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Luca Anderlini and Gaon Kim

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University of Naples Federico II **University of Salerno** **Bocconi University, Milan**

CSEF - Centre for Studies in Economics and Finance
Department of Economics and Statistics – University of Naples Federico II
80126 Naples - Italy
Tel. and fax +39 081 675372 – e-mail: csef@unina.it
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Luca Anderlini[*] and Gaon Kim[†]

Abstract
We examine "tournament" second-price auctions in which N bidders compete for the right to participate in a second stage and contend against bidder N +1. When the first N bidders are committed so that their bids cannot be changed in the second stage, the analysis yields some unexpected results. The first N bidders consistently bid above their values in equilibrium. When bidder N + 1 is sufficiently stronger than the first N, overbidding leads to an increase in expected revenue in comparison to the standard second-price auction when N is large.

JEL Classification: C70, C72, C79.

Keywords: Tournament Auctions, Overbidding, Revenue Equivalence.

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[*] Georgetown University, University of Naples Federico II and CSEF. Email: luca@anderlini.net (Corr*esponding author*)

[†] EIEF and LUISS University.

1. Introduction

We examine "tournament" second-price auctions in which N bidders compete for the right to participate in a second stage and contend against bidder $N+1$.

When the first N bidders can submit fresh bids in the second stage this is equivalent to a second price auction. When the first N bidders are committed so that their bids cannot be changed in the second stage, they consistently bid *above* their values in equilibrium.

When all bidders are ex-ante identical overbidding results in a decrease in expected revenue for the auctioneer. If instead bidder $N+1$ is sufficiently stronger than the first N overbidding *increases* expected revenue relative to the standard second-price auction when N is large.

All proofs are relegated to an Appendix. In the numbering of equations, definitions, lemmas etc prefix of "A" indicates that the relevant item is to be found in the Appendix.

2. Related Literature

Even attempting a review of the literature on auctions in a short note would be foolish. Given that one of our central results concerns expected revenue citing Vickrey (1961) cannot be avoided. The rest we leave to a survey by Klemperer (1999), to the colossal edited collection of papers found in Klemperer (2000), and a favorite textbook by Krishna (2009).

3. Setup

3.1. General

A single indivisible object is for sale, and there are $N \geq 2$ first-stage bidders and 1 second-stage bidder. First-stage bidders have independent private values $v_i \overset{\text{iid}}{\sim} F$ with support $[0, \bar{v}]$ with $\bar{v} > 0$. The distribution F is absolutely continuous with density $f(v) > 0 \; \forall v \in [0, \bar{v}]$.

The second-stage bidder $N+1$ has value $w \sim G$ with support $[0, \bar{w}]$ with $\bar{w} \geq \bar{v}$. The distribution G is absolutely continuous with density $g(w) > 0 \; \forall w \in [0, \bar{w}]$. For simplicity $g(\cdot)$ is assumed to be differentiable with g' locally bounded at 0. The value w is independent of the values v_i of the first stage bidders.

Bids in both first and second stage are sealed. Bidder $N+1$ competes with the highest bidder of the first stage.[1]

[1]It seems more appealing to assume, as we do, that $N+1$ does not observe the actual first stage winning bid although this is completely irrelevant. All our results holds unchanged if $N+1$ observes the actual bid of the winner of the first stage.

Some significant details of the two-stage auction will change as we consider different cases. In all cases, the values of all $N+1$ bidders are private information to each of the participants and are independently drawn *once and for all* at the beginning of the game.

Throughout, by "equilibrium" we mean a Perfect Bayesian Equilibrium of the two-stage incomplete information game at hand. We also restrict attention to equilibria with bidding functions that are non-decreasing in values and identical across the N first-stage bidders.[2] In one instance[3] it will be necessary to consider explicitly an upper bound on bids, which will be denoted by $\bar{b} > \bar{v}$ for the first stage bidders.[4] In all other cases this is immaterial just as it is in virtually all existing auction models.

The (identical) bidding functions of the first-stage bidders are denoted interchangeably by $b_i(\cdot)$ or $b(\cdot)$ while the bidding function for bidder $N+1$ is denoted by $b_{N+1}(\cdot)$.

Finally, we refer to the case in which *all* $N+1$ bidders with the values as described above bid simultaneously in a second price auction as the standard one-shot case.

3.2. No Commitment

This is our point of departure. In the setup we just described we consider a two-stage auction as follows. At $T = 1$ the N first-stage bidders submit sealed bids. The highest bidder i^* wins and pays a price equal to second highest bid.

Bidder i^* wins the right compete against $N+1$ in the second stage of the auction held at $T = 2$. In the second stage auction i^* and $N+1$ submit sealed bids.[5] The highest bidder wins the object and pays a price equal to the second highest bid — the bid of the only other second-stage participant.

3.3. Commitment With Symmetric Bidders

If the first stage bidders are committed to their bids in the sense that they cannot be changed in the second stage the picture changes considerably and several details begin to matter.

The first case we consider is that of symmetric bidders in the sense that the value of $N+1$ has the same distribution as the first N bidders. So, $G = F$.

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[2] Any ties are broken randomly, with all tied bids winning with equal probability.

[3] The case of Commitment with Symmetric Bidders described in Subsection 3.3 below.

[4] A bound $\bar{b} = \bar{v}$ does not pose a problem. If indeed $\bar{b} = \bar{v}$ then, for obvious reasons, (5) of Proposition 3 below only holds for $v \in (0, \bar{v})$ instead of any $v \in (0, \bar{v}]$.

[5] Bidder i^* submits a fresh bid in the second stage.

The highest bidder of the first stage i^* goes on to compete with bidder $N+1$ in the second stage without the possibility of changing the bid. In the second stage the highest bidder wins and pays a price equal to the bid of the other bidder.

