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Learning and Equilibrium Selection in a Monetary Overlapping Generations Model with Sticky Prices

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Abstract

This paper studies the properties of adaptive learning as a device to select amongst the rational expectations equilibria of a monetary overlapping generations model. It extends previous contributions by introducing monopolistic competition and improves upon them by analyzing learning in a model with a well-defined temporary equilibrium map, a coherent informational setup, and properly specified microfoundations. The main result is that adaptive learning is a robust selection mechanism that independent from the degree of imperfect competition always selects the same equilibrium. The indeterminate steady state and the non-stationary equilibria are never stable. The determinate low inflation steady state is the unique stable equilibrium; however, depending on how agents forecast, stability is found to be related to observable characteristics of the economy.

Keywords: adaptive learning, equilibrium selection, rational expectations, indeterminacy, stability

JEL-Class.: E31, D83, C62

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1 Introduction

In many macroeconomic models rational expectations are not powerful enough to deliver a unique equilibrium prediction. Monetary overlapping generations models are perhaps the widest-known model class illustrating the weakness of rational expectations along this dimension.

To sharpen the predictions of rational expectations it is -by now- standard to study the stability of the equilibria under adaptive learning schemes (e.g. Marcet and Sargent (1989b), Evans and Honkapohja (2001)). Consistent with experimental evidence (Marimon and Sunder (1993), (1994)) it is widely believed that equilibria whose expectations can be acquired via simple adaptive learning schemes constitute more plausible model predictions than equilibria that would require more sophisticated coordination devices.

This paper examines the stability of rational expectations equilibria under adaptive learning schemes in a monetary overlapping generations (OLG) model and contributes to the existing literature in the following way.

Firstly, the paper extends the standard OLG model with flexible prices and perfect competition to a setting with sticky prices and monopolistic competition (Dixit and Stiglitz (1977)). Sticky prices are widely believed to be an important ingredient to empirically plausible macroeconomic models (e.g. Galí and Gertler (1999)). Yet, their implications for learning based equilibrium selection have never been analyzed. Checking for the robustness of the equilibrium selection with respect to different price setting assumptions is therefore of interest.

Secondly, the current model improves upon previous work on equilibrium selection in standard monetary OLG models by analyzing learning in a model with a well-defined temporary equilibrium map, a coherent informational setup, and properly specified microfoundations. As explained further in section 2 below, previous work was falling short on at least one of these dimensions. It either lacked fully specified micro-foundations (Duffy (1994)), assumed a somewhat inconsistent informational setup (Marcet and Sargent (1989b)), or had to deal with multiplicities in the temporary equilibrium map (Lettau and van Zandt (1999)).

One can broadly summarize the results as follows. Adaptive learning schemes robustly select the same equilibrium independent from the degree of imperfect competition. In line with Marcet and Sargent (1989b) the determinate monetary steady state is the unique learnable equilibrium. The indeterminate steady state and the non-stationary equilibria turn out not to be learnable. These results are obtained although money demand in the present model depends on current endogenous variables, i.e. current prices, which is a feature that has been shown to be conducive to the learnability of indeterminate equilibria (e.g. Duffy (1994), Evans and Honkapohja (2001), chapter 12).

More precisely, the paper starts out by analyzing a deterministic model and studies the learnability of the monetary and non-monetary steady states under constant and decreasing gain learning rules. I find that both types of learning rules always select the same equilibrium.[1] Moreover, the determinate low inflation steady state is the unique learnable equilibrium.

However, stability of the determinate steady state is not warranted and is related to observable characteristics of the economy. In particular, this equilibrium becomes unstable when the elasticity of labor supply falls below one-half. When this is the case I show how demand shocks destabilize the equilibrium.

This feature implies that the degree of imperfect competition may influence the stability of the low inflation steady through its influence on equilibrium output and its effect on the elasticity of labor supply at the new output level. For plausible assumptions on the elasticity of labor supply, increasing competition between entrepreneurs can even cause a previously stable steady state to become unstable.

The paper then considers the learnability of stationary and non-stationary equilibria under decreasing gain learning rules in a stochastic environment. I find again that the low inflation steady state is the unique stable rational expectations equilibrium. Neither the indeterminate high inflation steady state nor any of the non-stationary equilibria are learnable.

Interestingly, in the stochastic model the low inflation steady state turns out to be stable irrespective of the elasticity of labor supply. A stochastic spending component prevents the money stock from settling down to its deterministic steady state value. In contrast to the deterministic setup, this forces agents to condition their inflation forecasts on the economy's money stock to acquire rational expectations. Condition on this additional variable renders the low inflation steady state stable with respect to the demand shocks that destabilize it in the deterministic setup.

[1] This contrasts to the results of Lettau and van Zandt (1999).

The paper is organized as follows. It starts out in section 2 by discussing the trade-off between informational consistency and well-defined temporary equilibrium maps that arises in monetary OLG models with flexible prices and shows how sticky prices can be thought of as resolving this inconsistency. Section 3 then introduces the OLG model with imperfect competition. Section 4 determines its rational expectations equilibria and shows that these approach the ones of the competitive model as the degree of imperfect competition vanishes. Section 5 analyzes learning of the steady states in the deterministic model and section 6 considers the learnability of stationary and non-stationary equilibria in a stochastic version of the model. The appendix contains most of the technical details and proofs.

2 Consistency and Multiplicity

This section shows that in monetary models where real money demand depends on expected future inflation there exists a trade-off between a model formulation with learning that achieves an informationally consistent setup and a formulation that avoids multiplicity in the temporary equilibrium map.

Suppose real money demand m_t^d depends negatively on expected future inflation Π_{t+1}^e:

$$m_t^d = m^d(\Pi_{t+1}^e) \tag{1}$$

Money demand functions of this form can be derived from an OLG model, see section 3, but also appear in Cagan (1956) and Sargent and Wallace (1987). Agents who wish to hold the real quantity m_t^d have to put

$$m_t^d P_t$$

nominal units of cash into their pockets. Thus, assuming that agents can hold the desired levels of real balances implies that these agents must observe current prices.[2]

In models which consider learning the expectations Π_{t+1}^e are determined through an explicit updating mechanism (e.g. Marcet and Sargent (1989b)). Consistency between the model setup and the learning setup then requires that Π_{t+1}^e must be allowed to depend on Π_t. Otherwise, agents would ignore current prices when determining

[2] This is just another way of saying that money is a nominal asset.

the desired level of real money balances but suddenly recall them when it comes to implementing the real balances through appropriate amounts of nominal money bills.

A problem arising with agents using current prices Π_t to update Π^e_{t+1} is that it generates multiplicity in the temporary equilibrium of the model because market clearing prices Π_t and expectations Π^e_{t+1} are then determined simultaneously.

To illustrate this point suppose that agent update according to the following rule

$$\Pi^e_{t+1} = \Pi^e_t + \gamma(\Pi_t - \Pi^e_t)$$

where Π^e_{t+1} are the time t expectations of $t+1$ inflation and where γ is a parameter determining how fast expectations are updated in response to past forecast errors. Moreover, suppose that real money supply is given by

$$\frac{m_{t-1}}{\Pi_t} + g$$

where m_{t-1} denotes the stock of real balances in $t-1$ and g denotes the time t increase in the real money stock.[3] Clearing of the time t money market implies that Π_t solves

$$\frac{m_{t-1}}{\Pi_t} + g = m^d_t(\Pi^e_t + \gamma(\Pi_t - \Pi^e_t))$$

Since money supply and money demand both decrease with the inflation rate, this equation might have multiple solutions. Figure 1 illustrates this situation for the case of a linear money demand function.[4] Given the $t-1$ expectations Π^e_t there exist two inflation rates that clear the time t money market.

In the light of the trade-off between informational consistency and well-defined temporary equilibrium maps the literature has typically opted for allowing agents to use only lagged endogenous variables to update expectations, which eliminates the simultaneity between prices and expectations (e.g. Marcet and Sargent ((1989b)), Evans and Honkapohja (2001) chapter 10.3).[5] Yet, this is a delicate operation because unchanged stability properties are only warranted if expectations based on contemporaneous data do not create multiplicity, see Marcet and Sargent (1989a) and Lettau and van Zandt

[3]For $g = 0$ the rule specifies a fixed nominal supply of money .

[4]The figure assumes $m^d(\Pi^e_{t+1}) = a - b\Pi^e_{t+1}$ with $a = 3$, $b = 1$. The other parameters are $g = .5$, and $\gamma = 0.2$.

[5]Clearly, this is motivated by the desire to avoid ad-hoc assumptions about the relevance of the respective market clearing prices, which would be an even more unpleasant feature given that the goal is equilibrium selection.

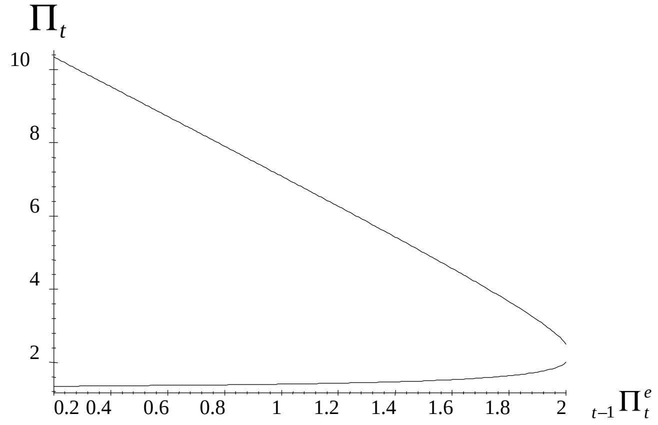

Figure 1: Multiple market clearing prices

(1999) . As figure 1 shows, OLG models with learning based on contemporaneous data easily generate such multiplicities. This casts doubts on the robustness of selection results which are based on lagged data.

The present paper overcomes the multiplicity and inconsistency problems by introducing price setting entrepreneurs that commit to prices at the beginning of each period. This causes contemporaneous inflation to be a predetermined variable. Therefore, agents can formulate the expectations that enter the money demand function based on contemporaneous prices without creating multiplicity.

3 OLG-Model with Imperfect Competition

I consider a simple overlapping generations model where each generation of agents lives for two periods, works when young, consumes when old, and may transfer wealth across time via fiat money. There is also an infinitely lived government that finances a constant real deficit through seignorage. This is essentially the setup of Sargent and Wallace (1987).

In each time period a new generation of agents is born. In contrast to standard models I assume that agents of a given generation are either born as workers or as entrepreneurs. The new generation consists of a unit mass of workers and a unit-mass of entrepreneurs.[6]

[6] Any numbers could be chosen as long as they are constant through time.

Workers are homogeneous and offer their labor force at a competitive labor market in return for a wage income.

Young entrepreneurs are in monopolistic competition with each entrepreneur $i \in [0,1]$ producing a good that is differentiated from the goods of other entrepreneurs (Dixit and Stiglitz (1977)). Entrepreneurs, therefore, earn monopolistic profits.

