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Federica Carannante[*], Marco Pagnozzi[†], and Elia Sartori[‡]

Abstract

We study the interim seller's revenue — the expected revenue conditional on the valuation of one bidder — in a class of sealed-bid auctions that are ex-ante equivalent by the Revenue Equivalence Theorem. Interim revenue differences across auction formats depend on the expected transfer of a generic bidder conditional on a competitor's valuation. The first-price auction yields higher (lower) interim revenue than the second-price auction if the valuation is below (above) a threshold. At the lowest possible valuation, the first-price auction also yields the highest interim revenue among all standard auctions. By contrast, at high valuations the first-price auction yields the lowest interim revenue, while the last-pay auction — an atypical mechanism where only the lowest bidder pays — allows the seller to extract arbitrarily large revenues.

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1 Introduction

Consider a seller who runs an auction to allocate a single object among risk-neutral bidders with independent private values. In this canonical setting, the Revenue Equivalence Theorem (RET, Vickrey (1961); Myerson (1981); Riley and Samuelson (1981)) establishes that any efficient equilibrium of any auction where the lowest bidder type obtains zero surplus yields the same expected revenue for the seller and results in the same expected payment by any type of bidder. As a result, both the seller and the bidders are indifferent among auction formats.

But suppose that, after bids have been submitted, the valuation of one of the bidders is revealed. This particular bidder remains indifferent across auction formats at this interim stage. The reason is that the bidder is already informed about his valuation when he submits his bid and still wins the object if and only if he has the highest valuation; hence his expected payment is the same in all auction formats by the RET. What about the other participants in the auction? In particular, is the seller still indifferent when the valuation of one of the bidders is revealed?

We characterize the *interim revenue* — the expected seller's revenue conditional on the valuation of one of the bidders — in the efficient equilibrium of a variety of sealed-bid auctions that are ex-ante equivalent by the RET. Our work is relevant to situations where a seller has private information about the valuation of one of the bidders, referred to as the *special bidder*. This information can arise in different settings, either exogenously — e.g., when a potential buyer has a score derived from past purchases (see, e.g. Bonatti and Cisternas (2020)) — or in repeated auctions where bidders reveal their valuations through past bids — e.g., in procurement auctions where the government has information about the costs of the current provider. Our analysis identifies the format preferred by the seller conditional on the valuation of the special bidder.

First, we compare the two most commonly-used sealed-bid formats, the first-price auction (FPA) and the second-price auction (SPA), and show that the FPA yields higher interim revenues if and only if the special bidder's valuation is below a threshold. A setting with two bidders provides a clear intuition for this result. By the RET, when the special bidder wins the auction, the expected revenue is the same in both auction formats. Hence, revenue differences only arise in the event that the other bidder wins. In this case, in the SPA the seller's revenue is the special bidder's bid, which is equal to his valuation, while in the FPA it is the other bidder's bid, which is shaded below his valuation. And the interim ranking between the FPA and SPA amounts to a race between the other bidder having a higher value than the special bidder when he wins (favoring the FPA) and him shading bid below valuation in the FPA (favoring the SPA).

A special bidder with valuation close to the the maximum is good news in both auction formats, but particularly so in the SPA: conditional on winning, the other bidder also has a valuation close to the maximum so the bid-shading effect of the FPA dominates. Conversely, a special bidder with valuation close to the minimum is bad news in both auction formats, but especially so in the SPA because the special bidder's valuation — which is the price paid by the other bidder the SPA — is lower than the expectation of the other bidder's bid in the FPA. Moreover, by iterated expectations, the expectation of the interim revenue with respect to the special bidder's value is the ex-ante revenue, which is the same in both formats. Hence, the reversed ranking at the extrema implies that the interim revenue functions of the two formats must cross. Under mild regularity assumptions, the crossing is unique and the result extends to multiple bidders.

