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Dynamic Diffusion in Production Networks

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Abstract

I show three properties in which a dynamic input-output economy with time to build differs from a static economy: first, a standard result in a Cobb-Douglas production networks is that productivity shocks diffuse downstream while demand shocks diffuse upstream. This fact interacts with the discount rate to generate a potentially quite different aggregate impact in different sectors. With time to build the direction of the diffusion is the opposite, and demand shocks also diffuse downstream. Second, I show that time to build leads to less comovement across sectors. Third, I provide bounds on the recovery time of the economy hit by a shock.

JEL Classification: D57, D85, E32.

Keywords: production networks, diffusion, propagation, shocks.

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Introduction

How do shocks to some economic sectors impact the rest of the economy? For many years a widespread view, exemplified by Lucas (1995)' argument, has been that when considering whole economies composed by a large number of agents idiosyncratic shocks should average out and not have a sizable aggregate impact. Recently, this view has been challenged, noting that the averaging out might not happen if the connections between sectors are sufficiently asymmetric, so that the very well connected sectors will have a sizable impact on aggregate output, as argued in the seminal paper Acemoglu et al. (2012). The understanding of such mechanisms is of crucial importance to understand business cycles and to evaluate and design policies directed to smooth or insure against shocks, such as bailouts or monetary policy.

A growing literature has indeed provided empirical grounding for the importance of idiosyncratic shocks in shaping aggregate outcomes.[1] Yet, most of the analyses have focused on static general equilibrium models or on steady states of the dynamics.[2] While this has certainly allowed many useful insights, production is essentially a dynamic phenomenon, as testified by the sizable literature that studies *time to build* in its own right.[3] It is therefore to be expected that the temporal dimension of the propagation of shocks contains many important features that a static analysis would miss. Some are classical questions pertaining to dynamic environments, such as what is the *persistence* of a shock, other are more specific to an input-output level analysis: which sectors are more affected by the shock in the short run rather than the long run? which sectors generate more short than long run impacts on the welfare of the consumers? All these questions simply can't be answered in a static model.[4]

In this work, I want to address these issues, analyzing a model that generates a *dynamic* diffusion, namely a propagation of shocks *over time* as well as over sectors. To generate a dynamic diffusion while keeping analytical tractability, I will follow the original input-output model by Long Jr and Plosser (1983) and in particular assume that the production of any good necessitates 1 period of time. This implies that the reaction of each sector to shocks will be lagged and diffusion will not be instantaneous: a shock to

[1] See section .

[2] There are exceptions, in particular Pasten et al. (2018), as explained in the literature section.

[3] A classic contribution is Kydland and Prescott (1982), while more recently Meier (2017).

[4] Note that, perhaps not surprisingly, also persistence will depend crucially on network characteristics in this setting.

a sector will trigger a reaction from the immediate neighbors, but in general not from the others. This will generate a dynamic diffusion of the impact, that will take time to spread to the whole economy, allowing us to analyze it in details.

There are two perspectives from which we can analyze such an environment: focusing on the properties of the stationary stochastic process generated by the uninterrupted random disruptions that hit the economy, or analyzing the impact of a single shock and the properties of the *transition* to the (possibly new) steady state - the *impulse response function*.[5]

My results show that the properties of a dynamic diffusion can depart substantially from a static benchmark, even in simple Cobb Douglas environments: productivity shocks proagate exclusively downstream, and an *unexpected* productivity shock has a cumulative welfare impact which is proportional to a *dynamic* version of Bonacich centrality, that takes into account the different value of consumption over time. Moreover, the cumulative impact is equal to the share of sales of the respective sector, its *Domar weight*. This is an analogous of Hulten (1978) theorem, stating that in an efficient economy the first order contribution of a small shock to a sector to aggregate GDP is exactly its sales share. The result, though, is not obvious: here I am considering an *unexpected* shock, which a priori needs not behave as Hulten theorem predicts.

Preference shocks, instead, have a radically different propagation behavior, that can be summarized as such: their *physical* impact propagates downstream, while the *information* impact propagates upstream. By physical impact I mean the impact working through the physical decrease in real output, that through a change in prices causes the customers to vary their purchases and so their production. The information impact is the update in expectations of future demand changes due to autocorrelations in preference shocks over time. The comparative difference in preference shocks with respect to productivity is a by-product of the Cobb-Douglas technology, that implies that productivity shocks do not have nominal effects, and I do not expect it to generalize.

Moreover, thanks to the linear nature of the problem, the dynamics is very close to the iteration of a Markov chain. So we can apply the ergodic theory of Markov chains to provide upper bounds on the time that the economy takes to recover from a negative shock (or to scale down from a positive one), in a spirit similar to Golub and Jackson (2012). These bounds depend

[5]Note that the transition I am referring to is always *along an equilibrium path*, although the analysis of out of equilibrium responses to disruptions is of great interest, and I am addressing it in ongoing work.

crucially on the network characteristics, such as (eigenvector) centrality, the labor share of technology, and community structure.

Finally, I show that the dynamic model with time to build generates systematically less comovement, measured as lag 0 autocorrelation. This happens because shocks take time to affect other sectors, so the effect can hit different sectors at lagged times, not generating contemporaneous comovement.

Outline In the next section I present the related literature, then the model and the implied diffusion dynamics. In section 4 I explore the welfare impacts, in section 5 the long run stationary properties of the model. In section 6 I present upper bounds on recovery time of the economy after a shock.

Related literature

This papers contributes to two literatures: the literature on the network origins of aggregate fluctuations, and the literature on the macroeconomic consequences of time to build and adjustment costs.

Despite the original contribution of Long Jr and Plosser (1983), the literature on the network origins of aggregate fluctuations has focused mainly on models without time to build. A recent exception is Liu and Tsyvinski (2024), that study a continuous time model with adjustment costs that can be thought as a continuous time version of Long Jr and Plosser (1983). They only focus on productivity shocks, and do not focus on comovement. Many models do not study an explicit dynamics, and the fluctuations are analyzed using comparative statics (as in Baqaee (2018), Acemoglu et al. (2016)), or in a statistical sense (as in Acemoglu et al. (2012)). Some papers explicitly study the dynamics, as Pasten et al. (2018), that analyzes analytically how the network affects the response of variables to monetary policy shocks.

The literature on time to build and adjustment costs has studied the implications for aggregate fluctuations and business cycles, from Kydland and Prescott (1982) to Meier et al. (2020), Bachmann et al. (2013): these papers do not consider the input-output dimension. Pellet and Tahbaz-Salehi (2023) study rigidity in adjustment of inputs in a production network but, apart from studying a static setting, in their paper they focus on the efficiency loss from the rigidity, whereas in our setting time to build is a feature of the technology, and is not inefficient per se.

1 Model

I adopt the setup in Long Jr and Plosser (1983), but I will depart from it in the case of stochastic preferences. I summarize it here:

1. Time is infinite and discrete. There are two vector Markov processes A_t and γ_t, which are the sources of stochasticity in the model. For simplicity I assume they have a finite state space $S \subset \mathbb{R}_+^N$. I will denote the history of realizations up to time t as $h^t = ((\gamma_1, A_1), \ldots, (\gamma_t, A_t))$. All the endogenous variables should be indexed by histories. When the context does not strictly require it, I abuse the notation by indexing just with t, as in γ_t.[6]

2. There is one infinitely lived representative consumer maximizing its expected discounted utility. Instantaneous utility is the logarithm of a Cobb-Douglas aggregator $C_t = \prod c_{i,t}^{\gamma_{i,t}}$. The intertemporal utility is a standard discounted sum $U = \sum_t \beta^t \ln C_t = \sum_t \beta^t \sum_i \gamma_{i,t} \ln c_{i,t}$, where $\beta < 1$ is the discount factor. The consumer will maximize the expectation of this intertemporal utility. She has an endowment of 1 unit of labor each period, and she supplies it inelastically.

3. There are N sectors, each producing a distinct good, acting as neoclassical firms, that maximize their intertemporal profits.[7] subject to a constant returns Cobb Douglas technology, with the important feature described in the next point.

$$\sum_t \left(p_{i,t} y_{i,t} - \sum_{j=1}^{N} p_{j,t} z_{ij,t} - w_t l_{i,t} \right)$$

4. Inputs need to be purchased one period in advance. The specific form of the production function is: $A_{i,t+1} \prod_{j=1}^{N} (z_{ij}^t)^{\alpha \omega_{ij}} l_{i,t}^{1-\alpha}$; the parameters ω_{ij} define a matrix Ω, that defines a directed weighted network which we call the *input-output* network of the economy and represents the strenghts of intersectoral linkages. In particular, due to the Cobb Douglas assumption, ω_{ij} is the share of revenues of sector i spent on input j.

[6]Note that the process for γ_t has to be such that the normalization $\sum_i \gamma_{i,t} = 1$ is true for all t.

[7]Note that the prices that appear in the profit expression are not in real terms, but are intertemporal prices, so they include the interest rate.

5. The consumer owns the firms, and each period receives or pays the necessary cash flow:

$$f_t = \sum_i f_{i,t} = \sum_i \left(p_{i,t} y_{i,t} - \sum_{j=1}^{N} p_{j,t} z_{ij,t} - w_t l_{i,t} \right)$$

   Despite the Cobb-Douglas assumption, this is not zero, because it is *not* the expected profit, for two reasons: it is a *realized*, not expected quantity, and second it is the sum of earnings today from inputs bought *yesterday*, and expenditure for inputs whose output will be sold *tomorrow*. Hence there is no reason to expect this quantity to be 0;

6. The intertemporal budget constraint of the consumer is:

$$\sum_{h^t} \sum_i p_{i,h^t} c_{i,h^t} \leq \sum_i p_{i,0} \omega_i + \sum_{h^t} w_{h^t} l_{h^t} + \sum_{h^t} f_{h^t}$$

   where $w_t l_{i,t}$ is labor income, f_t is the cash flow she receives from the firms, and ω_i is the endowment of the consumer at period 0. This endowment has to be introduced in order for the model to "kick off", otherwise in the first period there can be no production, but is otherwise unimportant and will not appear in any result.

