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Marcello D'Amato[*], Christian Di Pietro[†], and Marco M. Sorge[‡]

Abstract
We study a model of wealth accumulation in altruistic lineages, in which households face uninsurable risk, investment indivisibilities and credit market imperfections. A thick upper tail of the stationary distribution of wealth is shown to emerge as a robust prediction, irrespective of (i) the presence of multidimensional (wealth and ability) heterogeneity and non-convexities in human capital formation, and (ii) the nature of parental bequest motives (joy-of-giving vs. paternalism). Additionally, (iii) we identify conditions under which the unique, ergodic wealth distribution exhibits a mass point at the bottom of its support, where bequest incentives are inactive and social mobility can only occur via occupational upgrading within lineages. Our interest in the features of the left tail motivates the exploration of the effects of various frictions and fiscal measures on intergenerational wealth transmission and the persistence of inequality. We show that tax policies (e.g. capital income taxation) targeting top wealth inequality can dilate expected residence time of lineages in the lower states of the wealth space, providing a strong case for redistributive policies that favour access to education for the less wealthy.

JEL Classification: D31, H20, I24.

Keywords: Wealth distribution, Wealth inequality, Capital income risk, Credit market imperfections, Educational investment.

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"Money, says the proverb, makes money.
When you have a little, it is often easier to get more.
The great difficulty is to make that little."
A. Smith, The Wealth of Nations, Chapter XI, p. 111.

1 Introduction

Household wealth data from a large panel of countries reveal two fairly general and robust patterns: first, a tilt to the top in the concentration of wealth over the last decades (e.g. Wolff, 1987; Davies and Shorrocks, 1999; Klass et al., 2007: Piketty, 2014; Vermeulen, 2018); second, a sharp and growing divide among the richest and poorest households, with large shares of the population owning zero or negative non-housing wealth (e.g. Survey of Consumer Finances, 2017, 2019; Balestra and Tonkin, 2018).[1]

Common to many advanced market economies, such patterns appear to suggest a major role for general economic mechanisms, rather than specific political and/or institutional factors, in forging long-term trends in wealth inequality. In this paper, we develop a unified theoretical framework for assessing the relative importance of two well-established market imperfections (non-insurability of income shocks and borrowing constraints) in producing patterns of wealth evolution in line with the observed structure of the data. Focusing on educational investment and financial (voluntary) bequests as a mean of intergenerational wealth transmission, we explicitly address the determinants of unequal opportunities in the lower rungs of the wealth ladder, where the fortunes of children might be shaped by family characteristics and market forces other than those governing the process of wealth accumulation among the wealthiest.

While Marshall (1890) explicitly acknowledged the concern for children as a key reason for saving, Vilfredo Pareto (1902) was the first to suggest that preferences for altruism and bequest strategies operating in accordance with ordinary economic principles were the main driving force of the observed structure of wealth distributions. This view was later on taken by Gary Becker in a few contributions, formalized in Becker and Tomes (1979, 1986). Though lacking a formal derivation of the equilibrium distribution of wealth, these studies trace the origins of the inequality among families back to more fundamental heterogeneity in socio-economic inheritable characteristics ("endowments", "ability", "social connections") so that the economic status tends to persist at the lineage level. On top of the transmission of endowments and the inheritability of socio-economic connections, the main channel of the persistence of inequality was the acknowledged credit market imperfection in human capital formation. In this view, little or no role is played by heterogeneity in the returns to financial

[1]Survey information from the SCF 2017 (Federal Reserve Board) reveals that the wealth gap between US richest and poorer families more than doubled from 1989 to 2016, with a record high 30% of zero-wealth households in 2016. Exploiting the second wave of the OECD Wealth Distribution Database, Balestra and Tonkin (2018) find that up to a quarter of all households in a number of OECD countries report negative net worth (i.e. liabilities exceeding the value of their assets).

wealth.[2]

In a series of recent articles, Benhabib et al. (2011, 2015) have explicitly tackled the task of deriving rigorous micro-foundations for the empirically documented thick upper tail of wealth distributions. The fundamental driving force of the rise and fall of families in this case is shown to be the exposure to capital income risk, as determined by the absence of insurance markets; propagating through the economy via the life-cycle and bequest behaviour of rational agents, idiosyncratic shocks to financial returns accumulate multiplicatively into wealth, boosting its concentration in the upper tail of the stationary distribution.[3]

To study occupational mobility and its relation with the evolution of wealth inequality, we incorporate ability heterogeneity and indivisible human capital investment into an otherwise standard framework of intergenerational mobility. Specifically, we consider a simple dynamic economy populated by a large number of family lineages who differ with respect to innate ability ('ex ante heterogeneity') and wealth, due to the presence of uninsured income shocks ('ex post heterogeneity'). Parents decide both on financial bequests and investments in their children's education, based on their own preferences for altruism and investment costs. Given the presence of multiple channels of intergenerational wealth transmission, we explicitly study the implications of a *paternalistic bequest motive*, which engenders the enjoyment of a child's economic status through the lens of her parents' preferences. We show that, in sharp contrast with the implications of the *joy-of-giving* hypothesis (e.g. Andreoni, 1990), according to which parents derive direct utility from the pure act of giving, paternalistic altruism entails portfolio choice considerations on the part of risk-averse parents, that render the educational investment problem more complicated to solve, and the implications of capital income risk and ex-ante heterogeneity on wealth dynamics more interesting to evaluate.

In our model, the combined effects of investment indivisibilities and capital income risk propel heterogeneous responses of altruistic parents to uninsurable shocks. Our first contribution is to show that the force of capital income risk is strong enough to deliver a 'thick right tail' property resembling Benhabib et al. (2011)'s, notwithstanding the presence of multidimensional heterogeneity (wealth and ability) investment indivisibilities at the lineage level. However, these features crucially interact with the nature and extent of parental altruism to determine the transmission of economic status across generations *at the bottom* of the wealth distribution. Specifically, we establish rather general conditions on the model's fundamentals – conditions that admit natural economic interpretations – under which a unique, ergodic

[2]In Loury (1981), who provided the first elegant and rigorous formalization for the emergence of the limit distribution of earnings, financial wealth plays no role at all in the determination of the limit distribution of the socio-economic status of families.

[3]The applied literature has offered ample evidence of heterogeneous exposure to risk springing from private business equity and principal residence ownership (e.g. Bertaut and Starr-McCluer, 2002; Flavin and Yamashita, 2002; Wolff, 2006), as well as of high volatility of capital gains and earning on private equity (e.g. Moskowitz and Vissing-Jorgensen, 2002), pointing to limited diversification of risk associated with the returns to household wealth. See also Cao and Luo (2017) for a general equilibrium model incorporating persistent idiosyncratic returns on wealth for the analysis of wealth dynamics and top end inequality.

wealth distribution will emerge (Proposition (1)) that exhibits *an atom* – i.e. a mass point – at (almost) zero wealth, where upward mobility can only occur through occupational upgrading within lineages (Proposition (3)). The intuition for this result is as follows. When the bequest motive is sufficiently weak, the inability to borrow against their children's human capital and the presence of indivisibilities fully deter any kind of parental risky investment, inducing emergence of a mass of unskilled workers with no wealth inheritances at any time period – i.e. a mobility trap. Absent any direct wealth transfer in the group of the least wealthy families, the support of the stationary distribution of wealth will therefore inherit the structural properties of the left end of the support of the endogenous distribution of income as determined by the educational investment choices made by parents. As a result, the unique steady state of the wealth accumulation process will display a mass of households bunched at the bottom end of the stationary distribution.

Operating through capital market imperfections and the indivisibility of human capital formation, this mechanism ceases to operate over larger wealth holdings towards the top, where financial wealth transfers become an overwhelmingly important source of asset accumulation within lineages; the occurrence of high returns paths on financial bequests (market good luck) will then produce large and slowly declining wealth shares at the upper end of the stationary distribution (Proposition (2)).

We show that modeling family altruism in paternalistic form is crucial for the existence of an atom in the left tail of the stationary distribution. Under joy-of-giving, in fact, financial bequests have no compensatory goals in terms of consumption opportunities of later generations (e.g. De Nardi, 2004; Benhabib et al., 2011; De Nardi and Fella, 2017). Any degree of altruism will then trigger positive financial bequests all along the equilibrium path, and the mechanics of multiplicative shock accumulation will sustain upward mobility flows in all states of the wealth space, so that credit market imperfections and ability heterogeneity would not matter for the stationary distribution of wealth (Proposition 4).

Under paternalism, exposure to uninsurable shocks to both financial and human capital returns may force risk-averse parents not to engage in any kind of bequest choice against the expected benefits to their heirs. Intuitively, the presence of an option to build human capital affects the optimal (endogenous) investment choice of constrained households to financial risk via a saving composition effect. We identify conditions under which bequest incentives are not operative for a positive measure subset of lineages at each point in time, implying a positive probability for the wealth transitions to reach the lowest state of the wealth space. The distribution of abilities will then dictate the size of the atom in the left tail of the cross-sectional distribution of wealth: human capital formation among the group of constrained families, rather than exposure to uninsured financial risk, will then produce the necessary mobility across the lowest wealth levels, thereby warranting ergodicity of the wealth dynamics. In this environment – an economy with non-interactive agents and exogenous wages – the size of the atom in the left tail is not related to the size of the Pareto exponent of the right tail: while the latter is exclusively determined by capital income risk, the former ultimately depends on the the properties of the human capital formation process and thus on the ability heterogeneity across individuals.

When these latter forces contribute to shaping the long-run features of wealth inequality, a natural question revolves around the kinds of fiscal policies needed to enhance economic trajectories of the group of the poorest vis-à-vis instruments for taxing the rich. In this respect, of particular interest are the implications of multidimensional heterogeneity for the design of public interventions aimed at reducing inequalities along the cross-sectional distribution of wealth. In such a setting, in fact, policy design should reckon with changing incentives to saving by taxed households in the lower and middle class, in particular those populating wealth states where no intergenerational wealth transmission of any kind arises.

Our second contribution is to link the properties of the left tail of the wealth distribution to various structural parameters of the model, such as the intensity of parental altruism, the heterogeneity in educational investment opportunities and the working of fiscal policies that affect the bequest behaviour of altruistic individuals. A key finding here pertains to the impact of capital income taxes on the size of the atom. According to the recent narrative that emphasizes insurance market incompleteness as a key driver of top wealth concentration, a uniform tax rate on the realized rates of return on wealth reduces inequality in the upper tail by curbing exposure to uninsured investment risk (e.g. Zhu, 2019); numerical evidence from the simulation of calibrated models suggests, by contrast, that the sign of the response of the Gini coefficient for the whole distribution to a tax increase depends on the kind of fiscal instrument employed (estate versus capital income tax), see Benhabib et al. (2011). We complement these results by formally proving that estate and capital income taxes both have ambiguous effects on the size of the atom in the left tail of the stationary wealth distribution (Proposition (7)). Capital income and/or estate taxes trade off the distortions on the overall level of total bequests and the composition of individual savings, discouraging wealth-constrained individuals from substituting human by financial assets in response to changes in the risk-return structure of assets. If the enhanced process of occupational upgrading overcompensates the contraction in the intergenerational transmission of non-human wealth, upward movements of the least wealthy households in the cross-sectional distribution can improve up to the point of lowering the size of the atom in the long run. However, capital income (or estate) taxation also induces households near the borrowing constraint, for whom labour earnings are the main component of their permanent income, to decumulate their non-human wealth to a greater extent than lineages with larger wealth holdings: this wealth effect on consumption and savings can prove sufficiently strong to exacerbate downward mobility flows towards the bottom end of the wealth space, thereby increasing the measure of the most unwealthy. In general, the framework presented in this study highlights that the impact of capital income taxation on intergenerational mobility in the left tail of the stationary distribution depend critically on the economy's fundamentals. While positive in nature, these results also help us to conceive the design of public interventions targeted to the peculiar market failure that underpins the persistence of inequality.

The remainder of the paper is organized as follows. Section (2) provides the basic setup and assumptions of the framework of analysis. Sections (3) and (4) investigate the differential impact of bequest motives on the wealth accumulation process and the ensuing features of the stationary distribution of wealth. Section (5) studies the effects of various frictions and fiscal policies on intergenerational wealth transmission and the ensuing patterns of social mobility. Section (6) connects our work to the extant literature and discusses some potentially interesting extensions. Section (7) offers concluding remarks. For ease of exposition, all proofs are confined to the Appendix.

2 Model environment

We consider a simple heterogeneous-agent model with a measure-one continuum of individual lineages i in each period $t \geq 0$. In each lineage, parental wealth is allocated to current consumption needs as well as financial and educational bequests. Upon entering adulthood, agents inelastically supply labor in one of two different occupations, requiring different skill profiles (high and low). As in Loury (1981), the labor earnings technology depends on human capital; for simplicity, it is assumed that agents accessing (respectively, not accessing) education always become skilled (resp. unskilled) workers. Parents' own wealth thus is made of wealth inheritance, capital income (i.e. the return on financial bequests) and labor income (i.e. the occupation-specific wage).

Agent heterogeneity in the model comes in two non-trivial, concurrent forms: ex ante heterogeneity in abilities, which is relevant for educational investment decisions; and ex post return (hence wealth) heterogeneity, due to the occurrence of idiosyncratic income shocks against which agents cannot fully insure. Moreover, parents are unable to borrow against their children's future earnings or human capital (credit market imperfection), since inherited debts are not enforceable.