A competitive sector assembles the differentiated goods into an aggregate consumption good according to the following production function

$$c = \left(\int_{i \in [0,1]} \left(q^i\right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \qquad \text{with } 1 > \sigma \geq 0$$

where q^i denotes the input from entrepreneur i. The parameter σ determines the degree of substitutability between the different products. If $\sigma = 0$, goods are perfect substitutes and entrepreneurs are in perfect competition. As σ increases goods become less and less substitutable and competition between entrepreneurs decreases.

The timing of events is as follows. At the beginning of each period, young entrepreneurs commit to a price at which they are willing to sell the product. Then old agents, i.e. old workers and old entrepreneurs, spend all their money holdings to order goods. At the same time, the government orders goods for government consumption. Firms accept any amount of orders at the price they posted. When all orders are made, firms hire the work force that is necessary to produce the ordered quantities at a competitive labor market. The labor market clears and production takes place. Young workers are paid their wage, young entrepreneurs retain their profits, and the produced goods are delivered to the old agents for consumption. Then a new period starts.

Let P_t^i denote the price posted for good i at the beginning of period t. Profit maximization by the competitive sector implies that the price P_t for the composite good is given by

$$P_t = \left(\int_{[0,1]} \left(P_t^i\right)^{\frac{\sigma-1}{\sigma}} \mathrm{di} \right)^{\frac{\sigma}{\sigma-1}} \tag{2}$$

Let M_t denote the stock of nominal money at the end of period t and let $m_t = \frac{M_t}{P_t}$ denote its real value (in terms of the aggregate good). Since old agents die with certainty after their second period of life, m_t is held by the young agents at the end of period t, with young workers holding amount m_t^w and young entrepreneurs holding amount m_t^e, where $m_t = m_t^w + m_t^e$.

The government finances a constant real expenditure $g > 0$ through seignorage. Real balances therefore evolve according to

$$m_t = \frac{m_{t-1}}{\Pi_t} + g \tag{3}$$

where $\Pi_t = \frac{P_t}{P_{t-1}}$ is the inflation factor from $t-1$ to t.

3.1 Workers

The representative worker who is born in period t maximizes

$$\max_{n_t, c_{t+1}} E_t^* \left[-v(n_t) + \beta u(c_{t+1}^w) \right]$$

subject to

$$m_t^w = n_t w_t$$
$$c_{t+1}^w \leq \frac{m_t^w}{\Pi_{t+1}}$$

where $v(n_t)$ represents the disutility of supplying n_t hours of work in period t and $u(c_{t+1}^w)$ represents the utility from consuming c_{t+1}^w units of the composite good in period $t+1$. $\beta \in (0,1)$ is the discount factor, w_t is the time t hourly real wage, and m_t^w are the worker's end of period t real money holdings. The utility functions $v(\cdot)$ and $u(\cdot)$ are twice continuously differentiable with $v'(\cdot) > 0$, $v''(\cdot) \geq 0$ and $u'(\cdot) > 0$ and $u''(\cdot) < 0$.

Note that agents maximize utility with respect to some (potentially) subjective expectations operator E_t^* that is based on the information set

$$H_t = \sigma(P_t, P_{t-1}, \ldots ; m_t, m_{t-1}, \ldots)$$

which contains past and current values of prices and real money balances. Workers' expectations therefore depend on current prices, as required for an informationally consistent setup.

The first order conditions of the utility maximization problem implicitly define the workers' labor supply as a function of the current real wage and expected inflation:

$$n(w_t, E_t^*(\Pi_{t+1})) \tag{4}$$

Alternatively, the first order conditions define a real wage function

$$w(n_t, E_t^*(\Pi_{t+1})) \tag{5}$$

that determines the real wage that has to be paid to induce the representative worker to supply n_t units of labor when her inflation expectations are given by $E_t^*(\Pi_{t+1})$. The function $w(\cdot,\cdot)$ is continuously differentiable for $E_t^*(\Pi_{t+1}) > 0$ and all feasible levels n_t.

I require that the substitution effect of a relative price change between labor and consumption dominates the income effect, i.e.

Condition 1

$$\frac{u''(c)\cdot c}{u'(c)} > -1 \qquad \forall c \geq 0$$

Condition 1 insures that agents' labor supply (4) is strictly increasing in wages and strictly decreasing in the expected inflation tax $E_t^*[\Pi_{t+1}]$. It also insures that the real wage function (5) is increasing in both arguments.

Finally, I impose the following two conditions on the labor supply function (4):

Condition 2 $n(1-\sigma,1) > 0$ *and* $\exists\ \Pi^h \in (1,\infty)$ *s.t.* $n(1-\sigma,\Pi) = 0$

The first part of condition 2 requires workers to offer positive amounts of labor when the real wage is equal to $1-\sigma$ and when the expected inflation factor is equal to 1. This is simply a necessary condition for a monetary equilibrium to exist for some positive level of government expenditures. The second part requires that there is a finite relative price of consumption in terms of labor at which optimal behavior implies zero consumption.[7] This is a sufficient condition for the existence of a high inflation steady state.

3.2 Entrepreneurs

At the beginning of each time period young entrepreneurs simultaneously decide about their prices. Profit maximization by the competitive sector implies that the demand curve faced by entrepreneur j is given by

$$q_t^j = m_t \left(\frac{P_t}{P_t^j}\right)^{\frac{1}{\sigma}} \tag{6}$$

[7] A sufficient condition is given by $u'(0) < \infty$.

With a production technology that is linear in labor and product demand being given by (6), entrepreneur j maximizes:

$$\max_{P_t^j} E_{t-1}^* \left[m_t \left(\frac{P_t}{P_t^j} \right)^{\frac{1}{\sigma}} \left[P_t^j - P_t w_t \right] \right] \tag{7}$$

where

$$m_t = \frac{m_{t-1}}{\Pi_t} + g \tag{8}$$

$$w_t = w(n_t, \Pi_{t+1}) \tag{9}$$

Note that entrepreneurs maximize with respect to some (potentially) subjective expectations operator E_{t-1}^* that is based on the information set H_{t-1}, which does not contain the values of time t variables. Therefore, outside equilibrium, entrepreneurs choose prices without knowing the aggregate price level P_t.[8]

With the information set given by H_{t-1} entrepreneurs must formulate forecasts of P_t, m_t, and w_t. I require that these forecasts are consistent with the model structure, even when expectations are subjective. Moreover, I require that subjective expectations fulfill the law of iterated expectations. As shown below, these assumptions have the convenient implication that the forecasts of P_t, m_t, and w_t can be expressed as functions of the inflation forecast $E_{t-1}^* [\Pi_t]$ and $E_{t-1}^* [\Pi_{t+1}]$ and variables which are part of H_{t-1} and, thus, known to the entrepreneur.

First, consider the forecast $E_{t-1}^* [P_t]$. Clearly,

$$E_{t-1}^* [P_t] = E_{t-1}^*[\Pi_t] P_{t-1} \tag{10}$$

Second, consider the forecast $E_{t-1}^* [m_t]$. Using equation (8) one obtains:[9]

$$E_{t-1}^*[m_t] = \frac{m_{t-1}}{E_{t-1}^* [\Pi_t]} + g \tag{11}$$

Finally, consider the forecast $E_{t-1}^* [w_t]$. From equation (9) and the law of iterated expectations follows

$$E_{t-1}^* [w_t] = w \left(E_{t-1}^* [n_t], E_{t-1}^* [\Pi_{t+1}] \right) \tag{12}$$

[8] However, in a perfect foresight equilibrium where $E_{t-1}^* = E_{t-1}$ the future price level P_t is known and entrepreneurs act correspondingly.

[9] Following the learning literature I assume that agents have point expectations.

>From the linearity of the production technology and equation (6) it follows that labor demand can be expressed as

$$\begin{aligned} n_t &= \int q_t^i \mathrm{di} \\ &= m_t \left(P_t\right)^{\frac{1}{\sigma}} \left(\int \left(\frac{1}{P_t^i} \right)^{\frac{1}{\sigma}} \mathrm{di} \right) \end{aligned} \tag{13}$$

To simplify matters, I impose the restriction that entrepreneur j expects all other entrepreneurs to set the same price. By equation (2) this assumption implies $P_t^i = P_t$ (for all $i \neq j$) and equation (13) simplifies to

$$n_t = m_t \tag{14}$$

Combining this with equations (12) and (11) delivers the final result

$$E_{t-1}^* \left[w_t\right] = w \left(\frac{m_{t-1}}{E_{t-1}^* \left[\Pi_t\right]} + g, E_{t-1}^* \left[\Pi_t\right] \right) \tag{15}$$

Substituting (10), (11), and (15) into the objective function and taking the derivative with respect to P_t^j delivers the following expression for the profit maximizing price:

$$P_t^j = \frac{1}{1-\sigma} E_{t-1}^* (P_t w_t) \tag{16}$$

$$= \frac{1}{1-\sigma} E_{t-1}^* [\Pi_t] P_{t-1} w (\frac{m_{t-1}}{E_{t-1}^* \left[\Pi_t\right]} + g, E_{t-1}^* \left[\Pi_{t+1}\right]) \tag{17}$$

This is the familiar result that the optimal price P_t^j is a mark-up over expected costs where the mark-up factor depends on the degree of imperfect competition σ.

4 Rational Expectations Equilibria

This section characterizes the model's rational expectations equilibria (REE) and shows that the equilibria of the model with imperfect competition approach the ones of the competitive model as the degree of imperfect competition σ vanishes.

Equation (16) together with the fact that all firms charge the same price implies that current inflation can be expressed as

$$\Pi_t = \frac{1}{1-\sigma} E_{t-1} [\Pi_t w_t] \tag{18}$$

Imposing rational expectations delivers the equilibrium real wage

$$w_t = 1 - \sigma \tag{19}$$

Next, substitute (14) into (3), impose the market clearing condition $n_t = n(w_t, \Pi_{t+1})$ and use result (19) to substitute the wage. This delivers a single equation characterizing REE:

$$n(1-\sigma, \Pi_{t+1}) = \frac{n(1-\sigma, \Pi_t)}{\Pi_t} + g \tag{20}$$

Rational expectations equilibria can now be generated by choosing an initial inflation rate Π_0 and by iterating equation (20) forward.[10]

The REE in the competitive version of the model are characterized by an equation similar to (20) but with the equilibrium real wage given by $w_t = 1$. Since $n(\cdot, \cdot)$ is continuously differentiable and since the inflation rates in a REE must lie in the bounded interval $[0, \Pi^h]$, the REE of the model with imperfect competition will approach the ones of the perfectly competitive model as the degree of imperfect competition σ approaches zero.[11]

First, consider stationary REE. Imposing steady state conditions and $g = 0$ on equation (20) one finds two solutions, see condition 2. There is a low inflation steady state Π^l where money is valued and a high inflation steady state Π^h where money has no value:[12]

$$\Pi^l = 1 \quad \textit{and} \quad \Pi^h > 1$$

Since $\frac{\partial \Pi^h(g)}{\partial g} < 0$ money starts to become valued in the high inflation steady state for small positive levels of seignorage g. The steady state values of real balances m^l and m^h, which are equal to the steady state values of real output and labor input, are given by

$$m^l = n(1-\sigma, \Pi^l(g)) > m^h = n(1-\sigma, \Pi^h(g))$$

[10] The argument implicitly assumes that equation (20) has a solution for all t. As I will show below, this is the case for many initial inflation rates Π_0.