7. There are forward markets for any contingent commodity.

The equilibrium concept is the standard Arrow-Debreu equilibrium. I report here the definition for further clarity.

Definition 1.1 (Equilibrium)**.** An equilibrium of this economy is a vector of prices, consumptions, input demands for each history h^t such that

1. The consumer chooses streams of consumption optimizing its expected utility over its budget constraint, solving:

$$\max \mathbb{E}_0 \sum_t \beta^t \ln C_t = \sum_t \beta^t \sum_i \gamma_{i,t} \ln c_{i,t}$$

   subject to:

$$\sum_{h^t} \sum_i p^*_{i,h^t} c_{i,h^t} \leq \sum_i p^*_{i,0} \omega_i + \sum_{h^t} w_{h^t} l_{h^t} + \sum_{h^t} f_{h^t}$$

   where f_{h^t} is defined above.

2. Firms maximize their expected profits subject to the technology constraint:

$$\max_{(l_{i,t})_{t=0}^{\infty},(z_{ij,t})_{j=1,t=0}^{n,\infty}} \mathbb{E}_0 \sum_t \left(p_{i,t} A_{i,t+1} \prod_{j=1}^{N} (z_{ij}^t)^{\alpha\omega_{ij}} l_{i,t}^{1-\alpha} - \sum_{j=1}^{N} p_{j,t} z_{ij,t} - w_t l_{i,t} \right)$$

3. Prices clear the goods market and the labor market at each history:

$$y_{i,h^t} = c_{i,h^t} + \sum_j z_{ij,h^t} \qquad \sum_i l_{i,h^t} = 1 \quad \forall h^t$$

$$\omega_i = c_{i,0} + \sum_j z_{ij,0} \qquad \sum_i l_{i,0} = 1$$

2 Dynamics

In this section, I report the solutions of the model, respectively for productivity and preference shocks. As in other production network models, *Bonacich centrality* is crucial: we denote it as $d_i(\alpha\beta, \gamma)$, where $d(\alpha\beta, \gamma)$ is the vector such that:

$$d = (I - \alpha\beta\Omega')^{-1}\gamma$$

This also corresponds to what Baqaee and Farhi (2019) call the *Domar weight*.[8] When the coefficient is clear from the context I will omit the dependence. The next proposition follows Long Jr and Plosser (1983)

Proposition 2.1. *If productivity parameters follow a Markov process, while preferences are deterministic, in equilibrium the outputs follow:*

$$\ln y_{i,t+1} = const_i + \ln A_{i,t+1} + \sum_j \alpha\omega_{ij} \ln y_{j,t}$$

The sale shares are constant: $\frac{p_{i,t} y_{i,t}}{GDP_t} = d_i(\alpha\beta, \gamma)$, *where* $GDP_t = \sum_i p_{i,t} c_{i,t}$.

Proof. See Appendix. □

So we can see from the above proposition that productivity shocks diffuse through a very simple linear dynamics. In particular, the process of logarithms of productivity is a filter of the process of the errors, increasing its persistence. For example, if productivity shocks are i.i.d. across time,

[8] They distinguish between *revenue* and *cost* based Domar weights. Since the economy studied here is efficient, the two coincide and there is no ambiguity.

$\ln y = (I - \alpha\Omega L)(const + \ln A)$, where L is the lag operator. That is, log-output follows a VAR(1).

Moreover, sales share are constant in time and are equal to centralities, as in the static model. Yet, there are significant differences in that the relevant centrality here has as a discount coefficient $\alpha\beta$, as I will argue in section 3.1.

Proposition 2.2. *If preference parameters follow a Markov process, in equilibrium, the dynamics of output follows:*

$$\log y_{i,t+1} = const_i + \alpha \sum_j \omega_{ij} \log y_{j,t} - \alpha \sum_j \omega_{ij} \log(d_{i,t}) +$$

$$\log(\mathbb{E}_t d_{i,t+1})$$

where $d_{i,t} = \gamma_{j,t} + \sum_k \alpha^k \beta^k \sum_h \omega_{hj}^{(k)} \mathbb{E}_t \left[\gamma_h^{t+k}\right]$

The sale shares are: $\frac{p_{i,t} y_{i,t}}{GDP_t} = d_{i,t}$

If γ_t *are i.i.d., then:*

$$d_{i,t} = \Delta\gamma_i + d_i(\alpha\beta, \overline{\gamma})$$

where $\mathbb{E}\gamma_t = \overline{\gamma}$. *Hence the dynamics follows:*

$$\log y_{i,t+1} = const_i + \ln d_i(\alpha\beta, \gamma) + \alpha \sum_j \omega_{ij} \log y_{j,t} - \alpha \sum_j \omega_{ij} \log(\Delta\gamma_{j,t} + d_j(\alpha\beta, \overline{\gamma}))$$

Proof. See Appendix. □

One of the features of static models such as Huremovic and Vega-Redondo (2016) or Acemoglu et al. (2016) is that in a Cobb Douglas environment preference (more in general: demand) shocks diffuse downstream. From the dynamics above we can see that, contrary to the static model, here, despite the Cobb-Douglas assumption, preference shocks diffuse also *downstream*. This happens because when a positive taste shock hits any good j then prices adjust. In particular the price of j increases and so firm i is able to buy less of it, so it will have a (relative) negative impact on its production. There is also a direct effect hitting all firms if shocks are correlated over time: the anticipation of a future higher demand drives the sectors whose demand depend more on the relatively more preferred good to increase their production. Instead, when shocks are i.i.d, the realization of the shock does not give any information on the future, hence the *only* impact is downstream. Summing up: the impact of *realized* preference shocks acts downstream, while the impact of *anticipated* shocks acts both upstream and downstream.

To understand better this behaviour, consider the case in which $\Delta\gamma_{\hat{\imath}} = -\Delta\gamma_{\hat{\jmath}} = \varepsilon$, and all the other components are constant. Then:

$$\log y_i^{\hat{t}+1} = -\alpha\omega_{i\hat{\imath}} \log\left(1 + \frac{\varepsilon}{d_{\hat{\imath}}}\right) - \alpha\omega_{i\hat{\jmath}} \log\left(1 - \frac{\varepsilon}{d_{\hat{\jmath}}}\right)$$

$$\sim \alpha\varepsilon\left(\frac{\omega_{\hat{\imath}\jmath}}{d_{\hat{\jmath}}} - \frac{\omega_{\hat{\imath}\imath}}{d_{\hat{\imath}}}\right)$$

In this case we can see that the output of sector i increases if the good less preferred because of the shock (and hence costs less) is more important as an input than the good which is more preferred (and so costs more).

3 Welfare impact of shocks

In this section, I investigate the welfare impact of a productivity and a preference shock. As anticipated, productivity shocks behave in a way much analogous to the static case, while preference shocks do not. Temporary and permanent shocks behave alike.

3.1 Productivity shocks

In this section I investigate productivity shocks.

Definition 3.1 (Productivity shocks)**.** In the following, by a **permanent shock** at node $\hat{\imath}$ at time $\hat{t}$ I define an unanticipated change in parameters such that $\ln A_{\hat{\imath},t} \to \ln A'_{\hat{\imath},t}$, for all $t \geq \hat{t}$, and $\ln A'_{i,t} = \ln A_{i,t}$ for all $i \neq \hat{\imath}$ and for all t. By a **temporary shock** I define an unanticipated change in parameters such that $\ln A_{\hat{\imath},\hat{t}} \to \ln A'_{\hat{\imath},\hat{t}}$, and $\ln A'_{i,t} = \ln A_{i,t}$ for all $i \neq \hat{\imath}$ and for all $t \neq \hat{t}$.

Proposition 3.1. *Consider a **permanent shock** hitting node $\hat{\imath}$. The consequent impact for the consumer is:*

$$\lim_{\Delta \ln A_i \to 0} \frac{\Delta \ln U}{\Delta \ln A_{\hat{\imath}}} = \beta^{\hat{t}} v_{\hat{\imath}}(\alpha\beta) \tag{1}$$

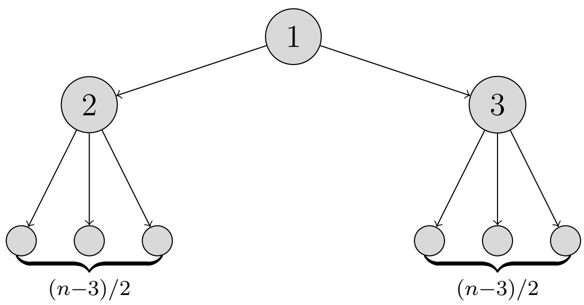
*Consider a **temporary shock** hitting node $\hat{\imath}$. The consequent impact for the consumer is:*

$$\lim_{\Delta \ln A_i \to 0} \frac{\Delta \ln U}{\Delta \ln A_{\hat{\imath}}} = \beta^{\hat{t}}(1-\beta) v_{\hat{\imath}}(\alpha\beta) \tag{2}$$

This result is an analogous of the well known Hulten Theorem: the impact of the shock in (log) utility is (proportional to) the sales share of the sector hit. Moreover, the dynamics of shocks is linear, so the impact of the realization of the stochastic productivity is identical to a variation in the parameter in a version without uncertainty. This feature depends heavily from the Cobb-Douglas technology assumption, and we do not expect it to be generalizable.

Nevertheless, there are significant differences with Acemoglu et al. (2012): longer paths are more heavily discounted, at a rate $\alpha\beta$ rather than β. This happens because in this model the impact of the shock accrues over time, hence the consumer will discount impacts that are further in the future with its intertemporal discount factor. This can result in changes in the importance of nodes, as in the following example.

Consider the following network, on n (even) nodes:



Centralities:

$$d_1 = \frac{1}{n} + \beta\alpha\frac{2}{n} + \beta^2\alpha^2\frac{n-3}{n} \tag{3}$$

$$d_2 = 1/n + \beta\alpha\frac{n-3}{2n} \tag{4}$$

If $\beta\alpha > 1/2$ then in the static model a consumer prefers a shock to 2 rather than 1. In the dynamic model instead, the loss in utility are:

$$\Delta U_1 = \varepsilon + \beta\varepsilon^{\frac{2\alpha}{n}} + \beta^2\varepsilon^{\alpha^2\frac{n-3}{n}} \tag{5}$$

$$\Delta U_2 = \varepsilon + \beta\varepsilon^{\alpha\frac{n-3}{2n}} \tag{6}$$

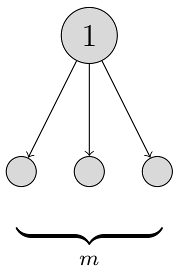
and, e.g. if $n = 6$, $\alpha = 0.6$, $\beta = 0.7$, nodes 2 and 3 are more important than node 1 in the dynamic version.