3.4. Commitment With Asymmetric Bidders

The auction procedure is exactly the same as in subsection 3.3. The first N bidders are ex-ante identical just as in the previous cases. In this case bidder $N+1$ is "stronger" in the sense that $E(w) > \bar{v}$.

4. Results

4.1. Equilibrium Behavior of Bidder $N+1$

It is clear from the description of our two-stage auction that from the point of view of bidder $N+1$ the set up always corresponds to a second price auction between $N+1$ and the first stage winner i^*.

By completely standard arguments we then know that "bidding his value" is a weakly dominant strategy for $N+1$. Hence, from now on we restrict attention to equilibria in which $N+1$ bids according to to this weakly dominant strategy. From now on we assume that in equilibrium $b_{N+1}(w) = w$.

4.2. Equilibrium in the No Commitment Case

Equilibrium for the model described in Subsection 3.2 is relatively straightforward to characterize.

Proposition 1. *Bidding Functions In The No Commitment Case: The following constitute an equilibrium in the case of no commitment.*

$$b_i^I(v_i) \;=\; \int_0^{v_i} (v_i - \xi) g(\xi)\, d\xi \quad \text{and} \quad b_i^{II}(v_i) = v_i, \tag{1}$$

where $b_i^I(\cdot)$ and $b_i^{II}(\cdot)$ are respectively the first and second stage (if he wins the first stage) bids for i. And for bidder $N+1$,

$$b_{N+1}(w) \;=\; w. \tag{2}$$

Remark 1. *Value of Second Stage Participation: By inspection of (1) it is clear that the N first stage bidders simply bid "truthfully" their value of participating in the second stage of the auction.*

Remark 2. *Revenue Equivalence with No Commitment: By inspection of (1) and (2) it is clear that the outcome of the two-stage auction without commitment is ex-post efficient. Therefore the Revenue Equivalence Theorem implies that the expected revenue in this case is the same as the expected revenue in the standard one-shot second price auction case.*

Proposition 2. *Expected Revenue in the Standard and No Commitment Case: Let $R^{NC}(N)$ be the expected revenue in the standard and in the no commitment case as a function of the number N of first stage bidders. Then*[6]

$$R^{NC}(N) \;=\; E[(v_1,...,v_N,w)_{(2:N+1)}] \tag{3}$$

When the values for all $N+1$ bidders are uniformly distributed on $[0,1]$, equation (3) becomes

$$R^{NC}(N) \;=\; \frac{N}{N+2}. \tag{4}$$

4.3. Equilibrium Overbidding With Commitment

The commitment case is substantially different from the no-commitment one and hence the model differs from the standard one-shot second price auction case.

Recall that in this case the first N bidders bid *only once*. The highest bidder among the first N, denoted i^* goes on to compete with $N+1$ without revising his bid. A second price contest with $N+1$ then decides who win the object and the price paid.

The two-stage structure of the auction with commitment generates increased competition in the first stage. The only possibility to achieve a positive payoff for the first-stage bidders is to win the first stage. This gives them an incentive to bid above their values in the first stage competition. Of course the possible downside of committing to an above-value bid is mitigated by the fact that, contingent on winning, the price they pay in the second stage is the second highest bid — namely the bid of $N+1$. Formally, we prove the following.

[6]Throughout, given a vector of random variables $x = (x_1, \ldots, x_L)$ of length L, we denote by $(x_1, ..., x_L)_{(q:L)}$ the q-th (descending) order statistic of x.

Proposition 3. *<u>Above Value Bidding with Commitment:</u> Consider distributions F and G as in Subsection 3.1 and the model with commitment and either symmetric or asymmetric bidders.*

Suppose that the (non-decreasing) bidding function $b(\cdot)$ for bidders $i = 1, ..., N$ induces an equilibrium in conjunction with $b_{N+1}(w) = w$. Then, for every $v \in (0, \bar{v}]$, we must have

$$b(v) \ > \ v \tag{5}$$

Proposition 3 establishes that the two-stage structure of the auction in the case of commitment generates bids that are above value for the first N bidders allowing for general distributions of values satisfying the conditions spelled out in Subsection 3. Further characterization of the bidding functions and of the expected revenue will differ in crucial respects in the two cases of symmetric and asymmetric bidders described above in Subsections 3.3 and 3.4 respectively.

Restricting attention to values that are uniformly distributed yields further worthwhile characterizations. When we say that values are uniformly distributed we mean that F is the uniform distribution on $[0, 1]$ and that G is the uniform distribution on $[0, \bar{w}]$ with $\bar{w} \geq 1$.

Before proceeding further, we pause to notice that whenever Proposition 3 holds the Revenue Equivalence Theorem cannot be invoked to pin down the expected revenue to the auctioneer.

Remark 3. *<u>Failure of Revenue Equivalence with Commitment:</u> Combining the fact that bidder $N + 1$ always bids his value truthfully with the overbidding highlighted in Proposition 3 it is clear that in all the cases we consider where there is commitment the outcome of the auction may be ex-post inefficient with the object failing to be allocated to the bidder with the highest value.*

In all these cases the Revenue Equivalence Theorem does not *pin down the expected revenue to the auctioneer to be the same as in the standard one-shot second-price auction as was the case for the no commitment model.*

Clearly, the failure of the Revenue Equivalence Theorem leaves open the possibility that the revenue with commitment may go up as well as down relative to the standard case.

4.4. Equilibrium with Commitment and Symmetric Bidders

We are now ready to characterize further the equilibrium of the model described in Subsection 3.3 above.