[11] More precisely: for any initial value Π_0 let Π_t^σ denote the path of inflation generated by iterating (19) forward with $w_t = 1 - \sigma$, then for any $\varepsilon > 0$ there exists a $\sigma > 0$ such that for all Π_0 : $\sup_t \left|\Pi_t^\sigma - \Pi_t^0\right| < \varepsilon$.

[12] Applying the implicit function theorem to (20), one can show that at $g = 0$: $\frac{\partial \Pi^l(g)}{\partial g} > 0$ and $\frac{\partial \Pi^h(g)}{\partial g} < 0$.

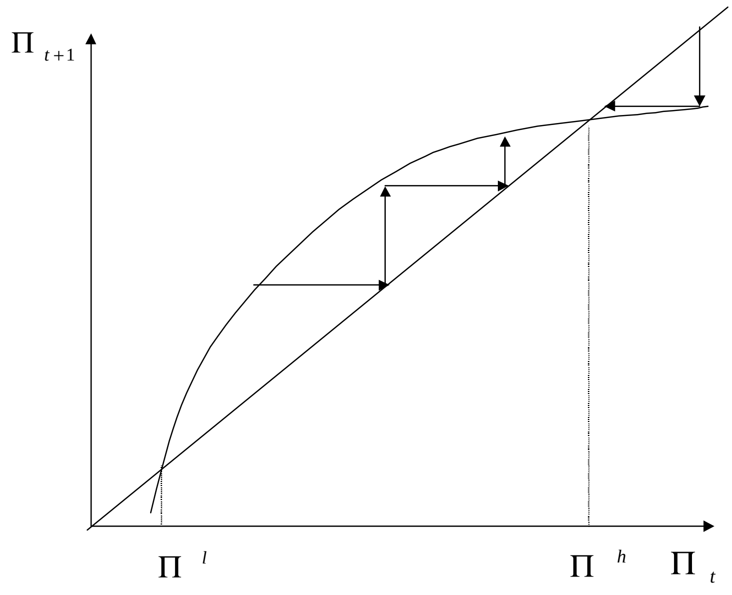

Figure 2: Rational Expectations Paths

As is easily verified, the low inflation steady state (Π^l, m^l) Pareto dominates the high inflation steady state (Π^h, m^h).

Next, consider nonstationary REE. Applying the implicit function theorem to equation (20), one finds that $\frac{d\Pi_{t+1}}{d\Pi_t} > 1$ at the low inflation steady state and $\frac{d\Pi_{t+1}}{d\Pi_t} < 1$ at the high inflation steady state. This implies that there exist initially non-stationary REE where Π_t either increases or decreases over time and where inflation asymptotically approaches Π^h, see figure (2) for an illustration of the situation.

5 Stability of Steady States under Adaptive Learning

This section analyzes the stability of the steady state equilibria under adaptive learning schemes. The analysis of non-stationary REE is deferred to section 6.

It is assumed that agents forecast future inflation rates according to the following simple adaptive rule where left-hand side subscripts denote the time at which expectations are formed and right-hand side subscripts the date for which the variable is forecasted:

$$ {}_t\Pi^e_{t+1} = {}_{t-1}\Pi^e_t + \gamma_t(\Pi_t - {}_{t-1}\Pi^e_t) \tag{21} $$

The new inflation forecast is equal to the previous forecast plus γ_t times the past forecast error, which is given by the term in the brackets. Note that the current inflation rate Π_t enters into the forecast made at time t, as required.

The gain parameter $\gamma_t \in (0,1)$ determines how fast expectations adapt to the forecast errors. Two kinds of gain sequences will be considered: constant gain learning rules where

$$\gamma_t = \gamma \text{ for all } t$$

and decreasing gain learning rules where

$$\lim_{t\to\infty} \gamma_t = 0 \qquad \text{and}$$
$$\sum_t \gamma_t = \infty$$

An example for a decreasing gain learning rule is given by $\gamma_t = \frac{1}{t}$. With this rule the forecast Π_t^e equals the average of past inflation rates, which is the least squares estimate of the steady state inflation rate.

Entrepreneurs must also forecast inflation rates two periods ahead, see equation (17). The forecast ${}_t\Pi_{t+2}^e$ is obtained by writing (21) for ${}_{t+1}\Pi_{t+2}^e$ and by taking time t expectations on both sides, which results in

$${}_t\Pi_{t+2}^e = {}_t\Pi_{t+1}^e$$

Learning rule (21), thus, imposes stationarity on agent's inflation expectations, which seems reasonable if agents believe that they have to learn a steady state.

5.1 The General (In-)Stability Result

This section derives the properties of steady state equilibria that guarantee that an economy that is populated by agents who forecast inflation according to rule (21) will converge to the steady state.

To derive these conditions I have to impose the following regularity conditions, which insure that various matrices are of full rank and that their eigenvectors do not lie right on the unit circle:

Condition 3 *At a stationary rational expectations equilibrium* (Π, m)

$$\frac{w_1}{1-\sigma}\frac{m}{\Pi^2} \neq \frac{w_2}{1-\sigma}\frac{\Pi-1}{\Pi}$$
$$\frac{w_1}{1-\sigma}\frac{m}{\Pi^2} \neq \frac{\Pi+1}{\Pi}$$
$$\gamma_t \neq \frac{m}{\Pi^2}\frac{w_1}{1-\sigma} - \frac{1}{\Pi} \quad \forall\ t$$

where

$$w_1 = \frac{\partial w}{\delta n}(m, \Pi) \quad \textit{and} \quad w_2 = \frac{\partial w}{\partial\, _{t-1}\Pi^e_{t+1}}(m, \Pi)$$

The following proposition states the main result of this section. The proof can be found in the appendix.

Proposition 1 *Consider a stationary rational expectations equilibrium* (Π, m) *and assume conditions 1, 2, and 3 hold. If at the steady state*

$$\frac{w_2}{1-\sigma}(\Pi - 1) < \frac{w_1}{1-\sigma}\frac{m}{\Pi} < \Pi + 1 \tag{22}$$

then there exists a $\overline{\gamma} > 0$ *such that the steady state is locally asymptotically stable*

1. *for all constant gain learning rules with adaptation rates* $0 < \gamma < \overline{\gamma}$.

2. *for all decreasing gain learning rules.*

If (22) does not hold, then there exists a $\overline{\gamma} > 0$ *such that the steady state is unstable*

1. *for all constant gain learning rules with adaptation rates* $0 < \gamma < \overline{\gamma}$.

2. *for all decreasing gain learning rules.*

The inequality on the right-hand side of (22) insures the stability of the steady state with respect to demand shocks while the left-hand side inequality insures the stability with respect to shocks in inflation expectations.

First, consider demand shocks. The term in the middle of (22) can be interpreted as the elasticity of the real wage with respect to the $t-1$ money stock m_{t-1}.[13] Due to mark-up pricing by entrepreneurs this elasticity is identical to the elasticity of the inflation rate with respect to m_{t-1}. From $w_1 > 0$ it then follows that inflation will

[13] Remember that the equilibrium real wage is given by $1 - \sigma$.

rise in response to positive demand shocks. In principal, this stabilizes the economy because it devaluates the excessive value of the money stock and thereby pushes the economy back towards the steady state. Yet, if the inflation reaction is too strong then a demand shock might create an even larger demand shock of the opposite sign in the subsequent period and the system might start to oscillate with increasing amplitude around the equilibrium. The term on the very right of (22) is a bound on the maximum inflation reaction which prevents this from happening.

Next, consider inflation expectations shocks and the following inequality

$$w_2 < w_1 \frac{m}{\Pi^2} \tag{23}$$

which is a sufficient condition for the inequality on the left-hand side of (22) to hold. A positive shock to agents' inflation expectations has two opposing effects on inflation.

Firstly, it causes firms to anticipate lower product demand because inflation devaluates old agents' real money balances. This causes a fall in expected labor demand and expected wages, which due to mark-up pricing puts downward pressure on inflation. This move down the labor supply function is captured by the term on the right of (23), which is the derivative of the real wage with respect to labor demand times the derivative of real money m_t (which is identical to labor demand n_t) with respect to Π_t.

Secondly, higher expected inflation taxes move the labor supply schedule upwards because workers have to be compensated with a higher real wage to offer any given amount of labor. This effect puts upward pressure on inflation and is captured by the term on the left of (23).

If the first effect dominates the second, then real wages will decrease in response to an increase in inflation expectations. The price setting equation (18) then implies that realized inflation will we lower than expected inflation. From the learning rule it then follows that inflation expectations will return over time to the steady state value.[14]

5.2 Learning the High Inflation Steady State

This section applies proposition 1 to study the stability of the high inflation steady state under adaptive learning.

[14] When equation 23) does not hold then the steady state is not necessarily unstable, see equation (22). The reason for this is that expectations feed also back on future expectations and thereby alter the expected money stock and future inflation rates. This is a channel which has been ignored in the previous argument and that is captured by the additional term on left of equation (22).

The following lemma shows that the high inflation steady state is unstable when agents forecast inflation according to equation (21):

Corollary 1 *For g sufficiently close to zero condition (22) never holds for the high-inflation steady state.*

Proof For contradiction suppose that the left-hand side of (22) holds. This requires

$$w_2\Pi^2 - w_2\Pi - w_1 m < 0$$

With $w_2 > 0$, a necessary condition for this is

$$\Pi < \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{w_1}{w_2}m}$$

As $g \to 0$, $\Pi \to \Pi^h$ and $m \to 0$. Since $w_2 > 0$ and $w_1 > 0$ at $(\Pi, m) = (\Pi^h, 0)$, this condition boils down to $\Pi < 1$, which contradicts $\Pi^h > 1$.

Importantly, instability is obtained independently from the degree of imperfect competition in the product market. Moreover, corollary 1 delivers the stability properties for the indeterminate steady state as reported by Marcet and Sargent (1989b) for a competitive economy with lagged information. However, the result contrast with the ones reported by Lettau and van Zandt (1999) for a competitive economy and contemporaneous information with multiplicities in the temporary equilibrium.[15] This

[15] To see why the stability properties of equilibria might differ in the three setups one has to compare the equations that determine the inflation rates: In Marcet and Sargent (1989b) inflation solves

$$n(w_t,\, {}_{t-1}\Pi^e_{t+1}) = \frac{m_{t-1}}{\Pi_t} + g$$

where the real wage is given by technology, i.e. $w_t \equiv 1$.

In Lettau and van Zandt (1999) inflation solves

$$n(w_t,\, {}_t\Pi^e_{t+1}) = \frac{m_{t-1}}{\Pi_t} + g$$

where also $w_t \equiv 1$.