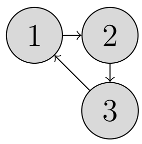
Another feature to be noted is that permanent and temporary shocks behave very much alike. This is due to the fact that permanent shocks converge to a different steady state, but the convergence process to the new steady state is very similar to the convergence back to the old steady state of a temporary shock.

3.1.1 Short run and long run

A possible interpretation of the Hulten-like result above is that, once we know the relative share of revenues, the specific network structure is irrelevant to the impact of the diffusion. In a dynamic setting, though, the same total impact can be achieved in very different ways: there can be shocks whose impact is very strong in the time periods immediately following the realization, but dies out quicly, and there can be shocks whose impact is mild, but diffuses a lot through the network, thereby achieving a high total impact over time nonetheless. The following example is meant to illustrate such behavior.

Example Consider the following two production networks: a circle with n nodes and a star with m leaves. In both cases, consider a temporary shock to node 1.





- a shock on node 1 in the star network exhausts after 1 period: impacts more in the **short run**;
- a shock on node 1 in the circle remains active forever: most important in the **long run**.

In particular, the welfare impacts are: $\frac{1+\alpha\beta m}{m+1}$ and $\frac{1}{n(1-\alpha\beta)}$ in the second. Parameters can be choosed such that the cumulative impacts are the same, despite the two very different structures.

3.2 Preference shocks

Definition 3.2 (Preference shocks)**.** In the following, by a **permanent shock** at node $\hat{\imath}, \hat{\jmath}$ at time $\hat{t}$ I define an unanticipated change in parameters such that $\gamma'_{\hat{\imath},t} - \gamma_{\hat{\imath},t} = -(\gamma'_{\hat{\jmath},t} - \gamma_{\hat{\jmath},t}) > 0$, for all $t \geq \hat{t}$, and $\gamma'_{i,t} = \gamma_{i,t}$ for all $i \neq \hat{\imath}$ and for all t. By a **temporary shock** I define an unanticipated change in parameters such that $\gamma'_{\hat{\imath},\hat{t}} - \gamma_{\hat{\imath},\hat{t}} = -(\gamma'_{\hat{\jmath},\hat{t}} - \gamma_{\hat{\jmath},\hat{t}}) > 0$, and $\gamma'_{i,t} = \gamma_{i,t}$ for all $i \neq \hat{\imath}$ and for all $t \neq \hat{t}$.

The next proposition describes the welfare impact of preference shocks.

Proposition 3.2. *Following a permanent shock to γ at time $\hat{t}$, the welfare impact is:*

$$\lim_{\Delta\gamma_{i,\hat{t}}\to 0} \frac{\Delta U}{\Delta\gamma_{\hat{\imath},\hat{t}}} = \beta^{\hat{t}}(\ln c_{\hat{\imath},\hat{t}} - \ln c_{\hat{\jmath},\hat{t}})$$

Following a transitory shock to γ at time $\hat{t}$, the welfare impact is:

$$\lim_{\gamma_{\hat{\imath},\hat{t}}\to 0} \frac{\Delta U}{\Delta\gamma_{\hat{\imath},\hat{t}}} = (1-\beta)\beta^{\hat{t}}(\ln c_{\hat{\imath},\hat{t}} - \ln c_{\hat{\jmath},\hat{t}})$$

Proof. See Appendix. □

Again, we see that transitory and permanent shocks behave in a similar way. And again, shocks with a very similar cumulative impact can differ greatly in the pattern of diffusion. Indeed, in the proof of the proposition we get the following expression:

$$\Delta U^{temp} = \underbrace{-\sum_j (v_j - \gamma_j)\ln\left(\frac{\Delta\gamma_j}{v_j} + 1\right)}_{\substack{\text{diffusion term}\\ \text{as in productivity shocks}}} \text{ periods } t \geq 1$$

$$\underbrace{\Delta\sum \gamma_i \ln\gamma_i + \sum \gamma_i' \ln\left(\frac{\Delta\gamma_i}{v_i} + 1\right)}_{\substack{\text{Direct impact trough revenues}\\ \text{specific to preferences shocks}}} \text{ period } t = \hat{t}$$

so we can see that the cumulative impact is the sum of two terms: one, labeled *diffusion term* above, is the analogous of productivity shocks: the variation in prices creates a chain reaction that affects all reached sectors with the appropriate lag. More interesting is the impact at period $\hat{t}$: this term is due to the adjustment of prices due to produced quantities being pre-determined. This is specific to the preference shock case: the price adjustment in the case of productivity shocks do not impact welfare.

Moreover, again as with productivity shocks, we see that once we know consumption the global impact does not depend on the network anymore. However, if we decompose the impact into a short and a long run impact, we see that centrality is an important modulation factor. Interestingly, the effect of centrality here is the reverse than with productivity shocks: very central nodes will have a small variation in prices, hence have a small utility impact, while nodes with a very low centrality will have a high impact because their prices will be more volatile.

4 Time to recovery

One issue of great practical importance about the impact of shocks is how much time does the economy take to absorb it and reach a new steady state (possibly identical to the one it started from). This is what in the following I call *recovery* or *convergence* time. It is a quantity of great interest to policy makers or stakeholders interested in predicting economic variables. One technical issue is that in a smooth equilibrium model as the one I am analyzing, shocks never *totally* die out. Hence we define recovery time as the time the economy takes to arrive ε-close to the steady state, as made precise in the following definition.

Definition 4.1 (Recovery time)**.** Given a shock to node k of magnitude $\Delta \ln A_k = 1$ and a bound ε, define the *time to recovery* of node i as the smallest time after which the output of node i differs less than ε from the new steady state. In formulas:

$$CT_{ki}(\varepsilon) = \min\{t : |\ln y_i^{t'} - \ln y_i^{SS}| < \varepsilon, \forall t' \geq t\}$$

Consider a shock, possibly to multiple sectors, satisfying the normalization $\|\ln A\|_2 = 1$. A global convergence time, independent of source and end node is:

$$CT_2(\varepsilon) = \min\{t : \| \ln y^{t'} - \ln y^{SS}\|_2 < \varepsilon, \forall t' \geq t\}$$

The particularly simple dynamics of the model allows to analyze the recovery time in detail. Indeed, since Ω is row stochastic, it can be seen as the transition matrix of a Markov chain, and we can apply the rich theory of mixing times of Markov chains to the task of bounding the recovery time. In order to do this, I maintain throughout the section two assumptions:

Strongly connected network Assume that the production network is strongly connected, meaning that for every pair of nodes i and j there exist a directed path $i_1, \ldots, i_k$ such that $i_1 = i$ and $i_k = j$.

Aperiodic network The minimum common denominator of the length of all cycles is 1.

The first assumption assures that the network cannot be split into separate classes that do not influence each other. If there is a group of sectors that sell output only to themselves a shock hitting one of them (directly or following diffusion) can be analyzed inside the group as a shock on a reduced production network formed just by those sectors, so this is without loss of generality. In particular, rules out sectors that sell only to consumers (i.e.

they are not connected to other sectors in the production network). If such a sector exist, a shock to it would just impact consumers and its effect would disappear at the next time period (remember that labor supply is inelastic), so it represents a rather non interesting case.

The second assumption assures that the diffusion of shocks does not feature cycles in such a way that the performance of nodes follows a

In the following, I present two simple results that provide bounds on the convergence time: the first is a global bound that has the advantage of using rather few assumptions, while the second is a sector specific bound, but has a limitation with respect to the first: it requires the Ω matrix to be *reversible*.

Eigenvector centrality The stochastic process for the difference of output from the steady state is defined by the iteration of a Markov chain: $\Delta y^t = \alpha^t \Omega^t \Delta \ln A_0$. Since we assume the matrix to be irreducible and aperiodic the convergence theorem guarantees that the chain converges to the Perron projection of the matrix Ω, that is the matrix with on the rows the leading eigenvalue, which is also the stationary distribution. In formulas:

$$\Omega^t_{ki} \to \pi_i$$

where π is the vector such that $\pi'\Omega = \pi'$. Since here Ω defines a network, π is also the (left) *eigenvector centrality*. The previous discussion shows that in the context of this model eigenvector centrality has the additional interpretation of representing the flow of revenues from nodes that are very far in the production network. Another interpretation can be the vector of revenues that results in the limit as α goes to 1, and so the importance of firms is given by purely network effects.

We note an interesting fact: the time of convergence is connected to eigenvector centrality, while the cumulative impact is connected to Bonacich centrality. These two measures are usually very correlated, but in this context there is an important difference: eigenvector centrality depends only on the *technology* parameters, while Bonacich centrality crucially depends also on the preference parameters of the consumer.

Reversibility An assumption that will be needed for some result in the following is *reversibility*. Consider Ω and π as above. Define $\Omega^*_{ij} = \Omega_{ji}\frac{\pi_j}{\pi_i}$, the *reversibilization* of Ω. A chain is called reversible if $\Omega^* = \Omega$. These concepts are well known in the literature on Markov chains.[9] In this context,

[9] In the context of Markov chains, this matrix represents the chain that would result if time would go from the future to the past: i.e., the probability of observing first i and then j is the same as the probability of observing first j and then i.

following the interpretation of eigenvector centrality as the vector of revenues in an economy where the share of labor goes to zero, reversibility asks that in the same limit the flow of funds from sector i to j be the same as the flow from j to i. This is a rather strong assumption in our context, as it assumes that each link is reciprocal (Ω_{ij} and Ω_{ji} have to be both positive at the same time). Unfortunately the sector-specific result relies on this assumption. That is the reason why I present results for global bounds, which are weaker but do not require reversibility.

