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Common Ownership in Production Networks

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Matteo Bizzarri* **and Fernando Vega-Redondo†**

Abstract
We characterize the firm-level welfare effects of a small change in ownership overlap, and how it depends on the position in the production network. In our model, firms compete in prices, internalizing how their decisions affect the firms lying downstream as well as those that have common shareholders. While in a horizontal economy the common-ownership effects on equilibrium prices depend on firm markups alone, in the more general case displaying vertical inter-firm relationships a full knowledge of the production network is typically required. Addressing then the normative question of what are the welfare implications of affecting the ownership structure, we show that, if costs of adjusting it are large, the optimal intervention is proportional to the Bonacich centrality of each firm in the weighted network quantifying interfirm price-mediated externalities. Finally, we also explain that the parameters of the model can be identified from typically available data, hence rendering our model amenable to empirical analysis.

JEL Classification: D43, D57, D85, L13, L16.

Keywords: production networks, network games, common ownership, oligopoly.

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Introduction

In this paper, we study the welfare effect of a change in common ownership, depending on the position in the input-output network.

By common ownership we mean the overlap of ownership structure across firms. Common ownership of firms has increased substantially in recent years, as documented in, for example, Backus et al. (2021b). The hypothesis that firms that share the same owners tend to compete less aggressively has been labelled the "Common ownership hypothesis" (Backus et al. (2021a)).[1] This fact, connected with the debate on the rise of market power, opens questions on what are the anticompetitive effects of common ownership, and how to mitigate them. Recently, both the European commission and the US Department of Justice have expressed interest in a better understanding of the antitrust implications of the phenomenon.

However, there is a basic economic trade-off, mutuated from merger theory, that makes the problem non trivial in an input-output network, featuring both vertical and horizontal connections. In such a situation firm coordination can be welfare damaging because of higher prices, but can be welfare improving if the firms are vertically related. Efficiency would require to decrease as much as possible any coordination among horizontally related firms, while at the same time maximizing the coordination between vertically related firms. This is not always possible: the input-output network structure of production creates a trade-off between increasing one and decreasing the other. Moreover, different firms, having different network position and pattern of connections, can have a very heterogeneus effect on welfare.

Our main contributions are three. First, we build a tractable model of oligopolistic price competition with common ownership, in which firms strategically set output prices, understanding the direct and indirect impacts that this has throughout the network: the model is solvable in closed form with general patterns of common ownership. Second, we characterize the effect of a small change in the distribution of ownership, leading to stronger or weaker incentives to coordinate: we show that, while for an horizontal economy the markups are sufficient statistics to rank the effects of a change in common ownership,[2] in more general economies this is not true anymore, and the network structure plays a crucial role. Third, we show that the parameters of our model can be uniquely identified from typically available

[1] There is currently ongoing debate on how much large institutional investors affect the managerial decisions: see the literature.

[2] Specifically, we assume that goods have the same substitutability, but can still be heterogeneous in marginal cost of production, or demand intercept.

data, at least when common ownership is weak enough.

To model the effect that overlapping ownership has on the competitive incentive of firms, we adopt a widely used framework introduced by Rotemberg (1984). This is a stylized description of the phenomenon, in which firms (or firms' managers) try to optimize a weighted average of investor income, while mantaining their decision decentralized. The model implies that the more similar the set of owners of firms, the more firms' pricing decisions are going to be coordinated. Namely, firms set their price optimizing a weighted average of the income of shareholders, taking into account how the downstream network affects the slope of the demand they face. The fact that shareholders have stakes in multiple firms means that firms do not optimize their profit, but a weighted average of profits, where the weights are the cosine similarities between ownership profiles of firms: if two firms are owned by the same investors, they have incentives to coordinate and soften competition.

Our model of competition makes firms decisions crucially related to network position. The network is exogenously given, and firms compete oligopolistically in prices. Firms' technology is Leontief, so the output quantity pins down the input requirement for each input. This allows firms (and us) to solve for the residual demand firms face as a function of the prices of (in principle) all other firms in the network, in a similar procedure as in Acemoglu and Tahbaz-Salehi (2020). Equipped with this residual demand, firms profits are just a function of prices. Firms compete a' la Bertrand with these payoff functions, internalizing the network effects via the residual demand computation.

In equilibrium, the elasticity of demand firms face depends on how strong are the direct and indirect connections of the firm to the consumer goods, where substitute goods increase the slope (decrease the markup), complement goods decrease the slope (increase the markup). Indeed, we show that the ability of firms to charge a large markup is crucially related to a measure called upstream similarity. Essentially, a firm can charge a high price if its output is hard to substitute. This measure highlights that, in an input-output network setting, the substitutability of final products for the consumers is not enough: if two firms produce close substitutes, but have a very similar set of suppliers, then they still charge a high price, because the upstream suppliers can charge a high markup.

To reduce the dimensionality of the problem, we parameterize common ownership assuming each firm shares are owned by a main investor and divided symmetrically among the others. We use the amount of shares owned by the main investor "ownership separation". This measure has as extreme cases an oligopoly with purely maximizing firms and a fully integrated monopoly. We characterize the effect of a small change in ownership separation for different firms in the network. We decompose this effect in two parts: one relative to price externalities (the cross derivatives of profits with respect to prices), the other connected also to *strategic externalities* (the *second* cross derivatives of profits with respect to prices).

The welfare effect of small changes in ownership separation depends crucially on a matrix of price externalities, weighted by the multipliers of the planner problem. The matrix characterizes the variation in the externality level that the planner would like to implement via a change in ownership, weighted by how costly it is to incentivize firms to do so. Moreover, for small but not too large adjustment cost, the optimal level of intervention is proportional to the Bonacich centrality of each firm in the network defined by these weighted price externality relations.

We show that the particular functional form used to describe the overlap in shares (cosine similarity across ownership vectors) means that the welfare effect of small ownership variations can be decomposed in directions along which the welfare variation is proportional to the vector of shares, using a diagonalization procedure similar to Galeotti et al. (2020).

We show that the effect of common ownership crucially depend on whether price externality exerted by firms on each other are positive or negative. These externalities are, in turn, crucially dependent on whether goods are substitutes or complements, once taking network effects into account. These effects can have different signs, depending on a demand effect and a cost effect. The cost effect is present only for direct customers, and is always negative, reflecting the fact that an increase in price damages the customers. The demand effect can be positive or negative, we label these two cases network substitutes and network complements. Such complementarity sums up the complementarity or substitutability of final outputs from the consumer perspective, and the production network effects. If a firm is upstream from another, its output is ceteris paribus a network complement to the price of the first; if two firms are horizontally related, in that they have no input-output connections but they share the same customers, they are network-substitutes.

Related literature

We contribute to the literature on production networks, on common ownership, and interventions in networks.

To the literature on common ownership we contribute an analysis of a full input-output economy. Various model study the welfare impact of common ownership, but without including the vertical dimension, such as Ederer and Pellegrino (2022), Azar and Vives (2021), Azar and Ribeiro (2021). The fact that large institutional investors have an impact on firms decisions is an empirical question addressed by a large literature. Azar and Ribeiro (2021) show that a similar pattern can arise a model with shareholder voting. Antón et al. (2018) find that in firms with high level of cross ownership managers pay are less sensitive to performance, Azar et al. (2018) finds evidence that common ownership is connected to anticompetitive behavior in the airline industry. Brav et al. (2020) finds that mutual funds exerts their voting rights in boards. More in general, the IO literature has studied the welfare impact of mergers depending on the vertical or horizontal interaction of firms. The formalization of common ownership we use can be seen as a "continuous" version of a merger, where the merger is represented by fully overlapping ownership. The fact that vertically related firms are analogous to complements is analogous to some analyses of the welfare impact of mergers, e.g. Asker and Nocke (2021).

To the literature on production networks we contribute the analyis of the interaction between network position and pricing decisions, and the level and structure of common ownership. Our model of competition follows Pellegrino (2019), and is a tractable alternative to Acemoglu and Tahbaz-Salehi (2020) or Grassi (2017) to endogenize markups is a production network. Many models of oligopolistic competition feature markups that do not really depend on network position, such as Grassi (2017), Kikkawa et al. (2019), Baqaee (2018), Baqaee and Farhi (2020). Carvalho et al. (2020) uses a similar model of competition in prices, but focuses on bottlenecks rather than markups. None of these papers consider common ownership.

To the literature on interventions in networks, sparked by Galeotti et al. (2020), we contribute the study of a intervention that is directly on the network of externalities, with the particular structure given by the production network. Our results complement Galeotti et al. (2021), that study interventions in the form of tax or subsidy, in a network that is determined by consumer demand rather than input-output connections. A similar exercise in an input-output network is performed by Liu (2019), in a model of constant markups. The main difference is that we ask how the planner would like to restrict the ownership structure of firms, manipulating the network of externalities rather than the firms incentives directly. Kor and Zhou (2022) study the problem of network intervention in an abstract setting.

The structure of the paper is as follows. Section 1 introduces the model. Section 1 characterizes the equilibrium in closed form, and interprets it in terms of upstream similarity. Section 3 characterizes the welfare effect of a change in common ownership, and its decomposition. Section 4 discusses identification. Section 5 extend some results to a more general technology, and to planner interventions with adjustment costs. Section 6 concludes.

1 The Model

The model is a simultaneous game played by the firms.

1.1 Setting

Firms and technology There are N firms, that produce distinct goods denoted q_i. To produce, each firm needs labor (labeled as good 0), and a subset of other goods as inputs $\mathcal{N}_i \subseteq \{0, 1, \dots, N\}$, denoted q_{ij} for $j \in \mathcal{N}_i$. The goods needed as inputs define the input-output network of this economy.

Firms have a Leontief production function: $q_i = \min\left\{\left(\frac{q_{ij}}{f_{ij}}\right)_j, \frac{\ell_i}{f_i^\ell}\right\}$. We denote their profit as $\pi_i = p_i q_i - \sum_{j\to i} p_j q_{ij} - w\ell_i$.