The corresponding equation in the current model can be obtained by solving equation (18) for ${}_{t-1}w^e_t$ and by applying the identity $n(w(x,y),y) \equiv x$:

$$n({}_{t-1}w^e_t, {}_{t-1}\Pi^e_{t+1}) = \frac{m_{t-1}}{{}_{t-1}\Pi^e_t} + g \quad \text{with}$$
$${}_{t-1}w^e_t \equiv \frac{\Pi_t}{{}_{t-1}\Pi^e_t}(1-\sigma)$$

As is apparent, there are important timing differences between the three setups, that do not disappear as the degree of imperfect competition vanishes.

suggests that well-defined temporary equilibria should be an essential ingredient to learning models.

5.3 Learning the Low Inflation Steady State

Applying proposition 1 to the low inflation steady state delivers the following result:

Corollary 2 *For government expenditures g close enough to zero, the stability condition (22) holds at the low inflation steady state if and only if*

$$\varepsilon_{n,w} > \frac{1}{2} \tag{24}$$

where $\varepsilon_{n,w}$ is the real wage elasticity of labor supply at $(\Pi, m) = (1, n(1-\sigma, 1))$.

Proof All the terms in (22) are continuous in g. Therefore, when (22) holds for $g = 0$, it will also hold for sufficiently small but positive g. At $g = 0$, one has $\Pi = 1$ and $m = n(1-\sigma, 1) > 0$. For these values $\frac{1+\Pi}{\Pi} = 2$ and $\frac{w_2}{1-\sigma}\frac{\Pi-1}{\Pi} = 0$. Since $w_1 > 0$ and $m > 0$, the inequality in the left of (22) holds. Since $w = 1-\sigma$ in equilibrium, the term in the middle of (22) is equal to $\varepsilon_{w,n} = \frac{1}{\varepsilon_{n,w}}$, which establishes the claim.

The elasticity condition (24) insures the stability of the equilibrium with respect to demand shocks. If labor supply is inelastic ($\varepsilon_{n,w} < \frac{1}{2}$) then inflation reacts too strongly to demand shocks and causes the economy's real money stock to oscillate with increasing amplitude around the equilibrium value, as described in section 5.1.

Clearly, the degree of imperfect competition might now have an impact on the stability of the steady state because a change in the degree of competition will alter firms' markup over costs and thereby affect the equilibrium output and the equilibrium labor supply.

When labor supply displays a constant elasticity, then the degree of imperfect competition has no impact on the stability of the low inflation steady state. However, if the elasticity of labor supply is decreasing in the amount of supplied labor, which seems not an unreasonable assumption, then increasing the competition between entrepreneurs can cause the low inflation equilibrium to become unstable. On the other hand, if the elasticity of labor supply was increasing, then making the economy more competitive would keep the low inflation equilibrium stable or even cause it to become so. Finally, non-monotonic labor supply elasticities may generate non-monotonic relationships between the degree of imperfect competition and the stability of the low inflation steady state.

6 A stochastic model

This section considers a stochastic version of the model with a government expenditure shock. Such a setup facilitates the stability analysis of non-stationary rational expectations equilibria.

Government seignorage is now composed of a fixed and a random component

$$g_t = g + v_t$$

with v_t being a mean zero random disturbance with small bounded support. Real money now evolves according to the following stochastic law of motion

$$m_t = \frac{m_{t-1}}{\Pi_t} + g + v_t \tag{25}$$

To determine the REE linearize (18) and (25) around the deterministic steady states (Π^n, m^n) with $n = l, h$ to obtain:[16]

$$\begin{pmatrix} \Pi_t \\ m_t \end{pmatrix} = \alpha^n + \beta_0^n E_{t-1}\left[\begin{pmatrix} \Pi_t \\ m_t \end{pmatrix}\right] + \beta_1^n E_{t-1}\left[\begin{pmatrix} \Pi_{t+1} \\ m_{t+1} \end{pmatrix}\right]$$
$$+ \delta^n \begin{pmatrix} \Pi_{t-1} \\ m_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ v_t \end{pmatrix} \tag{26}$$

As is shown in the appendix, all rational expectations solutions to (26) have a minimum state variable representation as an AR(1) process

$$\begin{pmatrix} \Pi_t \\ m_t \end{pmatrix} = a + B \begin{pmatrix} \Pi_{t-1} \\ m_{t-1} \end{pmatrix} \quad \text{with} \tag{27}$$

$$a = \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} \text{ and } B = \begin{pmatrix} b^{11} & b^{12} \\ b^{21} & b^{22} \end{pmatrix} \tag{28}$$

where $b_{12} \neq 0$. Since the real money stock is permanently shocked, rational expectations require that agents condition their inflation forecasts on these money balances.

As is also shown in the appendix, there are two REE in the neighborhood of the deterministic low inflation steady state: a stochastic steady state with coefficients $(a^{l,1}, B^{l,1})$ and a non-stationary solution with coefficients $(a^{l,2}, B^{l,2})$ that diverges from the low inflation steady state. Similarly, in the neighborhood of the deterministic high inflation steady state there exists a stochastic steady state with coefficients $(a^{h,1}, B^{h,1})$ and an initially non-stationary solution with coefficients $(a^{h,2}, B^{h,2})$ that approaches the high inflation steady state.[17]

[16] The linearization coefficients are stated in the appendix.

[17] The high inflation steady state $(a^{h,2}, B^{h,2})$ exists only for $g > 0$. See the appendix for details.

6.1 Learning Stationary and Non-Stationary REE

In this section I suppose that learning can be modeled by least squares estimation.[18]

In particular, I assume that agents estimate the coefficients (a, B) of equation (27) by ordinary least squares estimation. Since this equation denotes the minimum state variable solution of the model, it is the simplest equation that agents could estimate with the hope to acquire rational beliefs in the long run.

When agents have information H_t, the corresponding parameter estimates are denoted by a_t and B_t. To form their inflation expectations, agents use these estimates to iterate (27) into the future. Thus, the $t-1$ forecast of Π_t is given by

$$\Pi_t^e = a_{t-1}^1 + b_{t-1}^{11}\Pi_{t-1} + b_{t-1}^{12}m_{t-1}$$

and the $t-1$ forecast of Π_{t+1} by

$$_{t-1}\Pi_{t+1}^e = a_{t-1}^1 + b_{t-1}^{11}\,\Pi_t^e + b_t^{12}\,m_t^e$$

where

$$m_t^e = a_{t-1}^2 + b_{t-1}^{21}\Pi_{t-1} + b_{t-1}^{22}m_{t-1}$$

Whether the least squares estimates (a_t, B_t) locally converge to the values of the rational expectations solution is governed by the so-called E-stability criterion, see Evans and Honkapohja (2001) chapter 10. As the following proposition shows the low inflation steady state is the unique stable rational expectations equilibrium. The proof can be found in the appendix.

Proposition 2 *There exists a level of government expenditures $\overline{g} > 0$ such that for all levels $0 \leq g < \overline{g}$ the following holds. At the low inflation equilibrium*

- *the stochastic steady state $(a^{l,1}, B^{l,1})$ is E-stable,*
- *the non-stationary REE $(a^{l,2}, B^{l,2})$ is E-unstable,*

At the high inflation equilibrium

- *the stochastic steady state $(a^{h,1}, B^{h,1})$ is E-unstable,*

[18] I restrict consideration to decreasing gain learning rules since constant gain learning rules may generate 'escape dynamics', see Cho and Sargent (1999).

- *the non-stationary REE* $(a^{h,2}, B^{h,2})$ *is E-unstable.*

In the stochastic model the low inflation steady state is E-stable regardless of elasticity of labor supply at the steady state. The intuition for this result is straightforward: in the stochastic steady state $(a^{l,1}, B^{l,1})$ the impact of past money on current inflation is given by $b_{12} = \frac{1}{m^l}$. Therefore, shocks to m_{t-1} lead to a corresponding increase in inflation expectations that implies that future real balances are expected to be back at their steady state value. As a result, firms do not expect an increase in labor demand in response to past demand shocks, which makes the labor supply elasticity irrelevant for entrepreneur's price setting behavior.

Thus, the fact that agents condition their inflation expectations on past money balances improves the stability of the determinate steady state.

7 Conclusions

The present paper considered a monetary overlapping generations model with imperfect competition and studied the stability of its rational expectations equilibria under adaptive learning.

Adaptive learning has been found to be a powerful tool to select between the rational expectations equilibria of the model. First, out of the multitude of equilibria at most one equilibrium turned out to be stable. Second, the instability of the non-stationary equilibria and of the indeterminate high inflation steady state has been found to be independent from the degree of imperfect competition between entrepreneurs. Third, decreasing gain learning rules and constant gain learning rules with small gain parameter always selected the same equilibrium. Forth, although money demand could depend on contemporaneous prices the stability properties of the equilibria are virtually unchanged compared to the case of lagged information.

These results should give further confidence to the usefulness of adaptive expectations as an equilibrium selection device.

8 Appendix

8.1 Appendix to Section 5

Proof 4 (Proof of Proposition 1:) *Substitute (18) into (21):*

$$ {}_t\Pi^e_{t+1} = {}_{t-1}\Pi^e_t + \gamma_t \left(M({}_{t-1}\Pi^e_t, m_{t-1}) - {}_{t-1}\Pi^e_t \right) \quad \textit{with} \tag{29} $$

$$ M({}_{t-1}\Pi^e_t, m_{t-1}) \equiv {}_{t-1}\Pi^e_t \frac{w\left(\frac{m_{t-1}}{{}_{t-1}\Pi^e_t} + g,\ {}_{t-1}\Pi^e_t\right)}{1-\sigma} \tag{30} $$

The above equation describes the new inflation expectations as a function of past expectations and past real money holdings. Real money evolves according to

$$ m_t = \frac{m_{t-1}}{M({}_{t-1}\Pi^e_t, m_{t-1})} + g \tag{31} $$

Linearizing (29) and (31) around a steady state (Π, m) *yields*

$$ \begin{pmatrix} \theta_{1,t} \\ \theta_{2,t} \end{pmatrix} = A(\gamma_t) \begin{pmatrix} \theta_{1,t-1} \\ \theta_{2,t-1} \end{pmatrix} + \begin{pmatrix} \gamma_t r_{1,t} \\ r_{2,t} \end{pmatrix} \tag{32} $$

where

$$ \theta_{1,t} = {}_t\Pi^e_{t+1} - \Pi $$
$$ \theta_{2,t} = m_t - m $$

are the deviations from the equilibrium values, the $r_{i,t}$ *are second order approximation errors, and* $A(\gamma_t)$ *is a 2x2 matrix given by*

$$ A(\gamma_t) = \begin{pmatrix} 1 + \gamma_t(M_1(\Pi, m) - 1) & \gamma_t M_2(\Pi, m) \\ -\frac{m}{\Pi^2} M_1(\Pi, m) & \frac{1}{\Pi} - \frac{m}{\Pi^2} M_2(\Pi, m) \end{pmatrix} $$

where M_i *is the partial derivative of* M *with respect to the* i*-th argument.*

The eigenvalues of $A(0)$ *are given by*

$$ \lambda_1 = 1 $$
$$ \lambda_2 = \frac{1}{\Pi} - \frac{m}{\Pi^2} \frac{\omega_1}{(1-\sigma)} $$

and the eigenvalues of $A(\gamma_t)$ *by*

$$ \lambda_{1,t} = \lambda_1 + \frac{\partial \lambda_1}{\partial \gamma} \gamma_t + O(\gamma_t^2) \tag{33} $$

$$ \lambda_{2,t} = \lambda_2 + \frac{\partial \lambda_2}{\partial \gamma} \gamma_t + O(\gamma_t^2) \tag{34} $$

where the last terms are second order approximation errors.