4.1 Global bound

Adapting corollary 2.14 of Montenegro et al. (2006), we get:

Proposition 4.1. *Assume Ω is strongly connected and aperiodic. Then:*

$$CT_2(\varepsilon) \leq \max\left\{ \left\lceil \frac{1}{1-\|\Omega^*\|} \ln \frac{1}{\varepsilon \min_i \sqrt{\pi_i}} \right\rceil , \left\lceil \frac{\ln \frac{\varepsilon+1}{\varepsilon}}{\ln 1/\alpha} \right\rceil \right\}$$

where $\Omega^*_{ij} = \Omega_{ji}\frac{\pi_j}{\pi_i}$ is the reversibilization *of Ω.*

The threshold is:

1. decreasing in ε;
2. increasing (weakly) in α;
3. increasing (weakly) in $\|\Omega^*_{ij}\|$.

Property 1 is trivial. The second tells us that the more intermediate inputs are important, the longer the recovery time. This is because firms will rely more on produced goods, which are affected by the shock and its propagation, rather than labor (which is not affected by the shock). The third is harder to interpret in general. If Ω is reversible, though, it can be shown that $\|\Omega^*_{ij}\| = \lambda_2$, the second largest eigenvalue of Ω. Under some specific network formation models, in which nodes are partitioned in groups identified by some exogenous characteristics, this has been shown to represent a measure of *homophily*, the tendence of nodes of different groups to be connected together (Golub and Jackson (2012)) or, equivalently, a measure of how strong is the community structure of the network. This is out of this model, but a very interesting empirical as well as theoretical question: is there a community structure in the sectors of an economy? this could happen if for example more productive sectors tended to be comparatively have more exchanges among themselves than with others.

4.2 Sector-specific bound

Next, we look for bounds on the convergence time that are sector dependent, to answer the question: which sectors recover first in case of a disruption? which recover later? Unfortunately, since our transition matrix is only sub-stochastic, we can only obtain an upper bound on the convergence time, as the following proposition shows.

Proposition 4.2 (Sector specific convergence time)**.** *Assume* Ω *is* reversible*, aperiodic and irreducible. Then:*

$$CT_{ki}(\varepsilon) \leq \max\left\{\left\lceil \frac{\ln\left(\sqrt{\frac{\pi_i}{\varepsilon\pi_k}}\right)}{\ln 1/\lambda_2}\right\rceil, \left\lceil \frac{\ln\frac{\varepsilon+\pi_k}{\varepsilon}}{\ln 1/\alpha}\right\rceil\right\}$$

where λ_2 *is the second largest eigenvalue in absolute value of* Ω *and* π_i *is the eigenvector centrality of node* i*.*

The threshold is:

1. increasing (weakly) in λ_2;
2. (in general) u shaped in π_k;
3. decreasing (weakly) in π_i

The first property is just the adaptation of the result of the previous section to the reversible case, as explained before. The following are more interesting: they suggest that a shock to a more eigenvector central node will take more time to be absorbed, but more central nodes will go back to steady state quicklier than others. To quantify the heterogeneity across nodes, note that variation in eigenvector centrality yields a difference in (the upper bound on) convergence time that is proportional to the second eigenvalue/spectral gap: fix the centrality of the source, if the centrality of the objective is doubled, $\pi'_k = 2\pi_k$, then the convergence time is increased by $CT'_{ki} - CT_{ki} = 1/2 \ln 2/\ln(1/\lambda_2)$, which can be arbitrarily high if λ_2 is close to 1, or very small if λ_2 is far from 1.

5 Long run properties

In this section, I assume the (log) productivity shocks are i.i.d. and have mean 0. Then, because of the dynamics described above, (log) sectoral output follows a VAR(1), and analyze the stationary, or long run, properties of

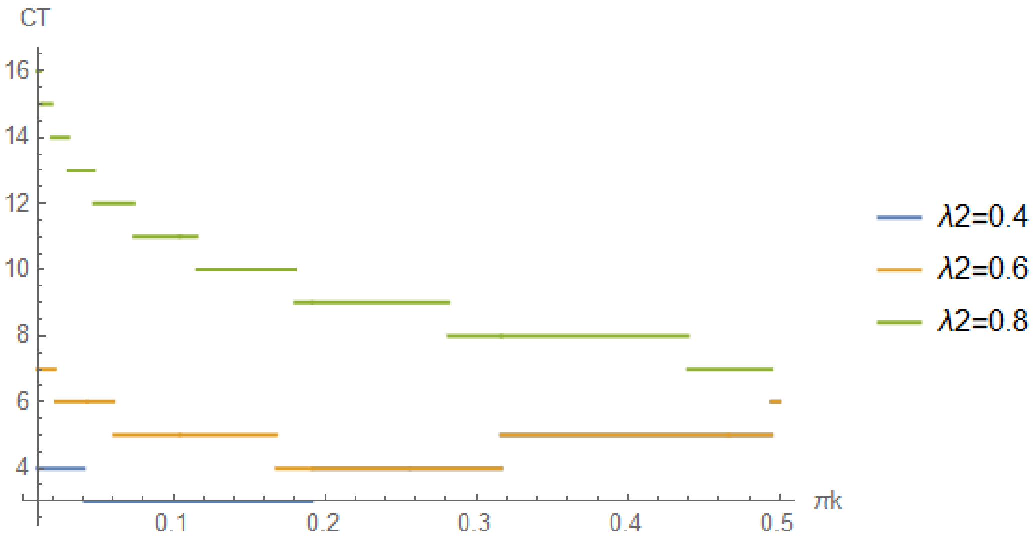


Figure 1: The sector-specific upper bound on convergence time as a function of centrality of source node π_k, for fixed centrality of end node π_i.

the output process, comparing with the static benchmark. To perform this analysis, we need the technical assumption that time starts at $-\infty$. This is not possible in the model analyzed, so we interpret this as a synonym of saying that we analyze the long run behavior of the model.

The aim is to show that the dynamic model generates sistematically less comovement. Foerster et al. (2011) show quantitatively that the Long Jr and Plosser (1983) generates less aggregate volatility than Acemoglu et al. (2012): here I show a stronger result, namely that Long Jr and Plosser (1983) generates less covariance among all the nodes. Let us first see an extreme example.

5.1 Example

If

$$\Omega = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

we have a cycle network. For any cycle network (indeed, any network whose adjacency matrix is orthogonal), the covariance in the dynamic model is a multiple of identity, while in the static model: in a connected network all nodes are correlated.

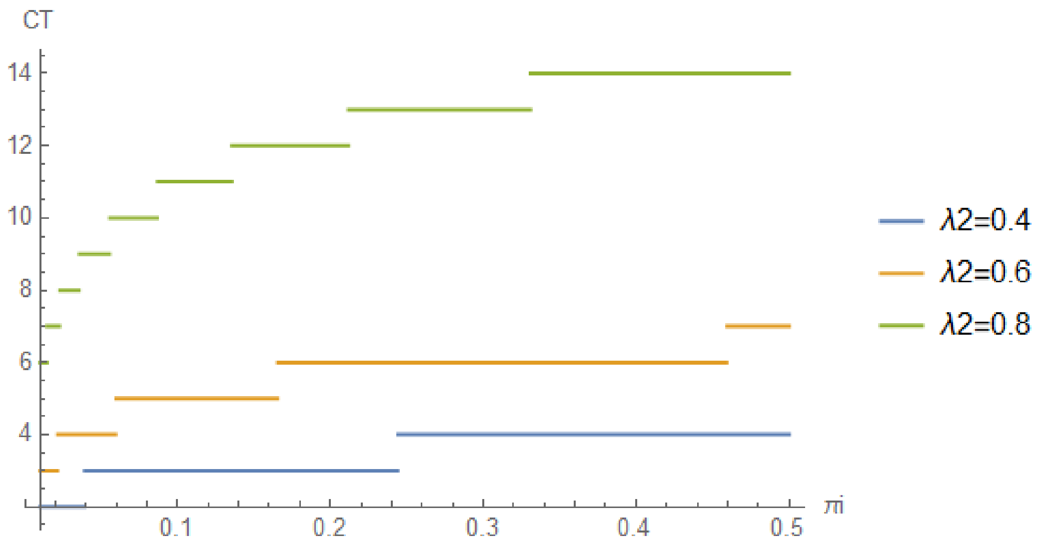


Figure 2: The sector-specific upper bound on convergence time as a function of centrality of end node π_i, for fixed centrality of source node π_k.

Static **Dynamic**

$$\begin{pmatrix} 1 & \frac{\alpha(1+\alpha+\alpha^2)}{1+\alpha^2+\alpha^4} & \frac{\alpha(1+\alpha+\alpha^2)}{1+\alpha^2+\alpha^4} \\ \frac{\alpha(1+\alpha+\alpha^2)}{1+\alpha^2+\alpha^4} & 1 & \frac{\alpha(1+\alpha+\alpha^2)}{1+\alpha^2+\alpha^4} \\ \frac{\alpha(1+\alpha+\alpha^2)}{1+\alpha^2+\alpha^4} & \frac{\alpha(1+\alpha+\alpha^2)}{1+\alpha^2+\alpha^4} & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The reason is best understood by looking at the elementwise expressions:

$$\mathrm{Cov}_{stat}(i,j) = \sum_k m_{ik} m_{jk}$$

so this covariance between sectors i, j is high when there are sectors k that are very (out)-connected to both i and j, and other sectors that are less connected (there needs to be asymmetry).

The dynamic covariance instead:

$$\mathrm{Cov}_{dyn}(i,j) = \sum_n \alpha^{2n} \sum_k \omega_{ik}^{(n)} \omega_{jk}^{(n)}$$

is high if there are sectors that are connected to both i and j, at *the same distance*. Otherwise shocks to k diffuse in the network but hit i and j at different times, causing less covariation.

The following proposition states the general result, proven in the Appendix.

Proposition 5.1. *Assume the* $\ln A_t$ *are a white noise process with covariance matrix* I*. Then for each* i *and* j $\mathrm{Cov}_{dyn}(i,j) \leq \mathrm{Cov}_{stat}(i,j)$*.*

Conclusion

In conclusion, if we model shocks as truly stochastic events and look at a model where they gradually spread across the network, different mechanisms for diffusion are at play for preference shocks, while the diffusion of productivity shocks follows the same principles. Moreover, it is possible to derive upper bounds on the recovery time of the economy after a shock, and these apply to both productivity and preference shocks, in both the temporary and permanent case.