For reasons discussed above, we let the economic environment be described by the following

Assumption 1. *(Capital income risk) The rate of return on wealth R is a stochastic process, independent and identically distributed (i.i.d.) over time and across lineages; it has a cumulative distribution function H and strictly positive density h on the bounded support $\Delta r \equiv [-1, \overline{r}]$, with $\overline{r}$ large enough and $\mathbb{E}[R] = \int r dH(r) > 0$;*

Assumption 2. *(Labour income risk) The wage in the high-skill occupation is a stochastic process, independent and identically distributed (i.i.d.) over time and across lineages; it has a cumulative distribution function F with strictly positive density f on the bounded support $\Delta y = [\underline{y}, \overline{y}]$ ($\underline{y} > 0$), with Y and R being mutually independent. The wage in the low-skill occupation is equal to $\underline{y}$.*

Assumption 3. *(Heterogeneity in abilities) Investment costs X – a measure of educational costs related to abilities – are heterogeneous across lineages and i.i.d. random variables drawn from distribution G with strictly positive density g on the bounded support $\Delta x = [0, \overline{x}]$, with $\overline{x} > \underline{y}$.*

The first and second assumptions are rather standard, and have found empirical support (e.g. Benhabib et al., 2011); a main implication of idiosyncratic shocks to capital income and labour earnings is the emergence of a positive ex-post correlation between realized returns/wages and wealth. The third assumption generalizes Galor and Zeira (1993)'s by introducing heterogeneous abilities in population, as, for example, in Mookherjee and Napel (2007) and D'Amato and Di Pietro (2014).[4]

According to Assumption 3, abilities are not correlated over time. Also, the most able individual is assigned a zero educational cost; this feature simplifies the analytical characterization of upward mobility via occupational upgrading but can be easily relaxed, and relatively sharper conditions devised for all of our results to hold true.

2.1 Preferences

Let $\omega_{i,t}$ denote parental wealth at time t within lineage i, to be allocated among current consumption $c_{i,t}$, financial bequests $b_{i,t}$ and educational investment $e_{i,t} \cdot x_{i,t+1}$, where $e_{i,t} = 0$ (resp. $e_{i,t} = 1$) when parents in generation t decide not to invest (resp. invest) in human capital of their children in generation $t+1$, who face known educational costs $x_{i,t+1}$.

Agents have homothetic preferences of the form

$$U(c_{i,t},\, b_{i,t},\, e_{i,t}) := u(c_{i,t}) + \chi \mathbb{E}[v(\omega_{i,t+1})], \tag{1}$$

under *paternalistic altruism*, where

$$\omega_{i,t+1} = (1 + r_{i,t+1})b_{i,t} + \underline{y} + e_{i,t} \cdot (y_{i,t+1} - \underline{y}). \tag{2}$$

where the parameter $\chi > 0$ captures the intensity of the bequest motive.

Observe that the expectation operator $\mathbb{E}$ is conditional on the educational choice given the probability distribution of their offspring's wealth components (financial and human capital): bequest strategies must be formulated prior to the realization of the two sources of risk.[5]

[4]Without any loss of generality, the wage in the low-skill occupation is taken to be deterministic.

[5]Since the high-skill occupation is only accessible when $e_{i,t} = 1$, expected utility from intergenerational altruism within lineage i is

$$\mathbb{E}[v(\omega_{i,t+1})] := \begin{cases} \int v(\underline{y} + (1 + r_{i,t+1})b_{i,t})\, h(r) dr & e_{i,t} = 0 \\ \int v(y_{i,t+1} + (1 + r_{i,t+1})b_{i,t})\, \Lambda(\omega) d\omega & e_{i,t} = 1 \end{cases} \tag{3}$$

where $\Lambda(\omega) = \int h(r)\, f(\omega - rb) dr$ by the independence assumption.

Different from Loury (1981), we assume that the child's ability is known to parents at the time of the educational investment decision, and that the latter produces a random rate of return which does not correlate with ability. Also, Loury (1981) allows the educational choice variable to be continuous rather than discrete, as we do in order to capture indivisibilities in the process of human capital formation.

An alternative formulation also considered in the ensuing analysis features a *joy-of-giving* bequest motive, whereby both financial and educational bequests enter the utility function directly (*bequest-as-last-consumption*), i.e.

$$U(c_{i,t},\, b_{i,t},\, e_i) := u(c_{i,t}) + \chi v\left(b_{i,t} + e_{i,t} \cdot x_{i,t+1}\right) \tag{4}$$

We further adopt the following

Assumption 4. *Preferences satisfy*

$$u := \frac{c_{i,t}^{1-\gamma}}{1-\gamma}, \qquad v := \begin{cases} \frac{\omega_{i,t+1}^{1-\gamma}}{1-\gamma} & \textit{under paternalism} \\ \frac{(b_{i,t}+e_{i,t}x_{i,t+1})^{1-\gamma}}{1-\gamma} & \textit{under joy-of-giving} \end{cases} \tag{5}$$

with elasticity $\gamma > 0$ *(*$\gamma \neq 1$*)*[6]

The class of CRRA preferences described in Assumption 4 represents a cornerstone of theoretical and applied models in finance and macroeconomics, and have been extensively used in scholarly work on wealth dynamics and social mobility, see e.g. Benhabib et al. (2011, 2015), Zhu (2019), Wan and Zhu (2019). While allowing for analytic expressions for some of the objects of interest in our analysis (such as e.g. stability conditions for the dynamics of the wealth distributions), Assumption (4) is rather weak and can be relaxed, at some computational cost, in favour of a general class of increasing, smooth and strictly concave functions u and v.[7]

2.2 Intergenerational transfers and wealth accumulation

In each lineage i parents face the same choice about whether to bequeath educational investment or financial bequests or both. Optimal bequests emerge from a portfolio choice among alternative investments where one form of investment (educational) has an indivisible component, the other (financial) has not. Wealth heterogeneity and heterogeneous educational investment costs will affect these optimal bequest strategies and will define the accumulation

[6]When $\gamma \to 1$, the limiting logarithmic forms for both u and v can be adopted.

[7]Loury (1981) employs a completely general specification of parental utility, as do e.g. Mookherjee and Napel (2007), Mookherjee and Ray (2003, 2010) for the study of inequality and mobility issues in different contexts. See also Cioffi (2022) for a model with non-homothetic preferences.

of wealth at the lineage level. Aggregating parental bequests at the economy level will then deliver the equilibrium dynamics of the wealth distribution.

For each proposed formulation of the bequest motive, the parents' utility maximization problem is

$$\max_{c_{i,t},\, b_{i,t},\, e_{i,t}} \quad U\left(c_{i,t},\, b_{i,t},\, e_{i,t}\right) \tag{6}$$

$$s.to \quad c_{i,t} + b_{i,t} + e_{i,t} \cdot x_{i,t+1} \leq \omega_{i,t} \tag{7}$$

$$b_{i,t} \geq 0 \tag{8}$$

$$c_{i,t} \geq 0 \tag{9}$$

$$e_{i,t} \in \{0,\, 1\} \tag{10}$$

where (7) is the resource constraint defining feasible choices, (8) underscores the inability of parents to borrow from their offspring, and (9) is a conventional non-negativity constraint for parents' own consumption.

Notice that, by assumption, the distributional features of exogenous variables (R, Y, X) are stationary over time. This makes the utility maximization problem (6)-(10) essentially static within each linage i, implying that optimal consumption and bequest policies are time invariant functions of the state variable ω_t. Keeping this in mind, we shall omit from now on the subscripts i and t on policy functions, in the interest of better readability.

In general, constrained optimization problems involving both discrete and continuous variable sets and nonlinear objectives such as (6)-(10) raise non-trivial solution challenges. Mixed-integer nonlinear programming procedures are usually brought to bear upon the combinatorial difficulty of tackling this class of problems (e.g. Floudas, 1995; D'Ambrosio and Lodi, 2013).[8]

In order to solve (6)-(10) and at the same time allow for an intuitive understanding of the properties of financial and educational bequest policies, we adopt a simple branch-and-bound approach by which (i) the feasible set is partitioned by fixing the binary variable e (to either 0 or 1), involving a corresponding partition of the constrained optimization problem into two distinct sub-problems; (ii) each of these sub-problems is solved for continuous *pseudo policy functions* (c_e^*, b_e^*) via Karush-Kuhn-Tucker (KKT) optimality conditions that handle non-negativity constraints, and finally (iii) the *optimal bequest plan* (b^*, e^*) – where $b^* = b_e^*$ when $e = e^*$ – is identified as the one producing the maximal objective value between the two partitions. For fixed e, the budget constraint (7) necessarily bites at the optimum and the constrained set of feasible points is convex; hence the pseudo consumption and pseudo bequest policies c_e^* and b_e^* solving the associated problems are fully characterized by the first-order KKT conditions. Direct comparison of indirect utilities induced by distinct educational

[8]Relative to classical portfolio theory, the economic interpretation of properties of optimal consumption and portfolio policies is also less straightforward in the presence of investment indivisibilites: efficient investment frontiers need not be convex (not even continuous), and the risk premium required on perfectly divisible assets may depend on a given investor's share of wealth invested in indivisible ones (Lazimy, 2007).

choices will then define the optimal bequest strategies as a function of parental wealth ω_t.

Once properly characterized, the set of optimal bequest policy functions will entail wealth transition patterns at the lineage level, whose limiting properties (ergodicity and structure of the tails of the stationary distribution) can be studied by standard techniques from the theory of Markov chains (e.g. Meyn and Tweedie, 2009).

3 Paternalistic altruism

3.1 Pseudo bequest policies

Consider the following sub-program obtained from (6)-(10) by fixing $e \in \{0, 1\}$:

$$\max_{c,\, b} \quad U\,(c,\, b;\, e) \tag{11}$$

$$s.to \quad c + b \leq \omega_{i,t} - e \cdot x \tag{12}$$

$$b \geq 0 \tag{13}$$

$$c \geq 0 \tag{14}$$

Using (12) to substitute consumption out, the KKT first-order conditions for each partition $e \in \{0,\, 1\}$ are as follows

$$\frac{d\,u(b_e^*; e)}{d\,b} + \chi \frac{d\,\mathbb{E}[v(\omega_{t+1}(b_e^*; e)]}{d\,b} = 0 \quad \text{and} \quad b_e^* \geq 0 \tag{15}$$

or

$$\frac{d\,u(b_e^*; e)}{d\,b} + \chi \frac{d\,\mathbb{E}[v(\omega_{t+1}(b_e^*; e)]}{d\,b} \leq 0 \quad \text{and} \quad b_e^* = 0 \tag{16}$$

with condition (15) pinning down all candidate interior solutions, and (16) implied by the corner solution $b_e^* = 0$ and $c_e^* = \omega_t - e \cdot x$.[9]

We next characterize the properties of the pseudo consumption and financial bequest policies:

Lemma 1. *For fixed $e \in \{0, 1\}$, there exist unique pseudo policy functions b_e^* and c_e^* solving (11)-(14); both functions are continuous and non-decreasing in ω_t.*

Proof. See the Appendix. □

To figure out conditions under which parents optimally choose to undertake educational investment, let us define *educational threshold cost* any $\tilde{x}(\omega_t) \in \Delta x$ at which parents with wealth ω_t would enjoy the same utility by undertaking educational investment as by not doing so, i.e. let $\tilde{x}$ solve

$$U\,(b_0^*;\, e = 0) = U\,(b_1^*;\, e = 1; \tilde{x}) \tag{17}$$

[9] Observe that, from $t = 1$ onward, parental wealth in each lineage is larger than or equal to $\underline{y}$, i.e. the wage in the low-skilled occupation. That is, $\omega_t > 0$ for all $t \geq 0$ and all $x \in \Delta x$.

Under our stated assumptions about the distribution of abilities, the threshold $\tilde{x}$ always exists as a function of parental wealth ω_t (call it $\tilde{x}(\omega_t)$), and entails the following: parents with equally talented kids but different wealth holdings can make different educational investment choices, so that heterogeneity in wealth background influences the distribution of earning capacities across workers; all else equal, the higher parental wealth, the larger the set of innate abilities conducive to human capital formation. The educational investment policy, part of the optimal bequest policy, can in fact be characterized as follows:

Lemma 2. *$e^* = 1$ if and only if $x < \tilde{x}(\omega_t)$, where $\tilde{x}(\omega_t) : \Re_+ \mapsto \Delta x$ is differentiable and satisfies (i) $\tilde{x}(\omega_t) \in (0, \omega_t)$ and (ii) $\frac{d\tilde{x}}{d\omega_t} \in (0,1)$ for all $\omega_t > 0$.*

Proof. See the Appendix. □

The solution to the expected utility maximization problem (6)-(10) is then given by (c_e^*, b_e^*, e) when $e = e^*$, or simply (c^*, b^*, e^*). The optimal bequest plan (b^*, e^*) exhibits the following property: at any given level of wealth, financial bequests of parents who also engage in human capital formation are never larger than those chosen by parents who rather abstain from educational investment. Intuitively, in the face of indivisibilities, parents use financial bequests to compensate kids with high educational costs, taking into account the risks and expected returns in either kind of investment. This results in a discontinuity in the optimal level of financial bequests given wealth, which in turn influences the transition of wealth within heterogeneous lineages. More formally, we state the following

Lemma 3. *Consider bequest policies b_e^* and, for fixed $\omega_t > 0$, define the mapping $b^*(x) : \Delta X \mapsto \Re_+$ as*

$$b^*(x) = \begin{cases} b_1^*(x), & \text{for} \quad x < \tilde{x}(\omega_t) \\ \\ b_0^*(x), & \text{for} \quad x \geq \tilde{x}(\omega_t) \end{cases} \tag{18}$$

Then (i) $b^*(0) = b_0^*$, (ii) $b^*(x)$ exhibits an upward jump discontinuity in $\tilde{x}$ and (iii) $b^*(x)$ is decreasing over $[0, \tilde{x})$ and constant over $[\tilde{x}, \overline{x}]$.

Proof. See the Appendix. □

We now turn to investigating how the economy's fundamentals (preferences and market characteristics) shape financial bequest strategies along the wealth distribution. A preliminary step of our analysis entails the identification of the wealth states for which optimal financial bequests are zero irrespective of the distribution of abilities and thus of the expected returns to human capital. In those states, borrowing constraints and indivisibilities do affect the intergenerational wealth transmission at the lineage level, and thus influence the evolution of the wealth distribution over time. The intuition is straightforward: whenever families in the lowest tiers of wealth find it optimal not to undertake risky financial investment choices, upward mobility can only occur via occupational upgrading based on human capital formation; the latter, in turn, requires the presence of sufficiently strong incentives to educational investment, which are provided by the expected private benefits of improved earnings capacities to parents of enough talented descendants.