If condition (22) holds, then $|\lambda_2| < 1$*. The regularity condition 3 implies* $\left|\frac{\partial \lambda_2}{\partial \gamma}\right| < \infty$ *at* $\gamma = 0$*. Thus,* $|\lambda_{2,t}| < 1$ *for small enough* γ_t*. Next, consider the eigenvalue* $\lambda_{1,t}$*. Use the characteristic polynomial of* $A(\gamma_t)$ *and apply the implicit function theorem to obtain* $\frac{\partial \lambda_1}{\partial \gamma}$ *at* $\gamma = 0$ *:*

$$\begin{aligned}\frac{\partial \lambda_1}{\partial \gamma} &= -\frac{\partial P(A(\gamma))/\partial \gamma}{\partial P(A(\gamma))/\partial \lambda} \\ &= \frac{\frac{1}{1-\sigma}\left(w_2(\Pi - 1) - w_1 \frac{m}{\Pi}\right)}{1 - \lambda_2}\end{aligned}$$

Condition (22) implies that $\frac{\partial \lambda_1}{\partial \gamma} < 0$ *and thereby* $|\lambda_{1,t}| < 1$ *for small enough* γ_t*. Thus, (22) implies that both eigenvalues of* $A(\gamma_t)$ *are within the unit circle for* γ_t *sufficiently small. Otherwise, at least one eigenvalue lies outside the unit circle. This establishes the stability and instability claims for the constant gain learning rules.*

The remaining part of the proof considers decreasing gain learning rules. The function $M(\cdot,\cdot)$ *as defined in (30) is continuously differentiable in both arguments in a neighborhood of* $\theta = (0, 0)$*. Therefore, for all constants* $K_1 > 0$ *and* $K_2 > 0$ *with* K_1 *and* K_2 *arbitrarily small there exists a neighborhood to* $(0, 0)$ *where the absolute values of the approximation errors in (32) are bounded by*

$$|r_{1,t}| \leq K_1 \left(|\theta_{1,t-1}| + |\theta_{2,t-1}|\right) \tag{35a}$$

$$|r_{2,t}| \leq K_2 \left(|\theta_{1,t-1}| + |\theta_{2,t-1}|\right) \tag{35b}$$

Next, consider the eigenvectors e_1 *and* e_2 *of* $A(0)$ *corresponding to the eigenvectors* λ_1 *and* λ_2*, respectively:*

$$\begin{aligned} e_1 &= \begin{pmatrix} e_{11} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1 - \frac{1}{\Pi} + \frac{m}{\Pi^2} M_2}{-\frac{m}{\Pi^2} M_1} \\ 1 \end{pmatrix} \\ e_2 &= \begin{pmatrix} e_{21} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

The eigenvectors $e_{1,t}$ *and* $e_{2,t}$ *of* $A(\gamma_t)$ *corresponding to the eigenvalues* $\lambda_{1,t}$ *and* $\lambda_{2,t}$*,* *respectively, are given by*

$$e_{1,t} = \begin{pmatrix} e_{11,t} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_{1,t} - \frac{1}{\Pi} + \frac{m}{\Pi^2} M_2}{-\frac{m}{\Pi^2} M_1} \\ 1 \end{pmatrix}$$

$$e_{2,t} = \begin{pmatrix} e_{21,t} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_{2,t} - \frac{1}{\Pi} + \frac{m}{\Pi^2} M_2}{-\frac{m}{\Pi^2} M_1} \\ 1 \end{pmatrix}$$

Now consider the vector base consisting of the eigenvectors $(e_{1,t}, e_{2,t})$ *of* $A(\gamma_t)$*. Let the vector* (θ_1, θ_2) *have representation* $(\rho_1, \rho_2)_{\gamma_t}$ *with this new base, i.e.*

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = (e_{1,t}, e_{2,t}) \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}_{\gamma_t} \tag{36}$$

where the subscript γ_t *indicates the base to which the coordinates refer. Then from (32)*

$$\begin{pmatrix} \rho_{1,t} \\ \rho_{2,t} \end{pmatrix}_{\gamma_t} = \begin{pmatrix} \lambda_{1,t} & 0 \\ 0 & \lambda_{2,t} \end{pmatrix} \begin{pmatrix} \rho_{1,t-1} \\ \rho_{2,t-1} \end{pmatrix}_{\gamma_t} + \begin{pmatrix} s_{1,t} \\ s_{2,t} \end{pmatrix}_{\gamma_t} \tag{37}$$

where the approximation errors are given by

$$\begin{pmatrix} s_{1,t} \\ s_{2,t} \end{pmatrix}_{\gamma_t} = \begin{pmatrix} \frac{1}{e_{11,t} - e_{21,t}} (\gamma_t r_{1,t} - e_{21,t} r_{2,t}) \\ \frac{1}{e_{11,t} - e_{21,t}} (-\gamma_t r_{1,t} + e_{11,t} r_{2,t}) \end{pmatrix}_{\gamma_t}$$

Using (35a), (35b), and (36) the approximation errors can be bounded as follows:

$$\begin{aligned} |s_{1,t}| &\le \frac{1}{|e_{11,t} - e_{21,t}|} (\gamma_t |r_{1,t}| + |e_{21,t}| |r_{2,t}|) \\ &\le \frac{1}{|e_{11,t} - e_{21,t}|} (\gamma_t K_1 + |e_{21,t}| K_2) ((|\theta_{1,t-1}| + |\theta_{2,t-1}|)) \\ &= \frac{1}{|e_{11,t} - e_{21,t}|} (\gamma_t K_1 + |e_{21,t}| K_2) (\left|\rho_{1,t-1} e_{11,t} + \rho_{2,t-1} e_{21,t}\right| + \left|\rho_{1,t-1} + \rho_{2,t-1}\right|) \\ &\le \frac{1}{|e_{11,t} - e_{21,t}|} (\gamma_t K_1 + |e_{21,t}| K_2) (\left|\rho_{1,t-1}\right| (|e_{11,t}| + 1) + \left|\rho_{2,t-1}\right| (|e_{21,t}| + 1)) \end{aligned}$$

Since $\lim_{t \to \infty} |e_{11,t} - e_{21,t}| > 0$, $|e_{21,t}| \sim O(\gamma_t)$, *and since* K_1 *and* K_2 *can be made arbitrarily small it follows that*

$$|s_{1,t}| \le \gamma_t K_1' (\left|\rho_{1,t-1}\right| + \left|\rho_{2,t-1}\right|) \tag{38}$$

for some $K_1' > 0$ *that can also be made arbitrarily small by considering a sufficiently* *small neighborhood around the steady state. Similarly,*

$$\begin{aligned}|s_{2,t}| &\leq \frac{1}{|e_{11,t}-e_{21,t}|}(\gamma_t |r_{1,t}| + |e_{11,t}| |r_{2,t}|) \\ &\leq \frac{1}{|e_{11,t}-e_{21,t}|}(\gamma_t K_1 + |e_{11,t}| K_2)(\left|\rho_{1,t-1}\right| (|e_{11,t}|+1) + \left|\rho_{2,t-1}\right| (|e_{21,t}|+1)) \\ &\leq K_2'(\left|\rho_{1,t-1}\right| + \left|\rho_{2,t-1}\right|) \end{aligned} \tag{39}$$

for some $K_2' > 0$ *arbitrarily small.*

An inconvenient feature of (37) is that the coordinates are expressed in terms of a different vector base for each γ_t. *Therefore, I rewrite (37) with coordinates from the vector base* (e_1, e_2). *This base is almost identical to the base* $(e_{1,t}, e_{2,t})$ *for small* γ_t.

Let $(\rho_1, \rho_2)_{\gamma_t}$ *have representation* $(\alpha_1, \alpha_2)_0$ *with base* (e_1, e_1), *i.e.*

$$\begin{aligned}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}_0 &= \begin{pmatrix} a_{11,t} & a_{12,t} \\ a_{21,t} & a_{22}, t\end{pmatrix}\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}_{\gamma_t} \\ &= (e_1, e_2)^{-1}(e_{1,t}, e_{2,t})\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}_{\gamma_t} \\ &= \begin{pmatrix} \frac{e_{11,t}}{e_{11}} & \frac{e_{21,t}}{e_{11}} \\ 1-\frac{e_{11,t}}{e_{11}} & 1-\frac{e_{21,t}}{e_{11}}\end{pmatrix}\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}_{\gamma_t}\end{aligned} \tag{40}$$

or conversely

$$\begin{aligned}\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}_{\gamma_t} &= \begin{pmatrix} b_{11,t} & b_{12,t} \\ b_{21,t} & b_{22,t}\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}_0 \\ &= \begin{pmatrix} \frac{e_{11}-e_{21,t}}{e_{11,t}-e_{21,t}} & -\frac{e_{21,t}}{e_{11,t}-e_{21,t}} \\ \frac{e_{11,t}-e_{11}}{e_{11,t}-e_{21,t}} & \frac{e_{11,t}}{e_{11,t}-e_{21,t}}\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}_0\end{aligned} \tag{41}$$