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Appendix

A Proof of Proposition 5.1

Assume the autocovariance function of $\ln A$ is $\Gamma^k_{\ln A}$, and its autocovariance generating function is $G(z) = \sum \Gamma^k_{\ln A} z^k$. Assume that $\Gamma^k_{\ln A}$ is absolutely summable (it is in our case, as the autocovariance generating function of a white noise is $G(z) = I$).

$\ln y_t$ is a filter of $\ln A_t$ by the filter $(I-\alpha L)^{-1}$, where L is the lag operator. Hence its autocovariance generating function is (Hamilton 10.3):

$$F(z) = (I - \alpha z\Omega)^{-1} G(z)(I - \alpha z^{-1}\Omega')^{-1}$$

From here, we can recover the autocovariance by integrating the spectrum: $\Gamma^k_{\ln y} = \int_{-\pi}^{\pi} F(e^{i\omega})\mathrm{d}\omega$ (recall that $F(e^{i\omega})$ is the spectrum of $\ln y$).

By expanding the three series and taking the Cauchy product (all are absolutely summable) we get:

$$F_{ij}(z) = \sum_n^\infty \sum_k \sum_m \sum_h^n \left(\sum_l^h \alpha^l \omega_{ik}^{(l)} z^l \alpha^{h-l} \omega_{jm}^{(h-l)} z^{l-h} \right) G_{km}^{n-h} z^{n-h}$$

$$F_{ij}(e^{i\omega}) = \sum_n^\infty \sum_k \sum_m \sum_h^n \alpha^h \left(\sum_l^h \omega_{ik}^{(l)} \omega_{jm}^{(h-l)} \right) G_{km}^{n-h} e^{i\omega(2(l-h)+n)}$$

Now since we assume that the autocovariance is absolutely summable then the series is finite and each partial sum is dominated by the total sum, so by the dominated convergence theorem we can exchange series and integral and then we are left with a combination of integrals of $e^{i\omega(2(l-h)+n)}$. These integrals are all zero (integrals of trigonometric functions over multiples of the domain) except the ones for $2(l-h)+n=0$, or $l = h - n/2$, n even, $h \geq n/2$. Hence we can write:

$$\mathrm{Cov}_{dyn}(i,j) = \Gamma^0_{\ln y} = \int_{-\pi}^{\pi} F(e^{i\omega})\mathrm{d}\omega =$$

$$\sum_n^\infty \sum_k \sum_m \sum_h^n \alpha^h \left(\sum_l^h \omega_{ik}^{(l)} \omega_{jm}^{(h-l)} \right) G_{km}^{n-h} \int_{-\pi}^{\pi} e^{i\omega(2(l-h)+n)}\mathrm{d}\omega$$

$$\sum_{n \text{ even}}^\infty \sum_k \sum_m \sum_{h=n/2}^n \alpha^h \left(\omega_{ik}^{(h-n/2)} \omega_{jm}^{(n/2)} \right) G_{km}^{n-h}$$

now we can redefine n as $n/2$, and h as $h - n/2$ to get:

$$\mathrm{Cov}_{dyn}(i,j) = \sum_n^\infty \sum_k \sum_m \sum_h^n \alpha^{h+n} \left(\omega_{ik}^{(h)} \omega_{jm}^{(n)} \right) G_{km}^{n-h}$$

Now we use the assumption that $\ln A$ is a $WN(I)$. In this case the autocovariance function satisfies $G^k = 0$ for any $k \neq 0$, $G^0 = I$. Then the only term surviving in the expression above is the one with $h = n$, hence:

$$\mathrm{Cov}_{dyn}(i,j) = \sum_{n \text{ even}}^\infty \sum_k \alpha^n \left(\omega_{ik}^{(n/2)} \omega_{jk}^{(n/2)} \right)$$

Now compare this to:

$$\text{Cov}_{static}(i,j) = \sum_n^{\infty} \alpha^n \sum_k \sum_l^n \omega_{ik}^{(l)} \omega_{jk}^{(n-l)}$$

In the expression for the dynamic covariance all the terms are zero, moreover the even terms are $\omega_{ik}^{(n/2)} \omega_{jk}^{(n/2)}$ which is just one addend of the corresponding term in the static expression $\sum_l^n \omega_{ik}^{(l)} \omega_{jk}^{(n-l)}$. Hence, the dynamic is smaller for any i and j.

B Proof of Proposition 4.1

From standard techniques, see e.g. Montenegro et al. (2006) for any irreducible, aperiodic Markov chain with stationary distribution π and transition matrix Ω:

$$CT_2^{Markov}(\varepsilon) \leq \left\lceil \frac{1}{1 - \|\Omega^*\|} \ln \frac{1}{\varepsilon \min_i \sqrt{\pi_i}} \right\rceil$$

These result applied to our setting yield:

$$CT_2^{Markov}(\varepsilon) \leq \left\lceil \frac{1}{1 - \|\Omega^*\|} \ln \frac{1}{\varepsilon \min_i \sqrt{\pi_i}} \right\rceil$$

here π is the eigenvector centrality.

So, if $t > CT^{Markov}(\varepsilon)$, then:

$$\alpha^t \omega_{ki}^t \leq \alpha^t (\pi_k + \varepsilon)$$

In turn, $\alpha^t(\pi_k + \varepsilon)$ is smaller than ε if and only if $t > \ln \frac{\varepsilon}{\varepsilon + \pi_k} / \ln \alpha$, so that:

$$CT(\varepsilon) = \max\{ \left\lceil \frac{\ln\left(\varepsilon \sqrt{\frac{\pi_k}{\pi_i}}\right)}{\ln \lambda_2} \right\rceil, \left\lceil \ln \frac{\varepsilon}{\varepsilon + \pi_k} / \ln \alpha \right\rceil \}$$

C Proof of Proposition 4.2

From Levin and Peres (2017) for any irreducible, aperiodic, reversible Markov chain :

$$CT_{ki}^{Markov}(\varepsilon) \leq \left\lceil \frac{\ln\left(\varepsilon \sqrt{\frac{\pi_k}{\pi_i}}\right)}{\ln \lambda_2} \right\rceil$$

These results applied to our setting yield:

$$CT_{ki}^{Markov}(\varepsilon) \leq \left\lceil \frac{\ln\left(\varepsilon\sqrt{\frac{\pi_k}{\pi_i}}\right)}{\ln \lambda_2} \right\rceil$$

So, if $t > CT_{ki}^{Markov}(\varepsilon)$, then, repeating the reasoning above we get:

$$CT_{ki}(\varepsilon) = \max\{ \left\lceil \frac{\ln\left(\varepsilon\sqrt{\frac{\pi_k}{\pi_i}}\right)}{\ln \lambda_2} \right\rceil, \left\lceil \ln \frac{\varepsilon}{\varepsilon + \pi_k} / \ln \alpha \right\rceil \}$$

Online Appendix

D Heterogeneous primary factor share

The bounds in the previous sections are likely to be strongly driven by α. For this reason in this section we analyze the time of recovery in the case in which the primary factor shares are heterogeneous. The precise meaning of which is the following.

Model with heterogeneous primary factor shares By the model with *heterogeneous primary factor* share, I mean the same model used until now, with one modification, that is the technology is defined as:

$$y_{i,t+1} = A_{i,t+1} \prod_{j=1}^{N} (z_{ij,t})^{\omega_{ij}} l_{i,t}^{1-\alpha_i}$$

where $\sum_j \omega_{ij} = \alpha_i$. That is, the primary factors (labor) in the model are heterogeneous. All expressions and dynamics derived in the special case extend to this case, with the only modification that the matrix $\alpha\Omega$ is replaced by Ω. All the same proofs and propositions go through with obvious modifications. I just report the dynamics of the shock, since is our current object of interest:

$$\ln y_{i,t+1} = \ln A_{i,t+1} + const + \sum_j \omega_{ij} \ln y_{j,t}$$

which implies a deviation from the steady state of:

$$\Delta \ln y_{i,t+1} = \sum_j \omega_{ij} \Delta \ln y_{j,t}$$

for periods following a productivity shock. Hence, the dynamics has a structure very similar to the one studied until now. In this section we exploit the fact that, if Ω is substochastic, nonnegative and irreducible then its eigenvalue maximum in absolute value is real, simple and has a positive left eigenvector (called Perron vector), $\lambda_1 \pi^L = \pi^L \Omega$. Then:

$$P = \lambda_1^{-1} D^{-1} \Omega' D$$

is row stochastic, where $D = diag(\pi_i^L)$ (see proof of the proposition). In this section we differentiate the left and right Perron vectors π^L and π^R because they are different and we will need them both.

Assumption: generalized reversibility The role of reversibility in the substochastic case is played by the condition $\pi_i^R \Omega_{ji} \pi_j^L = \pi_j^R \pi_i^L \Omega_{ij}$. I will call a matrix Ω reversible if it satisfies it. I could derive a global bound without assuming it (see appendix), but since I think the sector-specific bound is more interesting I show the sector specific bound here.

Proposition D.1. *Assume Ω is aperiodic, strongly connected and reversible. Then:*

$$CT_{ki}(\varepsilon) \leq \max \left\{ \left\lceil \frac{\ln \left(\frac{1}{\varepsilon} \sqrt{\frac{\pi_i^R \pi_i^L}{\pi_k^R \pi_k^L}} \right)}{\ln \lambda_1 / |\lambda_2|} \right\rceil, \left\lceil \frac{\ln \left(\frac{\pi_i^L}{\pi_k^L} \frac{\varepsilon + \pi_k^R \pi_k^L}{\varepsilon} \right)}{\ln 1/\lambda_1} \right\rceil \right\}$$

where λ_1 and λ_2 are respectively the first and second largest eigenvalues in absolute value of Ω, π^L and π^R are the left and right eigenvector centralities.