3.2 Bequest strategies and individual wealth transition

In order to characterize wealth transitions at the lineage level we evaluate the KKT conditions in (16) at the point in which the pseudo financial bequest is optimally chosen to be zero conditional on the indivisible education choice. For $x \in \Delta X$, let us denote as $\tilde{\omega}_e(x)$, for each $e = \{0, 1\}$, the supremum of the set of wealth states ω for which b_e^* is zero. From (16) and the mutual independence between R and Y, any such $\tilde{\omega}_e(x)$ (for $\gamma \neq 1$) must solve

$$\tilde{\omega}_e(x) = \chi^{-\frac{1}{\gamma}} \left(1 + \mathbb{E}[R]\right)^{-\frac{1}{\gamma}} \left(\mathbb{E}\left[\underline{y} + e\left(y - \underline{y}\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} + e \cdot x \tag{19}$$

where (as before) the expectation is taken with respect to the distributions of r and y. Notice that, by monotonicity of the pseudo policy functions, it holds $b_e^* = 0$ for all $\omega_t < \tilde{\omega}_e(x)$, $e \in \{0, 1\}$.[10]

Notice that variation in x induces a variation in the wealth thresholds above which investment is affordable. We can accordingly interpret $\tilde{\omega}_0$ as the threshold below which financial transfers are not part of the optimal bequest plan of parents, regardless of the educational choice of parents; and $\tilde{\omega}_1(x)$ as the cost-specific threshold above which both financial and educational investment occur whenever convenient. Intuitively, the lower part of the wealth space entailing no intergenerational wealth transfers shrinks with the relative degree of risk aversion γ and the intensity of parental altruism χ.

By the same logic, positive financial bequests require parental preferences to be sufficiently altruistic towards children and tolerant vis-à-vis the market risk and the rate of return attached to the financial bequest. When the choice also entails educational bequests, such returns and risks are relative to those in the skilled labor market. For those families who invest in education, financial bequests are part of the optimal bequest plan only at sufficiently large levels of wealth. When this is the case, wealthier households will leave higher bequests, producing persistence in wealth.

We formalize the foregoing arguments in the following

Lemma 4. *At each point in time, there exists a positive measure of lineages i for whom $b_0^* = b_1^* = 0$ if and only if $\chi(1 + \mathbb{E}[R]) < 1$. In this case, $\tilde{\omega}_0 \in \left(\underline{y}, \tilde{\omega}_1(x)\right)$ for all $x \in \Delta x$, and it decreases with the relative strength of risk aversion γ and the intensity of the bequest motive χ.*

Proof. See the Appendix. □

[10] From now on, we shall not consider the log-log preference specification characterized by $\gamma = 1$. All of our results can nonetheless be explicitly restated to encompass it.

Observe that, at each point in time, educational investment will be undertaken by a subgroup of those families who optimally decide not to bear the risk associated to financial bequests: $b^* = 0$ and $e^* = 1$ are chosen by parents in the lower states of the wealth space states ($\omega_t \in [\underline{y}, \tilde{\omega}_0)$), whose children have access to affordable education ($x < \tilde{x}(\underline{y})$). The ensuing analysis will clarify that this is a key result in order for an atom in the support of the stationary wealth distribution to exist and for its characterization.

Since bequest policy functions (b_e^*, e^*) are uniquely defined in terms of parental wealth ω_t, the wealth transition laws at the lineage level are in the form

$$\omega_{t+1} = \begin{cases} (1+r)\, b_1^*(\omega_t,\, x) + y, & x \leq \tilde{x} \\ (1+r)\, b_0^*(\omega_t) + \underline{y}, & x > \tilde{x} \end{cases} \tag{20}$$

where the first line refers to families whose children get into education and the second line to those who do not.

In the presence of paternalistic altruism, two sources of non-linearity in wealth dynamics emerge: one pertains to the threshold $\tilde{x}$, which induces a jump discontinuity in the financial bequest policy as a function of wealth; the other, more troublesome from the analytic point of view, hinges on the presence of zero bequest policies in the lower region of the wealth space. In terms of predictions on the evolution of the wealth distribution, this is the key difference between our model and Zhu's (2019), which entails a wealth accumulation process in (conditionally) linear form. These features of the policy functions will require some adaptations of the arguments put forward in Zhu (2019) for the establishment of the limit properties of the wealth distribution. This analysis is provided in the next section.

3.3 The stationary wealth distribution

The evolution of wealth (20) defines a Markov process $\{\omega_t\}$ evolving on the state space $W = [\underline{y}, \infty)$ according to some probability law μ, whereby the equilibrium wealth level ω_{t+1} achievable by agents receiving financial and educational bequests (b^*, e^*), as a function of current wealth ω_t, depends on i.i.d. shocks $(R,\, Y)$ hitting at time $t+1$, for different values of $x \in \Delta x$.

Our first characterization result is about existence of an ergodic, almost-surely unique limit distribution of wealth. Roughly speaking, ergodicity of the limit distribution requires that every wealth state in W is accessible from any other (*irreducibility*), and that states are visited at irregular times (*aperiodicity*), implying that the accumulation process is not permanently trapped in subsets of the wealth space; a suitable non-explosion (*bounded drift*) condition is also to be satisfied, so that the dynamics of wealth (20) converges almost surely to a unique invariant distribution. Formally, we state the following

Proposition 1. *Let $\chi(1 + \mathbb{E}[R]) < 1$. The process $\{\omega_t\}$ generated by (20) is geometrically ergodic, and thus converges to a unique stationary distribution ω_∞ with limit support S_∞,* *where*

$$S_\infty = \begin{cases} [\underline{y}, \infty[& \textit{if and only if} \quad \chi(1+\mathbb{E}[R]) > \left(\frac{y}{\overline{\overline{y}}}\right)^\gamma \\ \\ [\underline{y}, \overline{y}] & \textit{otherwise} \end{cases}$$

Proof. See the Appendix. □

Proposition (1) establishes two main results. First, when $\tilde{\omega}_0 \in (\underline{y}, \overline{y})$, at any point in time there exist parents for whom bequeathing wealth is optimal even when educational investment is undertaken. The law of compounded interest rate, amplified by the occurrence of idiosyncratic shocks to the return on financial inheritances, will then fuel wealth accumulation patterns along which lineages experiencing high realizations in random returns on capital income will move up towards infinitely large wealth states. A simple corollary of this result is that, when parents are sufficiently risk-averse and/or concerned with the wealth status of their descendants, the propagation of wealth via intergenerational financial transfers can produce power-law decay for the upper tail of the stationary wealth distribution, as in Benhabib et al. (2011). Remarkably, accumulation patterns producing large wealth shares at the top of the wealth distribution are consistent with the emergence of a temporary mobility trap at the bottom: by formulating bequest strategies entailing no financial transfers, a positive measure of lineages (those facing relatively higher educational costs) will necessarily reside for more than one period in the lower states of the wealth space (i.e. below the $\tilde{\omega}_0$ threshold); in this group, human capital investment by households with enough talented kids will then induce the necessary mobility across wealth levels, resulting in an ergodic wealth distribution in the long run.[11]

Second, notice that the same stability condition $\chi(1+\mathbb{E}[R]) < 1$ that guarantees that the wealth dynamics converges to a unique stationary distribution can also induce not enough large incentives to invest in financial bequests, a form of poverty trap where the wealth distribution coincides with the labor income distribution as produced by agents' educational choices. For a given structure of human and financial asset returns, when parental altruism is not sufficiently strong – i.e. if $\chi \leq \frac{1}{(1+\mathbb{E}[R])}\left(\frac{y}{\overline{\overline{y}}}\right)^\gamma$ – ergodicity implies that any household will visit the lower states of wealth space (those below the $\tilde{\omega}_0$ threshold) with probability one where financial transfers are not part of the optimal bequest plan; as a result, the stationary wealth distribution will be defined on the finite support $[\underline{y}, \overline{y}]$.

Our next step is to derive the tail behavior of the limit distribution of wealth. One major difficulty emerging from the analysis regards the analytic properties of (20): individual wealth transitions are in fact non-linear and major characterization theorems for the limit

[11]It is immediate to notice that any $\tilde{\omega}_e$ satisfying equation (19) must be larger than $\underline{y}$. Otherwise, at all times and for all lineages at the bottom left of the wealth distribution, optimal bequest strategies entail positive financial bequests in the absence of educational investment. Capital income risk will then fuel mobility patterns across wealth states supporting the emergence of an ergodic stationary distribution, as in Benhabib et al. (2011).

distribution require some adaptation to our case. We tackle this issue by developing a simple logic based on Jensen's inequality, that allows us to rely on natural restrictions on the effective rate of return on wealth, which in turn depends on the fundamentals of the economy, such as preference parameters, as clarified in Benhabib et al. (2011). Intuitively, for a given labor income distribution and average return on wealth, parental altruism should be *large enough* to guarantee positive wealth expansion through financial bequests; at the same time it cannot be *too large*, otherwise the underlying limit distribution would entail explosive behavior. The accumulation of uninsurable capital income shocks will then follow the standard growth law and hence a thick upper tail in the wealth distribution will emerge.

From a technical standpoint, the proof of this result requires us to overcome the difficulty inherent in the non-linearity of policy functions by demonstrating that bequests policies b^* are bounded from below by a linear (in wealth) function that embodies the optimal bequest choice of a *fictitious* altruistic agent faced with the same risky investment opportunities and yet concerned with the *average* wealth of their offspring. This modified relationship between the agent's own preference for altruism and their attitude toward risk will induce parents to abstain from over-compensating their children for the risk inherent in the intergenerational transmission of wealth via financial bequests. In other words, the optimal bequest strategies of the fictitious agent will not entail, at any wealth level, larger transfers than those induced under risk-aversion, whatever the intensity of the bequest motive and the associated incentive to invest in human capital formation. This allows us to construct an auxiliary individual wealth accumulation process in linear form

$$\omega_{t+1}^a = \alpha_{t+1}\omega_t^a + \beta_{t+1} \tag{21}$$

where $(\alpha_t, \beta_t) = (\alpha(r_t, \chi, \gamma)), \beta(r_t, y_t, \chi, \gamma))$ are i.i.d. processes induced by the fictitious bequest plan, both admitting a close form representation in terms of investment returns (R, Y) and the preference parameters (χ, γ) – see equations (42) and (46) in the Appendix. Following the analysis in Kesten (1973), standard regularity assumptions on (α_t, β_t) can then be invoked to ensure that the ergodic solution to the auxiliary random recurrence (21), and *a fortiori* the limit distribution of the actual transition process (20), exhibits power-law decay in the upper tail. Formally, we state the following

Proposition 2. *Let (α_t, β_t) in (21) satisfy Kesten (1973)'s regularity conditions (in the Appendix). Then the unique stationary distribution ω_∞ has a fat right tail, provided $\chi(1+\mathbb{E}[R]) > \left(\frac{y}{\underline{\underline{y}}}\right)^\gamma$.*

Proof. See the Appendix. □

The intuition for why the wealth accumulation equation (20) can converge to a fat-tailed distribution is the same as in Benhabib et al. (2011) and Zhu (2019). When the strength of parental altruism is sufficiently strong, positive intergenerational transfers occur over relatively large wealth holdings, see Lemma (4; exposure to capital income risk will then push lineages experiencing a long streak of high returns on financial wealth toward the upper end of the stationary distribution, provided the process does not grow without bounds (i.e. it contracts on average). This ensures that the process defined in (20) converges in probability to a limit distribution which is asymptotically bounded by a power law (Kesten, 1973).

We remark that the condition $\chi(1 + \mathbb{E}[R]) < 1$ that characterizes existence of wealth states entailing zero financial bequests (Lemma 4) implies that the dynamics of the auxiliary equation (21) obey a contraction on average property, which is part of the regularity conditions producing fat-tailed behavior of the stationary solution to the wealth accumulation process (20). As a consequence, for a sufficiently large intensity of altruism that induces strictly positive bequests over the whole wealth space and still preserves convergence of the wealth accumulation process – i.e. when $\frac{1}{1+\mathbb{E}[R]} \leq \chi < \bar{\chi}$ where $\bar{\chi}$ solves $\mathbb{E}[\alpha(r_t, \bar{\chi}, \gamma)] = 1$ – the unique stationary distribution would be atomless and its right tail asymptotically equivalent to a Pareto distribution.

By contrast, when altruism is weak enough to induce no accumulation of wealth in the lower states of the wealth space – i.e. when $\chi < \frac{1}{1+\mathbb{E}[R]}$ – the left tail of the stationary distribution will be populated by all those lineages who are temporarily stuck in the unskilled occupation and enjoy no wealth inheritances from their parents. As long as financial bequests are not part of the optimal bequest plan of constrained households, the stationary wealth distribution will necessarily exhibit a mass point at the lower end of its support, whose size is determined by the distribution of abilities across lineages. Formally it holds the following

Proposition 3. *Let $\chi(1 + \mathbb{E}[R]) < 1$. Then the stationary wealth distribution ω_∞ exhibits an atom in $\underline{y}$, i.e.*

$$\mu^*\left(\underline{y}\right) = \frac{\int_{\omega \in]\underline{y}, \tilde{\omega}_0]} [1 - G(\tilde{x}(\omega))] d\mu^*}{G(\tilde{x}(\underline{y}))} > 0 \tag{22}$$

where μ^* is the invariant measure of ω_∞.

Proof. See the Appendix. □

Propositions 2 and 3 jointly clarify that right-skewed and thick-tailed wealth distributions, as one of the most pervasive empirical features of market economies, can emerge in environments where wealth backgrounds, heterogeneous abilities and credit market imperfections all interact in producing a temporary mobility trap at the bottom of the support, with a mass of lineages bunched in the lowest state of the wealth space.

From a technical standpoint, an atom in the support of the distribution of the additive shock component (here, labour income, whose distribution over workers emerges as a result of educational choices made by parents) becomes an atom of the support of the stationary distribution of the state variable (here, wealth) since the random component embodying the multiplicative shock (here, the gross return on financial bequests) can be null with positive probability (as long as parents' optimizing behavior entails zero financial bequests for a positive measure of lineages).[12]

4 Altruism in joy-of-giving form

For the sake of comparisons with other contributions where intergenerational altruism is assumed in the form of joy-of-giving we now analyze the implications for wealth accumulation of this bequest motive, i.e. when both financial and educational bequests enter the utility function directly, see equation (4).[13]

Solving the parents' utility maximization problem for the system of the KKT conditions associated with each education choice $e \in \{0, 1\}$ delivers the following interior pseudo bequest functions

$$b_0^* = \left(1 + \chi^{-\frac{1}{\gamma}}\right)^{-1} \cdot \omega_t \tag{23}$$

$$b_1^* = b_0^* - x \tag{24}$$

Notice that in the presence of a joy-of-giving bequest motive, the utility flow obtained from leaving a bequest (of whatever nature) depends *only* on the size of the bequest; in the presence of a borrowing constraint, educational investment $e^* = 1$ in lineage i can thus be taken to occur if and only if $x \leq b_0^*$, where $b_0^* = 0$ never occurs at positive wealth levels.