One can express the bound on the approximation error in (38) in new coordinates

$$\begin{aligned}|s_{1,t}| &\leq \gamma_t K_1'(|b_{11,t}\alpha_{1,t-1} + b_{12,t}\alpha_{2,t-1}| + |b_{21,t}\alpha_{1,t-1} + b_{22,t}\alpha_{2,t-1}|) \\ &\leq \gamma_t K_1'\left((|b_{11,t}| + |b_{21,t}|)\,|\alpha_{1,t-1}| + (|b_{12,t}| + |b_{22,t}|)\,|\alpha_{2,t-1}|\right) \\ &\leq \gamma_t K_1''\left(|\alpha_{1,t-1}| + |\alpha_{2,t-1}|\right)\end{aligned} \tag{42}$$

for $K_1'' > 0$ *and arbitrarily small for a sufficiently small neighborhood. Similarly for the bound in (39)*

$$|s_{2,t}| \leq K_2''(|\alpha_{1,t-1}| + |\alpha_{2,t-1}|) \tag{43}$$

with $K_2'' > 0$, *arbitrarily small. From (37), (40), and (41)*

$$\begin{aligned}\alpha_{1,t} &= a_{11,t}\rho_{1,t} + a_{12,t}\rho_{2,t} \\ &= a_{11,t}\left(\lambda_{1,t}\rho_{1,t-1} + s_{1,t}\right) + a_{12,t}(\left(\lambda_2\rho_{2,t-1} + s_{2,t}\right) \\ &= a_{11,t}\left(\lambda_{1,t}\left(b_{11,t}\alpha_{1,t-1} + b_{12,t}\alpha_{2,t-1}\right) + s_{1,t}\right) \\ &+ a_{12,t}\left(\lambda_{2,t}\left(b_{21,t}\alpha_{1,t-1} + b_{22,t}\alpha_{2,t-1}\right) + s_{2,t}\right) \\ &= \left(a_{11,t}\lambda_{1,t}b_{11,t} + a_{12,t}\lambda_{2,t}b_{21,t}\right)\alpha_{1,t-1} \\ &+ \left(a_{11,t}\lambda_{1,t}b_{12,t} + a_{12,t}\lambda_{2,t}b_{22,t}\right)\alpha_{2,t-1} + a_{11,t}s_{1,t} + a_{12,t}s_{2,t} \end{aligned} \tag{44}$$

and similarly

$$\begin{aligned}\alpha_{2,t} &= \left(a_{21,t}\lambda_{1,t}b_{11,t} + a_{22,t}\lambda_{2,t}b_{21,t}\right)\alpha_{1,t-1} \\ &+ \left(a_{21,t}\lambda_{1,t}b_{12,t} + a_{22,t}\lambda_{2,t}b_{22,t}\right)\alpha_{2,t-1} + a_{21,t}s_{1,t} + a_{22,t}s_{2,t} \end{aligned} \tag{45}$$

Using (42), (43), and (44) one can construct upper and lower bounds for $|\alpha_{1,t}|$:

$$\begin{aligned}|\alpha_{1,t}| &\le \left|\left(a_{11,t}\lambda_{1,t}b_{11,t} + a_{12,t}\lambda_{2,t}b_{21,t}\right)\right| |\alpha_{1,t-1}| \\ &+ \left|\left(a_{11,t}\lambda_{1,t}b_{12,t} + a_{12,t}\lambda_{2,t}b_{22,t}\right)\right| |\alpha_{2,t-1}| + |a_{11,t}| |s_{1,t}| + |a_{12,t}| |s_{2,t}| \\ &\le \left(\left|\left(a_{11,t}\lambda_{1,t}b_{11,t} + a_{12,t}\lambda_{2,t}b_{21,t}\right)\right| + \gamma_t |a_{11,t}| K_1'' + |a_{12,t}| K_2''\right) |\alpha_{1,t-1}| \\ &+ \left(\left|\left(a_{11,t}\lambda_{1,t}b_{12,t} + a_{12,t}\lambda_{2,t}b_{22,t}\right)\right| + \gamma_t |a_{11,t}| K_1'' + |a_{12,t}| K_2''\right) |\alpha_{2,t-1}|\end{aligned}$$

Note that the terms $a_{12,t}$, $b_{12,t}$, *and* $b_{21,t}$ *are of order* $O(\gamma_t)$. *Moreover, using a Taylor series expansion*

$$\begin{aligned}a_{11,t}\lambda_{1,t}b_{11,t} &= \left(1 + C\frac{\partial\lambda_1}{\partial\gamma}\gamma_t + O(\gamma_t^2)\right)\left(1 + \frac{\partial\lambda_1}{\partial\gamma}\gamma_t + O(\gamma_t^2)\right)\left(1 - C\frac{\partial\lambda_1}{\partial\gamma}\gamma_t + O(\gamma_t^2)\right) \\ &= 1 + \frac{\partial\lambda_1}{\partial\gamma}\gamma_t + O(\gamma_t^2)\end{aligned}$$

Therefore

$$|\alpha_{1,t}| \le (1 + \gamma_t V_{11}) |\alpha_{1,t-1}| + \gamma_t V_{12} |\alpha_{2,t-1}|$$

where $V_{12} > 0$ *and* V_{11} *of the same sign as* $\frac{\partial\lambda_1}{\partial\gamma}$. *Also* V_{11} *and can be made arbitrarily close to* $\frac{\partial\lambda_1}{\partial\gamma}$ *by choosing a sufficiently small neighborhood.*

A lower bound for $|\alpha_{1,t}|$ *is given by*

$$\begin{aligned}|\alpha_{1,t}| &\ge \left|\left(a_{11,t}\lambda_{1,t}b_{11,t} + a_{12,t}\lambda_{2,t}b_{21,t}\right)\right| |\alpha_{1,t-1}| \\ &- \left|\left(a_{11,t}\lambda_{1,t}b_{12,t} + a_{12,t}\lambda_{2,t}b_{22,t}\right)\right| |\alpha_{2,t-1}| - |a_{11,t}| |s_{1,t}| - |a_{12,t}| |s_{2,t}| \\ &\ge \left(\left|\left(a_{11,t}\lambda_{1,t}b_{11,t} + a_{12,t}\lambda_{2,t}b_{21,t}\right)\right| - \gamma_t |a_{11,t}| K_1'' - |a_{12,t}| K_2''\right) |\alpha_{1,t-1}| \\ &- \left(\left|\left(a_{11,t}\lambda_{1,t}b_{12,t} + a_{12,t}\lambda_{2,t}b_{22,t}\right)\right| + \gamma_t |a_{11,t}| K_1'' + |a_{12,t}| K_2''\right) |\alpha_{2,t-1}| \\ &\ge (1 + \gamma_t W_{11}) |\alpha_{1,t-1}| - \gamma_t W_{12} |\alpha_{2,t-1}|\end{aligned}$$

with $W_{12} > 0$ *and* W_{11} *of the same sign and arbitrarily close to* $\frac{\partial \lambda_1}{\partial \gamma}$, *by the same arguments as above.*

Next use (42), (43), and (45) to get bounds for $|\alpha_{2,t}|$:

$$\begin{aligned} |\alpha_{2,t}| \leq & \left(|(a_{21,t}\lambda_{1,t}b_{11,t} + a_{22,t}\lambda_{2,t}b_{21,t})| + \gamma_t |a_{21,t}| K_1'' + |a_{22,t}| K_2'' \right) |\alpha_{1,t-1}| \\ & + \left(|(a_{21}\lambda_{1,t}b_{12,t} + a_{22,t}\lambda_{2,t}b_{22,t})| + \gamma_t |a_{21,t}| K_1'' + |a_{22,t}| K_2'' \right) |\alpha_{2,t-1}| \end{aligned}$$

Since $a_{21,t}$, $b_{12,t}$, *and* $b_{21,t}$ *are of order* $O(\gamma_t)$

$$|\alpha_{2,t}| \leq V_{21} |\alpha_{1,t-1}| + V_{22} |\alpha_{2,t-1}|$$

with $V_{21} > 0$, $V_{22} > 0$. *Moreover,by choosing a sufficiently small neighborhood and a* t *large enough one can choose* V_{21} *arbitrarily close to zero. Also, since*

$$\lim a_{22,t} = \lim b_{22,t} = 1$$

one can choose $V_{22} < 1$ *when* $|\lambda_2| < 1$ *and* $V_{22} > 1$ *when* $|\lambda_2| > 1$ *for all* t *sufficiently large and all sufficiently small neighborhoods.*

A lower bound for $|\alpha_{2,t}|$ *is given by*

$$\begin{aligned} |\alpha_{2,t}| \geq & - \left(|(a_{21,t}\lambda_{1,t}b_{11,t} + a_{22,t}\lambda_{2,t}b_{21,t})| + \gamma_t |a_{21,t}| K_1'' + |a_{22,t}| K_2'' \right) |\alpha_{1,t-1}| \\ & + \left(|(a_{21,t}\lambda_{1,t}b_{12,t} + a_{22,t}\lambda_{2,t}b_{22,t})| - \gamma_t |a_{21,t}| K_1'' - |a_{22,t}| K_2'' \right) |\alpha_{2,t-1}| \\ & \geq -W_{21} |\alpha_{1,t-1}| + W_{22} |\alpha_{1,t-1}| \end{aligned}$$

with $W_{21} > 0$, $W_{22} > 0$. *By the same arguments as above, for* t *sufficiently large and a sufficiently small neighborhood* $W_{22} < 1$ *if* $|\lambda_2| < 1$ *and* $W_{22} > 1$ *if* $|\lambda_2| > 1$.

Collecting the previous bounds we have

$$W_t \begin{pmatrix} |\alpha_{1,t-1}| \\ |\alpha_{2,t-1}| \end{pmatrix} \leq \begin{pmatrix} |\alpha_{1,t}| \\ |\alpha_{2,t}| \end{pmatrix} \leq V_t \begin{pmatrix} |\alpha_{1,t-1}| \\ |\alpha_{2,t-1}| \end{pmatrix} \tag{46}$$

$$\text{where} \quad W_t = \begin{pmatrix} 1 + \gamma_t W_{11} & -\gamma_t W_{12} \\ -W_{21} & W_{22} \end{pmatrix}$$

$$V_t = \begin{pmatrix} 1 + \gamma_t V_{11} & \gamma_t V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

where the inequalities should be interpreted component-wise. Now choose a time t^* *and a neighborhood* U *such that one can choose* $W_{22} < 1$ *(*$W_{22} > 1$*), and* $V_{22} < 1$ $(V_{22} > 1)$ *if* $|\lambda_2| < 1$ $(|\lambda_2| > 1)$.