In the following, we normalize the wage to 1.

Investors There are M investors. Each investor i owns s_{if} shares of firm f. We call the *ownership profile* of firm f the vector $s_f = (s_{1f}, \dots, s_{Mf})$. We assume investors care only about their financial wealth: $W_i = \sum_f s_{if}\pi_f$. The matrix stacking horizontally all the vectors s_i is called S.

Consumers There is a continuum of identical price-takers consumers or, equivalently, a representative consumer. Its utility is:

$$U(c) = b'\Sigma^{-1}c - \frac{1}{2}c'\Sigma^{-1}c - L$$

where c is the vector of quantity of goods consumed, L is the number of hours worked (b and c are $N \times 1$ vectors). The consumers own no shares of the firms, so their budget constraint is simply $p'c = L$. The demand for goods is, hence, $c(p) = b - \Sigma p$, where p is the price vector. The indirect utility of the consumer is $V(p) = \frac{1}{2}(b' - \Sigma p')\Sigma^{-1}(b - \Sigma p)$.

Remark 1.1. In the main text we assume that investors and consumers have a different utility: investors care only about monetary profits, while consumers derive utility from goods. We express the assumption in this way for simplicity. It is equivalent to the approach in Ederer and Pellegrino (2022), in which investors care about consumption, but they do not consume the same goods as the standard consumers, but rather a distinct good produced independently from the network, whose price is in fixed proportion with the wage.

Such a difference between investors and "standard" consumers is the polar opposite assumption to Azar and Vives (2021), in which owners have the same utility as consumers. Both are evidently abstractions. We follow the former to simplify the analysis keeping the incentives of owners separate from the incentives of consumers: in a sense, it allows to recover in this general equilibrium setting the classic IO dichotomy of consumer and producer surplus.

Welfare When we refer to total welfare in this model we refer to the total utilitarian welfare, that is the sum of the welfare of consumers and investors at the equilibrium prices p^*:

$$W = V(p^*) + \sum_i W_i(p^*) = V(p^*) + \sum_i \sum_j s_{ij}\pi_j(p^*) = V(p^*) + \sum_i \pi_i(p^*)$$

where the last step follows from the fact that the shares are normalized to 1.

1.2 The game

Firms (or managers thereof) play a simultaneous game where the strategic variables are *prices* p_i. To complete the definition of the game, we have to define the payoffs. To do this, we need to express the profit of firms as functions of the prices alone. In order to do this, we need the following assumption.

Assumption 1 for every firm i, its input demands as a function of the price vector p and the output level q_i are the quantities that solve the standard cost minimization problem of a neoclassical price taker firm. Hence, because of our technology: $q_{ij} = f_{ij}q_i$, and $\ell_i = f_i^\ell q_i$.

In the Appendix we show that, if the network is acyclic, this can be obtained as a result of the assumption that firms strategic variables are the output price, and the input quantities.

Residual demand Taking Assumption 1 into account, firms are able to compute the residual demand they face, solving the market clearing conditions. In vector form, these are:

$$q = c + F'q$$

In the next Section, we prove that under our assumptions on F these equations have a unique solution $q(p)$.

Payoffs As a consequence of the previous Assumption 1 and the definition of the residual demand, we can express the profit of firm i as a function of prices alone. That is, for each g define the function π_i as:

$$\pi_i(p) = p_i q_i(p) - \sum_j p_j q_{ij} - \ell_i = (p_i - \sum_j f_{ij} p_j - f_i^\ell) q_i(p)$$

and we denote the marginal cost $p_i - \sum_j f_{ij} p_j - f_i^\ell$ as MC_i.

Now, following a large literature starting with Rotemberg (1984), we define the payoffs as the weighted average of firms payoffs:

$$U_f = \sum_g (s_f \cdot s_g) \pi_g = \sum_g (s_f \cdot s_g) q_g(p)(p_g - MC_g)$$

where $(s_f \cdot s_g) = \sum_i s_{if} s_{ig}$ is the scalar product of the ownership profiles of the two firms f and g.

A standard way to rationalize this payoff is the following: we assume that the managers' objective function is a weighted average of the (indirect) utility of the owners: $\sum_i s_{ig} V_i$, where an investor with more shares has higher weight. In turn, since investors only care about financial wealth, their utility is a weighted sum of the profits of the firms they own shares of: $\sum_f s_{if} \pi_f$. Hence:

$$U_f = \sum_i s_{if} V_i = \sum_i s_{if} \left(\sum_g s_{ig} \pi_g \right) = \sum_g (s_f \cdot s_g) \pi_g$$

In the following, to economize on notation, we call $K_{ij} = s_i \cdot s_j$.

The game Finally, we are looking for a Nash equilibrium of the game G, which strategic variables are the prices $p_i \in [0, +\infty)$, and the payoffs are the profits π_i, expressed as functions of prices alone via the residual demand.

1.3 Interventions interpretation

Our goal is to characterize the effect of a change in ownership separation on welfare. That is, we want to compute the derivatives: $\frac{\partial W}{\partial \zeta_i}$. Denoting $\nabla_\zeta W$ the vector stacking all the derivatives $\frac{\partial W}{\partial \zeta_i}$ for $i = 1, \dots, n$, it is a standard result that:

$$\nabla_\zeta W = \arg\max_{v \in \mathbb{R}^n} \lim_{h \to 0} \frac{W(\zeta + hv) - W(\zeta)}{h}$$

that is, we can also interpret the gradient as the optimal *direction of intervention* that a planner might want to implement. Locally, a planner insterested in optimizing welfare would like to move along the $\nabla_\zeta W$ direction.

2 Equilibrium

The best reply problem of firm i is concave if

$$-2 \sum_k \ell_{ki} \sigma_{ki} + \sum_{j \neq i} K_{ij} \left(f_{ji} \sum_k \ell_{kj} \sigma_{ki} \right)$$

We are going to maintain this assumption on the parameters from now on.

It is useful to write the FOC in their general form that is:

$$(p_i - MC_i) = \frac{1}{K_{ii}\ell_{ii}} \left(q_i + \sum_{j \neq i} K_{ij} \left(\overbrace{\underbrace{\frac{\partial q_j}{\partial p_i}(p_j - MC_j)}_{\substack{\text{demand (downstream)} \\ \text{effect}}} - \underbrace{q_j \frac{\partial MC_j}{\partial p_i}}_{\substack{\text{cost (upstream)} \\ \text{effect}}}}^{\partial \pi_j / \partial p_i} \right) \right)$$

where we see that the markup charged by firm i depends on a *demand channel* $\frac{\partial q_j}{\partial p_i}$, and a *cost* channel. If the technology is constant returns, the cost channel can only be present for firms that are *immediately* downstream from i (i.e. such that there is a directed path from i to j) - remember that this is a best response, prices of others are taken as given. The cost channel is always negative, is always decreasing the profit of other firms.

Solving the equations implied by the FOCs, we arrive at the following Proposition.

Proposition 1 (Equilibrium)**.** *The Nash equilibrium of the game G, in case there* *are no corner solutions, is:*

$$p = \Sigma^{-1}(\Sigma^{-1} + \tilde{L})^{-1}(\tilde{b} + p^{eff})$$

where $\tilde{L} = L(K \circ (\Sigma L))^{-1}(K \circ (I - F'))L'$*,* $\circ$ *denotes the elementwise (Hadamard) product* $\tilde{b} = \tilde{L}b$ *and* $p^{eff} = Lf^{\ell}$*.*

Proof. See Appendix B.1

□

The vector p^{eff} represents the prices that would arise under perfect competition. The intuition of matrix $\tilde{L}$ is studied in the next paragraph.

Upstream similarity To gain some intuitions on the mechanisms of the equilibrium first focus on the case with no common ownership, $K_{ij} = 0$ for all $i \neq j$. In this case, the best reply equation is:

$$p_i = \frac{1}{2}\left(\sum_j f_{ij}p_j + f_i^{\ell} + \frac{1}{\ell_{ii}}\left(\sum_j \ell_{ji}b_j - \sum_{j \neq i} \ell_{ji}p_j\right)\right)$$

We can see that the price is high when the cost is high (upstream effect), and when the deman is high (downstream effect). Solving the upstream part we get that the equilibrium prices solve:

$$p = p^{eff} + Ldiag(L)^{-1}L'(b - p)$$

We can see that the departure from the perfect competition prices depends on the matrix $Ldiag(L)^{-1}L'$. This can be seen as the price impact $\partial p_i/\partial q_i = 1/\ell_{ii}$, weighted by all forward and backward connections. Inspecting the elements of the matrix, we see that the i, j entry is:

$$\sum_k \frac{\ell_{ik}\ell_{jk}}{\ell_{kk}}$$

that is a measure of similarity of the patterns of *upstream* connections of i and j, weighted for price impact.

The way to read the equation above is the following: firms charge high prices if their suppliers do, and a lower bound on such prices is to cover labor costs: this is the reason for the term Lf^{ℓ}. Then, under oligopoly prices are also affected by demand: as per the best reply equation, whenever firms are indirectly supplying firms with high demand (b) they are able to charge higher prices: this creates the term $diagL^{-1}L'b$. But the same mechanism is true for the suppliers of each firm, directly and indirectly: so this positive demand effect contributes to an increase in prices not only via own demand, but via the demand *of all upstream firms*: as a result, firms charge a high price when they have the same suppliers as other firms with high demand. We can call $LdiagL^{-1}L'$ the *upstream similarity*. This is the origin of the term $LdiagL^{-1}L'b + Lf^{\ell}$. All these considerations are neglecting the negative demand effect, given by strategic substitutability: strong downstream connections to firm with high prices reduce the residual demand and harm market power, and this adds a term to the effect: $LdiagL^{-1}L'b + Lf^{\ell} - LdiagL^{-1}L'(LdiagL^{-1}L'b + Lf^{\ell})$. Now this price effect is true not only of direct customers, but of all customers of customers, so there is an infinite sum of terms, that yields the inverse in the expression above.