We then extend our analysis to a broad class of sealed-bid auctions, referred to as "standard auctions." An auction in this class corresponds to a *who-pays-what* specification: for each order statistic of the bids, the auction specifies whether it pays or not and, if so, which bid it pays. This broad class includes common formats like the First-Price Auction (in *who-pays-what* terminology: first bid pays first bid), the Second-Price Auction (first bid pays second bid), the All-Pay Auction (all bids pay their bid), as well as rather uncommon formats that will play an important role in our analysis, like the Last-Pay Auction (last bid pays last bid).

As for the FPA and SPA, an iterated expectations argument implies that the interim revenue of each standard auction integrates to the same value. What factors determine which auction yields the highest interim revenue, depending on the valuation of the special bidder? What drives the differences in interim revenues across formats that ultimately average out? A key insight emerges when we express the interim seller's revenue as the sum of the transfers received from each bidder: The difference in interim revenues between auctions does not depend on the expected transfer of the special bidder, which is independent of the auction format by the RET, but only on the expected transfers of the other (interim symmetric) bidders, which depend jointly on the special bidder's valuation and the auction format. Hence, comparing two alternative auction formats when the special bidder has valuation v, the auction that yields the higher interim revenue is the one in whose efficient equilibrium a generic bidder expects to pay more, prior to drawing his type but conditioning on the information that a competitor has valuation v.

Some implications follow directly from this observation. When the special bidder's valuation is at the upper bound of the support, the FPA and the SPA — as well as all other auctions where only the winner pays (WPA) — are interim equivalent. This is because, in any such auction, the special bidder always wins and the expected payment of the other bidders is equal to zero. Not all standard auctions, however, are equivalent at the upper bound. For example, the All-Pay Auction yields a strictly higher interim revenue than a WPA because the other bidders make positive expected payments.

Leveraging on this observation, we prove (Proposition 10) that the FPA yields the highest (resp., lowest) interim revenue among all standard auctions when the special bidder's valuation is the lowest (resp. highest) possible one. When the special bidder has a valuation close to the highest possible one, the transfer of other bidders is equal to 0 in any WPA, since the special bidder wins almost surely (and losers pay nothing). Hence, WPAs yield a lower revenue than other formats where also losers pay. Moreover, extending the comparison with the SPA, we show that around the upper bound of valuations WPAs are ranked according to the order of the bid paid by the winner, indicating that the FPA yields the lowest interim revenue.

Conversely, when the special bidder has the lowest possible valuation, other bidders transfer more in the FPA than in any other standard auction. To get an intuition for why this holds, first consider a comparison between the FPA and other WPAs, hence maintaining that only the winner pays but changing which bid he pays. In other WPAs other bidders still pay only if they have the highest value, but their expected payment is lower because, conditional on the highest bid, the distribution of all other order statistics of bids are depressed when one of the bids is zero. This reflects a crucial property of the FPA, namely that it is a pay-as-bid auction (PBA) and hence that, conditional on paying, a bidder's transfer is unaffected by others' bids. Therefore, in contrast to a non-PBA, in a PBA a special bidder with value 0 has no negative effect on bidders' expected payments conditional on paying.

Second, consider a comparison between the FPA and other PBAs, hence changing the set of payers but maintaining that they pay their own bids. In a PBA, the other bidders' transfer is determined by a likelihood ratio that measures how the probability of the bidder belonging to the set of payers changes with the information that another bidder's valuation is v. Hence, the PBA that maximizes the interim revenue is the one with a set of payer that maximizes this likelihood ratio (given v). For v close to zero, this likelihood ratio is maximized when only the highest bidder pays. This is because, given that a competitor has value zero, the probability that a generic bidder is any order statistic other than the minimum increases, but the largest increase occurs for the first-order statistic.