I now assume (22) holds and will prove the stability part for decreasing gain learning rules. First, construct a matrix norm $\|\cdot\|_h$ *and a compatible vector norm* $|\cdot|_h$ *such that*

$$|Sx|_h \leq \|S\|_h \, |x|_h \tag{47}$$

for all 2×2 *matrices* S *and* 2×1 *vectors* x*. Define the matrix norm as follows*

$$\|S\|_h = \left\| D_h S D_h^{-1} \right\|_{\max} \text{ with}$$
$$D_h = \begin{pmatrix} h^2 & 0 \\ 0 & h \end{pmatrix}$$

where $\|\cdot\|_{\max}$ *is the maximum absolute norm defined by* $\|M\|_{\max} = \max_{i,j} |M_{i,j}|$*. A compatible vector norm is given by (see Horn and Johnson (1985), p.297)*

$$|x|_h = \|(x,x)\|_h$$

where (x,x) *is the matrix whose columns consist of the vectors* x*.*

With these definitions we have

$$\|V_t\|_h = \left\| \begin{pmatrix} 1 + \gamma_t V_{11} & \gamma_t h V_{12} \\ h^{-1} V_{21} & V_{22} \end{pmatrix} \right\|_{\max}$$

Now choose h *large enough such that*

$$h^{-1} V_{21} < V_{22}$$

and a time $t^{**} \geq t^*$ *large enough such that for all* $t \geq t^{**}$

$$\gamma_t h V_{12} < V_{22} < 1 + \gamma_t V_{11}$$

Then for $t \geq t^{**}$

$$\|V_t\|_h = 1 + \gamma_t V_{11}$$

and by (47)

$$\left\| V_t \cdot \begin{pmatrix} |\alpha_{1,t-1}| \\ |\alpha_{2,t-1}| \end{pmatrix} \right\|_h \leq (1 + \gamma_t V_{11}) \left| \begin{pmatrix} |\alpha_{1,t-1}| \\ |\alpha_{2,t-1}| \end{pmatrix} \right|_h \tag{48}$$

Since the vector norm $|\cdot|_h$ *is absolute, i.e.*

$$|x|_h = |\,|x|\ \ |_h$$

it follows (from Horn and Johnson (1985), p.285) that it is monotone. From (46) and (48) we therefore have that

$$\left|\left(\begin{array}{c}|\alpha_{1,t}|\\|\alpha_{2,t}|\end{array}\right)\right|_h - \gamma_t V_{11}\left|\left(\begin{array}{c}|\alpha_{1,t-1}|\\|\alpha_{2,t-1}|\end{array}\right)\right|_h \le \left|\left(\begin{array}{c}|\alpha_{1,t-1}|\\|\alpha_{2,t-1}|\end{array}\right)\right|_h \tag{49}$$

Since $V_{11} < 0$ when (22) holds, $|\alpha_t|$ is a strictly decreasing positive sequence. This implies that it has a limit $\alpha^* \ge 0$. I now show that $\alpha^* = 0$. Summing the left- and right-hand side of equation (49) for t to $t+s$ yields

$$\left|\left(\begin{array}{c}|\alpha_{1,t+s}|\\|\alpha_{2,t+s}|\end{array}\right)\right|_h - V_{11}\sum_{i=0}^{s}\gamma_{t+i}\left|\left(\begin{array}{c}|\alpha_{1,t-1+i}|\\|\alpha_{2,t-1+i}|\end{array}\right)\right|_h \le \left|\left(\begin{array}{c}|\alpha_{1,t-1}|\\|\alpha_{2,t-1}|\end{array}\right)\right|_h$$

Now assume $\alpha^* > 0$. Then we can divide the previous expression by the norm of $|\alpha_{t+s-1}|_h$ which together with the fact that $|\alpha_t|_h$ is decreasing yields

$$\frac{\left|\left(\begin{array}{c}|\alpha_{1,t+s}|\\|\alpha_{2,t+s}|\end{array}\right)\right|_h}{\left|\left(\begin{array}{c}|\alpha_{1,t-1+s}|\\|\alpha_{2,t-1+s}|\end{array}\right)\right|_h} - V_{11}\sum_{i=0}^{s}\gamma_{t+i} \le \frac{\left|\left(\begin{array}{c}|\alpha_{1,t-1}|\\|\alpha_{2,t-1}|\end{array}\right)\right|_h}{\left|\left(\begin{array}{c}|\alpha_{1,t-1+s}|\\|\alpha_{2,t-1+s}|\end{array}\right)\right|_h}$$

Since $\sum_t \gamma_t = \infty$ the left-hand side will increase without bound as s increases. But then $|a_{t-1+s}|_h$ must converge to zero, a contradiction. Therefore, $\lim_{t\to\infty}|\alpha_t|_h = \alpha^* = 0$.

This establishes that there exists a neighborhood U of $\alpha = (0,0)$ such that if $\alpha_t \in U$ at a time $t \ge t^{**}$, then $\alpha_t \to (0,0)$. By continuity of $M_i(\cdot,\cdot)$ and the fact that (32) has a fixed point at $(0,0)$ for all γ_t, α_t remains in U for $t \le t^{**}$ if the initial values α_0 are chosen from another sufficiently small neighborhood $U' \subset U$. This establishes the asymptotic stability result for decreasing gain learning rules.

I now proceed with the instability part of the proposition. When (22) does not hold then $\frac{\partial\lambda_1}{\partial\gamma} > 0$, or $|\lambda_2| > 1$, or both.

First suppose $|\lambda_2| > 1$. Then $W_{22} > 1$ for $t \ge t^*$. Consider the cone $C_\beta = \{(\alpha_1,\alpha_2)|\,|\alpha_2| \ge \beta\,|\alpha_1|\}$. I will show that there exists a finite time $t^{**} \ge t^*$ and a neighborhood $U' \subset U$ such that if $\alpha_t \in C_\beta \cap U'$ at a time $t \ge t^{**}$, it follows that $\alpha_{t+1} \in C_\beta$. In other words, α_t must leave U' before it can leave C_β.

>From (46) we have that for $\alpha_t \in C_\beta$ and β large enough

$$|\alpha_{2,t+1}| - |\alpha_{2,t}| \ge (-\frac{W_{21}}{\beta} + W_{22} - 1)\,|\alpha_{2,t}| = Z\,|\alpha_{2,t}| \tag{50}$$

For β large enough $Z > 0$. Also from (46)

$$
\begin{aligned}
|\alpha_{1,t+1}| - |\alpha_{1,t}| &\leq \gamma_{t+1} \left(V_{12} |\alpha_{1,t}| + V_{22} |\alpha_{2,t}| \right) \\
&\leq \gamma_{t+1} (\frac{V_{12}}{\beta} + V_{22}) |\alpha_{2,t}|
\end{aligned}
$$

Choosing t^{**} large enough such that for all $t \geq t^{**}$

$$
\beta \gamma_{t+1} (\frac{V_{12}}{\beta} + V_{22}) < Z
$$

we have

$$
\beta \left(|\alpha_{1,t+1}| - |\alpha_{1,t}| \right) \leq |\alpha_{2,t+1}| - |\alpha_{2,t}|
$$

>From $\alpha_t \in C_\beta$ we have

$$
\beta |\alpha_{1,t}| \leq |\alpha_{2,t}|
$$

Adding up the last two equations implies $\alpha_{t+1} \in C_\beta$.

Now note that from (50) it follows that for $t \geq t^{**}$ and any $\alpha_t \in C_\beta \cap U'$ with $\alpha_{2,t} \neq 0$ the sequence $\{\alpha_{t+i}\}_{i=1}^{\infty}$ will leave U' in finite time.

It remains to show that for any small neighborhood $U'' \subset U'$ there is a point $\alpha_0 \in U''$ that is mapped in a fixed number of steps t^{**} into a non-zero point $\alpha_{t^{**}} \in C_\beta \cap U'$. The mapping $M(\Pi, m)$ as defined in (30) is continuously differentiable for $\Pi > 0$. Furthermore, $M(\Pi, m) > 0$ for $\Pi > 0$. Since in any stationary rational expectations equilibrium (Π, m) with $g \geq 0$ we have $\Pi \geq 1$, (32) is continuously differentiable in a neighborhood of $\alpha = (0,0)$. By the regularity assumptions, the matrices $A(\gamma_t)$ are non-singular. The mapping (32), therefore, fulfills the assumptions of the inverse function theorem (see e.g. Hirsch and Smale (1974), p.337). Moreover, they have a fixed point at $(0,0)$ for all t. Therefore, the t^{**}-iterative map also fulfills the assumptions of the inverse function theorem and has a fixed point at $(0,0)$. Now fix an arbitrary $U'' \subset U'$ and choose a $\alpha_{t^{**}} \in C_\beta \cap U''$ with $|\alpha_{1,t^*}|$ and $|\alpha_{2,t^*}|$ sufficiently small. Then by the continuous differentiability of the t^{**}-iterative map and the fixed point property, the pre-image (θ_0, ρ_0) must be in U''. But I have shown that from t^{**} onwards one obtains a divergent trajectory.

Next, suppose $\frac{\partial \lambda_1}{\partial \gamma} > 1$ and without loss of generality $|\lambda_2| < 1$. Then $W_{11} > 0$ and $V_{22} < 1$ for $t \geq t^*$. Define the cone $C'_\beta = \{(\alpha_1, \alpha_2) | \, |\alpha_1| \geq \beta |\alpha_2| \}$. With $\alpha_t \in C_\beta \cap U$

equation (46) implies for $t \geq t^*$

$$|\alpha_{1,t+1}| - |\alpha_{1,t}| \geq \left(\gamma_{t+1} W_{11} |\alpha_{1,t}| - \gamma_{t+1} W_{12} |\alpha_{2,t}| \ \right)$$
$$\geq \gamma_{t+1} \left(W_{11} - \frac{W_{12}}{\beta} \right) |\alpha_{1,t}| \tag{51}$$

Similarly, (46) implies

$$\beta \left(|\alpha_{2,t+1}| - |\alpha_{2,t}| \right) \leq \beta \left(V_{21} |\alpha_{1,t}| + (V_{22} - 1) \right) |\alpha_{2,t}|$$
$$\leq \left(\beta V_{21} + (V_{22} - 1) \right) |\alpha_{1,t}| \tag{52}$$

Now choose a β such that

$$\left(W_{11} - \frac{W_{12}}{\beta} \right) > 0 \tag{53}$$

This implies that $|\alpha_{1,t}|$ is increasing in $C'_\beta \cap U$.

Restricting consideration to a sufficiently small neighborhood $U' \subset U$ one can choose V_{21} arbitrarily close to zero and V_{22} arbitrarily close to $|\lambda_2| < 1$. With a sufficiently small U' it holds that

$$\left(\beta V_{21} + (V_{22} - 1) \right) < 0$$

and $|\alpha_{2,t}|$ is decreasing in $C'_\beta \cap U'$. This implies that $\alpha_{t+1} \in C'_\beta$ whenever $\alpha_t \in C'_\beta \cap U'$ for $t \geq t^*$. As before, α_t must leave U' before it can leave C'_β. At the same time (51),(53), and the fact that $\sum \gamma_t = \infty$ imply that α_t will leave U' in finite time. Then choosing an $\alpha_{t^*} \in C'_\beta \cap U''$ sufficiently close to zero will insure that the pre-image α_0 of α_{t^*} will be from any arbitrarily small neighborhood $U'' \subset U'$. But from t^{**} onwards we will get a divergent trajectory.