In the equilibrium with common ownership, these effects have to be weighted by the common ownership weights K.

2.1 Price externalities

The impact of common ownership on the equilibrium depends crucially on the sign of the *price externality* $\partial\pi_j/\partial p_i$. In turn, this depends on the sign of $\partial q_j/\partial p_i$. This is a generalization of the concept of complement and substitute goods, taking into account the input-output connections.

Definition 2.1. We call goods i:

network-complement of j if $\partial q_j/\partial p_i = \frac{\partial q_i}{\partial p_j} = -\sum_k \ell_{ki}\sigma_{kj} < 0$;

network-substitute of j if $\partial q_j/\partial p_i = \frac{\partial q_i}{\partial p_j} = -\sum_k \ell_{ki}\sigma_{kj} > 0$.

We call the *matrix of price externalities* the matrix Π such that:

$$\Pi_{ij} = \frac{\partial \pi_i}{\partial p_j}$$

Notice that being network complement or substitute is not a symmetric property, as the following example shows.

Example 1 (Network complements and substitutes - line network)**.** Consider the line network illustrated in Figure 1. Here $b_1 = b_2 = b$, $\sigma_{01} = \sigma_{10} = -\sigma$, $f_0^{\ell} = f_1^{\ell}$, and $f = f_{10}$ for simplicity.

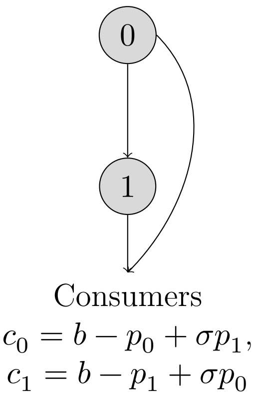

Figure 1: A simple supply chain in which the goods 0 and 1 have a degree of substitutability σ for the consumer.

The demand of the firms are:

$$c_0 = b - p_0 + \sigma p_1 + f c_1$$
$$c_1 = b - p_1 + \sigma p_0$$

Hence, here good 0 is a network substitute of 1 if $\sigma > 0$. Instead, 1 is a network complement of 0 if and only if $\sigma < f_{10}$. Notice that if $0 < \sigma < f_{10}$ goods are substitutes for the consumer, but *network-complements*.

We can characterize the price externalities as follows.

Lemma 2.1. In equilibrium the price externalities matrix is:

$$\Pi = -diag(\mu)L'\Sigma + diag(q)(I - F)$$

The cost effect always imposes a negative externality, while the demand effect imposes a positive externality if and only if goods are network-complements.

Similarly, the two externalities can be quite different, as the next example shows.

Example 2. In the line of the previous example, the price externalities are:

$$\frac{\partial \pi_1}{\partial p_0} = -f q_1 + \sigma(p_1 - MC_1)$$
$$\frac{\partial \pi_0}{\partial p_1} = (\sigma - f)(p_0 - MC_0)$$

So 1 has a positive price externality on 0 if and only if $\sigma > f$. That is, it is never the case if goods are network-complements. Moreover, from the FOCs of the firms:

$$\frac{\partial U_0}{\partial p_0} = c_0 + fq_1 + (-1 + f\sigma)(p_0 - MC_0) + K_{01}(-fq_1 + \sigma(p_1 - MC_1)) = 0$$

$$\frac{\partial U_1}{\partial p_1} = q_1 - (p_1 - MC_1) + (\sigma - f)K_{10}(p_0 - MC_0) = 0$$

we can deduce that $q_1 - (p_1 - MC_1) < 0$ if and only if $\sigma > f$, that implies that the price externality of 0 on 1, $-fq_1 + \sigma(p_1 - MC_1)$, is also positive if and only if $\sigma > f$.

Some of the properties of the previous example can be generalized. The general rule can be expressed as: "if firms are *vertically* related, goods are network-complements and price externalities are negative; if firms are *horizontally* related, goods are network-substitutes and price externalities are positive". Of course, in a general network the total effect will depend on the balance of the two.

Proposition 2. *Suppose $\Sigma = I$. If the* only *input-output connection between i and j is such that i is downstream from j, then: i is a network-complement of j ($\partial q_j / \partial p_i < 0$), while j is only weakly ($\partial q_i / \partial p_j = 0$) Since in this case j has a direct effect on the cost of i, then we can conclude that all goods have negative price externalities on each other.*

If $\sigma_{ij} < 0$, and there are no input-output connections (direct or indirect) between i and j, then they are (network) substitutes, and have positive price externality on each other.

2.2 Ownership separation

To reduce the degrees of freedom of the problem by looking at a configuration of shares that only depend on an *ownership separation* parameter.

Specifically, we assume that the investors are divided in N groups, and:

$$s_{iu} = \begin{cases} \zeta_i + \frac{1}{N} & i = u \\ \frac{1}{N} - \frac{\zeta_i}{N-1} & i \neq u \end{cases}$$

The parameter $\zeta_i \in [0, 1 - \frac{1}{N}]$ represents the degree of **ownership separation**: if $\zeta_i = \zeta$ for all i we say that there is *symmetric* common ownership; for $\zeta = 1 - \frac{1}{N}$ *fully divided* ownership; for $\zeta = 0$ means *full* common ownership.

3 The effect of common ownership

In this section we characterize the effect of a small change in ownership separation ζ_i. To understand the mechansim, we then decompose it in the effect of a planner that can directly affect the prices, and characterize the value of changing each price. This depends on an interplay of direct price externalities and strategic externalities. Then, we consider the effect of a small change in the common ownership weights K_{ij}, as if they were not constrained by being derived from shares. Then we study in detail the cases of the horizontal and vertical economy, and show that the markups are a sufficient statistic for the effect of ownership separation in the former but not the latter.

Definition 3.1. Define an auxiliary equilibrium in which the markups are constrained by $\overline{\mu}_i$. The equilibrium conditions become:

$$(p_i - MC_i) \leq \overline{\mu}_i - \frac{1}{\partial q_i/\partial p_i}\left(q_i + \sum_{j\neq i} K_{ij}\frac{\partial \pi_j}{\partial p_i}\right)$$

Now define:

$$\lambda_i = -\partial q_i/\partial p_i \frac{\partial W}{\partial \overline{\mu}_i} |_{\overline{\mu}_i=0}$$

The interpretation of the λs is the welfare effect of directly constraining the markups of firms. An alternative interpretation of the λs is as the *multipliers in a fictitius planner problem.* That is, we can represent the equilibrium as the constrained optimization of a planner, interested in maximizing welfare, constrained by the equilibrium optimality conditions of the firms:

$$\max_p V(p) + \sum_f \pi_f$$

subject to:

$$(p_i - MC_i) = \overline{\mu}_i - \frac{1}{\partial q_i/\partial p_i}\left(q_i + \sum_{j\neq i} K_{ij}\frac{\partial \pi_j}{\partial p_i}\right)$$

This problem is fictitious, in that the equilibrium conditions leave no room for further choices by the planner: but the Lagrange multipliers of this problem are exactly the λ of the previous definition.

Theorem 1. *The effect of a small change in ζ_i on welfare is given by:*

$$\frac{\partial W}{\partial \zeta_i}(\zeta^0) = H(\zeta^0)\zeta^0$$

where $H(\zeta^0)$ is the matrix defined as:

$$H_{uv} = \begin{cases} -2\frac{N}{N-1}\lambda_u \frac{\partial \pi_u}{\partial p_u} & u = v \\ \frac{N}{(N-1)^2}\left(\lambda_v \frac{\partial \pi_u}{\partial p_v} + \lambda_u \frac{\partial \pi_v}{\partial p_u}\right) & o.w. \end{cases}$$

where λ_i are the multipliers of the planner problem 1.

Define $\partial^{(2)}U$ as the matrix with in position i, j the entry $\frac{\partial^2 U_i}{\partial p_i \partial p_j}$, and $\nabla_p W$ the gradient of the welfare as a function of prices. The λ vector corresponds to:

$$\lambda = (\partial^{(2)}U)^{-1}\nabla_p W$$

Moreover, $\lambda_i \geq 0$, and it can be explicitly expressed as:

$$\lambda = (K \circ (L'\Sigma))^{-1}L'[\Sigma^{-1} + \tilde{L}']^{-1}L\mu$$

Proof. See Appendix B.2. □

Notice that fully integrated ownership is always a stationary point (but possibly a minimum!) so the proposition above does not give suggestions on the optimal direction of policy. In the following examples, we study the case of fully divided ownership, so that $\zeta^0 = \mathbb{1} - 1/N$. In this case, we know from Proposition 6 that fully integrated ownership is optimal, so all the derivatives are going to be negative. Nevertheless, the expression:

$$\frac{\partial W}{\partial \zeta_i}(\mathbb{1}) = (N-1)/NH(\mathbb{1})\mathbb{1}$$

is still interesting because it gives us a ranking of for which firms is more important to change the level of common ownership.

Example 3.1 (Two firms)**.** With fully divided ownership the matrix H has zero diagonal, because of the first order conditions. Hence, in case there are only two firms, with fully divided ownership the intervention on both is always symmetric.

Example 3.2 (Line network)**.** Consider a line network of 3 firms, 0, 1 and 2, and assume $b_i = b$ for all, and $\Sigma = I$. With fully divided ownership, $\mu = q$: this simplifies the analysis because the price and demand externalities have the same size. In this case, it is easy to prove that markups are increasing upstream. This is due to the fact that this demand has an increasing elasticity, so as we move up the production chain, when prices are smaller, markups are larger.[3]

The matrix of price externalities is:

$$\Pi = \begin{pmatrix} 0 & -f\mu_0 & -f^2\mu_0 \\ -f\mu_1 & 0 & -f\mu_1 \\ 0 & -f\mu_2 & 0 \end{pmatrix}$$

while the matrix of welfare-weighted price externalities:

$$H = \begin{pmatrix} 0 & f(\lambda_1\mu_0 + \lambda_0\mu_1) & f^2\lambda_2\mu_0 \\ f(\lambda_1\mu_0 + \lambda_0\mu_1) & 0 & f(\lambda_2\mu_1 + \lambda_1\mu_2) \\ f^2\lambda_2\mu_0 & f(\lambda_2\mu_1 + \lambda_1\mu_2) & 0 \end{pmatrix}$$

We see that in each case it is sufficient to compare two entries, thanks to symmetry.