In sharp contrast to the case of paternalistic altruism, educational and financial bequest choices are tightly intertwined when altruism is shaped by a joy-of-giving motivation. In fact, with an infinite marginal utility of zero bequests (either financial or educational or both), optimal bequests are always positive. Since the transition of wealth within each lineage is given by

$$\omega_{t+1} = \begin{cases} y - (1+r)\,x + (1+r)\,b_0^* & x \leq b_0^* \\ \underline{y} + (1+r)\,b_0^* & x > b_0^* \end{cases} \tag{25}$$

the following result is easily established

Proposition 4. *Whenever the wealth accumulation process (25) is ergodic, the unique stationary distribution ω_∞ exhibits no mass point in $\underline{y}$.*

Proof. See the Appendix. □

This result has a simple yet powerful implication for our understanding of the mechanisms through which wealth inequality tends to persist. In the presence of a joy-of-giving bequest

[12] See Buraczewski et al. (2016) for a discussion of the structure of the support of the stationary solution of linear random recurrences with multiplicative noise; and Zhu (2020) for a similar result in an incomplete-market model with endogenous labour supply, where agents solve an income-fluctuation problem over an infinite planning horizon.

[13] Under this formulation, financial and educational bequests both deliver an identical marginal benefit to altruistic parents. A different specification allowing for imperfect substitution between the two does not alter any of the qualitative results established next (details are available upon request).

motive, capital income risk would still provide the necessary mobility across wealth levels that prevents lineages them from being trapped in the lowest part of the wealth space, making credit market imperfections and indivisibilities in educational investment immaterial in this respect. Thus, any attempt to capture the qualitative features of the left tail of empirical cross-sectional distributions of wealth stated above, requires relaxing at least one of the two assumptions about the specification of preferences (homothetic utility in CRRA form and/or joy-of-giving bequest motive).

5 The size of the atom: some comparative statics

So far, our analysis has confirmed that capital income risk plays a fundamental role in generating fat-tailed behavior of the right tail of the stationary distribution of wealth, even when households exhibit a paternalistic bequest motive and face non-convexities in educational investment. It has further identified conditions under which occupational upgrading via human capital formation, rather than uninsured financial shocks, is the key driver of upward mobility for low-wealth lineages experiencing the vicious confluence of borrowing constraints and investment indivisibilities.

We now study the comparative statics of the size of the atom in the left tail of the stationary wealth distribution, i.e. $\mu^*(\underline{y})$, with respect to some of the structural parameters and fiscal policies that shape social mobility patterns in our economy. Clearly, we will only focus on cases where the support of the stationary distribution contains a mass point at its lower end, i.e. altruism is taken to be paternalistic and preference parameters/exogenous returns on investment are such that $\tilde{\omega}_0 > \underline{y}$ (see Lemma (4)).

5.1 Intensity of the bequest motive

As shown in Benhabib et al. (2011) and Zhu (2019), a stronger preference for altruism (i.e. a larger χ) in economies with uninsured investment risk will foster the accumulation behavior of lineages experiencing lucky streaks of high rates of return on financial bequests, thereby boosting wealth concentration in the right tail of the stationary distribution.

According to the same logic, a stronger concern for the wealth status of children entails, at any wealth level, stronger educational efforts, thereby reducing the probability of transitioning toward the lower states of the wealth space. As a result, the measure of the least wealthy households in the stationary distribution decreases, as stated next:

Proposition 5. *Let $\chi(1+\mathbb{E}[R]) < 1$. The measure $\mu^*(\underline{y})$ of the atom in the stationary wealth distribution decreases with the intensity of altruism χ, all else equal.*

Proof. See the Appendix. □

5.2 Educational costs

Ex-ante heterogeneity in abilities naturally influences upward mobility flows out of the lowest wealth states and the composition of optimal investment portfolios in the cross-section of lineages. We next study how the stochastic properties of the distribution of educational costs in the population affects the size of the mass point at the bottom of the support of the stationary distribution of wealth. Intuitively, the size of the atom (stochastically) increases in the cost of investment in education:

Proposition 6. *Let $\chi(1+\mathbb{E}[R]) < 1$. All else equal, consider two distinct distributions G and G' for the educational investment costs X such that G' first order stochastically dominates G, i.e. $G'(X \leq x) \leq G(X \leq x)$ for all $x \in \Delta x$; and let $\mu^*_{G'}(\underline{y})$ and $\mu^*_{G}(\underline{y})$ denote the measure of the atom in $\underline{y}$ in the stationary wealth distribution under G' and G, respectively. Then it holds $\mu^*_{G'}(\underline{y}) > \mu^*_{G}(\underline{y})$.*

Proof. See the Appendix. □

5.3 Fiscal policies

The presence of multiple dimensions of heterogeneity across individuals, such as in underlying labor earning abilities and investment returns, has been advocated as one of the main arguments in favour of positive capital income taxation, for both equity and efficiency reasons (e.g. Bastani and Waldenström, 2020). As emphasized in Benhabib et al. (2011), in economies with capital income risk, top wealth concentration proves highly sensitive to fiscal policies that curb the boosting effect of random rates of returns on the wealth accumulation process: all else equal, the higher the capital income and/or bequest taxes, the thinner the upper tail of the stationary distribution.

How do the properties of the left tail of the distribution depend on these fiscal measures? To answer this question, we adapt the model to alternatively encompass a flat tax $\tau_b \in (0,1)$ on financial bequests (an *estate tax*, following the terminology in Benhabib et al., 2011) or rather a uniform tax $\tau_r \in (0,1)$ issued on realized financial returns (*capital income tax*). In the former case, $(1-\tau_b)b^*$ will define post-tax optimal bequests enjoyed by children; in the latter, we simply re-define the random rate of return r_{t+1} as the pre-tax rate and introduce a (flat) capital income tax $\tau_r \in (0,1)$ so that $(1-\tau)(1+R)$ identifies the post-tax rate of return on financial bequests. By linearity, the dynamics of wealth at the lineage level under either type of tax is

$$\omega_{t+1} = \begin{cases} (1-\tau_j)\,(1+r)\,b_1^{*'}(\omega_t,\,x) + y, & x \leq \tilde{x}' \\ (1-\tau_j)\,(1+r)\,b_0^{*'}(\omega_t) + \underline{y}, & x > \tilde{x}' \end{cases} \tag{26}$$

where the subscript j denotes the operative tax rate (i.e. $j = r$ or $j = b$) and the superscript $'$ labels endogenous objects (bequest policies and thresholds) in the presence of positive taxation.

A positive tax on bequests (or on wealth returns) has three main effects on the individual wealth transition processes (26): first, it mechanically changes the average return on direct wealth transfers, as well as its variability (a standard first-order stochastic dominance effect); second, conditional on the actual degree of aversion to risk, it might trigger a *compensating effect* by inducing parents to bequeath, conditional on their educational choice, a larger share of their wealth to their offspring, to counter the depressive effects of taxes on post return inheritances of children, as in Becker and Tomes (1987); and third, it distorts parents' incentives to engage in financial vis-à-vis educational investment by modifying their expected risk-return profiles – a *saving composition effect*.

The first (exogenous) effect works through the accumulation law (26) by compressing the structure of returns on non-human wealth. The second and third (endogenous) effects both materialize through a change in agents' optimal bequest plans as follows: first, conditional on the educational choice, post tax pseudo financial bequests $(1-\tau_j)b_e^{*'}$ necessarily decrease, regardless of the strength of the compensating effect; second, the educational investment threshold $\tilde{x}'$ can increase with the tax rate and be strictly larger than that governing human capital formation in the the no-tax scenario, for a positive measure subset of wealth-constrained households. Thus, the introduction of an estate or capital income tax fosters the formation of human capital across lineages at middle wealth levels, and this capital can serve as a barrier preventing them from moving down to the bottom of the distribution. Formally

Lemma 5. *Let* $\chi(1+\mathbb{E}[R])<1$. *Then*
(i) for each educational cost $x \in \Delta x$ *and parental wealth* $\omega_t \geq \underline{y}$, *the pseudo financial bequest policies* $b_e^{*'}$ *under either tax* $\tau_j \in (0,1)$, $j=b,r$, *and their analogues* b_e^* *in the no-tax benchmark* ($\tau_j=0$) *satisfy*

$$(1-\tau_j)b_e^{*'} \leq b_e^*, \quad e \in \{0,1\};$$

(ii) if the maximal educational cost $\bar{x}$ *is sufficiently large, then there exist tax rates* $\tau_j \in (0,1)$, $j=b,r$ *under which the educational threshold cost* $\tilde{x}'(\omega_t)$ *and its analogue* $\tilde{x}(\omega_t)$ *in the no-tax benchmark* ($\tau_j=0$) *satisfy*

$$\tilde{x}'(\omega_t) > \tilde{x}(\omega_t), \quad \omega_t \in B$$

where B *is a positive measure subset of* $W=[\underline{y},\infty)$.

Proof. See the Appendix. □

The actual bearing of fiscal policies on the transmission of wealth at the lineage level naturally depends on the interplay across the three effects mentioned above. Capital income taxes expand the set of wealth states where financial bequests are not part of the optimal bequest plan of households; by the same token, they induce a portfolio re-allocation towards human capital investment that becomes relatively more attractive as a mean of intergenerational wealth transmission. If sufficiently strong, this effect can mitigate the inefficiencies produced by local non-convexities and borrowing constraints on human capital formation, and hence reduce expected residence time of lineages in the lowest states of the wealth space. The intuition is that capital income and/or estate taxes trade off the distortions on the overall level of total bequests and the composition of individual savings, discouraging wealth-constrained individuals from substituting human by financial assets in response to changes in the risk-return structure of assets. If the enhanced process of occupational upgrading and the ensuing persistence of human capital overcompensate the contraction in the intergenerational transmission of non-human wealth, this composition effect dominates the other two and ends up improving upward movements of the least wealthy households in the cross-sectional distribution of wealth. However, the wealth effect imparted by taxation on consumption and savings is relatively stronger for households who are relatively poorer in financial wealth, and might exacerbate downward mobility flows of lineages near the borrowing constraint towards the bottom end of the wealth space.

The next Proposition formalizes the foregoing arguments about the ambiguous effects of estate/capital income taxation on the left tail of the stationary distribution of wealth:

Proposition 7. *Let $\chi(1+\mathbb{E}[R]) < 1$. Depending on the risk-return structure of financial and educational investment, the measure of the atom in $\underline{y}$ can increase or decrease in response to the introduction of an estate tax τ_b or of a capital income tax τ_j.*

Proof. See the Appendix. □

From a policy perspective, this latter result suggests a word of caution in evaluating the effects on wealth inequality of fiscal policies that abstract from redistributive considerations. When wealth is observable, taxing wealth at the top and redistributing government revenue lump-sum among households at the bottom would in fact help to mitigate the effects of the distortions on the saving behavior of the poorest, for any given distribution of (unobservable) innate abilities among the tax-payers. Receipts from taxation of top wealth owners could also be used to design transfers to low-wealth families that are conditioned on educational investment by parents, or to lower their educational costs by e.g. promoting public schooling, with similar effects on the left tail of the wealth distribution. In light of the previous results, the analysis is straightforward and we do not provide it here.

6 Related literature and possible extensions

Our work is clearly inspired by the growing literature that relies on uninsurable capital income risk – as opposed to labour earnings uncertainty – to match the empirically documented features of the right tail of the wealth distribution in a large panel of countries. Within this context, it purports to explore the key features of the left tail of the stationary wealth distribution by investigating the dynamic implications of individual heterogeneity and credit constraints on social mobility (between occupations and across wealth levels) and thus on the process of wealth accumulation along the whole support; and the extent to which such implications depend on the essential ingredients of the framework of analysis (i.e. the formulation of parental preferences for altruism).

On the first point, a large variety of models – surveyed by e.g. De Nardi and Fella (2017) and Benhabib and Bisin (2018) – have been put forward to match the documented upsurge of wealth in the upper percentiles of the distribution. While early frameworks, such as Champernowne (1953)'s, were somewhat mechanical in nature and lacked explicit micro-foundations, relatively newer contributions have attempted to reconcile observed wealth accumulation patterns with the theoretical predictions of fully-fledged models of intergenerational transmission of human capital and financial bequests.

Albeit able to match empirical measures of inequality (e.g. the Gini coefficient) over the full distribution of wealth, a crucial implication of models of life-cycle behavior in the presence of uninsurable stochastic earnings – e.g. Banerjee and Newman (1991), Aiyagari (1994), Hugget (1996), Krussel and Smith (2006) – is that the predicted wealth distribution fails to display a fat enough upper tail compared to the data, since saving incentives are eroded over sufficiently high wealth levels – see Benhabib et al. (2015) for a discussion of this point.[14] Quadrini (2000) and Cagetti and De Nardi (2006), by contrast, show that quantitative dynamic stochastic models endowed with investment risk are able to generate sufficient cross-sectional dispersion in the wealth distribution for entrepreneurs and workers which appears to match fairly well a number of empirical facts in the U.S. macroeconomic history.

Based on substantial evidence on the impact of capital income on wealth concentration in recent decades (e.g. Bertaut and Starr-McCluer, 2002; Moskowitz and Vissing-Jorgensen, 2002; Wolff, 2006), the work of Benhabib et al. (2011) was the first to investigate wealth dynamics in an overlapping generations (OLG) economy where individual wealth accumulates as a result of the operation of a multiplicative shock (on wealth holdings) and an additive shock (stochastic earnings), and the intergenerational transmission of wealth occurs in the form of voluntary bequests. Under appropriate (and empirically plausible) restrictions on the random process generating financial returns, the upper tail of the stationary wealth distribution is analytically shown to approximately follow a Pareto distribution. Among the main features of the model above is the occurrence of scale invariance (the random shocks are anonymous i.e. the idiosyncratic shocks do not depend on the position of households in the wealth distribution) and ergodicity (i.e. the initial distribution of wealth becomes irrelevant as time goes by). This result has been generalized in the literature on stochastic process to show that thick-tailed distributions can emerge when both a multiplicative and an additive shock drive the accumulation process, i.e. the so-called *proportional random growth* mechanism based on

[14]Kuhn (2013) and Light (2020) address equilibrium existence and/or uniqueness problems in Aiyagari-style models.