8.2 Appendix to Section 6

8.2.1 Linearization Coefficients

For $g = 0$, the coefficient matrices for the linearization around the low inflation equilibrium (Π^l, m^l) are given by

$$\alpha^l = \begin{pmatrix} -\frac{w_2^l}{1-\sigma} \\ m^l(1 + \frac{w_2^l}{1-\sigma}) \end{pmatrix} \tag{54a}$$

$$\beta_0^l = \begin{pmatrix} 1 - \frac{m^l w_1}{1-\sigma} & 0 \\ -m^l + \left(m^l\right)^2 \frac{w_1^l}{1-\sigma} & 0 \end{pmatrix} \tag{54b}$$

$$\beta_1^l = \begin{pmatrix} \frac{w_2^l}{1-\sigma} & 0 \\ -m^l \frac{w_2^l}{1-\sigma} & 0 \end{pmatrix} \tag{54c}$$

$$\delta^l = \begin{pmatrix} 0 & \frac{w_1^l}{1-\sigma} \\ 0 & 1 - m^l \frac{w_1^l}{1-\sigma} \end{pmatrix} \tag{54d}$$

where

$$w_1^l = \frac{\partial w}{\partial n_t}(m^l, \Pi^l) \text{ and } w_2^l = \frac{\partial w}{\partial E_{t-1}[\Pi_t]}(m^l, \Pi^l)$$

Similarly, the coefficients for the high inflation equilibrium (Π^h, m^h) at $g = 0$ are given by

$$\alpha^h = \begin{pmatrix} -\left(\Pi^h\right)^2 \frac{w_2^h}{1-\sigma} \\ 0 \end{pmatrix} \tag{55a}$$

$$\beta_0^h = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{55b}$$

$$\beta_1^h = \begin{pmatrix} \Pi^h \frac{w_2^h}{1-\sigma} & 0 \\ 0 & 0 \end{pmatrix} \tag{55c}$$

$$\delta^h = \begin{pmatrix} 0 & \frac{w_1^h}{1-\sigma} \\ 0 & \frac{1}{\Pi^h} \end{pmatrix} \tag{55d}$$

where

$$w_1^h = \frac{\partial w}{\partial n_t}(m^h, \Pi^h) \text{ and } w_2^h = \frac{\partial w}{\partial E_{t-1}[\Pi_t]}(m^h, \Pi^h)$$

8.2.2 Minimum State Variable Solutions

Consider a stochastic linear expectational difference equation of the form

$$x_t = k + B_0 E_{t-1}[x_t] + B_1 E_{t-1}[x_{t+1}] + D x_{t-1} + u_t \tag{56}$$

with $x_t, u_t, k \in R^n$, $B_0, B_1, D \in R^{n\times n}$, and $B_1 \neq 0$, $D \neq 0$. The minimum state variable solutions of (56) take the form

$$x_t = a + Bx_{t-1} + u_t$$

provided there exists a real solution to the matrix quadratic equation

$$B_1 B^2 - (B_0 - I)B + D = 0 \tag{57}$$

see chapter 10 in Evans and Honkapohja (2001). Then a is given by

$$(I - B_0 - B_1(1 + B))a - k = 0 \tag{58}$$

The minimum state variable rational expectations solutions can be calculated by solving the matrix equations (57) for B and then using (58) to calculate a where k, B_1, B_2, and D are given by the linearization coefficients calculated in appendix 8.2.1. Some lengthy algebra shows that around the low inflation steady state (Π^l, m^l) there are two AR(1) rational expectations solutions given by

$$\begin{aligned}\begin{pmatrix}\Pi_t\\ m_t\end{pmatrix} &= a^{l,1} + B^{l,1}\begin{pmatrix}\Pi_{t-1}\\ m_{t-1}\end{pmatrix} + \begin{pmatrix}0\\ v_t\end{pmatrix}\\ &= \begin{pmatrix}0\\ m^l\end{pmatrix} + \begin{pmatrix}0 & \frac{1}{m^l}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\Pi_{t-1}\\ m_{t-1}\end{pmatrix} + \begin{pmatrix}0\\ v_t\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix}\Pi_t\\ m_t\end{pmatrix} &= a^{l,2} + B^{l,2}\begin{pmatrix}\Pi_{t-1}\\ m_{t-1}\end{pmatrix}\\ &= \begin{pmatrix}1 + m^l\frac{w_1{}^l}{w_2{}^l}\\ -\left(m^l\right)^2\frac{w_1{}^l}{w_2{}^l}\end{pmatrix} + \begin{pmatrix}0 & -\frac{w_1{}^l}{w_2{}^l}\\ 0 & 1 + m^l\frac{w_1{}^l}{w_2{}^l}\end{pmatrix}\begin{pmatrix}\Pi_{t-1}\\ m_{t-1}\end{pmatrix} + \begin{pmatrix}0\\ v_t\end{pmatrix}\end{aligned}$$

Around the high inflation steady state (Π^h, m^h) there is a single AR(1) rational expectations solution given by

$$\begin{aligned}\begin{pmatrix}\Pi_t\\ m_t\end{pmatrix} &= a^{h,2} + B^{h,2}\begin{pmatrix}\Pi_t\\ m_t\end{pmatrix} + \begin{pmatrix}0\\ v_t\end{pmatrix}\\ &= \begin{pmatrix}\Pi^h\\ 0\end{pmatrix} + \begin{pmatrix}0 & -\frac{w_1^l}{w_2^l}\\ 0 & \frac{1}{\Pi^h}\end{pmatrix}\begin{pmatrix}\Pi_{t-1}\\ m_{t-1}\end{pmatrix} + \begin{pmatrix}0\\ v_t\end{pmatrix}\end{aligned}$$

The fact that there is only a single solution shows that the linearization around the high inflation steady state must be degenerate for $g = 0$, because equations (57) and

(58) generically possess two solutions. For $g > 0$ one has the following linearization coefficients around (Π^h, m^h):

$$\alpha^h = \begin{pmatrix} -\left(\Pi^h\right)^2 \frac{w_2^h}{1-\sigma} \\ m^h + \frac{m^h w_2}{(1-\sigma)} \end{pmatrix} \tag{59a}$$

$$\beta_0^h = \begin{pmatrix} 1 - \frac{m^h w_1^h}{\Pi^h(1-\sigma)} & 0 \\ -\frac{m^h}{\left(\Pi^h\right)^2} + \frac{\left(m^h\right)^2 w_1^h}{\left(\Pi^h\right)^3 (1-\sigma)} & 0 \end{pmatrix} \tag{59b}$$

$$\beta_1^h = \begin{pmatrix} \Pi^h \frac{w_2^h}{1-\sigma} & 0 \\ -\frac{m^h w_2^h}{\Pi^h(1-\sigma)} & 0 \end{pmatrix} \tag{59c}$$

$$\delta^h = \begin{pmatrix} 0 & \frac{w_1^h}{1-\sigma} \\ 0 & \frac{1}{\Pi^h} - \frac{m^h w_1}{\left(\Pi^h\right)^2 (1-\sigma)} \end{pmatrix} \tag{59d}$$

It is easy to check that for the above coefficients the following is the second rational expectations solution:

$$\begin{aligned} \begin{pmatrix} \Pi_t \\ m_t \end{pmatrix} &= a^{h,1} + B^{h,1} \begin{pmatrix} \Pi_t \\ m_t \end{pmatrix} + \begin{pmatrix} 0 \\ v_t \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ m^h \end{pmatrix} + \begin{pmatrix} 0 & \frac{\Pi^h}{m^h} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Pi_{t-1} \\ m_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ v_t \end{pmatrix} \end{aligned}$$

It exists only for $g > 0$, not for $g = 0$.

8.2.3 Proof of Proposition 2

Agents have a perceived law of motion

$$\begin{pmatrix} \Pi_t \\ m_t \end{pmatrix} = a_{t-1} + B_{t-1} \begin{pmatrix} \Pi_{t-1} \\ m_{t-1} \end{pmatrix}$$

where a_{t-1} and B_{t-1} represent the agents' least squares estimates based on information H_{t-1}. Substituting the perceived law of motion into (26) yields the actual law of motion given by

$$\begin{aligned} \begin{pmatrix} \Pi_t \\ m_t \end{pmatrix} &= (\alpha + \beta_0 a_{t-1} + \beta_1 a_{t-1} + \beta_1 B_{t-1} a_{t-1}) \\ &+ \left(\beta_0 B_{t-1} + \beta_1 \left(B_{t-1}\right)^2 + \delta\right) \begin{pmatrix} \Pi_t \\ m_t \end{pmatrix} + \begin{pmatrix} 0 \\ v_t \end{pmatrix} \\ &= T^a(a_{t-1}, B_{t-1}) + T^b(B_{t-1}) \begin{pmatrix} \Pi_t \\ m_t \end{pmatrix} + \begin{pmatrix} 0 \\ v_t \end{pmatrix} \end{aligned}$$

A rational expectations solution (a^*, B^*) is E-stable (E-unstable) if the following differential equation is asymptotically stable (unstable) at (a^*, B^*):

$$\begin{pmatrix} \frac{\partial a}{\partial \tau} \\ \frac{\partial\, vec\, B}{\partial t} \end{pmatrix} = \begin{pmatrix} T^a(a, B) \\ vec\, T^b(B) \end{pmatrix} - \begin{pmatrix} a \\ vec\, B \end{pmatrix}$$

where vec is the vectorization operator. Clearly, this differential equation is asymptotically stable at (a^*, B^*) if all the eigenvalues of

$$\frac{\partial \begin{pmatrix} T^a(a, B) \\ vec\, T^b(B) \end{pmatrix}}{\partial\, (a, vec\, B)}$$

are smaller than one. Since $\frac{\partial\, vec\, T^b(B)}{\partial a} = 0$, this is the case if the eigenvalues of $\frac{\partial\, vec\, T^b(B)}{\partial\, vec\, B}$ and $\frac{\partial T^a(a,B)}{\partial a}$ are smaller than one. The following table lists the eigenvalues of these two matrices for the respective rational expectations solutions at $g = 0$:[19]

RE-Solution	EV's of $\frac{\partial\, vec\, T^b(B)}{\partial\, vec\, B}$	EV's of $\frac{\partial T^a(a,B)}{\partial a}$
$a^{l,1}, B^{l,1}$	$\lambda_1 = \lambda_2 = 0, \lambda_3 = \lambda_4 = 1 - \frac{w_2^l}{1-\sigma} - m^l \frac{w_1^l}{1-\sigma}$	$\lambda_5 = 0, \lambda_6 = 1 - m^l \frac{w_1^l}{1-\sigma}$
$a^{l,2}, B^{l,2}$	$\lambda_1 = \lambda_2 = 0, \lambda_3 = 1, \lambda_4 = 1 + \frac{w_2^l}{1-\sigma} + m^l \frac{w_1^l}{1-\sigma}$	$\lambda_5 = 0, \lambda_6 = 1 + w_2^l$
$a^{h,1}, B^{h,1}$	$\lambda_1 = \lambda_2 = 0, \lambda_3 = \lambda_4 = 1$	$\lambda_5 = 0, \lambda_6 = 1 + (\Pi^h - 1)\frac{w_2^h}{1-\sigma}$
$a^{h,2}, B^{h,2}$	$\lambda_1 = \lambda_2 = 0, \lambda_3 = 1, \lambda_4 = 1 + \frac{w_2^h}{1-\sigma}$	$\lambda_5 = 0, \lambda_6 = 1 + \Pi^h \frac{w_2^h}{1-\sigma}$

Since $w_1^n > 0$, $w_2^n > 0$ for $n = l, h$ and $\Pi^h > 1$, it follows that the rational expectations solution $(a^{l,1}, B^{l,1})$ is E-stable and the solutions $(a^{l,2}, B^{l,2})$, $(a^{h,1}, B^{h,1})$ and $(a^{h,2}, B^{h,2})$ are E-unstable.

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[19] For the solution $(a^{h,1}, B^{h,1})$ it lists the limits of the eigenvalues for $g \to 0$.

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