Pure externality To understand the mechanism, let us focus on the pure externality effect. To do that, we look at the knife edge case in which $\lambda = \ell\mathbb{1}$. This is true when:

$$\ell\mathbb{1} = \lambda = (I + (I - F)(I - F'))^{-2}(I - F)c^{eff}$$

that can be solved to find the level of $c^{eff} = b - Lf^L$ that yields the result.

In such a case, $H = \ell(\Pi + \Pi')$. Hence:

$$\partial W/\partial\zeta_0 > \partial W/\partial\zeta_2 \iff \mu_0 > \mu_2$$

which again is true. If firms are symmetric in terms of the network, what matters for the intensity of the externalities is the market power they have.

$$\partial W/\partial\zeta_0 > \partial W/\partial\zeta_1 \iff f^2\mu_0 > f(\mu_1 + \mu_2)$$

that depends on the input-output weight. This is because here there are two effects together: the direct market power effect, that says that firms with larger markup exert stronger externalities, and the network effect, that firms that are more periferic exert smaller externalities, while firms that are more central receive more externalities: here firm 0 only receives the demand externality from 2, while

[3] It can be deduced directly from:

$$\mu = (I + (I - F)(I - F'))^{-1}(I - F)c^{eff}$$

noting that c^{eff} is increasing upstream, hence also Fc^{eff}, and the other matrix has a positive inverse which is diagonally dominant of its column entries.

firm 1 receives both the demand ext from 2 and the cost ext from 0. Further, the demand externality of 2 on 0 has lower weight f^2 because of the distance. If $f \to 0$, the markups tend to be equal, since there is no asymmetry in firms anymore, hence the inequality fails. Otherwise, things depend on magnitudes, for example of b. Under symmetry probably this is never true.

Finally:

$$\partial W/\partial \zeta_1 > \partial W/\partial \zeta_2 \iff f(\mu_0 + \mu_1) > f^2 \mu_0$$

that is always true. This asymmetry with the previous case is due to the fact that here we are comparing the externality exerted by 2 on 1 and 0, and is stronger on 1. In the previous case instead 2 both exerts and receives an externality from 1.

Hence, we conclude that, if the welfare cost of markups of firms is the same, 2 is the the last firm to focus on, while the balance between 0 and 1 depends on the parameters, but typically favors 1.

Pure welfare effect Now we shut down the effect of markups, and consider only the multipliers. To do that, we look for the parameters that solve:

$$m\mathbb{1} = \mu = (I + (I - F)(I - F'))^{-1}(I - F)c^{eff}$$

this implies

$$\lambda = (I + (I - F)(I - F'))^{-1}((1 - \eta)(I - F)c + m\eta\mathbb{1})$$

and since for fully divided ownership $\mu = q = L'c$, then $c = m(I - F')\mathbb{1}$, so:

$$\lambda = m(I + (I - F)(I - F'))^{-1}((1 - \eta)(I - F)(I - F') + \eta I)\mathbb{1}$$

In the case 3by 3 we can show that λ_0 is always the largest, while e.g. if $\eta = 1$, λ_1 is larger than λ_2 only for large f, while it is the opposite if f small. So the ranking is different from the markups: this highlights the fact that the planner incentives are more complicated.

In this case, inspecting the matrix we find that it is always the case that the ranking of importance is $2 > 0 > 1$.

3.1 The effect of changing prices

We can think of the multipliers λ as capturing the welfare effect of directly intervening on the prices. To understand better the intuition, let us focus on a 2 by 2 case:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \frac{\partial^2 U_2}{\partial p_1 \partial p_N} \Big/ \frac{\partial^2 U_1}{\partial p_1^2} \\ \frac{\partial^2 U_1}{\partial p_1 \partial p_2} \Big/ \frac{\partial^2 U_2}{\partial p_2^2} & 1 \end{pmatrix}^{-1}}_{\text{Strategic externalities}} \underbrace{\begin{pmatrix} \frac{\partial W}{\partial p_1} \Big/ \frac{\partial^2 U_1}{\partial p_1^2} \\ \frac{\partial W}{\partial p_2} \Big/ \frac{\partial^2 U_2}{\partial p_2^2} \end{pmatrix}}_{\text{direct welfare effect, >0}}$$

3.2 The effect of changing the weight K_{ij}

Now let us consider a change in the common ownership weights K_{ij}, but let us neglect the constraints on the K weights given by their definition as a function of the shares. By the envelope theorem, we can express the derivative as:

$$\frac{\partial W}{\partial K_{ij}} = \frac{\partial \mathcal{L}}{\partial K_{ij}}$$

where $\mathcal{L}$ is the Lagrangian of the fictitious planner problem. Such Lagrangian is:

$$\mathcal{L}(p, S) = C(p) + \sum_f \pi_f - \sum_i \lambda_i \sum_g K_{fg} \frac{\partial \pi_g}{\partial p_i}$$

Now, the derivative of the Lagrangian is:

$$\frac{\partial \mathcal{L}}{\partial K_{ij}} = -\lambda_i \frac{\partial \pi_j}{\partial p_i} = -\lambda_i \frac{\partial \pi_j}{\partial p_i}$$

This is positive if and only if i has a negative price externality on j: $\frac{\partial \pi_j}{\partial p_i} < 0$. This corresponds to a standard intuition: since the planner goal is to decrease prices, if the price externality is negative, then the planner wants this externality to be taken into account more by firms, thereby reducing their prices.

The price externality is negative when:

$$\frac{\partial q_j}{\partial p_i}(p_j - MC_j) < q_j \frac{\partial MC_j}{\partial p_i}$$

that is, if the **cost effect** dominates the **demand effect**.

Example 3. If goods are (weak) complements for the consumer, then $\frac{\partial \pi_g}{\partial p_i} < 0$, and so the derivative above is positive.

If $\sigma < f_{10}$ in the previous example, the derivative above is always positive.

If firms have no input-output connections, and their outputs are substitutes

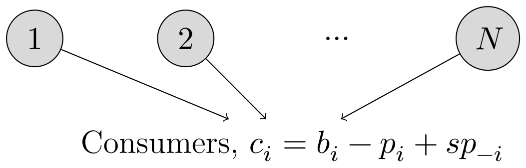

Figure 2: The horizontal economy.

for the consumer, the derivative is negative.

3.3 The horizontal and vertical economies

In this section we solve the problem in detail for two example economies: the horizontal economy and the vertical economy. We show that, while in the horizontal economy the markups are a sufficient statistic to understand where the effect of common ownership is stronger, in the vertical economy this is not true anymore.

3.3.1 The horizontal economy

The horizontal economy is the special case of the model corresponding to a simple Bertrand oligopoly with common ownership. It is represented in the above figure. There are no input output connections, so $F = 0$. Moreover, we assume $\sigma_{ij} = -s < 0$.

Here the direct welfare effect is high when markup high: $\dfrac{\partial W}{\partial p_i} = \mu_i - s\sum_{j\neq i}\mu_j$. The strategic effects are positive, since prices are strategic complements.

Proposition 3. *Assume that there are no input-output connections, and that the cross-derivatives of demand are all homogeneous: $\partial_i c_j = -s$, with $s > 0$ (goods are substitutes).*

Then, $\frac{\partial W}{\partial \zeta_i} > \frac{\partial W}{\partial \zeta_j}$ if and only if $\mu_i > \mu_j$.

Proof. See Appendix B.4

□

The result follows from showing that in this example $\lambda_i > \lambda_j \iff \mu_i > \mu_j$. Actually, more is true. Indeed, for some constant $S > 0$:

$$\lambda_i = \mu_i - S\sum_j \mu_j \implies \lambda_i - \lambda_j = \mu_i - \mu_j$$

To understand the intuition, consider an example with 2 firms. The effect of common ownership is stronger in i if:

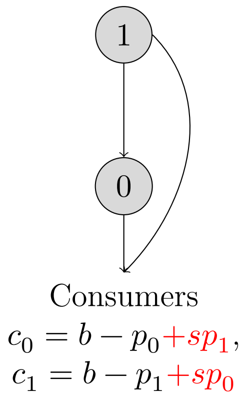

Figure 3: The vertical economy

$$\frac{\partial W}{\partial \zeta_i} > \frac{\partial W}{\partial \zeta_j}$$

$$-2\lambda_i \frac{\partial \pi_i}{\partial p_i} + \frac{1}{(M-1)} \left(\lambda_i \frac{\partial \pi_j}{\partial p_i} + \lambda_j \frac{\partial \pi_i}{\partial p_j} \right) >$$

$$-2\lambda_j \frac{\partial \pi_j}{\partial p_j} + \frac{1}{(M-1)} \left(\lambda_i \frac{\partial \pi_j}{\partial p_i} + \frac{\partial \pi_i}{\partial p_j} \right)$$

$$-\lambda_i \frac{\partial \pi_i}{\partial p_i} > -\lambda_j \frac{\partial \pi_j}{\partial p_j}$$

Now the FOCs imply: $\frac{\partial \pi_i}{\partial p_i} = -K \frac{\partial \pi_j}{\partial p_i}$, so:

$$-\lambda_i \frac{\partial \pi_i}{\partial p_i} > -\lambda_j \frac{\partial \pi_j}{\partial p_j}$$

$$\lambda_i K \frac{\partial \pi_j}{\partial p_i} > \lambda_j K \frac{\partial \pi_i}{\partial p_j}$$

$$\lambda_i K s \mu_j > \lambda_j K s \mu_i$$

$$(\mu_i - S) K s \mu_j > (\mu_j - S) K s \mu_i \iff \mu_i > \mu_j$$

which is what we wanted to show.