Proposition 4. *Commitment, Symmetric Bidders and Uniform Values: Consider the case of Commitment, Symmetric Bidders and Uniform Values. In other words assume that the values for all $N+1$ bidders are uniformly distributed on $[0,1]$. Bids are restricted to be in $[0,\bar{b}]$ with $\bar{b} > \bar{v} = \bar{w}$.*

Then there exists a $\hat{v}_N \in (1/2, (N+1)/2N)$ such that the following constitute an equilibrium.

$$b_i(v_i) \; = \; \begin{cases} \dfrac{2N}{N+1}\, v_i & \text{if} \;\; v_i \in [0, \hat{v}_N] \\ \\ \bar{b} & \text{if} \;\; v_i \in (\hat{v}_N, 1] \end{cases} \qquad \text{and} \quad b_{N+1}(w) \; = \; w \tag{6}$$

As expected, equation (6) confirms that in equilibrium the first N bidders bid above their values. Indeed from (6) we observe that they all bid $\bar{b}$ — the maximum allowed — whenever their value exceeds the cutoff value $\hat{v}_N$. While the cutoff value $\hat{v}_N$ depends on N, the bid always jumps to the same value $\bar{b}$ after the cutoff is exceeded.

As we noted in Subsection 4.1, bidder $N+1$ always bids his value in equilibrium. So in the case of symmetric bidders considered here, the bid of $N+1$ will never exceed 1. This in turn means that the first N bidders have "nothing to lose" in bidding arbitrarily high once their bid exceeds 1. Because the context against $N+1$ is a second price one, bidding any amount above 1 will not change the price paid. This coupled with the general overbidding feature of Proposition 3 generates the need for the explicit bidding cap that we use here.

Our next concern is the effect on the auctioneer's expected revenue of the equilibrium overbidding in this case. The expected revenue falls below the standard case and can be characterized as follows.

Proposition 5. *Revenue with Commitment and Symmetric Bidders: Let $R^{CS}(N)$ be the expected revenue for the case of commitment, symmetric bidders and uniformly distributed values on $[0,1]$. Then*

$$R^{CS}(N) \; < \; 1/2 \; \leq \; R^{NC}(N) \qquad \forall \, N \; \geq \; 2 \tag{7}$$

In the case of a general distribution of values $F = G$ our conclusion is limited to large values of N.

$$\lim_{N \to \infty} R^{CS}(N) \; < \; \lim_{N \to \infty} R^{NC}(N) \tag{8}$$

4.5. Equilibrium with Commitment and Asymmetric Bidders

We now turn to the model with Commitment and Asymmetric bidders described in Subsection 3.4. Bidder $N+1$ is stronger in the sense that F and G are such that $E(w) > \bar{v}$. Proposition 3 still applies in this case; the first N bidders bid above their value. The fact that $N+1$ is sufficiently stronger guarantees that overbidding always has a potential cost for the first N bidders. This in turn means that in this case there is no need to consider explicitly a bidding cap $\bar{b}$ as we did for the symmetric bidders case.

Proposition 6. *Commitment, Asymmetric Bidders and Uniform Values: Assume that the bidders are asymmetric with $N+1$ being sufficiently stronger than the first N. Assume further that all values are uniformly distributed, the first N on* $[0,1]$.

Then the following constitute an equilibrium.

$$b_i(v_i) \;=\; \frac{2N}{N+1}v_i \quad \text{and} \quad b_{N+1}(w) \;=\; w \tag{9}$$

The bidding above value generated by the two-stage structure of the auction with commintment has a very different and "unexpected" effect in the case of asymmetric bidders. As we noted in Remark 3 since the outcome fails to be ex-post efficient in this case, we cannot appeal to the Revenue Equivalence Theorem to pin down the auctioneer's expected revenue relative to the standard one-shot second-price auction case.

Proposition 7. *Revenue with Commitment and Asymmetric Bidders: Let $R^{CA}(N)$ be the expected revenue for the case of commitment and asymmetric bidders with bidder $N+1$ stronger than the first N so that $E(w) > \bar{v}$. Assume general distributions F and G for the bidders' values as in Subsection 3.1. Assume further that the bidding functions for the first N bidders are differentiable.*[7] *Then,*

$$\lim_{N\to\infty} R^{CA}(N) \;>\; \lim_{N\to\infty} R^{NC}(N) \tag{10}$$

For large N the two-stage structure of the auction yields an *increase* in the auctioneer's expected revenue relative to the standard one-shot second-price case. We believe this to be "unexpected" in the following sense.

[7] From Proposition 6 we know this to be the case for uniformly distributed values on $[0,1]$.

Imagine adding a *single* bidder $N + 1$ that is stronger than the first N to the standard one-shot second-price auction. This clearly has a vanishingly small effect on revenue. Bidder $N + 1$ is indeed stronger, but of course he pays a price equal to the second highest bid. As N becomes large, the second highest value of the $N + 1$ bidders, is almost surely arbitrarily close to (and just below) $\bar{v}$. Hence bidder $N + 1$ in the limit has no effect on revenue.

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Appendix

Proof of Proposition 1: Equation (2) follows from our remarks about the bidding behavior of $N+1$ in Subsection 4.1).

Taking this as given, the expected benefit from participating in the second stage for a bidder $i = 1, ..., N$ with value $v_i \in [0, \bar{v}]$ is given by

$$\theta(v_i) \equiv \int_0^{v_i} (v_i - \xi) g(\xi)\, d\xi \tag{A.1}$$

Observe that the first stage auction is equivalent to a standard one-shot second price auction in which bidders have values $\{\theta(v_i)\}_{i=1}^N$. By standard arguments bidding one's own value is a weakly dominant strategy. Hence (1) follows immediately. ∎

Proof of Proposition 2: Equation (3) follows from the observation that revenue equivalence with respect to the standard one-shot second-price auction holds in this case. Equation (4) is a standard result about order statistics with uniform distributions. ∎

Proof of Proposition 3: Since in some cases constructing an equilibrium requires imposing a bidding cap on the first N bidders we let $\bar{b} > \bar{v}$ be such cap, with the understanding that we may be in the case in which no such cap is imposed which here will correspond to setting $\bar{b} = \infty$.