Kesten recursions (e.g. Kesten, 1973; Gabaix, 2009).[15]

There is by now a huge body of empirical work suggesting that educational attainments and skills are associated with beneficial labour market outcomes such as earnings, work environment and career opportunities across the world (e.g. Hanushek et al., 2015). Since the seminal contributions of Becker and Tomes (1979), Banerjee and Newman (1993) and Galor and Zeira (1993), scholars have been interested in studying the relationship between initial wealth inequality, educational investment, human capital accumulation and the ensuing (social and occupational) mobility patterns, laying differential emphasis on the role of investment indivisibilities, credit market imperfections and ex-post uninsurable shocks.[16] However framed, economic environments leading to the emergence of poverty traps have represented a fertile ground for the investigation of policies focused on stimulating human capital formation and/or supporting poverty eradication (see e.g. Piketty, 2000; Azariadis and Stachurski, 2005; Barret et al., 2016).

We are of course not the first ones to study the predictions of dynamic models in which altruistic parents may transfer resources to their offspring by providing education and by leaving bequests. Ishikawa (1975) sets up an integrated framework of life-cycle savings and investment in human capital to investigate the role of family generational structures and family values in shaping the size distribution of income in the short and long run. Blinder (1976) studies intergenerational transfers and life cycle consumption decisions, with a focus on the implications of differential tax treatments for the optimal mix of financial bequests and human capital investment. Grossmann and Poutvaara (2009) develop an OLG model in which parental utility depends on both financial bequests and educational investment, and identify conditions under which, in the presence of wage taxation, a sufficiently small bequest tax proves Pareto-improving for it enhances private incentives to invest in human capital formation. Dávila (2023) provides a formal analysis of the market outcome in terms of individually optimal financial bequests and educational investment under laissez-faire, showing that it generally fails to replicate the planner's allocation due to the inability of parents to internalize the amplifying effects of human capital on total factor productivity.

Our contribution stands out as we delve into a different dimension, i.e. the interplay of capital income risk, credit market imperfections and investment indivisibilities in shaping the bequest behavior of paternalistic individuals, a facet untouched in previous works. This novel perspective enriches our understanding of the tail behavior of the stationary wealth

[15]This is an instance of Champernowne (1953)'s seminal result, that random growth processes with multiplicative components endowed with a reflecting lower barrier produces a power law at steady state. Other micro-foundations able to generate Pareto-tailed wealth and/or income distributions are studied by e.g. Nirei and Aoki (2016), where it is business productivity shocks in a world with safe and risky investment technologies that push concentration of income at the top; and Toda (2019), who exploit random discount factors to generate heterogeneity in saving rates.

[16]Loury (1981) and Mookherjee and Napel (2007) are notable exceptions in this respect, for they investigate implications of some form of ex-ante heterogeneity on wealth dynamics and the emergence (or the lack thereof) of history dependence in OLG settings.

distribution for the impact of fiscal policies which are designed to affect the intergenerational transmission of wealth and ultimately the long-run wealth inequality. We remark that our characterization results about the tail behaviour of the stationary wealth distribution pertain to a partial equilibrium, non-interactive model with exogenous wages; extending the analysis to endogenous factor prices and general equilibrium effects is arguably a fruitful research avenue to follow, in particular to understand whether labour market dynamics has the potential to establish a connection between the right and left tail of the wealth distribution, clarifying the relative contribution of the many forces involved in shaping social mobility and the persistence of inequality.

Settings with rationally optimizing agents exhibiting constant relative risk aversion (CRRA) are widespread in theoretical studies of income and wealth dynamics, for they are known to generate asymptotically linear policy rules, see e.g. Nirei and Aoki (2016), Benhabib et. al (2011, 2015), Zhu (2019). Coupled with homotheticity in the utility aggregator of consumption and bequests, the CRRA assumption does not prevent the emergence of ex post positive correlation between wealth holdings and returns, without requiring heterogeneity in the marginal propensities to save across rich and poor. As a result, even when bequests are not taken to be luxury goods, capital income risk is able to induce top wealth concentration patterns consistent with those observed in real-world data.[17]

In a Becker and Tomes (1979)'s type of framework augmented with idyiosincratic investment shocks, Zhu (2019) derives an exact expression of the stationary wealth distribution, showing that its upper tail is asymptotically equivalent to a Pareto distribution. Consistent with Zhu (2019)'s results, we do find that thick-tailness of the unique, ergodic wealth distribution is a robust prediction of heterogeneous-agent frameworks with capital income risk, even when allowing for indivisibilities in human capital formation – a generalization result in a well-defined class of models. Differently from Zhu (2019), our analysis explicitly addresses the properties of the left tail of the stationary distribution, and reveals that no mass point at the bottom of its support is bound to emerge under homothetic CRRA utility when agents *also* display a joy-of-giving bequest motive, a prediction at odss with one of the most dramatic features of the empirical wealth distribution in advanced economies. This finding appears to suggest that constructing a model of wealth accumulation that aims at rationalizing empirical facts about the two tails of the wealth distribution requires relaxing at least one of the two assumptions about the specification of preferences (homothetic utility in CRRA form or joy-of-giving formulations of the bequest motive).

Achdou et al. (2022) cast the Aiyagari-Bewley-Huggett model in continuous time and pro-

[17] A number of recent studies (e.g. Cioffi, 2021) have questioned the homotheticity assumption on the basis of empirical evidence about heterogeneous saving rates between entrepreneurs and workers (e.g. Quadrini, 1999), cross-sectional correlation between wealth and equity shares (e.g. Waachter and Yogo, 2010) and decreasing risk aversion in total wealth and (e.g. Meeuwis, 2022). Ma and Toda (2021) challenge this view by theoretically showing that standard income fluctuations models with homothetic CRRA preferences and general shock specifications can admit zero asymptotic marginal propensity to consume in the group of the wealthiest, under empirically plausible paramaterizations.

vide, among other results, a full characterization of saving policy functions near the borrowing constraint, showing that agents necessarily hit the latter in finite time upon experiencing a long streak of adverse income shocks. As a consequence, the stationary wealth distribution features a Dirac point mass at the borrowing constraint. While delivering a similar property about the behaviour of the left tail of the wealth distribution, our analysis is conducted in a standard discrete-time economy with capital income risk, credit market imperfections and indivisible human capital investment, which allows us to link the shape of the left tail of the wealth distribution to the structure of occupational mobility.

The role of alternative bequest formulations in shaping the intergenerational transmission of wealth has only recently been addressed from an analytical standpoint. Pestieau and Thibault (2012) explore features of the limit wealth distribution in economies populated by agents with heterogeneous bequest motives. Zhu (2019) establishes that the impact of estate taxes on the long-run wealth inequality do not depend on whether bequest motives are taken to be of the worm-glow (joy of giving) type or not. What contributes to shaping bequest motivations of altruistic agents mostly is an empirical question, which still lacks a conclusive answer (Kopczuk, 2013). From a methodological perspective, our investigation brings us fresh insights into how preferences for family altruism affect the intergenerational transmission of wealth and are therefore relevant for the exploration of the sources of long-run wealth inequality. Due to the structure of paternalistic altruism, we conjecture that our characterization of the mass point in the left tail the stationary distribution generalizes to (separable) dynastic preferences *à la* Barro (1974), by which parents are concerned with the welfare of their children and credit market imperfections may prevent low-wealth households from implementing positive intergenerational transfers of non-human wealth.[18]

Owing to its flexibility, our framework of analysis can also be employed to study conditions under which a thick right tail of the stationary distribution co-exists with permanent poverty traps at the bottom of the support (non-ergodicity). As is known, standard poverty trap models typically call for temporary large scale interventions, for they display permanent effects on both investment and growth. If the economy is rich enough in wealth, then tax/transfer policies can free constrained agents from the investment trap in one period. The impact of redistributive policies are by contrast more complex in the presence of multidimensional heterogeneity (wealth and ability). In line with D'Amato and Di Pietro (2014), we conjecture that heterogeneity in individual abilities may call for policies exhibiting higher persistence than in the standard homogeneous ability case, for they have to account for downward mobility flows. At the same time, redistributive policies might have larger scope, i.e. they can be effective in declining economies whereas they would not absent heterogeneity in educational costs. We leave this aspect to future research.

[18]Wan and Zhu (2019) develop a general equilibrium model without capital income risk where alternative bequest motives (joy of giving vs. altruism) are shown to generate different redistributive effects of bequest taxation, on the assumption that labour productivity is subject to i.i.d. shocks.

7 Concluding remarks

When credit market imperfections constrain educational investment opportunities, individual abilities and wealth background may exert a significant influence on the formation of human capital and the ensuing patterns of social mobility within and across family lineages (e.g. Becker and Tomes, 1986). The present paper has identified simple conditions on the actual nature and intensity of intergenerational altruism under which households in the lowest rungs of the wealth ladder are more likely to arrange their bequest strategies according to the mechanisms emphasized in Loury (1981) rather than those highlighted in Benhabib et al. (2011).

Our analysis has shown that the unique time invariant distribution of wealth can well exhibit a mass point at (almost) zero wealth, where upward wealth mobility is solely driven by occupational upgrading. We have then explored some comparative statics with respect to fiscal policies. Capital income taxation, aimed at mitigating top wealth inequality, is found to possibly hamper upward mobility flows at the bottom of the distribution, thereby incresing the measure of the least wealthy. In this setting, a simple tax-transfer scheme can be envisaged under which revenues from a small wealth tax levied on top holders are redistributed to so as to uniformly lower educational costs and fuel human capital formation as the prime engine of social mobility in the left tail.

Appendix

Proof of Lemma 1

For fixed $e = \{0, 1\}$, the KKT first-order conditions (15) and (16) are sufficient for existence of a global maximum, being the objective function (twice-differentiable and) strictly concave, and the inequality constraints continuously differentiable convex functions. Existence, uniqueness and differentiability (hence continuity) of pseudo policy functions b_e^* and c_e^* are therefore obtained.

Since the cross partial derivative of $u(c_e)$ with respect to b_e and ω_t is equal to minus the second-order derivative of $u(c_e)$ with respect to b_e, applying the implicit function theorem to (15) delivers

$$\frac{db_e^*}{d\omega_t} = -\frac{\frac{d^2u(b_e^*;e)}{db_e d\omega_t}}{\frac{d^2u(b_e^*;e)}{db_e^2} + \chi\mathbb{E}\left[(1+r)^2 \frac{d^2v(\omega_{t+1}(b_e^*;e))}{db_e^2}\right]} \in (0,1)$$

implying c_e^* is non-decreasing in wealth as well.

Proof of Lemma 2

Consider the indifference condition (17). Fix $\omega_t > 0$. The LHS does not depend on x (since $e^* = 0$), whereas the RHS continuously decreases in $x \in \Delta x$, since by the envelope theorem

$$-\frac{d\,u(b_1^*;1)}{d\,b_e} \cdot \left(\frac{\partial b_1^*}{\partial x} + 1\right) + \chi \frac{d\,\mathbb{E}v(b_1^*;1)}{d\,b_e}\frac{\partial b_1^*}{\partial x} < 0 \tag{27}$$

Observe that, when $x = 0$, one has $e^* = 1$ at any wealth level $\omega_t > 0$. By the same token, when $\tilde{x}(\omega_t) > \overline{x}$, parents with wealth ω_t will always find it profitable to engage in educational investment (without loss of generality in this specific case we set $\tilde{x} = \overline{x}$). Then, for given $\omega_t > 0$, there exists a unique $\tilde{x}(\omega_t) > 0$ such that $e^* = 1$ if and only if $x < \tilde{x}(\omega_t) < \omega_t$ (recall that $x > \omega_t$ implies $e^* = 0$ under the no-borrowing constraint).

The indifference condition (17) defines a continuously differentiable function $F(\omega_t, \tilde{x}) = 0$. By virtue of the implicit function theorem, $\tilde{x}(\omega_t)$ is a well-defined continuous and (continuously) differentiable function satisfying

$$\frac{d\tilde{x}}{d\omega_t} = -\frac{F_{\omega_t}}{F_{\tilde{x}}} = \frac{\frac{du(b_1^*;1;\omega_t,\tilde{x})}{db_e} - \frac{du(b_0^*;0;\omega_t)}{db_e}}{\frac{du(b_1^*;1;\omega_t,\tilde{x})}{db_e}} < 1 \tag{28}$$

and

$$sign\,\frac{d\tilde{x}}{d\omega_t} = -sign\,F_{\omega_t} \tag{29}$$

Using the first-order condition (15) we have

$$F_{\omega_t} = (\omega_t - b_0^*)^{-\gamma} - (\omega_t - b_1^*(\tilde{x}) - \tilde{x})^{-\gamma} \tag{30}$$

and thus $\frac{d\tilde{x}}{d\omega_t}$ in its domain if and only if $b_1^*(\tilde{x}) + \tilde{x} > b_0^*$ for all ω_t. Suppose not, i.e. assume $b_1^*(\tilde{x}) + \tilde{x} \leq b_0^*$ for some ω_t. Then from the indifference condition (17) it must hold at $\tilde{x}$

$$\mathbb{E}\left[v\left((1+r)b_1^*(\tilde{x}) + y\right)\right] \leq \mathbb{E}\left[v\left((1+r)b_0^* + \underline{y}\right)\right] \tag{31}$$

while (15) and the concavity of v jointly imply

$$\mathbb{E}\left[\frac{d\,v}{d\,b_1}\left((1+r)b_1^*(\tilde{x}) + y\right)\cdot(1+r)\right] \leq \mathbb{E}\left[\frac{d\,v}{d\,b_0}\left((1+r)b_0^* + \underline{y}\right)\cdot(1+r)\right] \tag{32}$$

i.e. a generic contradiction (which is obtained, *a fortiori*, when the borrowing costraint bites and (16) is the relevant optimality condition). Thus, the threshold $\tilde{x}(\omega_t)$ must increase monotonically in wealth.

