})=\frac{p\left(N-N^{\lambda_{jk}}\right)}{N^{\delta+\lambda_{jk}}}.$$

Consider the case $0\le\lambda_{jk}<1$, for $j=1,2$. For some $0<C_3,C_4<\infty$, we then have

$$\begin{aligned}
&\frac{\varphi_N(\lambda_{1k})^{-1/2}\left\{[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau)-\lambda_{1k}\right]\right\}}{2^{1/2}}-\frac{\varphi_N(\lambda_{2k})^{-1/2}\left\{[\ln(N)]\left[\hat{\lambda}_{2kNT}(\tau)-\lambda_{2k}\right]\right\}}{2^{1/2}} \\
=\;&\frac{\varphi_N(\lambda_{1k})^{-1/2}\left\{O_p\left(N^{1/2-\delta/2-\lambda_{1k}}\right)+O(1)\,\zeta_N(\lambda_{1k})+O\left[\exp\left(-C_1T^{C_2}\right)\right]\right\}}{2^{1/2}} \\
&-\frac{\varphi_N(\lambda_{2k})^{-1/2}\left\{O_p\left(N^{1/2-\delta/2-\lambda_{2k}}\right)+O(1)\,\zeta_N(\lambda_{2k})+O\left[\exp\left(-C_3T^{C_4}\right)\right]\right\}}{2^{1/2}}.
\end{aligned}$$

Under $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k} = \lambda_1$, it follows that $\varphi_N(\lambda_{1k}) = \varphi_N(\lambda_{2k}) = \varphi_N(\lambda_k)$, and $\zeta_N(\lambda_{1k}) = \zeta_N(\lambda_{2k}) = \zeta_N(\lambda_k)$, and

$$\frac{[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau) - \hat{\lambda}_{2kNT}(\tau)\right]}{[2\varphi_N(\lambda_k)]^{1/2}} \xrightarrow{d} \mathcal{N}(0,1) :$$

the result in (a) follows since

$$\left\{2\max\left\{\varphi_N\left(\hat{\lambda}_{1k}\right), \varphi_N\left(\hat{\lambda}_{2k}\right)\right\}\right\} \xrightarrow{p} 2\varphi_N(\lambda_k)$$

and

$$\left[\varphi_N\left(\hat{\lambda}_{1k}\right) + \varphi_N\left(\hat{\lambda}_{2k}\right)\right] \xrightarrow{p} 2\varphi_N(\lambda_k)$$

as $N \to \infty$ under $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k} = \lambda_k$. Under $\mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}$ it follows that

$$\begin{aligned} & \varphi_N(\lambda_{1k})^{-1/2}[\ln(N)]\hat{\lambda}_{1kNT}(\tau) - \varphi_N(\lambda_{2k})^{-1/2}[\ln(N)]\hat{\lambda}_{2kNT}(\tau) \\ = & \left\{\varphi_N(\lambda_{1k})^{-1/2}\{[\ln(N)]\lambda_{1k} + O(1)\zeta_N(\lambda_{1k})\} - \varphi_N(\lambda_{2k})^{-1/2}\{[\ln(N)]\lambda_{2k} + O(1)\zeta_N(\lambda_{2k})\}\right\} \\ & + O_p(1) + \left\{\varphi_N(\lambda_{1k})^{-1/2} O\left[\exp\left(-C_1 T^{C_2}\right)\right] - \varphi_N(\lambda_{2k})^{-1/2} O\left[\exp\left(-C_3 T^{C_4}\right)\right]\right\}, \end{aligned}$$

and

$$\left|\varphi_N(\lambda_{1k})^{-1/2}\{[\ln(N)]\lambda_{1k} + O(1)\zeta_N(\lambda_{1k})\} - \varphi_N(\lambda_{2k})^{-1/2}\{[\ln(N)]\lambda_{2k} + O(1)\zeta_N(\lambda_{2k})\}\right| \to \infty$$

as $N \to \infty$, which is sufficient to prove (b). This completes the proof of the theorem. ■

Proof of Theorem 3.2. By Corollary 1 in Qu and Perron (2007), the limiting distribution of the betas is the same as it would be if τ was known. The result in the theorem then follows from the same steps as in the Proof of Theorem 3.1. ■

References

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Qu, Z. and P. Perron (2007). Estimating and testing structural changes in multivariate regressions. *Econometrica 75*, 459–502