Fix a bidding function for the first N players forming an equilibrium together with $b_{N+1}(w) = w$ as in the statement of the proposition. Let $P(b_i, b(\cdot))$ be the probability that i wins the first stage given bid b_i and that all others bid according to $b(\cdot)$.

The expected payoff of a bidder $i = 1, .., N$ with value v_i in this equilibrium is then given by

$$P(b(v_i), b(\cdot)) \int_0^{b(v_i)} (v_i - \xi) g(\xi)\, d\xi. \tag{A.2}$$

First note that we must have $P(b(v_i), b(\cdot)) > 0$ for all $v_i > 0$ because $b(\cdot)$ is assumed to be non-decreasing ans ties are broken randomly.[8] Note also that the value of integral in (A.2) must also be positive otherwise $b(v_i)$ could not possibly be an optimal bid. Hence, overall the expected payoff in (A.2) must be positive.

The remainder of the proof is in three steps.

Step 1: *$b(v_i) \geq v_i$ for all $v_i \in [0, \bar{v}]$*

By way of contradiction suppose that $b(v_i) < v_i$ for some $v_i \in (0, \bar{v}]$. Recall that i's expected payoff in equilibrium conditional of having value v_i is given by (A.2). Note that $P(b_i, b(\cdot))$ is non decreasing in b_i and that since $b(v_i) < v_i$

$$\frac{\partial}{\partial b_i} \int_0^{b_i} (v_i - \xi) g(\xi)\, d\xi \;=\; (v_i - b_i)\, g(b_i) \;>\; 0 \tag{A.3}$$

[8] See footnote 2.

Hence it follows that (without violating the bid cap, if any) i can strictly increase his expected payoff by increasing his bid to $b(v_i)+\epsilon$ for a sufficiently small $\epsilon>0$.

Step 2: *If $v_1<v_2$ and $b(v_2)<\bar{b}$, then $b(v_1)<b(v_2)$.*[9]

If $v_1=0$ then it must clearly be that $b(v_1)=0$, and if $v_2>0$ then $b(v_2)$ must be positive since the interim payoff in (A.2) must be positive. So, there is nothing more to prove in this case.

Hence, it is sufficient to derive a contradiction from the case $0<v_1<v_2$ and $b(v_1)=b(v_2)<\bar{b}$.

Since $b(v_1)$ must be optimal given v_1 we must have for $\Delta>0$

$$P(b(v_1),b(\cdot))\int_0^{b(v_1)}(v_1-\xi)\,g(\xi)\,d\xi \;\geq\; P(b(v_1)+\Delta,b(\cdot))\int_0^{b(v_1)+\Delta}(v_1-\xi)\,g(\xi)\,d\xi. \tag{A.4}$$

where the left-hand side must be positive since $v_1>0$.

Furthermore, since by our contradiction hypothesis $b(v_1)=b(v_2)$, it must be that

$$\lim_{\Delta\downarrow 0}P(b(v_1)+\Delta,b(\cdot))>P(b(v_1),b(\cdot)) \tag{A.5}$$

However (A.5) implies that as we take the limit for $\Delta\downarrow 0$ inequality (A.4) must be false.

Step 3: We can now conclude the proof of Proposition 3.

For any $v\in(0,\bar{v}]$ such that $b(v)=\bar{b}$, the claim in the proposition obviously holds. Hence consider $v\in(0,\bar{v}]$ such that $b(v)<\bar{b}$. Note that from Step 2 we know that

$$P(b(v),b(\cdot)) \;=\; [F(v)]^{N-1} \tag{A.6}$$

By way of contradiction suppose now that $b(v)=v$. Then, using Step 2 for any $\epsilon>0$, interim optimality implies

$$F(v-\epsilon)^{N-1}\int_0^{b(v-\epsilon)}(v-\epsilon-\xi)g(\xi)\,d\xi\geq F(v)^{N-1}\int_0^{v}(v-\epsilon-\xi)g(\xi)\,d\xi$$

which we can rearrange as

$$\Big[F(v-\epsilon)^{N-1}-F(v)^{N-1}\Big]\int_0^{b(v-\epsilon)}(v-\epsilon-\xi)g(\xi)\,d\xi\geq F(v)^{N-1}\int_{b(v-\epsilon)}^{v}(v-\epsilon-\xi)g(\xi)\,d\xi \tag{A.7}$$

Since by Step 1 we know that $b(v-\epsilon)\geq v-\epsilon$ (Step 1), we now must have

$$\begin{aligned}\big[F(v-\epsilon)^{N-1}-F(v)^{N-1}\big]\textstyle\int_0^{b(v-\epsilon)}(v-\epsilon-\xi)g(\xi)\,d\xi &\geq F(v)^{N-1}\textstyle\int_{v-\epsilon}^{v}(v-\epsilon-\xi)g(\xi)\,d\xi\\ &= F(v)^{N-1}\left[\textstyle\int_{v-\epsilon}^{v}(v-\xi)g(\xi)\,d\xi-\epsilon[G(v)-G(v-\epsilon)]\right]\end{aligned} \tag{A.8}$$

[9]In the case of no bid cap $b(v_2)<\bar{b}$ is automatically satisfied.