This result is reversed around the upper bound, where all order statistics except the maximum become more likely, with the largest change occurring for the minimum. This explains why the APA dominates the FPA: the likelihood ratio becomes zero for the FPA but it remains constant (since all bidders always pay regardless of the information about competitors) in the APA. Consequently, revenues are higher in the format that relies on transfers from non-winning bidders. The interim-dominant PBA is one where only the *lowest* bidder pays. In general, whenever the seller relies *solely* on the transfers from non-winning bidders, she achieves unbounded interim revenues.

This insight offers a new perspective on why the FPA might be considered the less risky auction format for the seller: The FPA produces the narrowest range of possible interim revenues among standard auctions and maximizes the minimum interim revenue (Corollary 4.4).

However, for this interpretation to be valid, bidders must play the symmetric equilibrium of the selected format. This in turns requires that bidders take the auction format as given and don't make inference about the valuation of one competitor from the seller's choice.Such assumption is implicitly maintained in the analysis of any specific format, but becomes problematic when the format itself is endogenous. This poses a threat to the applicability of our analysis to settings in which the seller chooses an auction format based on the information about a bidder.[1] In this work, we do not explicitly address the plausibility of this assumption, which clearly depends on the specifics of the application at hands.[2] For this reason, although we believe that our analysis gives important insights to this problem, we refrain from addressing them explicitly and phrase our analysis as as an exploration of the properties of equilibria in a set of ex-ante equivalent formats.

The rest of the paper is organized as follows. After describing the environment in Section 2, we compare the FPA and the SPA in Section 3: Section 3.1 describes an example with uniform distributions, Section 3.2 presents the general analysis and Section 3.3 considers the presence of a reserve price. We then introduce the broad class of standard auctions in Section 4. Sections 4.2 and 4.3 consider the winner-pay

[1] A preliminary analysis of a setting where bidders infer the value of a competitor from the choice of the auction format and adjust their bid accordingly indicates that the ability to leverage such information is limited: When the seller can choose between FPA and SPA, there exists an equilibrium in which the SPA (the format where the margin for bid adjustment is negligible) is selected for nearly all values of the special bidder. See Appendix 6.1 for a formal description of the environment with savvy-bidders and this preliminary results.

[2] It is evident that such requirement is not needed for the special bidder: as long as other bidders play the symmetric equilibrium, his best response is, tautologically, the equilibrium bid. This observation possibly makes our unawareness assumption more natural in repeated settings where the other bidders are first-time participants (like in procurement settings where new firms face an incumbent provider of the service) and take the symmetric environment as a reasonable initial point of their learning process, while the special bidder is not "systematically fooled".

and the pay-as-bid auctions, and Section 4.4 analyzes the auctions that maximizes the interim revenue when the special bidder's value is close to the extrema. The last section concludes. Some extensions and applications of our analysis, omitted proofs and derivations are in the Appendix.

2 Environment

There are n risk-neutral bidders with independent private valuations for an object. Valuations are drawn from the CDF $F(\cdot)$ with support $[0,1]$ and continuous density f.

Throughout the paper, we denote $\boldsymbol{v} \in [0,1]^n$ a realization of bidders' values, $v_i(\boldsymbol{v})$ the valuation of bidder i and $v_{(i)}(\boldsymbol{v})$ the i^{th}-order statistic, suppressing the argument $\boldsymbol{v}$ for convenience. Moreover, expectation operators $\mathbb{E}, \mathbb{E}_{\boldsymbol{v}}, \mathbb{E}_{\boldsymbol{v}|v}$ integrate, respectively, over F, the product distribution $\times_1^n F$, and the product distribution conditional on the event $v_1 = v$.[3] $\mathbb{P}, \mathbb{P}_{\boldsymbol{v}}, \mathbb{P}_{\boldsymbol{v}|v}$ denote the associated probability measures.