Proof of Lemma 3

We first show that the optimal financial bequest b_1^* with $e^* = 1$ decreases monotonically in the educational cost $x \in \Delta X$, with slope in $(-1, 0]$. Differentiating the relevant first-order condition with respect to x one obtains

$$\frac{\partial b^*}{\partial x}(x) = -\frac{\frac{d^2 u}{db_1^2}}{\frac{d^2 u}{db_1^2} + \chi\frac{d^2\mathbb{E}v}{db_1^2}} < 0$$

provided the denominator is non-zero.

Suppose now there exists some wealth level $\omega_t \geq \underline{y}$ such that, at $x \in \Delta X$, it holds $b_0^* = b_1^*(x)(x) = b > 0$. It would necessarily follow

$$\chi\frac{d\,\mathbb{E}v}{d\,b_e}(b;0) = \frac{d\,u}{db_e}(\omega_t - b) < \frac{d\,u}{d\,b_e}(\omega_t - b - x) = \chi\frac{d\,\mathbb{E}v}{db_e}(b;1)$$

Since $\omega_t = \underline{y}$ implies

$$\frac{d\,\mathbb{E}v}{d\,b_e}(b;0) > \frac{d\,\mathbb{E}v}{db_e}(b;1)$$

i.e. contradiction. It must then be the case that, for all $x \in \Delta X$, it holds

$$b_1^*(x) \leq b_1^*(0) < b_0^*$$

as asserted.

Proof of Lemma 4

Inspection of (19) and Lemmata 2 to 3 reveal that $b_e^* = 0$ for each $e \in \{0,1\}$ requires parental wealth ω_t to belong to the set $[\underline{y}, \tilde{\omega}_0]$. Since by (19) one has

$$\tilde{\omega}_0 = \underline{y}\,[\chi\,(1+\mathbb{E}[R])]^{-\frac{1}{\gamma}}$$

we obtain

$$\tilde{\omega}_0 > \underline{y} \quad \Longleftrightarrow \quad \chi(1+\mathbb{E}[R]) < 1$$

whence the assertion.

Proof of Proposition 1

We adapt a proof in Zhu (2019) for a similar, albeit not identical, setup, which in turn exploits Theorem 15.0.1, part (iii) in Meyn and Tweedie (2009).[19] Formally, Meyn and Tweedie (2009)'s argument relies on establishing three key properties for the Markov process $\{\omega_t\}$ generated by (20): (1) irreducibility; (2) aperiodicity, and (3) bounded drift.

To facilitate sailing the technical details, we consider two distinct cases:

a. $\chi\,(1+\mathbb{E}[R]) > \left(\frac{\underline{y}}{\overline{\overline{y}}}\right)^{\gamma}$ (or equivalently $\tilde{\omega}_0 < \overline{y}$).

1. Irreducibility. We first recall that consumption and bequest policies (c^*, b^*) are continuous functions in ω_t, and that $\omega_{t+1} \geq \underline{y}$ for all $\omega_t > 0$ and $x \in \Delta X$, as per the equilibrium transition (20). Notice also that $Pr\left(\omega_t < \tilde{\omega}_0 \,\middle|\, \omega_1 > \underline{y}\right) > 0$ at some time $t \geq 1$ since

$$(1+\min{(\Delta r)})b_e^*(\omega_t) + \underline{y} + e^*\left(\min{(\Delta y)} - \underline{y}\right) = \underline{y} < \tilde{\omega}_0$$

for all $\omega_t > \underline{y}$ and $x \in \Delta X$. Since $b_e^* = 0$ for each $e \in \{0,1\}$ and $\omega_t \leq \tilde{\omega}_0$, and yet $Pr\,(X < \tilde{x}(\omega_t)\,|\,\omega_t < \tilde{\omega}_0) = G(\tilde{x}(\omega_t)) > 0$ by virtue of Assumption (3) and Lemma (2), implying $e^* = 1$ and $\omega_s = y_s > \underline{y}$ at some $s > t$, any set A such that

$$\int_A f(z)dz > 0 \tag{33}$$

can be reached in finite time with a positive probability. Letting $\varphi(A) = \int_A f(z)dz$ define a measure on $[\underline{y}, \infty)$, the process $\{\omega_t\}$ is therefore φ-irreducible, and thus ψ-irreducible for some other measure ψ on $[\underline{y}, \infty)$, which necessarily exists (see Proposition 4.2.2 of Meyn and Tweedie, 2009).

[19] In Section 7 of his manuscript, Zhu (2019) generalizes his baseline OLG model with uninsurable income risk and intergenerational transfers due to altruism to the case of unobservable labour and capital income rates of return and a no-borrowing constraint. While this endows parents with a precautionary motive for saving, there is no educational investment choice to make: each and every agent receives a random labour income which is subject to idiosyncratic shocks. A main consequence of this modeling structure is that the limit distribution of wealth is atomless.

2. Aperiodicity. Consider the set $C = [\underline{y}, \tilde{\omega}_0]$. For $\omega_t \in C$, $b_e^*(\omega_t) = 0$ and $e^* = 1$ for all $x < \tilde{x}(\omega_t)$, which occurs with probability $G(\tilde{x}(\omega_t)) > 0$. As a result, one has $\int_C f(z)dz > 0$; this allows constructing a non-trivial measure $v_1(C) := \int_C f(z)dz$ on the Borel σ-field of W, denoted with $\mathcal{B}(W)$, satisfying

$$v_1(B) \leq P(x, B), \quad \forall x \in C, \ B \in \mathcal{B}(W) \tag{34}$$

where $P(x, B) = \mu(\omega_t \in B \mid \omega_t = x)$ is the one-step transition probability kernel. Hence, C is a so-called v_1-small set. Since $\{\omega_t\}$ is φ-irreducible, existence of a small set C and of a positive measure $v_1(C) > 0$ imply that $\{\omega_t\}$ is also strongly aperiodic (see Meyn and Tweedie, 2009, p. 114).

Bounded drift. This condition ensures that the Markov process is stable in the sense of exhibiting inward drift to some small (typically compact) subset of W, i.e. for some measurable function $V \geq 1$, finite at some $x_0 \in W$, the drift of $V(\omega_t)$ at x defined as

$$\Delta V(x) = \int P(x, dz)V(z) - V(x)$$

satisfies

$$\Delta V(x) \leq -\beta V(x) + b\mathbb{1}_C(x), \quad x \in W$$

where C is a "petite" set, $\beta > 0$ and $b < \infty$ are constants, and $\mathbb{1}_C$ is the characteristic function associated to C (i.e. $\mathbb{1}_C = 1$ if and only if $x \in C$). By Proposition 5.5.3 in Meyn and Twediee (2009), a v_m-small set for some $m \geq 1$ is also petite (for some well-defined sampling distribution).[20]

Fix an arbitrary $\underline{\omega} > \underline{y}$, and consider the compact set $C = [\underline{y}, \underline{\omega}]$. Since

$$(1 + \min(\Delta r))b_e^*(\omega_t) + \underline{y} + e^* \left(\min(\Delta y) - \underline{y}\right) = \underline{y} < \tilde{\omega}_0$$

for all $\omega_t > \underline{y}$, there exists a common (across lineages) m, however big, such that $Pr(\omega_m < \tilde{\omega}_0 \mid \omega_1) \geq \epsilon > 0$. Since $b_e^* = 0$ for each $e = \{0, 1\}$ if and only if $\omega_t \in [\underline{y}, \tilde{\omega}_0]$, we have

$$Pr\left(\omega_{m+1} \in C \mid \omega_1\right) \geq Pr(\omega_{m+1} \in C \mid \omega_m < \underline{y}) \times Pr(\omega_m < \underline{y} \mid \omega_1) \geq \epsilon \int_C f(x)dx > 0$$

which implies that C is v_{m+1}-small and therefore petite.

Let us now observe that, for all ω_t, the pseudo financial bequest policy b_e^* for $e = \{0, 1\}$

[20] From (34), by analogy, a v_m-small set C satisfies $v_m(B) \leq P^m(x, B), \quad \forall x \in C, \ B \in \mathcal{B}(W)$, where $P^m(x, B)$ is the m-step transition kernel.

satisfies the first-order conditions (15)-(16). Thus either $b_e^* = 0$, or $b_e^* > 0$ solves

$$(\omega_t - b_e^* - e \cdot x)^{-\gamma} = \chi \mathbb{E}\left[(1+r)\left((1+r)b_e^* + \underline{y} + e \cdot (y - \underline{y})\right)^{-\gamma}\right]$$

or equivalently

$$(c_e^*)^{-\gamma} = \chi \mathbb{E}\left[(1+r)\left((1+r)(\omega_t - c_e^* - e \cdot x) + \underline{y} + e \cdot (y - \underline{y})\right)^{-\gamma}\right]$$

When $e^* = 0$ we have

$$\begin{aligned}
1 &= \chi \mathbb{E}\left[(1+r)\left((1+r)\left(\frac{\omega_t - c_0^*}{c_0^*}\right) + \frac{\underline{y}}{c_0^*}\right)^{-\gamma}\right], \\
&\leq \chi \mathbb{E}\left[(1+r)\left((1+r)\left(\frac{\omega_t - c_0^*}{c_0^*}\right)\right)^{-\gamma}\right] \\
&= \chi \left(\frac{\omega_t - c_0^*}{c_0^*}\right)^{-\gamma} \mathbb{E}\left[(1+r)^{1-\gamma}\right]
\end{aligned}$$

from which $c_0^* \geq \phi \cdot \omega_t$ – and thus $b_0^* \leq (1-\phi) \cdot \omega_t$ – where

$$\phi := \frac{1}{1 + \left(\chi \mathbb{E}\left[(1+r)^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}} \in (0,1)$$

Similarly, when $e^* = 1$

$$\begin{aligned}
1 &= \chi \mathbb{E}\left[(1+r)\left((1+r)\left(\frac{\omega_t - c_1^* - x}{c_1^*}\right) + \frac{\underline{y}}{c_1^*}\right)^{-\gamma}\right], \\
&\leq \chi \mathbb{E}\left[(1+r)\left((1+r)\left(\frac{\omega_t - c_1^* - x}{c_1^*}\right)\right)^{-\gamma}\right] \\
&= \chi \left(\frac{\omega_t - x}{c_0^*} - 1\right)^{-\gamma} \mathbb{E}\left[(1+r)^{1-\gamma}\right]
\end{aligned}$$

whereby $c_1^* \geq \phi \cdot (\omega_t - x)$, and hence $b_1^* \leq (1-\phi)(\omega_t - x)$.

Notice that $\chi(1 + \mathbb{E}[R]) < 1$ (as we assume) implies $\chi < 1$. Hence, from the wealth accumulation equation (20) we can write

$$\omega_{t+1} = \begin{cases} (1+r)\, b_1^*(\omega_t,\, x) + y \leq \chi(1-\phi)(1+r)\omega_t + y, & x \leq \tilde{x} \\ (1+r)\, b_0^*(\omega_t) + \underline{y} \leq \chi(1-\phi)(1+r)\omega_t + \underline{y}, & x > \tilde{x} \end{cases} \tag{35}$$

where apparently $\mathbb{E}\left[\chi(1-\phi)(1+r)\right] < 1$.

Define

$$
\begin{aligned}
&V(\omega_t) = \omega_t + 1, \quad \omega_t \in W; \\
&\beta = 1 - \chi(1-\phi)(1+\mathbb{E}[R]) - q, \quad q \in (0, 1 - \chi(1-\phi)(1+\mathbb{E}[R])); \\
&b = \beta + q + \mathbb{E}[Y],
\end{aligned}
$$

where $E[Y]$ is the unconditional average of the random process for labour income $\{y_t\}$. Pick any finite $\underline{\omega}$ such that $b \leq q \cdot (\underline{\omega}+1)$; then $C = [\underline{y}, \underline{\omega}]$ is a petite set, and (35) implies

$$
\begin{aligned}
&\mathbb{E}_t\left[V(\omega_{t+1}) - V(\omega_t)\right] \\
&\quad = \mathbb{E}_t[\omega_{t+1}] - \omega_t \\
&\quad \leq \mathbb{E}_t[\chi(1-\phi)(1+r)]\omega_t + \mathbb{E}[Y] - \omega_t \\
&\quad = -\omega_t + (1-1) + \mathbb{E}[\chi(1-\phi)(1+r)](\omega_t + 1) - \mathbb{E}[\chi(1-\phi)(1+r)] + \mathbb{E}[Y] \\
&\quad = -V(\omega_t) + \mathbb{E}[\chi(1-\phi)(1+r)]V(\omega_t) + 1 - \mathbb{E}[\chi(1-\phi)(1+r)] + \mathbb{E}[Y] \\
&\quad = -\left[1 - \mathbb{E}[\chi(1-\phi)(1+r)]\right]V(\omega_t) + (1 - \mathbb{E}[\chi(1-\phi)(1+r) + \mathbb{E}[Y]]) \\
&\quad \leq -\beta V(\omega_t) + b\mathbb{1}_C(\omega_t), \quad \omega_t \in W
\end{aligned}
\tag{36}
$$

showing that the drift condition is satisfied.

The support of the stationary distribution is $S_\infty = [\underline{y}, \infty]$ since

$$
Pr\left(\omega_{t+1} > \omega_t \mid \omega_t\right) > 0, \quad \forall \omega_t > \tilde{\omega}_0
$$

by Assumption (1), and

$$
Pr\left(\omega_{t+1} > \omega_t \mid \omega_t\right) > 0, \quad \forall \omega_t \in \left[\underline{y}, \tilde{\omega}_0\right]
$$

since in our case $\tilde{\omega}_0 < \overline{y}$.

b. $\chi\left(1 + \mathbb{E}[R]\right) \leq \left(\frac{\underline{y}}{\overline{y}}\right)^\gamma$ (or equivalently $\tilde{\omega}_0 \geq \overline{y}$).
The proof is analogous to that developed above, the only difference being that

$$
Pr\left(\omega_{t+1} < \overline{y} \mid \omega_t\right) > 0, \quad \forall \omega_t > \underline{y}
$$

and

$$
Pr\left(\omega_{t+1} > \overline{y} \mid \omega_t\right) = 0, \quad \forall \omega_t \in \left[\underline{y}, \tilde{\omega}_0\right]
$$

implying a bounded support $S_\infty = \Delta y$.