3.3.2 The vertical economy with 2 firms

In this section we show that for the vertical economy markups are not a sufficient statistics anymore.

Consider first the case of fully divided ownership for all i.

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \frac{1}{4+f^2} \underbrace{\begin{pmatrix} 2-2fs & s-f \\ f+s & 2 \end{pmatrix}}_{\text{strategic externalities}} \underbrace{\begin{pmatrix} \mu_0 + (f-s)\mu_1 \\ -s\mu_0 + (1-fs)\mu_1 \end{pmatrix}}_{\text{demand externalities}}$$

To understand the effects, consider the case in which the markups are the same, $\mu_0 = \mu_1$. In this case, **demand** externalities tend to make more important the firm **downstream**, while the **strategic** externalities tend to make more important the firm **upstream** (since p_0 is a strategic complement to p_1.). In this case, $\lambda_1 > \lambda_0$ if and only if $s < f$.

The welfare effect is:

$$\frac{\partial W}{\partial \zeta_0} > \frac{\partial W}{\partial \zeta_1} \iff -2(s-f)\lambda_0\mu_1 > -2(s-f)\lambda_1\mu_2$$

If markups are the same, this shows that λ determines the sign. In particular, this is still true if markups are very close to be the same but are not exactly: and their ranking is irrelevant in that case. So we conclude that **markups are not a sufficient statistic** anymore.

4 Identification

In this section we extend the approach of Ederer and Pellegrino (2022) to show that the show that at least for small common ownership the model parameters are uniquely identified provided the econometrician has data on firm to firm transactions, in addition to the requirements of Ederer and Pellegrino (2022). Intuitively, the firm to firm trade data allow to identify the matrix F (and L). The proposition below states the result precisely.

Proposition 4. *For K_{ij} small enough, the model parameters can be uniquely identified having data on:*

- *Revenues $R_i = p_i q_i$, profits π_i (and so costs $TC_i = R_i - \pi_i$);*
- *transactions between firms: $\hat{f}_{ij} = p_j q_{ij}$;*
- *demand cross-derivatives Σ (as in Pellegrino (2019), Ederer and Pellegrino (2022));*

Proof. See Appendix B.5

□

5 Extensions

5.1 Large interventions

Given the equilibrium characterization above, we can rewrite the planner problem as follows. Our planner solves:

$$\max_{p,S} C(p) + \sum_f \pi_f \tag{1}$$

subject to:

$$\sum_g K_{fg} \frac{\partial \pi_g}{\partial p_i} = 0 \qquad (\lambda_i)$$
$$\sum_i s_{iu} = 1 \qquad (\rho_i)$$
$$s_{iu} \geq 0 \qquad (\theta_{iu})$$

A preliminary question to ask is whether the optimal configuration of shares can achieve the first best. From the expression of the first order conditions of the firms (the constraints in the above problem), we can immediately conclude that the answer is negative, as formalized by the next Proposition.

Proposition 5. *In any network, if some firm only sells to consumers only, then there is no configuration of shares that yields the perfect competition outcome.*

What can we say on the form of the solution of the planner problem in general? we already know from standard IO that, in the simple example of two competitor firms with substitutable goods, fully divided ownership must be optimal: that is, the planner problem reaches the optimum at the corner solution: $s_0 = (1, 0)$, $s_1 = (0, 1)$. This suggests already that in case of network-substitutes, the problem is bound to often present corner solutions. The case of network-complements is simpler, as the next result shows.

Proposition 6. *Consider symmetric common ownership, $K_{ij} = k$ for all $i \neq j$.*

If price externalities are all negative, the fully integrated ownership ($k = 1$) is the optimum among all interior configurations.

If the price externalities are all positive, the fully divided ownership ($k = 0$) is the unique global optimum.

For example, the matrix above characterizes the behavior for the case of a Leontief technology with no substitution for the consumer ($\Sigma = I$), or the case of a completely horizontal economy with substitute goods ($F = 0$ and $\sigma_{ij} < 0$ for $i \neq j$). The next section shows some solved examples when vertical and horizontal relationships coexist.

This extends the usual intution that vertical mergers in absence of competition are welfare-improving. In the case of complements, any increase of common ownership puts a downward pressure on prices, because firms internalize the negative price externality they exert on others. Hence, the optimum must be when all the K weights are zero.

5.2 Principal components approach

With the ζ parameterization, we have:

$$s_i \cdot s_j = \begin{cases} \frac{1}{N-1}(N\zeta_i^2 - 2\zeta_i + 1) & i = j \\ \frac{1}{(N-1)^2}(-N\zeta_i\zeta_j + \zeta_i + \zeta_j + N - 2) \end{cases}$$

Hence, we can rewrite the Lagrangian of the "dummy" optimization problem as:

$$\begin{aligned}
\mathcal{L} &= W - \sum_i \lambda_i \left[\sum_j s_i \cdot s_j \frac{\partial \pi_j}{\partial p_i} \right] \\
&= W - \sum_i \lambda_i \frac{N}{N-1} \left[\zeta_i^2 \frac{\partial \pi_i}{\partial p_i} - \frac{1}{N-1} \sum_{j \neq i} \zeta_i \zeta_j \frac{\partial \pi_j}{\partial p_i} \right] \\
&- \sum_i \lambda_i \frac{1}{N-1} \left[-2\zeta_i \frac{\partial \pi_i}{\partial p_i} + \frac{1}{N-1} \sum_{j \neq i} (\zeta_i + \zeta_j) \frac{\partial \pi_j}{\partial p_i} \right] \\
&- \sum_i \lambda_i \frac{1}{N-1} \left[\frac{\partial \pi_i}{\partial p_i} + \frac{N-2}{N-1} \sum_{j \neq i} \frac{\partial \pi_j}{\partial p_i} \right] \\
&= W + \zeta' diag(\lambda) \Pi' \zeta - 1/N(\lambda' \Pi' \zeta + \zeta' diag(\lambda) \Pi' \mathbb{1}) + const \\
&= W + \frac{1}{2} \zeta' H \zeta - 1/N \mathbb{1}' H \zeta + const
\end{aligned}$$

where the last step is because $H = diag(\lambda)\Pi' + \Pi diag(\lambda)$.

Now for each given ζ, p and λ, since H is symmetric we can find a basis that diagonalizes it. Hence there is a matrix function $P(p, \lambda)$ such that the (orthogonal) change of variables $\bar{\zeta} = P(p, \lambda)\zeta$ that makes the problem diagonal in ownership separation parameters, so that $H = P' \Lambda P$:

$$W + \frac{1}{2} \sum_i \Lambda_i(p, \lambda) \bar{\zeta}_i^2 - \bar{\zeta}' \Lambda P 1/N \mathbb{1}$$

The matrix P has the eigenvectors of H as rows. The important part is that the eigenvalues Λ_i and the eigenvectors P do NOT depend on ζ directly. Hence, by the envelope theorem:

$$\partial W/\partial\bar{\zeta}_i = \Lambda_i(\bar{\zeta}_i^0 - 1/N\sum_k P_{ik})$$

where v_i' is the eigenvector corresponding to Λ_i (normalized to have norm equal to 1). this reveals that the optimal small intervention can be decomposed in an intervention along each eigenvector of the price externality matrix H, and the loading is proportional to its eigenvalue.

Example 4. Consider always the case of fully divided ownership. In this case $\zeta = \mathbb{1}$, so $\bar{\zeta} = P\mathbb{1}$. Hence the formula above can be specialized to:

$$\partial W/\partial\bar{\zeta}_i = \frac{N-1}{N}\Lambda_i\bar{\zeta}_i^0$$

In the Line network with 2 firms there are 2 distinct eigenvalues: $\Lambda_0 = H_{01} < 0$ and $\Lambda_1 = -H_{01} > 0$ (notice that the Perron eigenvalue is not unique, because the matrix is not irreducible), relative to, respectively, $u_0 = (1,1)$ and $u_1 = (-1,1)$. The status quo weights are $\bar{\zeta}_0^0 = 1$, $\bar{\zeta}_0^1 = 0$: hence the interventions will also be along the first eigenvalue alone, and the loading is negative, because the eigenvalue is negative: so the intervention decreases ownership division, uniformly. This is a different way to derive a result of a previous example.

5.3 Small but not infinitesimal interventions

We can formalize the idea of a small change from the status quo by modifying the planner problem, including a convex cost of implementing the preferred policy. The planner problem becomes:

$$\max_{p,\zeta} C(p) + \eta Pro - \frac{1}{2\tau}\sum_i(\zeta_i - \zeta_i^0)^2$$

subject to:

$$\sum K_{ij}\frac{\partial \pi_j}{\partial p_i} = 0 \quad (\lambda_i)$$

The parameter τ represents the adjustment cost/availability of public funds. In this problem, the optimum is easier to characterize, and the next Proposition shows its form.

Proposition 7. *The problem is concave if τ is small enough. At an interior optimum, the ownership concentration parameters satisfie the equation:*

$$\zeta = (I - \tau H(\zeta^0))^{-1}\zeta^0$$

The RHS of the equation for ζ is positive if τ is small enough. Hence, we can interpret the optimal level of ownership concentration for a firm as the *Bonacich centrality* of that firm in the network defined by the adjacency matrix H. This is an undirected network reflecting the strength of the price externalities between firms, weighted by the social costs of market power (the multipliers λ_i).

5.4 Technology

The Leontief functional form is useful to isolate the mechanisms, but quite restrictive, excluding a lot of the anticompetitive effects of common ownership. This is because firms have incentives to decrease markups whenever their output is a substitute and they do not have aligned inccentives with their competitors. In this section we show that Leontief technology is the limit case of a more general technology, introduced in Bizzarri (2022).