Hence we obtain that the following inequality must hold

$$\begin{aligned}
&\frac{F(v-\epsilon)^{N-1} - F(v)^{N-1}}{\epsilon} \int_0^{b(v-\epsilon)} (v-\epsilon-\xi) g(\xi)\, d\xi \\
&\geq F(v)^{N-1} \left[\frac{\int_{v-\epsilon}^{v} (v-\xi) g(\xi)\, d\xi}{\epsilon} - [G(v) - G(v-\epsilon)] \right]
\end{aligned} \tag{A.9}$$

Taking the limit as $\epsilon \downarrow 0$ on both sides of (A.9), we now obtain a contradiction since

$$-(N-1)F(v)^{N-2} f(v) \int_0^v (v-\xi) g(\xi)\, d\xi < 0$$

Proof of Proposition 4: We begin by defining the functions

$$\begin{aligned}
P_N(v) &\equiv \sum_{i=0}^{N-1} \binom{N-1}{i} v^{N-1-i} (1-v)^i \frac{1}{1+i} \\
A_N(v) &\equiv v^{N+1} \frac{2N}{(N+1)^2} \\
B_N(v) &\equiv (v - \frac{1}{2}) P_N(v)
\end{aligned} \tag{A.10}$$

Since

$$\binom{N-1}{i} \frac{1}{1+i} = \binom{N}{i+1} \frac{1}{N}$$

we then immediately obtain

$$P_N(v) = \frac{1}{N} \frac{1}{1-v} \left[\sum_{j=1}^{N} \binom{N}{j} (1-v)^j v^{N-j} \right] = \frac{1}{N} \frac{1-v^N}{1-v} = \frac{1}{N} \sum_{j=0}^{N-1} v^j. \tag{A.11}$$

Our candidate $\hat{v}_N$ as in the statement of the proposition is

$$\hat{v}_N = \min \left\{ v \in \left(\frac{1}{2}, \frac{N+1}{2N} \right) \quad \text{such that} \quad A_N(v) = B_N(v) \right\} \tag{A.12}$$

To see that the quantity in (A.12) is actually well defined, note that $A_N(1/2) > B_N(1/2) = 0$ and

$$B_N((N+1)/2N) = \frac{1}{2N} \cdot P_N((N+1)/2N) > \frac{1}{2N} \left(\frac{N+1}{2N} \right)^{N-1} = A_N((N+1)/2N)$$

So that we know that $\hat{v}_N$ as above it is in fact interior to the interval in (A.12), as required.

Given that all other players bid according to $b_{-i}(\cdot)$ as in equation (6) in the statement of the proposition, the interim expected payoff of a bidder $i = 1, ..., N$ with bid b_i and value v_i is

$$\pi(b_i, v_i, b_{-i}(\cdot)) = \begin{cases} \left(\frac{N+1}{2N} b_i\right)^{N-1} \cdot b_i \cdot \left(v_i - \frac{b_i}{2}\right) & \text{if} \quad b_i \leq \frac{2N}{N+1} \hat{v}_N \\ \hat{v}_N^{N-1} \cdot b_i \left(v_i - \frac{b_i}{2}\right) & \text{if} \quad \frac{2N}{N+1} \hat{v}_N < b_i < 1 \\ \hat{v}_N^{N-1} \left(v_i - \frac{1}{2}\right) & \text{if} \quad 1 \leq b_i < \bar{b} \ \ (\text{void if } \ \bar{b} = 1) \\ P(\hat{v}_N) \left(v_i - \frac{1}{2}\right) & \text{if} \quad b_i = \bar{b} \end{cases} \tag{A.13}$$

To see why (A.13) holds, notice that the first term in each line corresponds to the probability of winning the first stage. Also, notice that the second term in each of the top two lines corresponds to the probability of winning the second stage. Finally, the last term in each of the top two lines is the expected payoff conditional on winning the second stage.

We can proceed to establish the interim optimality of the bidding function $b_i(\cdot)$ explicited in (6). Before proceeding to examine specific cases, we note that it will never be an interim best reply to choose bid $b_i \in [1, \bar{b})$ because bidding $b_i = \bar{b}$ strictly improves i's payoff.[10] We consider two cases separately.

Case 1: $v_i \in [0, \hat{v}_N]$.

We begin by examining the optimal bid among the range $b_i \in [0, 1)$. First notice that

$$\sup_{b_i \in [0,1)} \left(\frac{N+1}{2N} b_i\right)^{N-1} \cdot b_i \cdot \left(v_i - \frac{b_i}{2}\right) \geq \sup_{b_i \in [0,1)} \pi(b_i, v_i, b_{-i}(\cdot)). \tag{A.14}$$

Hence, if there is some bid $b_i \in [0, 1)$ such that $\pi(b_i, v_i, b_{-i}(\cdot))$ equals the supremum on the *LHS* of (A.14), then this b_i must attain the supremum for the *RHS* as well. With this in mind, let us first consider the *LHS* of (A.14). From its first derivative, it is clear that the supremum is obtained at $b_i(v_i) = v_i 2N/(N+1) < 1$. Then, notice that for all $v_i \in [0, \hat{v}_N]$, we have $b_i(v_i) \leq \hat{v}_N 2N/(N+1)$ and hence the first line of (A.13) is the relevant one when computing $\pi(b_i(v_i), v_i, b_{-i}(\cdot))$. It then follows that this bid $b_i(v_i)$ attains the supremum for the righthand side of (A.14) as well. Hence $b_i(v_i)$ as in (6) is the optimal bid among bids $b_i \in [0, 1)$.

Our next step is to compare this bid $b_i(v_i)$ with $\bar{b}$. The expected payoff given bid $b_i(v_i)$ is $A_N(v_i)$, whereas the expected payoff given bid $\bar{b}$ is $P(\hat{v}_N)(v_i - 1/2)$. Notice that these expected payoffs are equal for $v_i = \hat{v}_N$. Observe next that for all $v_i \in [0, \hat{v}_N]$,

$$\frac{\mathrm{d}}{\mathrm{d}v_i} A_N(v_i) = v_i^N \frac{2N}{N+1} < \hat{v}_N^{N-1} < P_N(\hat{v}_N). \tag{A.15}$$

[10] The bid range $[1, \bar{b})$ is obviously empty if $\bar{b} = 1$, so it this is irrelevant in this case.

Hence it must be that $A_N(v_i) > P(\hat{v}_N)(v_i - 1/2)$ for all $v_i \in [0, \hat{v}_N)$. Thus, it is clear that for Case 1, $b_i(\cdot)$ as in (6) is indeed the interim best reply.