The intuitions are very similar to the homogeneous case. λ_1 plays the role of α, and is a measure of typical out-degree: it is a classical result that $\sum \alpha_k / N \leq \lambda_1 \leq \max_k \alpha_k$. $|\lambda_2|/\lambda_1$ is still a measure of community structure, normalized by the typical degree.

Centralities role is more complex here. In most terms, we could consider as the "relevant" centrality measure $\pi_i L \pi_i^R$, which ranks nodes according to the fact that they have both out and in-centralities high. This reasoning fails due to the term

E Proof of Proposition D.1

For P defined as in the text:

$$\sum_j P_{ij} = \lambda_1^{-1} \sum_j \Omega_{ji} \frac{\pi_j^L}{\pi_i^L} = \lambda_1^{-1} \frac{\lambda_1 \pi_i^L}{\pi_i^L} = 1$$

Note that the eigenvalues of P are those of Ω divided by λ_1. Our goal is to find t such that $\Omega^t_{ki} < \varepsilon$. But $\Omega^t = \lambda_1^t D P^t D^{-1}$, and the invariant distribution of P is $\pi^R_i \pi^L_i$:

$$\sum_i \pi^R_i \pi^L_i P_{ij} = \lambda_1^{-1} \sum_i \pi^R_i \pi^L_i \Omega_{ji} \frac{\pi^L_j}{\pi^L_i} = \lambda_1^{-1} \sum_i \pi^R_i \Omega_{ji} \pi^L_j = \lambda_1^{-1} \lambda_1 \pi^R_j \pi^L_j = \pi^R_j \pi^L_j$$

Moreover, P is irreducible and aperiodic if and only if Ω is irreducible and aperiodic, since the powers of Ω are positive whenever the ones of P are. Hence $P^t \to \pi^R (\pi^L)^\intercal$.

To proceed further we need P to be reversible, that is:

$$\pi^R_i \pi^L_i \Omega_{ji} \frac{\pi^L_j}{\pi^L_i} = \pi^R_j \pi^L_j \Omega_{ij} \frac{\pi^L_i}{\pi^L_j} \iff \pi^R_i \Omega_{ji} \pi^L_j = \pi^R_j \pi^L_i \Omega_{ij}$$

which is our assumption. So, reasoning as in Proposition 4.2 we get that for t high enough $P^t_{ik} < \pi^R_k \pi^L_k + \varepsilon$. So if $\lambda_1^t \frac{\pi^L_k}{\pi^L_i} (\pi^R_k \pi^L_k + \varepsilon) < \varepsilon$ then

$$\Omega^t_{ki} = \lambda_1^t \frac{\pi^L_k}{\pi^L_i} P^t_{ik} < \lambda_1^t \frac{\pi^L_k}{\pi^L_i} (\pi^R_k \pi^L_k + \varepsilon) < \varepsilon$$

so get:

$$CT_{ki}(\varepsilon) \leq \max \left\{ \left\lceil \frac{\ln \left(\frac{1}{\varepsilon} \sqrt{\frac{\pi^R_i \pi^L_i}{\pi^R_k \pi^L_k}} \right)}{\ln \lambda_1 / |\lambda_2|} \right\rceil , \left\lceil \frac{\ln \left(\frac{\pi^L_i}{\pi^L_k} \frac{\varepsilon + \pi^R_k \pi^L_k}{\varepsilon} \right)}{\ln 1/\lambda_1} \right\rceil \right\}$$

Proof of Proposition ??

To avoid clutter, I omit the explicit dependence on the history h^t. All variables are to be intended history - dependent.

The firms problems' are essentially static. FOCs:

$$z^t_{ji} : \quad \alpha \omega_{ij} \sum_{h^{t+1}|h^t} p^{t+1}_i y^{t+1}_i = p^t_j z^t_{ji}$$

$$l^t_i : \quad (1-\alpha) \sum_{h^{t+1}|h^t} p^{t+1}_i y^{t+1}_i = w^t l^t_i$$

The transversality is not needed for firms, because it's a sequence of static problems. (Or, equivalently said, the transversality is trivially satisfied.)

Consumer FOCs are:

$$\beta^t \pi(h^t)\gamma_i = \lambda p_i^t c_i^t$$

summing over goods and histories and using the budget constraint we get:

$$\frac{1}{1-\beta} = \lambda \left[\sum_t w^t + \sum_i p_i^0 \omega_i + \sum_{h^t} f_{h^t} \right]$$

Let us write, for brevity, W for $\sum_t w^t$, E for $\sum_i p_i^0 \omega_i$ and f for $\sum_{h^t} f_{h^t}$. From the expression above since in the homogeneous case $U = \lambda(W + E + f)$, we get that in equilibrium $U = \frac{1}{1-\beta}$. Then, substitute the multiplier above into the demand:

$$c_i^t = \beta^t (1-\beta) \frac{\gamma_i}{p_i^t} (W + E + f)\pi(h^t)$$

Goods market clearing yields (for $t > 0$):

$$\alpha \sum_{h^{t+1}|h^t} \sum_{j \in N_i^{in}} \omega_{ji} s_j^{t+1} + \beta^t (1-\beta)\gamma_i (W + E)\pi(h^t) = s_i^t \tag{7}$$

Now, consider goods market clearing. We denote for simplicity the revenues $p_{i,t} y_{i,t}$ as $s_{i,t}$.

$$s_i^t = \pi(h^t)\beta^t(1-\beta)(W + E + f)\gamma_i^t + \alpha \sum_{h^{t+1}|h^t} \sum_{j \in N_i^{in}} \omega_{ji} s_j^{t+1}$$

Normalize by the fraction of wealth allocated to time t and call $d_i^t = \frac{s_i^t}{\beta^t \pi(h^t)(1-\beta)(W+E+f)}$ the "per-period" (revenue based) Domar weight:

$$d_i^t = \frac{s_i^t}{\beta^t \pi(h^t)(1-\beta)(W + E + f)} = \gamma_i + \alpha \sum_{h^{t+1}|h^t} \sum_{j \in N_i^{in}} \omega_{ji} \frac{\beta^{t+1}\pi(h^{t+1})(1-\beta)(W + E + f)}{\beta^t \pi(h^t)(1-\beta)(W + E + Pro)}$$

$$\times \frac{s_j^{t+1}}{\beta^{t+1}\pi(h^{t+1})(1-\beta)(W + E + f)}$$

$$= \gamma_i + \alpha\beta \sum_{h^{t+1}|h^t} \sum_{j \in N_i^{in}} \omega_{ji} \pi(h^{t+1}|h^t) d_j^{t+1}$$

So d follows this difference equation:

$$d_i^t = \gamma_i + \alpha\beta \mathbb{E} \left[\sum_{j \in N_i^{in}} \omega_{ji} d_j^{t+1} | h^t \right]$$

iterating forward, since the expectation is bounded (because the state space is finite) and all eigenvalues of Ω are smaller than 1 we obtain that:

$$d^t = (I - \alpha\beta\Omega')^{-1}\gamma = d^*$$

are constant over time.

E.0.1 Wage

The amount spent by consumer each period is $\pi(h^t)\beta^t(1-\beta)$, normalizing total wealth $(W+E+f)=1$. This, by market clearing, has to come from the wage and the profit of the firms.

$$\pi(h^t)\beta^t(1-\beta) = \sum_i \pi_{i,t} + w(h^t) = \sum_i \left(p_{i,t}y_{i,t} - \sum_j p_{j,t}z_{ji,t} - w(h^t)l_{i,t} \right) + w(h^t)$$

note that profits are not zero because of two reasons: these are the *realized*, not expected, profits, and moreover the profit is computed using contemporaneous values of sales and purchases. These are not related by optimization, since purchases at time t will generate revenues next period, hence there is no reason to think that profits will be zero.

Note also that:

$$\pi(h^t)\beta^t(1-\beta) = \sum_i \left(p_{i,t}y_{i,t} - \sum_j p_{j,t}z_{ji,t} \right)$$

so first we see that wage payments and earnings (of course) cancel out. Hence the consumer expenditure comes from the value added on intermediate inputs.

Moreover, using the FOCs:

$$\pi(h^t)\beta^t(1-\beta) = \pi(h^t)\beta^t(1-\beta) \sum_i \left(d_i - \alpha \sum_j \omega_{ij}\beta d_i - (1-\alpha)\beta d_i \right) + w(h^t)$$

$$\pi(h^t)\beta^t(1-\beta) = \pi(h^t)\beta^t(1-\beta) \sum_i d_i \left(1-\beta\right) + w(h^t)$$

hence:

$$w(h^t) = \pi(h^t)\beta^t(1-\beta)\frac{\beta(1-\alpha)}{1-\alpha\beta}$$

and so:

$$f_{i,t} = \pi(h^t)\beta^t(1-\beta)(1-\beta)d_i$$

where we can see that the profit is positive because of the intertemporal dimension: the value of purchases equals the discounted value of revenues tomorrow, which, being discounted, is smaller than the revenues accrued today, even if Cobb-Douglas technology forces everything else to be constant. Moreover, as expected, the profit of each firm is proportional to its dimension, measured by revenues.

Moreover, call $e = \frac{1-\alpha\beta}{\beta(1-\alpha)}$, we get that consumer expenses are: $GDP_{i,t} = \sum p_i c_i = w(h^t)e = \pi(h^t)\beta^t(1-\beta)$, total profit $Pro_t = w(h^t)(e-1)$, and $\sum p_i c_i = \frac{e}{e-1} Pro_t$, so that wage, profit, and GDP are all proportional.