The technology we use here is such that the profit of firm i can be written as:

$$\pi_i = p_i \left(\sum \omega_{ij} q_{ij} + \alpha_i \ell_i \right) - \sum_j p_j q_{ij} - \left(\frac{1}{2} \sum_{j,k=0}^{n} \sigma_{i,jk} q_{ij} q_{ik} \right)$$

where the term $\left(\frac{1}{2} \sum_{j,k} \sigma_{i,jk} q_{ij} q_{ik} \right)$ represents the labor payments of firm i. They include quantities of labor specifically needed to complement each input good, and also a quantity of labor used for generic purposes, that we denote q_{i0} as "input zero". We can think of these quantities of labor as representing the cost of "handling" each input, including all the labor needed to store, transport, do inventories and so on.

The matrix Σ_i is supposed symmetric and positive semidefinite. It encodes the patters of substitutability or complementarity across inputs. To see this, the next example illustrates the solution of the profit maximization for a price taking firm with such technology.

Example 5 (Perfect competition)**.** The expression of the vector of inputs of a price-taking firm using the technology above is:

$$\boldsymbol{q}_i = \Sigma_i^{-1}(p_i \omega_i - \boldsymbol{p}_i)$$

If $\Sigma = I$ we can see that the demand for each input does not depend from the demand for other inputs.

If Σ is not invertible in general there is no solution, contrary to the imperfect competition case.

The idea is that for a suitable choice of the matrix Σ_i input goods are complement. If, further, we take the limit for the quadratic part of the cost going to infinity, we obtain that such complementarity dominates any price difference. The next examples illustrates this point.

Example 6 (Two inputs)**.** If there are two inputs 1 and 2, and $\sigma_{i,11} = k = \sigma_{i,22}$, and $\sigma_{i,12} = -k$, then the cost term is only $k(q_{i1} - q_{i2})^2$. Hence, if $k \to \infty$, the firm will buy the same of both goods, no matter the prices. In such a case $q_i = q_{i1} \sum_j \omega_{ij}$, so $f = (\sum_j \omega_{ij})^{-1}$.

If $\sigma_{i,11} = k/(f'_{i1})^2$, $\sigma_{i,22} = k/(f'_{i2})^2$, and $\sigma_{i,12} = -k/(f'_{i1}f'_{i2})$ then we can get the cost term to be $k(q_{i1}/f'_{i1} - q_{i2}/f'_{i2})^2$, so $q_i = q_{i1}(\omega_{i1} + \omega_{i2}f'_{i1}/f'_{i2})$, so $f_{i1} = (\omega_{i1} + \omega_{i2}f'_{i1}/f'_{i2})^{-1}$.

The following proposition formalizes the reasoning in the example in full generality.

Proposition 8. *The factor demands arising with Leontief technology with coefficient matrix F are the same as those arising in the limit for $k \to \infty$ in the above quadratic technology with matrix Σ such that:*

$$\begin{aligned}
\sigma_{i,00} &= k/(f'_{i0})^2 \\
\sigma_{i,d_i d_i} &= k/(f'_{id_i})^2 \\
\sigma_{i,jj} &= 2k/(f'_{ij})^2 \quad 0 < j < d_i \\
\sigma_{i,j,j+1} &= \sigma_{i,j+1,j} = -k/(f'_{ij}f'_{i,j+1})
\end{aligned}$$

where the coefficients f' are such that: $f_{ih} = \frac{f'_{ih}}{\sum_j \omega_{ij} f'_{ij}}$.

Proof. See Appendix B.8.

□

6 Conclusions

We develop a tractable framework to analyze how the impact of common ownership can depend on the position in a production network, and we show how to understand the effects in terms of the structure of externalities embedded both in the production and demand networks, and the common ownership network. Despite the tractability we show that the model can be brougth to the data and the parameters identified. Thus, this model can be a valuable tool to analize anticompetitive effects of common ownership, a much pressing topic.

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A Pareto weight

Suppose in the society there is a mass M of consumers without shares of firms, and a mass m of investors, that are partitioned in m groups, so that investors in group i hold s_{ij} shares of firm j. Suppose they all have the same utility. Then, if η_i: is the Pareto weight of investors in group i:

$$W = mV(p,0) + \sum_{i=1}^{M} \eta_i V(p, \sum_j s_{ij}\pi_j)$$

where V indicated the indirect utility, that is, for an agent with non-labor income I:

$$V(p,I) = \sum_k b_k c_k - \frac{1}{2}\sum_k c_k^2 - L$$

where L is labor, and (normalizing the wage to 1) the budget constraint reads: $L = \sum_k p_k c_k - I$, so that the indirect utility is quasi-linear in income:

$$V(p,I) = \sum_k b_k c_k - \frac{1}{2}\sum_k c_k^2 - \sum_k p_k c_k + I = \frac{1}{2}\sum_k c_k^2 + I$$

where the last step is obtained using the expression for the demand. Call the consumption part $C(p)$: it is independent of income. Hence:

$$W = MC(p) + \sum_{i=1}^{m} \eta_i C(p) + \sum_{i=1}^{m} \eta_i \sum_j s_{ij}\pi_j$$

Now, for simplicity assume that all investors have the same Pareto weight $\eta' = \eta_i$, then we obtain:

$$W = (M + m\eta)C(p) + \eta \sum_j \pi_j = (M + m\eta')\left(C(p) + \frac{\eta'}{M + m\eta'} Pro\right)$$

Where we relabel $\eta = \frac{\eta'}{M+m\eta'}$ the relative Pareto weight of profit income with respect to consumption.

B Proofs

B.1 Proof of Proposition 1

The FOCs are:

$$(p_i - MC_i) = \frac{1}{\sum_j \ell_{ji}\sigma_{ji}} \left(q_i + \sum_{j\neq i} K_{ij} \left(-\sum_k \ell_{kj}\sigma_{ki}(p_j - MC_j) - q_j f_{ji} \right)\right)$$

in terms of the markups $\mu_i = p_i - MC_i$ (remember that $\sigma_{ij} = \sigma_{ji}$):

$$\mu = diag(\Sigma L)^{-1}\left(q - \mathring{K} \circ (\Sigma L)\mu - K \circ F'q\right)$$

$$(K \circ (\Sigma L))\mu = (I - K \circ F')q = (I - K \circ F')L'(b - \Sigma L(\mu + f^\ell))$$

$$\mu = (K \circ (\Sigma L) + (I - K \circ F')L'\Sigma L)^{-1}(I - K \circ F')L'(b - \Sigma L f^\ell)$$

so that:

$$p = L(K \circ (\Sigma L) + (I - K \circ F')L'\Sigma L)^{-1}(I - K \circ F')L'(b - \Sigma L f^\ell) + L f^\ell$$

$$= L(I + (K \circ (\Sigma L))^{-1}(I - K \circ F')L'\Sigma L)^{-1}((K \circ (\Sigma L))^{-1}(I - K \circ F')L'b + f^\ell)$$

$$= \Sigma^{-1}(\Sigma^{-1} + L(K \circ (\Sigma L))^{-1}(I - K \circ F')L')^{-1}(\tilde{b} + p^{eff})$$

$$= \Sigma^{-1}(\Sigma^{-1} + \tilde{L})^{-1}(\tilde{b} + p^{eff})$$

where $\tilde{L} = L(K \circ (\Sigma L))^{-1}(I - K \circ F')L'$ is the matrix of the (adjusted) input-output similarities, $\tilde{b} = \tilde{L}b$.

The markup can be written as:

$$\mu = (I - F)\Sigma^{-1}(\Sigma^{-1} + \tilde{L})^{-1}\tilde{L}c^{eff}$$

again highlighting the role of input-output similarities.

If we consider the non-normalized Ks we get a similar expression but with $\tilde{L} = L(K \circ (\Sigma L))^{-1}(K \circ (I - F'))L'$. We can see that, anyway, the diagonal simplifies away.

B.2 Proof of Theorem 3

The derivative The Lagrangian of the planner problem is:

$$\mathcal{L} = C(p) + \eta Pro - \sum_i \lambda_i \left[\sum_j s_i \cdot s_j \frac{\partial \pi_j}{\partial p_i}\right] + \sum_i \theta_i \zeta_i$$

we consider here a generalized problem, where the profit part of welfare has a general Pareto weight η. The version in the text can be obtained setting $\eta = 1$.

Now consider an interior solution, with $\theta_i = 0$. Otherwise, it would not make sense to take the derivative. The derivative of the ownership weights is:

$$\frac{\partial(s_i \cdot s_j)}{\partial \zeta_i} = \begin{cases} 2\frac{N\zeta_i}{N-1} & \text{if } i = j \\ -\frac{N\zeta_j}{(N-1)^2} & \text{otherwise} \end{cases}$$

By the envelope theorem, we have:

$$\begin{aligned}
\frac{\partial W}{\partial \zeta_u} &= \frac{\partial \mathcal{L}}{\partial \zeta_u} \\
&= -\sum_i \frac{\partial(s_i \cdot s_j)}{\partial \zeta_u} \lambda_i \frac{\partial \pi_j}{\partial p_i} \\
&= -\sum_i \lambda_i \left(\delta_{iu} 2\frac{N\zeta_u}{N-1} \frac{\partial \pi_i}{\partial p_i} - \sum_{j \neq i} \delta_{ju} \frac{\zeta_i N}{(N-1)^2} \frac{\partial \pi_j}{\partial p_i} - \sum_{j \neq i} \delta_{iu} \frac{\zeta_j N}{(N-1)^2} \frac{\partial \pi_j}{\partial p_i} \right) \\
&= -\lambda_u \frac{N\zeta_u}{N-1} \left(2\frac{\partial \pi_u}{\partial p_u} - \sum_{i \neq u} \lambda_i \frac{1}{N-1} \frac{\partial \pi_u}{\partial p_i} - \lambda_u \sum_{j \neq u} \frac{1}{N-1} \frac{\partial \pi_j}{\partial p_u} \right)
\end{aligned}$$

that is exactly the expression in the text of the Theorem.