Case 2*:* $v_i \in (\hat{v}_N, 1]$.

We begin by considering the optimal bid b_i in the range $[0, \hat{v}_N 2N/(N+1)]$. Taking the first derivative of the first line in (A.13), it is clear that the optimal bid is always $\hat{v}_N 2N/(N+1)$. Therefore

$$\max_{b_i \in [0, \hat{v}_N 2N/(N+1)]} \pi(b_i, v_i, b_{-i}(\cdot)) = \frac{2N}{N+1} \hat{v}_N^N (v_i - \frac{N}{N+1} \hat{v}_N)$$

We can then establish that

$$\hat{A}_N(v_i) \equiv \sup_{b_i \in [0,1)} \pi(b_i, v_i, b_{-i}(\cdot)) =$$

$$\sup_{b_i \in [\frac{2N}{N+1}\hat{v}_N, 1)} \pi(b_i, v_i, b_{-i}(\cdot)) = \begin{cases} \frac{2N}{N+1} \hat{v}_N^N (v_i - \frac{N}{N+1} \hat{v}_N) & \text{if} \quad v_i \in (\hat{v}_N, \dfrac{2N}{N+1}\hat{v}_N] \\ \hat{v}_N^{N-1} \dfrac{v_i^2}{2} & \text{if} \quad v_i \in (\dfrac{2N}{N+1}\hat{v}_N, 1] \end{cases}$$

We then observe that

$$\frac{\mathrm{d}}{\mathrm{d}v_i} \hat{A}_N(v_i) = \begin{cases} \dfrac{2N}{N+1} \hat{v}_N^N & \text{if} \quad v_i \in (\hat{v}_N, \dfrac{2N}{N+1}\hat{v}_N] \\ \hat{v}_N^{N-1} v_i & \text{if} \quad v_i \in (\dfrac{2N}{N+1}\hat{v}_N, 1] \end{cases} \tag{A.16}$$

so that $\hat{A}_N(v_i)$ is continuously differentiable on $v_i \in (\hat{v}_N, 1]$. In addition, notice that the expected payoff given bid $\bar{b}$ is $P(\hat{v}_N)(v_i - 1/2)$, and observe that this is equal to $\hat{A}_N(\hat{v}_N)$ when $v_i = \hat{v}_N$. Finally observe that

$$P(\hat{v}_N) > \frac{\mathrm{d}}{\mathrm{d}v_i} \hat{A}_N(v_i) \tag{A.17}$$

for all $v_i \in (\hat{v}_N, 1]$. Therefore the expected payoff given bid $\bar{b}$ is strictly better than $\hat{A}_N(v_i)$ for all $v_i \in (\hat{v}_N, 1]$. Hence it follows that $b_i(\cdot)$ as in (6) is the interim best reply for Case 2 as well.

This is clearly enough to prove the claim. ∎

Proof of Proposition 5: Since $N+1$ bids truthfully and there are only two bidders in the second stage, the second-price nature of the contest immediately implies that the realized revenue must be weakly less than w for any distribution of values across all $N+1$ bidders. In addition, for the case of uniformly distributed values on $[0, 1]$, inspection of the equilibrium in Proposition 4 immediately reveals that with non-zero probability, the realized revenue must be *strictly* less than w. Thus, for the case of uniformly distributed values on $[0, 1]$, we must have $R^{CS}(N) < 1/2$ since the expected value of a uniform distribution on $[0, 1]$ equals $1/2$. The remainder of the proposition for the uniform case follows directly from Proposition 2.

For the case of general distributions case, observe once again that regardless of values across all bidders, the realized revenue must be weakly less than w.

It follows that $R^{CS}(N) \leq E(w)$ for all N. Then, notice that given our assumptions on the distribution F, the second-order statistic $(v_1, ..., v_N, w)_{2:N} \overset{\text{a.s.}}{\to} \bar{v} = \bar{w}$. Thus, $\lim_{N\to\infty} R^{NC}(N) = \bar{w} > E(w)$, from which (8) follows directly. ∎

Proof of Proposition 6: Notice that since $E(w) = \bar{w}/2$, we have that $\bar{w}/2 > \bar{v} = 1$. Next, suppose that all bidders other than $N+1$ bid as in (9). Then if $i \neq N+1$ has value v_i and bids b_i his expected payoff is given by

$$\pi[b_i, v_i, b_{-i}(\cdot)] = \begin{cases} \left[\frac{N+1}{2N} b_i\right]^{N-1} \frac{b_i}{\bar{w}}\left[v_i - \frac{b_i}{2}\right] & \text{if } b_i \in \left[0, \frac{2N}{N+1}\right] \\ \frac{b_i}{\bar{w}}\left(v_i - \frac{b_i}{2}\right) & \text{if } b_i \in \left(\frac{2N}{N+1}, \bar{w}\right] \\ v_i - \frac{\bar{w}}{2} & \text{if } b_i \in (\bar{w}, \infty) \end{cases} \tag{A.18}$$

To see why (A.18) holds observe that the first term of the top line is the probability that i by bidding b_i wins the first stage of the tournament. Then, $b_i/\bar{w}$ is the probability that i by bidding b_i wins the second stage of the tournament. Finally, $v_i - b_i/2$ is i's expected payoff if he wins the second stage.

Now consider maximizing the top row of (A.18) by choice of $b_i \in [0, \infty)$. From the first order conditions the maximum is attained by setting $b_i(v_i)$ as in (9). By inspection of the second and third row of (A.18), it is then clear that i's overall expected payoff as in (A.18) is maximized by setting $b_i(v_i)$ as in (9), and this is sufficient to prove the claim. ∎

Proof of Proposition 7: The proof is divided into three separate steps. The second step in turn is divided into three substeps.

Step 1: We prove that $\sup\limits_{N\geq 2} b(\bar{v}, N) < \bar{w}$.