We consider sealed-bid auction formats that satisfy the assumptions of the Revenue Equivalence Theorem. For a generic format a, let $\Pi^a : [0,1]^n \to \mathbb{R}$ be the seller's revenue as a function of bidders' valuations in its efficient equilibrium. The RET states that in any such equilibrium, each bidder with valuation v will make the same expected transfer

$$t(v) \equiv \int_0^v x \mathrm{d}F^{n-1}(x),$$

and the seller obtains the same ex-ante expected revenue

$$\bar{\Pi} = \mathbb{E}_{\boldsymbol{v}}[\Pi^a].$$

We study the interim revenue function (IRF)

$$\mathit{\Pi}^a(v) \equiv \mathbb{E}_{\boldsymbol{v}|v}[\Pi^a],$$

which represents the expected seller's revenue in the efficient equilibrium of auction a conditional on one bidder (WLOG, the first) having valuation v.

3 FPA vs. SPA

In this section, we focus on the two most common sealed-bid auctions: the FPA and the SPA. First, we build intuition in an example where the interim revenue can be computed in closed form. Then, we

[3] For generic functions $g : [0,1] \to \mathbb{R}$, $G : [0,1]^n \to \mathbb{R}$, we denote $\mathbb{E}[g(v)] = \int_0^1 g(v) \mathrm{d}F(v)$,

$$\mathbb{E}_{\boldsymbol{v}}[G(\boldsymbol{v})] = \underbrace{\int_0^1 \cdots \int_0^1}_{n \text{ times}} G(v_1, \ldots, v_n) \mathrm{d}F(v_1) \times \cdots \times \mathrm{d}F(v_n),$$

and

$$\mathbb{E}_{\boldsymbol{v}|v}[G(\boldsymbol{v})] = \underbrace{\int_0^1 \cdots \int_0^1}_{n-1 \text{ times}} G(v, v_2, \ldots, v_n) \mathrm{d}F(v_2) \times \cdots \times \mathrm{d}F(v_n).$$

generalize the analysis and show that the FPA yields higher interim revenues than the SPA if and only if the valuation of the special bidder is below a threshold.

The analysis in this section leverages the characteristics of the auction formats under consideration to obtain a tight characterization of the IRF, but is not applicable to all standard auctions discussed in Section 4. In particular, the analysis of this section relies on the observation that in the SPA bidders employ the dominant strategy equilibrium $b^S(x) = x$, whereas in the FPA they bid

$$b^F(x) = \frac{t(x)}{F^{n-1}(x)} = x - \int_0^x y \frac{F^{n-1}(y)}{F^{n-1}(x)} \mathrm{d}y.$$

3.1 Example: 2 uniform bidders

Suppose there are two bidders with valuation drawn from the uniform distribution, so $b^F(x) = \frac{x}{2}$. Let v be the valuation of the special bidder (known ex-interim), and x the valuation of the other bidder (unknown to the seller). Anticipating the construction of Lemma 1, we compute the interim revenue functions (IRFs) separating the cases in which v wins (i.e. $x < v$) and v loses (i.e. $x > v$) in either format.

If v wins, the seller receives $b^F(v)$ in the FPA, while in the SPA she receives the expected bid (equal to the value) of the other bidder, x, conditional on $x < v$. If v loses, the SPA yields deterministic revenues $b^S(v) = v$, while in the FPA the seller receives the expected bid (half the value) of the other bidder, x, conditional on $x > v$. In summary,

Auction Format	v wins	v loses
FPA	$b^F(v) = \frac{v}{2}$	$\mathbb{E}\left[b^F(x) \mid x > v\right] = \frac{1+v}{4}$
SPA	$\mathbb{E}\left[b^S(x) \mid x < v\right] = \frac{v}{2}$	$b^S(v) = v$

Table 1: Interim revenue in the FPA and SPA when the special bidder wins or loses

The IRFs are obtained by weighting the events that the special bidder wins and loses by their likelihood (respectively, v and $1 - v$):

$$\begin{aligned} \varPi^S(v) &= \mathbb{P}(x < v) \cdot \mathbb{E}\left[b^S(x) \mid x < v\right] + \mathbb{P}(x > v) \cdot b^S(v) \\ &= v \cdot \frac{v}{2} + (1 - v) \cdot v = v\left(1 - \frac{v}{2}\right) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \varPi^F(v) &= \mathbb{P}(x < v) \cdot b^F(v) + \mathbb{P}(x > v) \cdot \mathbb{E}\left[b^F(x) \mid x > v\right] \\ &= v \cdot \frac{v}{2} + (1 - v) \cdot \frac{1 + v}{4} = \frac{1 + v^2}{4}. \end{aligned} \tag{3.2}$$

Inspecting equations (3.2) and (3.1), graphically represented in Figure 3.1, we make the following observations.