The multipliers First of all, by Lagrange theorem, the multipliers in equilibrium are nonnegative.

The FOCs with respect to the prices are:

$$p_k: \quad -c_k + \eta \sum_i \frac{\partial \pi_i}{\partial p_k} - \sum_i \lambda_i \sum_j K_{ij} \frac{\partial^2 \pi_j}{\partial p_i \partial p_k} = 0$$

$$p_k: \quad -c_k + \eta \sum_i \frac{\partial \pi_i}{\partial p_k} -$$

$$\sum_i \lambda_i \sum_j K_{ij} \left(\frac{\partial q_k}{\partial p_i} \delta_{kj} + \frac{\partial q_i}{\partial p_k} \delta_{ij} - \frac{\partial q_j}{\partial p_i} \frac{\partial MC_j}{\partial p_k} - \frac{\partial q_j}{\partial p_k} \frac{\partial MC_j}{\partial p_i} \right) = 0$$

$$0 = -c_k + \eta \sum_i \left(\delta_{ik} q_i - \sum_s \ell_{si} \sigma_{sk} (p_i - MC_i) - q_i f_{ik} \right) -$$

$$\sum_i \lambda_i \sum_j K_{ij} \left(-\sum_s \ell_{sk} \sigma_{si} \delta_{kj} - \sum_s \ell_{si} \sigma_{sk} \delta_{ij} + \sum_s \ell_{sj} \sigma_{si} f_{jk} + \sum_s \ell_{sj} \sigma_{sk} f_{ji} \right)$$

and in matrix form:

$$-c' + \eta(q'(I - F) - p'(I - F')L'\Sigma + f^\ell L'\Sigma) -$$

$$\lambda'[-K \circ (\Sigma L) - diag(K)L'\Sigma + (K \circ (\Sigma L))F + (K \circ F')L'\Sigma] = 0$$

$$-c' + \eta(q'(I-F) - p'(I-F')L'\Sigma + f^\ell L'\Sigma) + \lambda'[(K \circ (I-F'))L'\Sigma + (K \circ (\Sigma L))(I-F)] = 0$$

Now the symmetric part of $K \circ (I - F')$ is positive definite, and also the symmetric part of $K \circ (L)$. Hence both matrices in the sum of the RHS are invertible; the first is positive.

$$\lambda = [\Sigma L(K \circ (I-F)) + (I-F')(K \circ (L'\Sigma))]^{-1} \left(c - \eta((b - \Sigma p) - \Sigma p + \Sigma L f^\ell)\right)$$

$$= [\Sigma L(K \circ (I-F)) + (I-F')(K \circ (L'\Sigma))]^{-1} \left((1-\eta)c + \eta\Sigma(p - p^{eff})\right)$$

$$= (K \circ (L'\Sigma))^{-1} L' [\Sigma^{-1} + \tilde{L}']^{-1} \Sigma^{-1} \left((1-\eta)c + \eta\Sigma(p - p^{eff})\right)$$

that highlights how in such an expression the *transpose* of the input-output similarity matrix appears. For η large and small the vector is, respectively, negative and positive. When goods are complements the vector is positive at least for all $\eta \leq 1$.

For $\eta = 1$:

$$\lambda = (K \circ (L'\Sigma))^{-1} L' [\Sigma^{-1} + \tilde{L}']^{-1} L\mu$$

that highlights how the multipliers are weighted sums (or differences) of the markups, weighted by a matrix that involves also the I-O similarities.

B.3 Proof of Example 5

The FOCs are:

$$p_i \omega_{ij} - p_j + \frac{1}{2}\left(\sum_j \sigma_{i,jk} q_{ik} + \sum_k \sigma_{i,kj} q_{ik}\right) = 0 \quad \ell_i = \lambda_i \alpha_i$$

Hence:

$$\boldsymbol{q}_i = \Sigma_i^{-1}(p_i \omega_i - \boldsymbol{p}_i)$$

B.4 Proof of Proposition 3

$$\frac{\partial W}{\partial \zeta_u} = -2\lambda_u M \frac{\zeta_u - 1/M}{M-1} \frac{\partial \pi_u}{\partial p_u} + \sum_{i \neq u} \lambda_i M \left(\frac{\zeta_i - 1/M}{(M-1)^2} \frac{\partial \pi_u}{\partial p_i}\right) + M\lambda_u \sum_{j \neq u} \left(\frac{\zeta_j - 1/M}{(M-1)^2} \frac{\partial \pi_j}{\partial p_u}\right)$$

so if $\zeta_i = \zeta_j$ we get:

$$\frac{\partial W}{\partial \zeta_i} > \frac{\partial W}{\partial \zeta_j} \iff$$

$$-2\lambda_i \frac{\partial \pi_i}{\partial p_i} + \frac{1}{(M-1)}\lambda_i \sum_{u \neq i} \left(\frac{\partial \pi_u}{\partial p_i}\right) + \frac{1}{(M-1)} \sum_{u \neq i} \lambda_u \left(\frac{\partial \pi_i}{\partial p_u}\right) > -2\lambda_j \frac{\partial \pi_j}{\partial p_j} + \frac{1}{(M-1)}\lambda_j \sum_{u \neq j} \left(\frac{\partial \pi_u}{\partial p_j}\right.$$

so:

$$-2\lambda_i \frac{\partial \pi_i}{\partial p_i} + \frac{1}{(M-1)}\lambda_i \sum_{u \neq i,j} \left(\frac{\partial \pi_u}{\partial p_i}\right) + \frac{1}{(M-1)} \sum_{u \neq i,j} \lambda_u \left(\frac{\partial \pi_i}{\partial p_u}\right) >$$

$$-2\lambda_j \frac{\partial \pi_j}{\partial p_j} + \frac{1}{(M-1)}\lambda_j \sum_{u \neq i,j} \left(\frac{\partial \pi_u}{\partial p_j}\right) + \frac{1}{(M-1)} \sum_{u \neq i,j} \lambda_u \left(\frac{\partial \pi_j}{\partial p_u}\right)$$

$$\left(2k + \frac{1}{(M-1)}\right) \lambda_i \sum_{u \neq i} s\mu_u + \frac{1}{(M-1)} \sum_{u \neq i} \lambda_u s\mu_i > \left(2k + \frac{1}{(M-1)}\right) \lambda_j \sum_{u \neq j} s\mu_u +$$

Now, to cancel out terms on both sides we isolate the sums $\sum_{u \neq i,j} \mu_u$ and $\sum_{u \neq i,j} \lambda_u$:

$$\left(2k + \frac{1}{(M-1)}\right) s\lambda_i \mu_j + \left(2k + \frac{1}{(M-1)}\right) \lambda_i s \sum_{u \neq i,j} \mu_u + s\mu_i \frac{\lambda_j + \sum_{u \neq i,j} \lambda_u}{(M-1)} >$$

$$\left(2k + \frac{1}{(M-1)}\right) s\lambda_j \mu_i + \left(2k + \frac{1}{(M-1)}\right) \lambda_j s \sum_{u \neq i,j} \mu_u + s\mu_j \frac{\lambda_i + \sum_{u \neq i,j} \lambda_u}{(M-1)} \iff$$

$$2ks\left(\lambda_i \mu_j - \lambda_j \mu_i\right) + (\lambda_i - \lambda_j)s\left(2k + \frac{1}{(M-1)}\right) \sum_{u \neq i,j} \mu_u + \frac{(\mu_i - \mu_j)s}{(M-1)} \sum_{u \neq i,j} \lambda_u > 0$$

$$2ks\left(\lambda_i \mu_j - \lambda_j \mu_i\right) + \frac{(\mu_i - \mu_j)s}{(M-1)} \sum_{u \neq i,j} \mu_u + \frac{(\mu_i - \mu_j)s}{(M-1)} \sum_{u \neq i,j} \lambda_u > 0$$

$$2ks^2 K\left(\mu_i - \mu_j\right) + \frac{(\mu_i - \mu_j)s}{(M-1)} \sum_{u \neq i,j} \mu_u + \frac{(\mu_i - \mu_j)s}{(M-1)} \sum_{u \neq i,j} \lambda_u > 0$$

If $s > 0$ this is positive whenever $\mu_i > \mu_j$: so the markups are sufficient statistics.

B.5 Proof of Proposition 4

We can rewrite:

$$\begin{aligned}
f_{ij} &= \frac{p_i}{p_j}\frac{p_j q_{ij}}{p_i q_i} \implies \\
p_j f_{ij} &= p_i \frac{p_j q_{ij}}{p_i q_i} \\
F &= diag(p)\hat{F}diag(p)^{-1} \\
L &= (I - diag(p)\hat{F}diag(p)^{-1})^{-1} = \\
&= diag(p)(I-\hat{F})^{-1}diag(p)^{-1}
\end{aligned}$$

Call the diagonal of $diag(L\Sigma) = D$. We have: $D_i = \sum_j L_{ij}\sigma_{ij} = \sum_j \frac{p_i}{p_j}\hat{L}_{ij}\sigma_{ij} = p_i\hat{D}_i$

For $K = I$, the equation for profits becomes: $p_i^2\pi_i = R_i^2/D_i = R_i^2/(p_i\hat{D}_i)$

where $\hat{D}_i$ does not depend on p_i, and is decreasing in the other prices. Then we can solve the equation:

$$p_i = \left(\frac{R_i^2}{\pi_i\hat{D}_i}\right)^{1/3}$$

The function $F_i(p_{-i}) = \left(\frac{R_i^2}{\pi_i\hat{D}_i}\right)^{1/3}$ is increasing in the other prices. The equilibrium is then defined by:

$$F(p) = p$$

Moreover, for $p_j \to \infty$ for all j we have $\hat{D}_i \to \hat{L}_{ii}$, so there is the upper bound $p_i \leq \left(\frac{R_i^2}{\pi_i\tilde{L}_{ii}}\right)^{1/3}$. So, by Topkis theorem, there exist a solution.