Proof of Proposition 2

Recall that $\chi(1+\mathbb{E}[R]) \leq \left(\frac{\underline{y}}{\bar{y}}\right)^{\gamma}$ is equivalent to $\tilde{\omega_0} \geq \bar{y}$; by Proposition 1 the ergodic stationary distribution of wealth has bounded support, and hence cannot exhibit a fat upper tail.

Let now the restriction $\chi(1+\mathbb{E}[R]) > \left(\frac{\underline{y}}{\bar{y}}\right)^{\gamma}$ be satisfied. Consider the following fictitious utility maximization problem

$$\max_{c,\,b,\,e} \quad u(\omega_t - b - e\cdot x) + \chi v\left(\mathbb{E}\left[(1+r)b + \underline{y} + e\cdot(y-\underline{y})\right]\right) \tag{37}$$

$$s.to: \qquad c + b + e\cdot x \leq \omega_t \tag{38}$$

$$b \geq 0 \tag{39}$$

$$c \geq 0 \tag{40}$$

$$e \in \{0,\,1\} \tag{41}$$

where it is assumed that bequest decisions are optimally taken with respect to the average rate of return on financial and human capital investment.

As in the actual problem, optimal financial bequests are zero when $\omega_t \leq \tilde{\omega}_0$ irrespective of the educational choice; and $\tilde{x}(\underline{y}) < \underline{y}$ continues to guarantee human capital investment of a positive measure of lineages in the group of the least wealthy households. By the same token, an interior solution $b^a > 0$ to the auxiliary problem (37)-(41) will necessarily satisfy the first-order condition

$$(\omega_t - b^a_e - e\cdot x)^{-\gamma} = \chi(1+r^M)\left[(1+\mathbb{E}[R])\,b^a_e + \underline{y} + e(E(y)-\underline{y})\right]^{-\gamma}$$

Solving explicitly for the financial bequest policies yields

$$b^a_e = \underbrace{\left(\frac{1}{1+\chi^{-\frac{1}{\gamma}}(1+\mathbb{E}[R])^{1-\frac{1}{\gamma}}}\right)}_{\phi(\gamma,\chi,\mathbb{E}[R])}\cdot\omega_t - \underbrace{\frac{e\cdot x + \chi^{-\frac{1}{\gamma}}(1+\mathbb{E}[R])^{-\frac{1}{\gamma}}\left(\underline{y} + e\cdot(\mathbb{E}[Y]-\underline{y})\right)}{1+\chi^{-\frac{1}{\gamma}}(1+\mathbb{E}[R])^{1-\frac{1}{\gamma}}}}_{\kappa_e} \tag{42}$$

or in compact form

$$b^{\circ}_{t,e} = \phi(\gamma,\chi,\mathbb{E}[R])\cdot\omega_t - \kappa_e \tag{43}$$

where $\phi(\cdot)$ and κ_e are constants that depend on the model's fundamentals.

From the CRRA specification (5) and the fact that future wealth is a non-decreasing function of bequests, we have

$$v_b(\omega_{t+1}(b,e)) \equiv \frac{\partial v}{\partial b}(\omega_{t+1}(b,e)) = \frac{\partial v}{\partial\omega_{t+1}}\frac{\partial\omega_{t+1}}{\partial b}(\omega_{t+1}(b,e)) = (1+r)v'(\omega_{t+1}(b,e))$$

and hence

$$\frac{\partial^2 v_b}{\partial b^2}(\omega_{t+1}(b,e)) \equiv \frac{\partial^2}{\partial b^2}\left(\frac{\partial v}{\partial \omega_{t+1}}\frac{\partial \omega_{t+1}}{\partial b}(\omega_{t+1}(b,e))\right) = (1+r)^3 v'''(\omega_{t+1}(b_t,e)) > 0$$

i.e. v_b is a convex function. By Jensen's inequality

$$v_b\left(\int \omega_{t+1}(b,e)d\mu\right) \leq \int v_b \circ \omega_{t+1}(b,e)d\mu$$

and thus

$$u'(\omega_t - b_e^* - e\cdot x) \geq v_b\left(\int \omega_{t+1}(b,e)d\mu\right)$$

which in turn implies $b_e^a \leq b_e^*$ for all $\omega_t > \tilde{\omega}_o$ and $e \in \{0,1\}$.

Consider now the auxiliary wealth accumulation process $\{\omega_t^a\}$ generated by the linear random recurrence

$$\omega_{t+1}^a = (1+r)b_e^a + \underline{y} \tag{44}$$

where b_e^a solve (37)-(41). This auxiliary process (44) can be equivalently rewritten as

$$\omega_{t+1}^a = \alpha_{t+1}\omega_t^a + \beta_{t+1} \tag{45}$$

where

$$\alpha_{t+1} = (1+r_{t+1})\cdot\phi(\gamma,\chi,\mathbb{E}[R]), \quad \beta_{t+1} = \underline{y} - (1+r_{t+1})\cdot\kappa_e \tag{46}$$

By construction, the sequences (α_t) and $(\beta)_t$ are bounded, stationary and ergodic. The set of regularity conditions (5) identified in Kesten (1973) – and reported below for the sake of completeness – are then sufficient for the upper tail of the stationary solution ω_∞^a to (45) to be asymptotic to a Pareto law with tail index $\nu \in (1,2)$, i.e. for

$$\bar{\omega}^\nu \cdot Pr(\omega_\infty^a > \bar{\omega}) \to c$$

as $\bar{\omega} \to +\infty$ for some constant $c > 0$, see e.g. Kesten (1973), Brandt (1986) and Goldie (1991).

Consider now the actual wealth process $\{\omega_t\}$ generated by (20) and the auxiliary process $\{\omega_t^a\}$ evolving according to (45). Pick $\omega_0 = \omega_0^a$. For any path $\{r_t, y_t\}$ we have $\omega_t \geq \omega_t^a$ for all $t \geq 1$, and hence

$$Pr\left(w_t \geq \bar{\omega}\right) \geq Pr\left(w_t^a \geq \bar{\omega}\right), \quad \forall \bar{\omega} > \underline{y}, \quad \forall t \geq 1$$

which in turn, by ergodicity, implies $Pr\left(w_\infty \geq \bar{\omega}\right) \geq Pr\left(w_\infty^a \geq \bar{\omega}\right)$, and hence

$$\lim \inf_{\bar{\omega}\to\infty} \bar{\omega}^\nu \cdot Pr\left(\omega_\infty > \bar{\omega}\right) \geq \lim \inf_{\bar{\omega}\to\infty} \bar{\omega}^\nu \cdot Pr\left(\omega_\infty^a > \bar{\omega}\right) = c$$

It follows that the stationary distribution ω_∞ has a right fat tail.

For completeness, below we report Kesten (1973)'s conditions for asymptotic power-law behavior of the axuiliary process (44):

Assumption 5. *In (44)*

(i) the support of the distribution of $\ln(\alpha)$ *is not contained in* $\lambda\mathbb{Z}$ *for some* λ,

(ii) $(\beta/(1-\alpha))$ *is non-degenerate, and*

(iii) there exists $\nu > 1$ *such that* $E\left[|\alpha|^{\nu}\right] = 1$ *and* $E\left[(\alpha^{\nu})\max\{\ln(\alpha), 0\}\right] < \infty$.

Proof of Proposition (3)

Since $\tilde{x}(\underline{y})$ is larger then zero, it holds $0 < G(\tilde{x}(\underline{y}))$. We also know that

$$G(\tilde{x}(\underline{y}))\mu^*(\underline{y}) = \int_{\omega_t \in (\underline{y}, \tilde{\omega}_0]} (1 - G(\tilde{x}(\omega_t)))d\mu^*(\omega) \tag{47}$$

since $Pr\left(\omega_{t+1} = \underline{y} \mid \omega_t > \tilde{\omega}_0\right) = 0$.
The invariant measure μ^* is such that $0 < \mu^*([\underline{y}, \tilde{\omega}_0])$, given that $\tilde{\omega}_0 > \underline{y}$. Since $1 - G(\tilde{x}(\omega_t)) > 0$, the RHS of (47) is larger than zero: there exists a strictly positive probability for a generic lineage i with wealth in $]\underline{y}, \tilde{\omega}_0]$ at time t to display a wealth level equal to $\underline{y}$ in the next period $t+1$. It follows that $\underline{y}$, i.e. $\mu^*(\underline{y}) > 0$.
To characterize the mass point, let us consider the following sets of events that collect lineages i with the lowest level of wealth:

$$N_1 = \{i \mid \omega_i = \underline{y},\, e^* = 0,\, b_0^* = 0\}, \tag{48}$$

$$N_2 = \{i \mid \omega_i = \underline{y},\, e^* = 0,\, 1 + r = 0\}, \tag{49}$$

$$N_3 = \{i \mid \omega_i = \underline{y},\, e^* = 1,\, b_1^* = 0,\, y = \underline{y}\}, \tag{50}$$

$$N_4 = \{i \mid \omega_i = \underline{y},\, e^* = 1,\, 1 + r = 0,\, y = \underline{y}\}. \tag{51}$$

Since the sets N_j, $j = 2, 3, 4$ have zero measure, it follows that the mass point at the bottom of the stationary wealth distribution has measure $\mu^*(N_1)$.

Proof of Proposition 4

Since $0 < \underline{y} \leq \omega_t$ for all t, one has $b_0^* > 0$ for all levels of parental wealth. Using the same argument as in the proof of Proposition (3) delivers the result.

Proof of Proposition 5

From the optimality conditions (15) it is easy seen that an infinitesimal increase in the intensity of the bequest motive, from χ to $\chi' > \chi$, increases, all else equal, the marginal utility from financial bequests relative to consumption, at any cost $x \in \Delta x$. Therefore, $\partial b_e^*/\partial \chi > 0$, $e \in \{0,1\}$ whenever $b_e^* > 0$. Also, from (17), one has $\partial \tilde{x}/\partial \chi > 0$ for all $\omega_t \geq \underline{y}$ if and only if $b_0^* < b_1^*(\tilde{x}) + \tilde{x}$, which is always the case – see Lemma (2). Finally, at any cost $x \in \Delta x$, the wealth thresholds $\tilde{\omega}_e(x)$ decrease with χ; hence, the support $(\underline{y}, \tilde{\omega}_0]$ producing downward mobility flows towards the lowest wealth state shrinks.

Consider now the individual wealth transition (20) for lineage i, letting the superscript $'$ label optimal bequest choices made by parents in lineage i when the intensity of altruism is χ':

- for lineage i with educational cost $x_i \in \Delta x$ and wealth $\omega_{i,t} \leq \tilde{\omega}_0$, whose optimal bequest plan entails zero financial bequests, we have

$$G\left(x_i \leq \tilde{x}(\omega_{i,t}) \,|\, \omega_{i,t} \leq \tilde{\omega}_0\right) < G\left(x_i \leq \tilde{x}'(\omega_{i,t}) \,|\, \omega_{i,t} \leq \tilde{\omega}_0\right);$$

- for lineage i with educational cost $x_i \in \Delta x$ and wealth $\omega_{i,t} > \tilde{\omega}_1(x_i)$, whose optimal bequest plan does entail positive financial bequests irrespective of their educational investment choice, relatively larger altruism produces larger financial transfers, hence

$$Pr\left(\omega_{i,t+1}(b_e^{*'} > 0) \geq \underline{y} \,|\, \omega_{i,t} > \tilde{\omega}_1(x_i)\right) > Pr\left(\omega_{i,t+1}(b_e^* > 0) \geq \underline{y} \,|\, \omega_{i,t} > \tilde{\omega}_1(x_i)\right),$$

  for $e \in \{0,1\}$;

- for lineage i with educational cost $x_i \in (\tilde{x}(\omega_{i,t}), \tilde{x}'(\omega_{i,t}]$ and wealth $\omega_{i.t} \in (\tilde{\omega}_0, \tilde{\omega}_1(x_i)]$, relatively larger altruism entails a switch from the optimal bequest plan $(b_{i.0}^* > 0, e_i^* = 0)$ to the optimal bequest plan $(b_{i,1}^{*'} = 0, e_i^{*'} = 1)$, with the property that $\tilde{x}'(\omega_{i,t}) > b_{i,0}^*$ (see Lemma (1)). From the indifference condition (17), for any such i at time t, it must hold

$$\mathbb{E}\left[v(y)\right] > \mathbb{E}\left[v((1+r)b_{i,0}^*)\right] \tag{52}$$

  where, as usual, the expectation is taken with respect to the distributions of R and Y. Since the latter inequality holds for any increasing and concave function v, it implies

$$\mathbb{E}[\omega_{i,t+1}(b_{i,1}^{*'} = 0, e_i^{*'} = 1)] \geq \mathbb{E}[\omega_{i,t+1}(b_{i,0}^* > 0, e_i^* = 0)]$$

  by the properties of the increasing concave order (see Shaked and Shanthikumar, p. 182). That is, the average wealth (averaging over all possible realizations of random returns) of the infra-marginal households i who substitute financial bequests with human capital investment under stronger intensity χ' of altruism cannot be lower than its counterpart under χ.

Integrating over i, it follows that, at any time period $t > 1$, the cross-sectional distribution of wealth under stronger intensity χ' (i) second-order stochastically dominates its counterpart under χ, and (ii) if the two ever cross over the support $[0, \infty)$, then at the first crossing point the former must cross the latter from below to the right of $\tilde{\omega}_0$. By ergodicity and Theorem 4.A.8(c) of Shaked and Shantikumar (2007), these properties also characterize the invariant measures $\mu^*_{\chi'}$ and μ^*_{χ}, implying that

$$\int_{\omega\in]\underline{y},\tilde{\omega}_0]}[1-G(\tilde{x}(\omega))]d\mu^*_{\chi} > \int_{\omega\in]\underline{y},\tilde{\omega}'_0]}[1-G(\tilde{x}'(\omega))]d\mu^*_{\chi'} \tag{53}$$

which, coupled with $G(X \leq \tilde{x}(\underline{y})) < G(X \leq \tilde{x}'(\underline{y}))$, delivers the assertion.