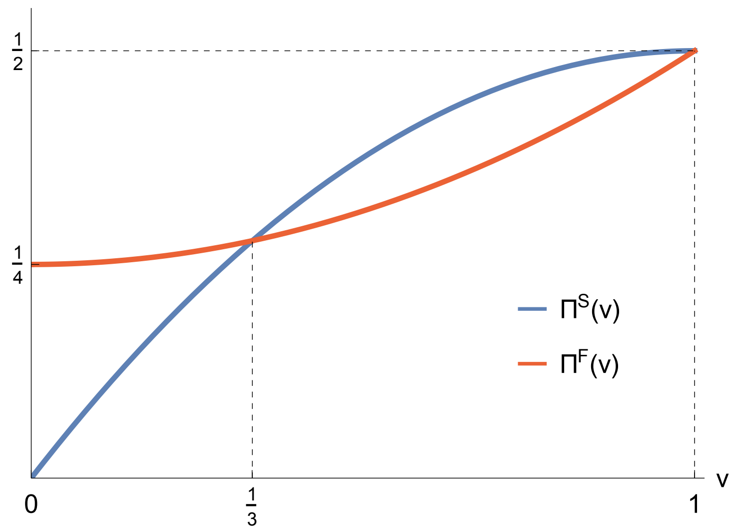

Figure 3.1: Interim revenue functions in the FPA and SPA with $n = 2$ and $v \sim \mathcal{U}[0,1]$.

Ranking at the extrema If the special bidder has valuation $v = 0$, the interim revenue is equal to 0 in the SPA and strictly positive in the FPA. This is natural: in both formats the special bidder loses for sure, however, in the SPA he sets the auction price (to zero), while in the FPA the auction price is set by the bid of the other bidder, which is, in expectation, $\mathbb{E}\left[\frac{x}{2}\right] = \frac{1}{4}$.

If the special bidder has valuation $v = 1$, the interim revenue is the same in FPA and SPA, and equal to $\frac{1}{2}$. In both formats the special bidder wins for sure, and he pays his bid $b^F(1) = \frac{1}{2}$ in the FPA and the bid of the other bidder in the SPA. The latter is equal to $\frac{1}{2}$ in expectation.

From Table 1, observe that whenever the special bidder wins the seller's revenue is the same in the two auction formats. This is an implication of the RET: to ensure that the expected payment of a bidder with value v is ex-ante the same in the two formats, the expected transfer conditional on winning must be the same in the two formats. In the FPA, this transfer is deterministic and equal to $\frac{v}{2}$, which equals the expected bid in the SPA of a losing competitor (distributed uniformly with support truncated at v). When $v = 1$, the special bidder wins with probability 1, implying interim revenue equivalence between the two auction formats.

Crossing The FPA interim dominates the SPA at $v = 0$, and the two are equivalent at $v = 1$. However, the FPA cannot dominate the SPA over the whole support, because the expected value of the two IRFs must be the same. The latter is another consequence of the RET: by iterated expectations the expected interim revenue is the ex-ante expected revenue, and therefore it is the same in any auction format.[4] Hence, the two IRFs must cross at least once for interior valuations. In particular, Table 1 shows that that the FPA interim dominates whenever $\frac{v+1}{4} > v$, that is whenever $v < \frac{1}{3}$.Thus, the interior crossing is unique.