In particular by the calculations above $d_i = \frac{p_{i,t}y_{i,t}}{\pi(h^t)\beta^t(1-\beta)} = \frac{p_{i,t}y_{i,t}}{GDP_t}$

E.0.2 Dynamics

Now, by FOCs input choices are:

$$z_{ji}^{t-1} = \alpha\omega_{ij}\frac{\sum_{h^{t+1}|h^t} d_i^* \beta^{t+1}\pi(h^{t+1})}{p_j^{t-1}} = \alpha\omega_{ij}\frac{d_i^*\beta^{t+1}\pi(h^t)}{p_j^{t-1}}e$$

and in the same way:

$$l_i^{t-1} = (1-\alpha)\frac{\sum_{h^{t+1}|h^t} d_i^* \beta^{t+1}\pi(h^{t+1})}{\beta^t\pi(h^t)} = (1-\alpha)d_i^*\frac{\beta\pi(h^t)}{\pi(h^t)} = (1-\alpha)\beta d_i^* e$$

So, output follows:

$$\ln y_i^{t+1} = \ln A^{t+1} + \sum_j \alpha\omega_{ij}\ln z_{ji}^t + (1-\alpha)\ln l_i^t =$$

$$\ln A^{t+1} + \sum_j \alpha\omega_{ij}\ln\alpha\omega_{ij} - \sum_j \alpha\omega_{ij}\ln p_j^t + \alpha\ln\beta^t\pi(h^t) + \ln\beta + \ln d_i^* + (1-\alpha)\ln(1-\alpha) + \ln e =$$

$$\ln A^{t+1} + \sum_j \alpha\omega_{ij}\ln\alpha\omega_{ij} + (1-\alpha)\ln(1-\alpha) - \sum_j \alpha\omega_{ij}\ln d_j^* + \sum_j \alpha\omega_{ij}\ln y_j^t + \ln\beta + \ln d_i^* + \ln e =$$

and finally:

$$\ln y_i^{t+1} = \ln A_i^{t+1} + C_i + \sum_j \alpha\omega_{ij}\ln y_j^t$$

where $C_i = C_i(\gamma, \alpha, \Omega, \beta) = c_i - \sum_j \alpha\omega_{ij}\ln(d_j^*) + \ln d_i^* + \ln e$, and $c_i = \ln\beta + (1-\alpha)\ln(1-\alpha) + \sum_j \alpha\omega_{ij}\ln(\alpha\omega_{ij})$. Iterating, we can get the relationships between any two outputs at different time periods.

F Proof of Proposition 3.1

We first need a lemma.

Lemma F.1. *Be $(a_k)_{k\in\mathbb{N}}$ a sequence of nonnegative real numbers, and be ρ and σ real numbers in the interval (0,1). If $\sum_{k=0}^{\infty} a_k$ converges, then:*

$$\sum_k^{\infty} \rho^k \sum_n^k \sigma^n a_{k-n} = \frac{1}{1-\rho\sigma}\sum_k^{\infty}\rho^k a_k$$

$$\sum_{k}^{\infty} \rho^k \sum_{n}^{k} \sigma^n a_n = \frac{1}{1-\rho} \sum_{k}^{\infty} \rho^k \sigma^k a_k$$

The flow utility of the consumer is, because of homotheticity of the utility:

$$U(c^t) = \sum_i \gamma_i \ln c_i^t = -\ln P^t + \ln w^t$$

hence:

$$U(c^t) = -\sum_i \gamma_i \ln p_i^t + \ln w^t = -\sum_i \gamma_i \ln \frac{p_i^t}{\beta^t \pi(h^t) e} + \ln\left(\frac{w^t}{\beta^t \pi(h^t) e}\right) = -\sum_i \gamma_i \ln \mathcal{P}_i^t$$

Then it is:

$$U^t = \alpha^t \sum_j g_{ji}^t \log \mathcal{P}_j^0 + \sum_{k=0}^{t-1} \alpha^k \sum_j \omega_{ij}^k \Big(-\log A_j^{t-k} - c_j \Big)$$

hence:

$$U = \sum_t \beta^t U^t = \sum_t \beta^t \left(\alpha^t \sum_j g_{ji}^t \log \mathcal{P}_j^0 + \sum_{k=0}^{t-1} \alpha^k \sum_j \omega_{ij}^k \Big(-\log A_j^{t-k} - c_j \Big) \right)$$

Consider a shock at 0. The impact on utility is:

$$\Delta U = \sum_{t \geq \hat{t}} \beta^t \Delta U_t = \sum_t \beta^t \sum_j \gamma_j (c_{j,t}^{\text{shock to } \hat{\imath}} - c_{j,t}) =$$

$$\sum_{t \geq \hat{t}} \beta^t \sum_j \gamma_j (\ln y_{j,t}^{\text{shock to } \hat{\imath}} - \ln y_{j,t}) = \sum_{t \geq \hat{t}} (\beta\alpha)^t \sum_j \omega_{j\hat{\imath}}^{(k)} \gamma_j \Delta \ln A_{\hat{\imath}}$$

$$= \beta^{\hat{t}} \vec{e}_{\hat{\imath}}' (I - \alpha\beta\Omega')^{-1} \gamma \Delta \ln A_{\hat{\imath}}$$

because $\ln c_{j,t}^{\text{shock to } \hat{\imath}} - \ln c_{j,t} = -(\ln p_{j,t}^{\text{shock to } \hat{\imath}} - \ln p_{j,t}) = \ln y_{j,t}^{\text{shock to } \hat{\imath}} - \ln y_{j,t}$. So

$$\frac{\Delta U / U}{\Delta \ln A_{\hat{\imath}}} = (1-\beta)\beta^{\hat{t}} v_{\hat{\imath}}(\alpha\beta) = \frac{p_{\hat{t},\hat{\imath}} y_{\hat{t},\hat{\imath}}}{U}$$

$$\lim_{\Delta \ln A_{\hat{\imath}} \to 0} \frac{\Delta \ln U}{\Delta \ln A_{\hat{\imath}}} = \lim_{\Delta \ln A_{\hat{\imath}} \to 0} \frac{\Delta U / U}{\Delta \ln A_{\hat{\imath}}} = (1-\beta) v_{\hat{\imath}}(\alpha\beta)$$

Consider now the case of a permanent shock

$$\Delta U=\sum_{t\geq\hat{t}}\beta^t\sum_j\gamma_j(\ln y_{j,t}^{\text{shock to }\hat{i}}-\ln y_{j,t})=\sum_{k=0}(\beta)^k\sum_j\sum_{h=0}^{k}\alpha^h g_{\hat{i}j}^{(h)}\gamma_j\Delta\ln A_{\hat{i}}=$$

$$\frac{1}{1-\beta}\sum_{h=0}(\alpha\beta)^h\omega_{j\hat{i}}^{(h)}\gamma_j\Delta\ln A_{\hat{i}}$$

using the lemma F.1.

Remember that utility U is exactly $\frac{1}{1-\beta}$. Hence we get another analogous to Hulten:

$$\frac{\Delta U/U}{\Delta\ln A_{\hat{i}}}=v_{\hat{i}}(\alpha\beta)=p_{\hat{t},\hat{i}}y_{\hat{t},\hat{i}}$$

hence, taking the limit:

$$\lim_{\Delta\ln A_{\hat{i}}\to 0}\frac{\Delta U/U}{\Delta\ln A_{\hat{i}}}=\lim_{\Delta\ln A_{\hat{i}}\to 0}\frac{\Delta\ln U}{\Delta\ln A_{\hat{i}}}=v_{\hat{i}}(\alpha\beta)$$

Proof of Proposition 2.2

Consumer demand is:

$$c_i^t=\beta^t(1-\beta)\frac{\gamma_{i,t}}{p_i^t}(W+E)\pi(h^t)$$

FOCs:

$$z_{ji}^t:\quad \alpha\omega_{ij}\sum_{h^{t+1}|h^t}p_i^{t+1}y_i^{t+1}=p_j^t z_{ji}^t$$

$$l_i^t:\quad (1-\alpha)\sum_{h^{t+1}|h^t}p_i^{t+1}y_i^{t+1}=w^t l_i^t$$

Hence market clearing:

$$p_i^t y_{i,t}=\beta^t(1-\beta)\gamma_{i,t}(W+E)\pi(h^t)+\alpha\sum_{h^{t+1}|h^t}\sum_j\omega_{ji}s_j^{t+1}$$

Normalize and get:

$$d_i^t=\frac{s_i^t}{\beta^t(1-\beta)\pi(h^t)(W+E)}=\gamma_{i,t}+\alpha\sum_{h^{t+1}|h^t}\sum_{j\in N_i^{in}}\omega_{ji}\frac{\beta^{t+1}\pi(h^{t+1})}{\beta^t\pi(h^t)}\frac{s_j^{t+1}}{\beta^{t+1}\pi(h^{t+1})(1-\beta)(W+E))}$$

$$=\gamma_{i,t}+\alpha\beta\sum_{h^{t+1}|h^t}\sum_j\omega_{ji}\pi(h^{t+1}|h^t)d_j^{t+1}$$

So d follows this difference equation:

$$d_i^t = \gamma_{i,t} + \alpha\beta\mathbb{E}\left[\sum_j \omega_{ji} d_j^{t+1} | h^t\right]$$

and iterating forward and passing to the limit we get:

$$d_i^t = \gamma_{i,t} + \sum_k \alpha^k \beta^k \sum_j \omega_{ji}^{(k)} \mathbb{E}\left[\gamma_j^{t+k} | h^t\right]$$

$$z_{ji,t} = \alpha\beta\omega_{ij} \frac{\mathbb{E}_t d_{i,t+1}}{d_{j,t}} y_{j,t}$$

The wage is

$$w_t = (1-\alpha)\beta\mathbb{E}_t \sum d_i^{t+1} = \frac{(1-\alpha)\beta}{1-\alpha\beta},$$

hence:

$$l_i^t = (1-\alpha)\beta \frac{\mathbb{E}_t d_{i,t+1}}{w_t} = (1-\alpha\beta)\mathbb{E}_t d_{i,t+1}$$

so profits are:

$$Pro_{i,t} = d_{i,t} - \beta\mathbb{E}_t d_{i,t+1}$$

General dynamics:

$$\log y_{i,t+1} = const + \alpha \sum_j \omega_{ij} \log y_{j,t} - \alpha \sum_j \omega_{ij} \log(\gamma_{j,t} + \sum_k \alpha^k \beta^k \sum_{h \in N_j^{in}} g_{jh}^{(k)} \mathbb{E}\left[\gamma_h^{t+k} | h^t\right]) +$$

$$\log(\sum_k \alpha^k \beta^k \sum_{j \in N_i^{in}} g_{ij}^{(k)} \mathbb{E}\left[\gamma_j^{t+1+k} | h^t\right])$$

or

$$\ln y_{i,t+1} = const + \alpha \sum_j \omega_{ij} \log y_{j,t} - \alpha \sum_j \omega_{ij} \ln d_{i,t} + \ln \mathbb{E}_t d_{i,t+1}$$

that for prices (in GDP units) yields:

$$\ln p_{i,t+1} = const + \alpha \sum_j \omega_{ij} \log p_{j,t} + \ln \mathbb{E}_t d_{i,t+1} - \ln d_{i,t+1}$$