Proof of Proposition 6

Notice first that, at any cost $x \in \Delta x$, the optimality conditions (15)-(16) as well as the wealth thresholds $\tilde{\omega}_e(x)$ do not depend on G, meaning that the individually optimal bequest policies b^*_e (for $e \in \{0,1\}$) do no vary with the distribution of educational costs in population. Also, the investment threshold cost $\tilde{x}(\omega_t)$ solving (17) is invariant with respect to G, implying that the threshold rule for optimal educational investment (as expressed in Lemma 2) does not vary either. As a consequence (i) for each and every lineage i, the individual wealth transition (20) is unaffected by the properties of G, and yet (ii) at each time period $t > 1$ the cross-sectional distribution of wealth entails a smaller measure of parents investing in human capital formation under G' than under G, all else equal, since $G'(X \leq \tilde{x}(\omega_t)) \leq G(X \leq \tilde{x}(\omega_t))$ for all $\omega_t \geq \underline{y}$. By ergodicity, it follows that the invariant measure $\mu^*_{G'}$ of the stationary wealth distribution under G' first-order stochastically dominates its counterpart μ^*_G under G; coupled with $G'(X \leq \tilde{x}(\underline{y})) \leq G(X \leq \tilde{x}(\underline{y}))$, and the fact that the support $(\underline{y}, \tilde{\omega}_0]$ producing downward mobility flows towards the lowest wealth state is unchanged, the assertion follows immediately from (22).

Proof of Lemma 5

We first notice that, all else equal, for all $\tau_j \in (0,1)$ $j = b, r$ we have

$$\tilde{\omega}'_0 = \chi^{-\frac{1}{\gamma}}\left[(1-\tau_j)(1+\mathbb{E}[R])\right]^{-\frac{1}{\gamma}}\underline{y} > \tilde{\omega}_0 \tag{54}$$

and

$$\tilde{\omega}'_1(x) = \chi^{-\frac{1}{\gamma}}\left[(1-\tau_j)(1+\mathbb{E}[R])\right]^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[y^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} + x > \tilde{\omega}_1(x), \quad x \in \Delta x \tag{55}$$

i.e. the introduction of estate/capital income taxes increases the wealth thresholds below which pseudo optimal financial bequests are zero.
(i) Fix e to either 0 or 1. From (54) and (55) it follows that $b^{*'}_e = 0$ whenever $b^*_e = 0$ (and the assertion follows trivially), so that $b^{*'}_e$ can only be positive if b^*_e are. We thus focus on establishing the assertion when pseudo optimal financial bequests are strictly positive with and without taxes. Suppose, to the contrary, that $(1-\tau_r)b^{*'}_e > b^*_e$. Then, by the relevant

first-order conditions

$$\left(\omega_t - b_e^{*'} - e\cdot x\right)^{-\gamma} = \chi\mathbb{E}\left[(1-\tau_j)(1+r)\left(\omega_{t+1}(b_e^{*'},e)\right)^{-\gamma}\right]$$
$$(\omega_t - b_e^{*} - e\cdot x)^{-\gamma} = \chi\mathbb{E}\left[(1+r)\left(\omega_{t+1}(b_e^{*},e)\right)^{-\gamma})\right]$$

and exploiting the concavity of v, we obtain

$$\mathbb{E}\left[(1-\tau_j)(1+r)\left(\omega_{t+1}(b_e^{*'},e)\right)^{-\gamma}\right] < \mathbb{E}\left[(1+r)\left(\omega_{t+1}(b_e^{*},e)\right)^{-\gamma})\right]$$

for $e\in\{0,1\}$. Since $\tau_j<1$, $(1-\tau_r)b_e^{*'} > b_e^*$ implies $b_e^{*'} > b_e^*$ and thus $c_e^{*'} < c_e^*$, from which

$$\mathbb{E}\left[(1-\tau_j)(1+r)\left(\omega_{t+1}(b_e^{*'},e)\right)^{-\gamma}\right] > \mathbb{E}\left[(1+r)\left(\omega_{t+1}(b_e^{*},e)\right)^{-\gamma})\right]$$

i.e. a contradiction.

(ii) Assume $\bar{x}>0$ is sufficiently large. Then, for some tax rates $\tau_j\in(0,1)$, there exists a positive measure set $\Delta'\subseteq\Delta X$ such that

$$\tilde{\omega}_0' < \tilde{\omega}_1(x), \quad x\in\Delta' \tag{56}$$

Fix now any such $x\in\Delta'$ and $t_j\in(0,1)$. For $\omega_t\in B=(\tilde{\omega}_0', \tilde{\omega}_1(x)]$ we have $b_0^*>0$, $b_0^{*'}>0$, $b_1^*=b_1^{*'}=0$, and the educational threshold $\tilde{x}'(\omega_t)$ cost under estate/capital income taxation solves

$$u(\omega_t - b_0^{*'}) + \chi\mathbb{E}\left[v\left((1-\tau_j)(1+r)b_0^{*'} + \underline{y}\right)\right] = u(\omega_t - \tilde{x}') + \chi\mathbb{E}\left[v\left(y\right)\right] \tag{57}$$

Applying the implicit function theorem in the neighborhood of $(\tilde{x}',\tau_j)$ fulfilling (57) shows that $d\tilde{x}/d\tau_j$ has the same sign as the following expression

$$\chi\cdot\mathbb{E}\left\{(1+r)\left[\left((1+r)b_0^{*'} + \underline{y}\right)^{-\gamma}\cdot b_0^{*'}\right]\right\}$$

which is clearly positive. Since $\lim_{\tau_j\to0^+} b_0^{*'} = b_0^*$ and $\lim_{\tau_j\to0^+} d\tilde{x}/d\tau_j > 0$ it follows that $\tilde{x}'(\omega_t) > \tilde{x}(\omega_t)$ for all $\omega_t\in B$.

Proof of Proposition 7

Let $\mu{*}'(\underline{y})$ denote the measure of the atom in the stationary wealth distribution in the presence of estate/capital income taxation. We next identify two simple sets of conditions pertaining to ex-ante heterogeneity in agents' characteristics and the expected returns on human capital versus financial investment under which the introduction of a tax τ_b on bequests or a tax τ_r on financial returns increases the size of the atom in the left tail of the stationary distribution (i.e. $\mu{*}'(\underline{y}) > \mu^*(\underline{y})$, case (i)) or rather lowers it (i.e. $\mu{*}'(\underline{y}) < \mu^*(\underline{y})$, case (ii)).

(i) Define $\hat{x}$, if it exists, as the solution to the following equation

$$(\tilde{\omega}_0)^{1-\gamma} + \chi \underline{y}^{1-\gamma} = (\tilde{\omega}_0 - \hat{x})^{1-\gamma} + \chi \mathbb{E}\left[y^{1-\gamma}\right] \tag{58}$$

where, we recall, $\tilde{\omega}_0 = \underline{y}\left[\chi\left(1 + \mathbb{E}[R]\right)\right]^{-\frac{1}{\gamma}}$. Notice that $\hat{x} = \tilde{x}(\tilde{\omega}_0)$: if there is no $\hat{x} \in \Delta x$ solving (58), then all parents with wealth $\tilde{\omega}_0$ will undertake educational investment, irrespective of the actual cost they face. Notice also that $\tilde{x}(\omega_t) > \hat{x}$ for all $\omega_t > \tilde{\omega}_0$ by Lemma (2); and that $\tilde{x}'(\omega_t) = \tilde{x}(\omega_t)$ for all $\omega_t \leq \tilde{\omega}_0$, with $\tilde{x}'(\omega_t)$ increasing in ω_t. It follows that, if $\bar{x} \leq \hat{x}$, at each time period t all the parents with wealth $\omega_t > \tilde{\omega}_0$ (and *a fortiori* those with wealth $\omega_t > \tilde{\omega}_0'$) will invest in human capital formation (i.e. $e^* = 1$) irrespective of the actual investment cost they face.
Assume now $\bar{x} \leq \hat{x}$. It follows that the introduction of a tax τ_j (whether on bequests or financial returns) does not modify the educational investment choices of households (i.e. $e^{*'} = e^* = 1$) for all wealth levels, while implying a contraction in the pseudo post-tax financial bequests $(1-\tau_j)b_e^{*'}$ for all wealth levels. For any given initial distribution of wealth and for any tax rate $\tau_j \in (0,1)$, $j = b, r$ we therefore have

$$Pr\left((1-\tau_j)(1+r)b_e^{*'} + \underline{y} + e^{*'}(y - \underline{y}) \leq \bar{\omega}\right) \geq Pr\left((1+r)b_e^{*} + \underline{y} + e^{*}(y - \underline{y}) \leq \bar{\omega}\right)$$

for all $\bar{\omega} > \underline{y}$ and time $t \geq 1$. By ergodicity we obtain $Pr(\omega_\infty' \leq \bar{\omega}) \geq Pr(\omega_\infty \leq \bar{\omega})$, where ω_∞' denotes the stationary distribution to which the wealth accumulation process converges in the presence of estate/capital income taxation. Notice that $G(\tilde{x}(\underline{y})) = G(\tilde{x}'(\underline{y}))$ and $G(\tilde{x}(\omega_t)) = G(\tilde{x}(\omega_t))$ for all $\omega_t \leq \tilde{\omega}_0'$. Then it must be the case that if the maximal educational cost is sufficiently small (i.e. if $\bar{x} \leq \hat{x}$) then $\mu^{*'}(\underline{y}) > \mu^*(\underline{y})$ for all $\tau_j \in (0,1)$, $j = b, r$.
(ii) With little loss of generality, let us assume that the wage in the high-skill occupation is deterministic and equal to $\overline{y} > \underline{y}$. Let also the economy's fundamentals be such that $\tilde{\omega}_0 < \overline{y}$, i.e. let

$$\underline{y}\chi^{-\frac{1}{\gamma}}(1 + \mathbb{E}[R])^{-\frac{1}{\gamma}} < \overline{y}$$

Define $\check{x}$ as the solution, if it exists, to the following equation

$$\overline{y}^{1-\gamma} + \chi \underline{y}^{1-\gamma} = (\overline{y} - \check{x})^{1-\gamma} + \chi \overline{y}^{1-\gamma} \tag{59}$$

Notice that if there is no $\check{x} \in \Delta x$ solving (59), then all individuals with wealth $\overline{y}$ who optimally choose not to leave financial bequests (i.e. for whom $b_e^* = 0$ for each $e = \{0, 1\}$) will undertake educational investment, irrespective of the actual cost they face.
Notice also that $\hat{x} < \check{x}$, provided both thresholds exist. Assume then $\hat{x} < \bar{x} \leq \check{x}$, and let actual

the tax rate τ_j $(j = b, r)$ implemented in the economy satisfy

$$\tau_j \geq \bar{\tau} = 1 - \left(\frac{\underline{y}}{\overline{y}}\right)^{\gamma} \cdot \frac{1}{\chi(1+\mathbb{E}[R])} \quad \iff \quad \tilde{\omega}_0' \geq \overline{y}$$

where the lower bound $\bar{\tau} \in (0,1)$ always exists for any finite $\overline{y}$ as $\lim_{\tau_j \to 0^+} \tilde{\omega}_0' = \tilde{\omega}_0 < \overline{y}$ and $\lim_{\tau_j \to 1^-} \tilde{\omega}_0' = \infty$.
Under these circumstances, Proposition (1), part b. applies: with probability one, any lineage will find herself in the lower states of the wealth space ($\omega_t \in [\underline{y}, \tilde{\omega}_0']$) entailing $b_e^* = 0$ for $e = \{0,1\}$, and will not manage to exit those states by means of human capital formation (since $\overline{y} \leq \tilde{\omega}_0'$). Since educational investment remains affordable by all households in the high skill occupation (as $\bar{x} \leq \check{x}$), the economy will converge to a one-point stationary distribution localized at $\omega_t = \overline{y}$, meaning no mass point at $\omega_t = \underline{y}$, i.e. $\mu^{*'}(\underline{y}) = 0$.
Consider now tax rates $\tau_j < \bar{\tau}$ (including the no tax scenario). In this case we have (see (55))

$$\underline{y} < \tilde{\omega}_0' < \overline{y} < \tilde{\omega}_1'(\bar{x})$$

and hence $b_0^{*'}(\overline{y}) > 0$ and $b_1^{*'}(\overline{y}) = 0$. As a result we obtain

$$\begin{aligned} &(\overline{y} - \tilde{x}'(\overline{y}))^{1-\gamma} + \chi\overline{y}^{1-\gamma} = \\ &\quad (\overline{y} - b_0^{*'})^{1-\gamma} + \chi\mathbb{E}\left[\left((1+r)b_0^{*'} + \underline{y}\right)^{1-\gamma}\right] > \\ &\quad \overline{y}^{1-\gamma} + \chi\underline{y}^{1-\gamma} = \\ &\quad (\overline{y} - \check{x})^{1-\gamma} + \chi\overline{y}^{1-\gamma} \end{aligned} \tag{60}$$

where the first equality follows from the definition of the investment cost threshold $\tilde{x}'(\omega_t)$, the inequality by the optimality condition, and the last equality from the definition of $\check{x}$.
From (60) we immediately obtain $\tilde{x}'(\overline{y}) < \check{x}$. As a main consequence, when $\tau_j < \bar{\tau}$ and provided $\bar{x} > \tilde{x}'(\overline{y})$, there exists a positive measure subset of households – those with eralth $\omega_t = \overline{y}$ and facing investment costs $x \in [\tilde{x}'(\overline{y}), \bar{x})$ – who optimally choose to refrain from educational investment while substituting it with financial bequests, whose rate of return is lower than $\overline{y}$ with positive probability. Proposition (1) part a. applies, and the economy will converge to a fully-fledged distribution with limit support $S_\infty = [\underline{y}, \infty)$ and a mass point at $\omega_t = \underline{y}$: a reduction of the tax rate increases wealth inequality and involves the emergence of an atom in the left tail of the stationary distribution of wealth. When $\tau_j = 0$, $j = b, r$, we thus have $\mu^{*'}(\underline{y}) < \mu^*(\underline{y})$.

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