By assumption $E(w) > \bar{v}$. Therefore there exists $\epsilon > 0$ such that

$$E(w|w \leq \bar{w} - \epsilon) = \frac{1}{G(\bar{w}-\epsilon)} \int_0^{\bar{w}-\epsilon} \xi g(\xi)\, d\xi > E(w) - \int_{\bar{w}-\epsilon}^{\bar{w}} \xi g(\xi)\, d\xi > \bar{v}. \tag{A.19}$$

Then, choose any $N \geq 2$ and suppose by way of contradiction that $b(\bar{v}, N) \geq \bar{w} - \epsilon$. Then, the expected payoff of a bidder $i = 1, ..., N$ with value $\bar{v}$ in equilibrium is $G(b(\bar{v}, N))[\bar{v} - E(w|w \leq b(\bar{v}, N))] \leq G(b(\bar{v}, N))[\bar{v} - E(w|w \leq \bar{w} - \epsilon)] < 0$. Since any bidder can achieve nonnegative payoffs for sure by bidding $b_i = 0$, this contradicts interim optimality. Hence, the claim made in Step 1 has been established.

Step 2: If $M > N$, then $b(v, M) > b(v, N)$ for all $v \in (0, \bar{v}]$.

This step is proved in three distinct substeps.

Step 2-a: $b(\cdot, N)$ satisfies the ODE

$$\frac{\partial}{\partial v} b(v, N) = H^N(b(v, N), v) \quad \text{for all} \quad v \in (0, \bar{v}] \tag{A.20}$$

where

$$H^N(b, v) \equiv (N-1) \cdot \frac{1}{b-v} \cdot \frac{f(v)}{F(v)} \cdot \frac{G(b)}{g(b)} \cdot E_w[v - w | w \leq b] \tag{A.21}$$

for all $(b, v) \in (0, \bar{w}] \times (0, \bar{v}]$ such that $b > v$.

The claim follows directly from rearranging the FOC of the first N bidders, and using Step 1.

Step 2-b: $b(0, N) = 0$.

Denote

$$T(b, b(\cdot, N)) \equiv F(b^{-1}(b_i, N))^{N-1} \quad \text{and} \quad S(b_i, v_i) \equiv \int_0^{b_i} (v_i - \xi) g(\xi)\, d\xi.$$

Then, let $v_i \in (0, \bar{v}]$. It immediately follows that $T(b(v_i, N), b(\cdot, N)) = F(v_i)^{N-1} > 0$. Next, by way of contradiction suppose that $b(0, N) > 0$. Then

$$S(b(v_i, N), v_i) \;=\; G(b(v_i, N))\,[v_i - E_w[w | b(v_i, N) > w]] \leq G(b(v_i, N))\,[v_i - E_w[w | b(0, N) > w]] \tag{A.22}$$

Now choose any $v_i \in (0, E_w[w | b(0, N) > w])$. It follows that $T(b(v_i, N), b(\cdot, N)) > 0$ and $S(b(v_i, N), v_i) < 0$. Therefore $\pi(b(v_i, N), v_i, b_{-i}(\cdot, N)) < 0$. This contradicts interim optimality and hence establishes the claim.

Step 2-c: Fix $M > N$. Then there exists a $v_\epsilon > 0$ such that $b(v, M) > b(v, N)$ for all $v \in (0, v_\epsilon)$.

We begin by observing that using (A.20) and (A.21)

$$H^N(b, v) \;<\; H^M(b, v) \;\; \forall\, b \in (v, b(v, M)], \;\; \forall\, v \in (0, \bar{v}] \tag{A.23}$$

moreover

$$\frac{\partial}{\partial b} H^N(b, v) = (N-1) \frac{f(v)}{F(v)} \;\cdot\; \frac{\partial}{\partial b} \left[\frac{1}{b-v} \, \frac{1}{g(b)} \, \int_0^b (v - \xi) g(\xi)\, d\xi \right] \tag{A.24}$$

From (A.24) we get

$$\text{sign} \left\{ \frac{\partial}{\partial b} H^N(b, v) \right\} = \text{sign} \left\{ \frac{\partial}{\partial b} \left[\frac{1}{b-v} \frac{1}{g(b)} \int_0^b (v - \xi) g(\xi)\, d\xi \right] \right\} \tag{A.25}$$

and

$$\operatorname{sign}\left\{\frac{\partial}{\partial b}H^N(b,v)\right\} = \operatorname{sign}\left\{-1 - \frac{1}{b-v}\frac{1}{g(b)}\left[\frac{g'(b)}{g(b)} + \frac{1}{b-v}\right]\int_0^b (v-\xi)g(\xi)\,d\xi\right\} \tag{A.26}$$

By assumption g' is bounded at 0 and $g(0) > 0$. Therefore there exists a $b_\delta > 0$ and $K < \infty$ such that

$$\left|\frac{g'(b)}{g(b)}\right| < K \qquad \forall\, b \in [0, b_\delta) \tag{A.27}$$

From Step 2-b we know that that $\lim_{v\downarrow 0} b(v,N) - v = 0$. It then follows that there exists $v_\epsilon > 0$ such that for all $v \in (0, v_\epsilon)$,

$$\frac{1}{b(v,N)-v} > K \qquad \text{and} \qquad b(v,N) < b_\delta \tag{A.28}$$

A direct implication of interim optimality is that

$$\int_0^{b(v,N)} (v-\xi)g(\xi)\,d\xi > 0 \qquad \forall v > 0$$

Hence

$$\frac{\partial}{\partial b}H^N(b,v) < 0 \qquad \forall\, b \in (v, b(v,N)],\ v \in (0, v_\epsilon) \tag{A.29}$$