F.0.1 i.i.d. case

If γs are i.i.d.:

$$d_i^t = \gamma_{i,t} + \sum_{k=1} \alpha^k \beta^k \sum_{j \in N_i^{in}} g_{ij}^{(k)} \mathbb{E}\left[\gamma\right] = \Delta\gamma_{i,t} + v_i(\alpha\beta, \overline{\gamma})$$

Then plug this into the FOCS:

$$z_{ji,t} = \alpha\omega_{ij} \frac{\sum_{h^{t+1}|h^t} p_i^{t+1} y_i^{t+1}}{p_{j,t}} = \alpha\beta\omega_{ij} \frac{v_i(\alpha\beta, \overline{\gamma})}{\Delta\gamma_{j,t} + v_j(\alpha\beta, \overline{\gamma})} y_{j,t}$$

The wage is the same.

$$l_i^t = (1-\alpha) \frac{\sum_{h^{t+1}|h^t} p_i^{t+1} y_i^{t+1}}{w_t} = (1-\alpha)\beta v_i(\alpha\beta, \overline{\gamma})$$

So the quantity dynamics becomes:

$$\log y_{i,t+1} = c_i + \ln e + \ln v_i + \alpha \sum_j \omega_{ij} \log y_{j,t} - \alpha \sum_j \omega_{ij} \log(\Delta\gamma_{j,t} + v_j)$$

where c_i is defined as above. This captures the idea that when a positive taste shock hits good j then its price increases and so firm i is able to buy less of it, so it will have a (relative) negative impact on its production.

Dynamics of prices:

$$\ln \mathcal{P}_{i,t+1} = \ln(\Delta\gamma_{i,t+1} + v_i) + \alpha \sum_j \omega_{ij} \ln \mathcal{P}_{j,t} - c_i - \ln e - \ln v_i$$

This captures the idea that current prices are directly affected by the realization of γ of the corresponding sector.

profits:

$$Pro_{i,t} = d_{i,t} - \beta \mathbb{E}_t d_{i,t+1} = \Delta\gamma_{i,t} + (1-\beta) v_i(\alpha\beta, \overline{\gamma})$$

G Proof of Proposition 3.2

The expected utility is:

$$U = \sum_{h^t} \beta^t \pi(h^t) \left(\sum_i \gamma_i^t \ln \gamma_i^t - \sum_i \gamma_i^t \ln \mathcal{P}_{i,t} + c_i \right)$$

Now assume to fix ideas that the only stochastic state is 1. All the others are fixed to the average γ. The utility once at 1 has been realized state γ' is:

$$U = \sum_{t>1} \beta^t \pi(h^t) \left(\sum_i \gamma_i \ln \gamma_i - \sum_i \gamma_i \ln \mathcal{P}_{i,t} + c_i \right) + \sum_i \gamma_i' \ln \gamma_i' - \sum_i \gamma_i' \ln \mathcal{P}_{i,1} + c_i$$

Transitory preference shock The impact of a realization γ' at time t is:

$$\Delta U = -\sum_{t>1} \beta^t \pi(h^t)\gamma_i \left(\ln \mathcal{P}'_{i,t} - \ln \mathcal{P}_{i,t}\right) + \sum_i \gamma'_i \ln \gamma'_i - \sum_i \gamma_i \ln \gamma_i - \sum_i \left(\gamma'_i \ln \mathcal{P}'_{i,1} - \gamma_i \ln \mathcal{P}_{i,1}\right)$$

Now:

$$\Delta \ln \mathcal{P}_{i,t} = \alpha^{t-1} \sum_j \omega_{ij} \Delta \ln \mathcal{P}_{j,1}$$

$$\ln \mathcal{P}'_{i,1} = \ln(\Delta\gamma_{i,1} + v_i) + \alpha \sum_j \omega_{ij} \mathcal{P}_{j,0} - c_i - \ln e - \ln v_i = \ln\left(\frac{\Delta\gamma_{i,1}}{v_i} + 1\right) - c_i - \ln e =$$

$$\ln\left(\frac{\Delta\gamma_{i,1}}{v_i} + 1\right) + \ln \mathcal{P}_{i,1}$$

so:

$$\Delta \ln \mathcal{P}_{i,1} = \ln\left(\frac{\Delta\gamma_{i,1}}{v_i} + 1\right)$$

hence:

$$\Delta U_{t>1} = -\sum_t \beta^t \Delta \ln \mathcal{P}_{i,t} = -\sum_t \gamma_i \alpha^{t-1} \beta^{t-1} \sum_j g^{(t)}_{ji} \Delta \ln \mathcal{P}_{j,1} = -\sum_j (v_j - \gamma_j) \ln\left(\frac{\Delta\gamma_j}{v_j} + 1\right)$$

$$\sim -\sum_j (v_j - \gamma_j) \frac{\Delta\gamma_j}{v_j} = -\sum_j \Delta\gamma_j - \sum_j \gamma_j \Delta\gamma_j = \sum_j \frac{\gamma_j \Delta\gamma_j}{v_j}$$

Instead the terms of the first period can be rewritten as:

$$\Delta \sum \gamma_i (\ln \gamma_i - \ln p_{i,1}) =$$

$$\Delta \sum \gamma_i \ln \gamma_i - \sum_i \left(\gamma'_i \ln \mathcal{P}'_{i,1} - \gamma_i \ln \mathcal{P}_{i,1}\right) = +\Delta \sum \gamma_i \ln \gamma_i - \sum \gamma'_i \Delta \mathcal{P}_i - \sum \Delta\gamma_i \ln \mathcal{P}_i$$

so:

$$\Delta U = -\sum_j (v_j - \gamma_j) \ln\left(\frac{\Delta\gamma_j}{v_j} + 1\right) - \sum_j \gamma_j \ln\left(\frac{\Delta\gamma_j}{v_j} + 1\right) +$$

$$\Delta \sum \gamma_i \ln \gamma_i - \sum \Delta\gamma_i \ln \mathcal{P}_i \sum \Delta\gamma_i \ln\left(\frac{\Delta\gamma_i}{v_i} + 1\right)$$

$$= -\sum_j v_j \ln\left(\frac{\Delta\gamma_j}{v_j} + 1\right) + \Delta \sum \gamma_i \ln \gamma_i - \sum \Delta\gamma_i \ln \mathcal{P}_i + \sum \Delta\gamma_i \ln\left(\frac{\Delta\gamma_i}{v_i} + 1\right)$$

Now note that:

$$\sum \Delta\gamma_i \ln\left(\frac{\Delta\gamma_i}{v_i} + 1\right) \sim \sum \Delta\gamma_i^2 / v_i$$

is second order in the size of the shock, and always at the first order

$$-\sum_j v_j \ln\left(\frac{\Delta\gamma_j}{v_j}+1\right) \sim -\sum_j \Delta\gamma_j = 0$$

So at the first order:

$$\Delta U = -\sum \Delta\gamma_i \ln \mathcal{P}_i + \Delta \sum \gamma_i \ln \gamma_i$$
$$= -\sum \Delta\gamma_i \ln \mathcal{P}_i + \sum \Delta\gamma_i \ln \gamma_i$$
$$\sum \Delta\gamma_i \ln c_i$$

Permanent preference shock Note that the revenues have a direct impact on the price dynamics only at the moment of the impact. Beyond that, things are equivalent to a productivity shock, of size modified according to centrality. So now:

$$\Delta U = \sum_t \beta^t \left(\Delta \sum \gamma_i \ln \gamma_i - \sum_i \left(\gamma_i' \ln \mathcal{P}_{i,t}' - \gamma_i \ln \mathcal{P}_{i,t}\right)\right)$$

The second part can be rewritten as before $\sum \gamma_i' \Delta\mathcal{P}_{i,t} + \sum \Delta\gamma_i \ln \mathcal{P}_{i,t}$, that is:

$$\sum \gamma_i' \alpha^t \sum_j \omega_{ij}^t \Delta\mathcal{P}_{j,0} + \sum \Delta\gamma_i \alpha^t \sum_j \omega_{ij}^t \ln \mathcal{P}_{j,0} + const$$

Apply lemma F.1 to this and get:

$$\frac{1}{1-\beta}\left(\sum_j v_j(\alpha\beta,\gamma')\Delta \ln \mathcal{P}_{j,0} + \sum_j v_j(\alpha\beta,\Delta\gamma) \ln \mathcal{P}_{j,0}\right) + const$$

so in total we get:

$$\Delta U = \frac{1}{1-\beta}\Delta \sum \gamma_i \ln \gamma_i - \frac{1}{1-\beta}\left(\sum_j v_j(\alpha\beta,\gamma')\Delta \ln \mathcal{P}_{j,0} + \sum_j v_j(\alpha\beta,\Delta\gamma) \ln \mathcal{P}_{j,0}\right)$$

$$= \frac{1}{1-\beta}\Delta \sum \gamma_i \ln \gamma_i - \frac{1}{1-\beta}\left(\sum_j v_j(\alpha\beta,\gamma') \ln\left(\frac{v_j(\alpha\beta,\Delta\gamma)}{v_j(\alpha\beta,\gamma)}+1\right) + \sum_j v_j(\alpha\beta,\Delta\gamma) \ln \mathcal{P}_{j,0}\right)$$

Also in this case, at the first order $\sum_j v_j(\alpha\beta,\gamma')\Delta \ln \mathcal{P}_{j,0} = \sum_j v_j(\alpha\beta,\gamma')(\ln v_i(\alpha\beta,\gamma') - \ln v_i(\alpha\beta,\gamma)) \sim \sum_j v_j(\alpha\beta,\gamma')\frac{v_j(\alpha\beta,\Delta\gamma)}{v_j(\alpha\beta,\gamma')} = 0$ (remember that the centrality is a linear combination of the preference parameters).