Moreover, now we prove uniqueness by proving that the Jacobian of the function defining the equilibrium is positive definite and applying Gale-Nikaido's theorem (Gale and Nikaido (1965)). Define $G(p) = p_i^2\pi_i - R_i^2/D_i$. Then:

$$JG = \begin{pmatrix} 2p_i\pi_i + R_i^2/(p_iD_i) & \dots & -R_i^2/D_i^2\frac{p_i\tilde{L}_{ij}\sigma_{ij}}{p_j^2} \\ & & \\ & & \end{pmatrix}$$

Sum of first row:

$$\begin{aligned}
&2p_i\pi_i + R_i^2/(p_iD_i) - \frac{R_i^2}{D_i^2}\sum_{j\neq i}\frac{\tilde{L}_{ij}\sigma_{ij}}{p_j^2} = \\
&2p_i\pi_i + \frac{R_i^2}{D_i}\left(\frac{1}{p_i} - \frac{1}{D_i}\sum_{j\neq i}\frac{p_i\tilde{L}_{ij}\sigma_{ij}}{p_j^2}\right)
\end{aligned}$$

Divide and multiply by $diagp$. Then:

$$\begin{pmatrix} 2p_i\pi_i + R_i^2/(p_iD_i) & ... & -R_i^2/D_i^2\frac{\tilde{L}_{ij}\sigma_{ij}}{p_j} \\ & & \\ & & \end{pmatrix}$$

Now $\sum_{j\neq i}\frac{1}{p_i}\frac{p_i\tilde{L}_{ij}\sigma_{ij}}{p_j} < \frac{1}{p_i}D_i$, so:

$$\begin{aligned} 2p_i\pi_i + \frac{R_i^2}{D_i}\left(\frac{1}{p_i} - \frac{1}{D_i}\sum_{j\neq i}\frac{p_i^2\tilde{L}_{ij}\sigma_{ij}}{p_j}\right) > \\ 2p_i\pi_i + \frac{R_i^2}{D_i}\left(\frac{1}{p_i} - \frac{1}{p_i}\right) = 0 \end{aligned}$$

So JG is diagonally dominant, so it is positive definite, so there is a unique solution.

B.6 Proof of Proposition 7

Using the first derivatives derived in the proof of Theorem 3, we can find the second derivatives:

$$\zeta_u, \zeta_v: \quad = -2\lambda'_u\frac{N}{N-1}\delta_{uv}\frac{\partial\pi_u}{\partial p_u} - (1-\delta_{uv})\lambda'_v\left(\frac{-N}{(N-1)^2}\frac{\partial\pi_u}{\partial p_v}\right) - (1-\delta_{uv})\lambda'_u\left(\frac{-N}{(N-1)^2}\frac{\partial\pi_v}{\partial p_u}\right) - \frac{1}{\tau}$$

We can rewrite them in terms of the H matrix. The derivatives are:

$$\frac{\partial W}{\partial\zeta} = -\frac{1}{\tau}(\zeta - \zeta^0) + H\zeta$$

The Hessian matrix of the planner problem is: $H - 1/\tau I$, and is always negative definite if τ is small enough.

So at the optimum, it must be:

$$\zeta = (I - \tau H(\zeta^0))^{-1}\zeta^0$$

B.7 Proof of Proposition 6

Lemma B.1. If price externalities are all positive, and $K \neq I$, the Hessian matrix $\tilde{H}$ has all negative diagonal entries and nonpositive off-diagonal entries.

If price externalities are all negative, and $K \neq I$, the signs are the opposite.

Proof. The case $K = I$ has to be excluded because the diagonal is zero, because of the first order conditions.

If we assume the input-output network is strongly connected, or each connected components have at least two goods that are not independent for the consumer, then it follows that all firm pairs have some price externalities on each other, because all firms affect the demand of others. Hence H has all nonzero elements and in particular is irreducible.

□

Proof of the Proposition

The first order conditions for an optimum with respect to the shares s_{iu} for all i are:

$$\tilde{H}\tilde{s}_u = \rho - \theta_u$$

where: $\tilde{s}_u$ is the investment profile of investor u, and:

$$\tilde{H} = -\left(\lambda_i' \frac{\partial \pi_j}{\partial p_i} + \lambda_j' \frac{\partial \pi_i}{\partial p_j}\right)$$

At the optimum, the multipliers must be nonnegative. This means that:

- if all the price externalities are nonnegative, $\frac{\partial \pi_i}{\partial p_j} \geq 0$ for $i \neq j$, then by the FOC it must be $\frac{\partial \pi_i}{\partial p_j} \leq 0$: so $-\tilde{H}$ has positive diagonal and nonpositive off-diagonal entries (is a Z-matrix);
- if all the price externalities are nonpositive, $\frac{\partial \pi_i}{\partial p_j} \leq 0$ for $i \neq j$, then by the FOC it must be $\frac{\partial \pi_i}{\partial p_j} \geq 0$: so $\tilde{H}$ has positive diagonal and nonpositive off-diagonal entries (is a Z-matrix).

To see that $\mathbb{1}$ is a solution of the FOCs, it is sufficient to choose $\rho = \tilde{H}\mathbb{1}$, since the $\theta_u = 0$. It is the only interior solution, because any interior solution must satisfie $s_u = s_{u'}$ for any u, u'.

Moreover, note that with positive externalities $-\Pi$ is a Z-matrix, and $\Pi\mathbb{1} = 0$ (because of the first order conditions). Hence it follows that it is a singular M-matrix.[4] Since the multipliers λ are positive, $\Pi\Lambda$ is also a singular M matrix. (to verify, but should be true). Hence, H is a singular M matrix, and in particular the second order conditions are satisfied. Hence, it is the unique maximum among interior solutions.

[4] If $M\mathbb{1} = 0$ and $M = sI - B$ with B positive, then $B\mathbb{1} = s\mathbb{1}$, so $\rho(B) = s$, and so M is a singular M matrix.

When externalities are all negative, all signs are reversed, and so this is the only *minimum* among interior solutions: hence the maximum must be at a corner. The fully divided ownership, in case of network-substitutes, presents strategic complementarities for the Ks close enough: prices are decreasing in all K. For η small enough we should be able to prove that the welfare is decreasing in prices, and so for each K to be at 0 is a local maximum. But there is no interior maximum, hence this is the global maximum.

B.8 Proof of Proposition 8

The first order conditions of the cost minimization problem are:

$$\begin{aligned} q_{i1}: &\quad k(q_{i0}/(f'_{i0})^2 - q_{i,1}/(f'_{i0}f'_{i1})) = \lambda_i\omega_{i0} - p_0 \\ q_{id_i}: &\quad k(q_{id_i}/(f'_{id_i})^2 - q_{i,d_i-1}/(f'_{id_i}f'_{id_i-1})) = \lambda_i\omega_{id_i} - p_{d_i} \\ q_{ij}: &\quad k(2q_{ij}/(f'_{ij})^2 - q_{i,j+1}/(f'_{ij}f'_{ij+1}) - q_{i,j-1}/(f'_{ij}f'_{ij-1})) = \lambda_i\omega_{ij} - p_j \end{aligned}$$

We can reorder them as:

$$\begin{aligned} q_{i0}/f'_{i0} - q_{i,1}/f'_{i1} &= f'_{i0}\frac{\lambda_i\omega_{i0} - p_0}{k} \\ q_{id_i}/f'_{id_i} - q_{i,d_i-1}/f'_{id_i-1} &= f'_{id_i}\frac{\lambda_i\omega_{id_i} - p_{d_i}}{k} \\ 2q_{ij}/f'_{ij} - q_{i,j+1}/f'_{ij+1} - q_{i,j-1}/f'_{ij-1} &= f'_{ij}\frac{\lambda_i\omega_{ij} - p_j}{k} \end{aligned}$$

Now we can see that summing all the LHS we get 0, and from this we can solve for the marginal cost: $\lambda_i = \frac{\sum f'_{ij}p_j}{\sum f'_{ij}\omega_{ij}}$.

Now, passing to the limit for $k \to \infty$, we get:

$$\begin{aligned} &q_{i0}/f'_{i0} - q_{i,1}/f'_{i1} \to 0 \\ &q_{id_i}/f'_{id_i} - q_{i,d_i-1}/f'_{id_i-1} \to 0 \\ &2q_{ij}/f'_{ij} - q_{i,j+1}/f'_{ij+1} - q_{i,j-1}/f'_{ij-1} \to 0 \end{aligned}$$

From the first two equations we get: $\frac{q_{i0}}{f'_{i0}} = \frac{q_{i1}}{f'_{i1}}$ and $\frac{q_{i1}}{f'_{i1}} = \frac{1}{2}\left(\frac{q_{i0}}{f'_{i0}} + \frac{q_{i2}}{f'_{i2}}\right)$. Using the first in the second we get: $\frac{q_{i1}}{f'_{i1}} = \frac{1}{2}\left(\frac{q_{i1}}{f'_{i1}} + \frac{q_{i2}}{f'_{i2}}\right)$ that implies $\frac{q_{i2}}{f'_{i2}} = \frac{q_{i1}}{f'_{i1}}$. Iterating the reasoning we obtain that $\frac{q_{ij}}{f'_{ij}}$ is constant for any j. But then:

$$q_i = \sum_j \omega_{ij} q_{ij} = q_{ih} \sum_j \omega_{ij} f'_{ij} / f'_{ih}$$

so: $q_{ih} = \frac{f'_{ih} q_i}{\sum_j \omega_{ij} f'_{ij}}$. So, setting $f_{ih} = \frac{f'_{ih}}{\sum_j \omega_{ij} f'_{ij}}$, we obtain exactly the same equations that define the factor demands of the Leontief technology:

$$\lambda_i = \frac{\sum f'_{ij} p_j}{\sum f'_{ij} \omega_{ij}} = \sum f_{ij} p_{ij}$$
$$q_{ij} = f_{ij} q_i \quad \forall j$$