By way of contradiction now suppose that we can find a $v^* \in (0, v_\epsilon)$ such that

$$b(v^*, M) \ \leq\ b(v^*, N)$$

Then, using (A.20), (A.21), (A.23) and (A.29), we get

$$\frac{\partial}{\partial v}b(v^*,N) = H^N(b(v^*,N),v^*) \leq H^N(b(v^*,M),v^*) < H^M(b(v^*,M),v^*) = \frac{\partial}{\partial v}b(v^*,M).$$

Hence there exists $\tilde{v} \in (0, v^*)$ arbitrarily close to v^* such that

$$b(\tilde{v}, M) < b(\tilde{v}, N) \tag{A.30}$$

Since $b(0,M) = b(0,N) = 0$, there exists $\hat{v} \in (0, \tilde{v})$ such that

$$\frac{\partial}{\partial v}b(\hat{v},M) < \frac{\partial}{\partial v}b(\hat{v},N) \tag{A.31}$$

from which we have

$$H^N(b(\hat{v},M),\hat{v}) < H^M(b(\hat{v},M),\hat{v}) = \frac{\partial}{\partial v}b(\hat{v},M) < \frac{\partial}{\partial v}b(\hat{v},N) = H^N(b(\hat{v},N),\hat{v}) \tag{A.32}$$

and hence

$$b(\hat{v}, M) > b(\hat{v}, N) \tag{A.33}$$

Hence $v^{**} \equiv \sup\{v \in (0, \tilde{v}) | b(v, M) > b(v, N)\} \in (\hat{v}, \tilde{v})$. Also, by continuity, $b(v^{**}, M) = b(v^{**}, N)$. However, this in turn implies that

$$\frac{\partial}{\partial v} b(v^{**}, M) \ > \ \frac{\partial}{\partial v} b(v^{**}, N) \tag{A.34}$$

This clearly contradicts the definition of v^{**}, and hence establishes the claim.

We can now conclude the proof of the claim in Step 2.

Firstly, from Step 2-c, there exists $v_\epsilon > 0$ such that $b(v, M) > b(v, N)$ for all $v \in (0, v_\epsilon]$. By way of contradiction suppose that $\{v \in (v_\epsilon, \bar{v}] \,|\, b(v, M) \leq b(v, N)\} \neq \emptyset$. Then, denote $v^{***} \equiv \inf\{v \in (v_\epsilon, \bar{v}] \,|\, b(v, M) \leq b(v, N)\}$. Since $b(v_\epsilon, M) > b(v_\epsilon, N)$, we must have $v^{***} \in (v_\epsilon, \bar{v}]$. Then, by continuity of $b(\cdot, N)$ and $b(\cdot, M)$, we must have $b(v^{***}, N) = b(v^{***}, M)$. However, this in turn implies

$$\frac{\partial}{\partial v} b(v^{***}, M) \ > \ \frac{\partial}{\partial v} b(v^{***}, N) \tag{A.35}$$

This contradicts the definition of v^{***} and hence establishes the claim.

Step 3*:* This step simply concludes the proof of Proposition 7.

From Steps 1 and 2 we have that the limit $\bar{b}_\infty \equiv \lim_{N\to\infty} b(\bar{v}, N)$ is well-defined with $\bar{b}_\infty \in (\bar{v}, \bar{w})$ A sharper characterization of $\bar{b}_\infty$ will allow us to complete the proof.

Define the value function

$$W^N(v) \equiv \pi^N(b(v, N), v, b_{-i}(\cdot, N)) = F(v)^{N-1} \int_0^{b(v,N)} (v - \xi) g(\xi)\, d\xi \quad \forall v \in [0, \bar{v}]. \tag{A.36}$$

By the envelope theorem,

$$\frac{\partial}{\partial v} W^N(v) \ = \ F(v)^{N-1} \cdot G(b(v, N))$$

and therefore

$$W^N(\bar{v}) = \int_0^{\bar{v}} F(v)^{N-1} \cdot G(b(v, N))\, dv$$

Then, by the dominated convergence theorem,

$$\lim_{N\to\infty} W^N(\bar{v}) = 0.$$

However, by definition (A.36) we also have that[11]

$$W^N(\bar{v}) = E_w\left[\mathbb{1}(b(\bar{v},N) \geq w) \cdot (\bar{v} - w)\right]$$

By the dominated convergence theorem again,

$$\lim_{N\to\infty} W^N(\bar{v}) = E_w\left[\mathbb{1}(\bar{b}_\infty \geq w) \cdot (\bar{v} - w)\right]$$

Hence

$$E_w[w \,|\, w \leq \bar{b}_\infty] \;=\; \bar{v}.$$

Denote as $\mathcal{R}^{CA}(N)$ the random variable of the revenue from the tournament auction with commitment and N bidders. Furthermore, denote as $\mathcal{R}^{NC}(N)$ the random variable of the revenue from the one-shot second-price auction with N bidders.

Then, observe that

$$\mathcal{R}^{CA}(N) = \min\left\{b\left(\max_{i=1,\ldots,N}\{v_i\}, N\right), w\right\} \overset{\text{a.s.}}{\to} \min\left\{\bar{b}_\infty, w\right\}$$

$$\mathcal{R}^{NC}(N) \overset{\text{a.s.}}{\to} \bar{v} \tag{A.37}$$

Using once more the monotone convergence theorem, it then follows that

$$\begin{gathered}\lim_{N\to\infty} E\left[\mathcal{R}^{CA}(N)\right] \;=\; E\left[\min\{\bar{b}_\infty, w\}\right] \;=\\ \left[1 - G(\bar{b}_\infty)\right]\bar{b}_\infty + G(\bar{b}_\infty)E_w\left[w \,|\, w \leq \bar{b}_\infty\right] \;>\; \bar{v}\end{gathered} \tag{A.38}$$

which, using (A.37), is clearly enough to complete the proof. ∎

[11] As is standard $\mathbb{1}(\cdot)$ denotes the indicator function.