In summary, the difference in interim revenues $\Pi^F(v) - \Pi^S(v) = (1-v)\left(\frac{1-3v}{4}\right)$ is zero either if $v = 1$, because the special bidder always wins, or if $v = \frac{1}{3}$, because the two formats yield the same expected revenue also in the event the special bidder loses. Moreover, the difference between interim revenues in

[4] For the case under discussion, $\mathbb{E}\left[\Pi^F(v)\right] = \int_0^1 \frac{1+v^2}{4} \mathrm{d}v = \frac{1}{3} = \int_0^1 v\left(1 - \frac{v}{2}\right) \mathrm{d}v = \mathbb{E}\left[\Pi^S(v)\right]$.

the FPA and SPA is maximized at $v = 0$, minimized at $v = \frac{2}{3}$, and strictly positive if and only if $v < \frac{1}{3}$.

3.2 General Analysis

We now show that the properties of the IRFs of the FPA and SPA discussed before extend to the case with general distribution F and number of bidders n. We begin by providing a decomposition of the IRF that will be useful in establishing single crossing. The decomposition highlights that the FPA and SPA yield different revenues only in the event the special bidder loses the auction.

Lemma 1. *If a is a FPA or SPA,*

$$\Pi^a(v) = t(v) + \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} \neq v\right) \mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^a | v_{(1)} \neq v\right] \tag{3.3}$$

where $t(v)$ is the transfer of the special bidder and $\mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^a | v_{(1)} \neq v\right]$ is the revenue in auction a when the special bidder is not the winner of the auction (he does not have the highest has valuation).

Lemma 1 decomposes the IRFs in the event the special bidder wins and loses the auction, and highlights that FPA and SPA are interim equivalent when the special bidder is the winner. When the special bidder wins, the seller's revenue is the expected payment of the special bidder and hence independent of the auction format. A fortiori, the two auction formats are interim equivalent when $v = 1$, as the special bidder wins for sure.

By Lemma 1,

$$\begin{aligned} \Delta(v) &\equiv \Pi^F(v) - \Pi^S(v) \\ &= \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} \neq v\right)\left(\mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^F | v_{(1)} \neq v\right] - \mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^S | v_{(1)} \neq v\right]\right). \end{aligned} \tag{3.4}$$

The following theorem shows that $\Delta(v)$ is positive — i.e., the FPA yields a higher interim revenue than the SPA — if and only if the value of the special bidder is lower than a threshold (see Figure 3.2 for an example).

Theorem 2. *Let $\psi(v) \coloneqq v - \frac{1-F(v)}{f(v)}$ be the virtual value function and suppose there is a unique $\hat{v}$ such that $b^F(\hat{v}) = \psi(\hat{v})$. Then, there is a unique $\tilde{v}$ such that*

$$\begin{cases} \Delta(v) > 0 & \text{if } v \in [0, \tilde{v}) \\ \Delta(v) < 0 & \text{if } v \in (\tilde{v}, 1) \\ \Delta(v) = 0 & \text{if } v \in \{\tilde{v}; 1\} \end{cases}$$

Moreover, $\Delta(v)$ has a global maximum at $v = 0$ and global minimum at $\hat{v}$.

Proof. By (3.4), the Interim revenue difference between the FPA and SPA is

$$\begin{aligned} \Delta(v) &= \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} \neq v\right) \cdot \left(\mathbb{E}_{\boldsymbol{v}|v}\left[b^F\left(v_{(1)}\right) | v_{(1)} \neq v\right] - \mathbb{E}_{\boldsymbol{v}|v}\left[v_{(2)} | v_{(1)} \neq v\right]\right) \\ &= \int_v^1 b^F(x)\, \mathrm{d}F^{n-1}(x) - \int_v^1 \int_0^x \max\{v, y\} \frac{\mathrm{d}F^{n-2}(y)}{F^{n-2}(x)} \mathrm{d}F^{n-1}(x). \end{aligned} \tag{3.5}$$

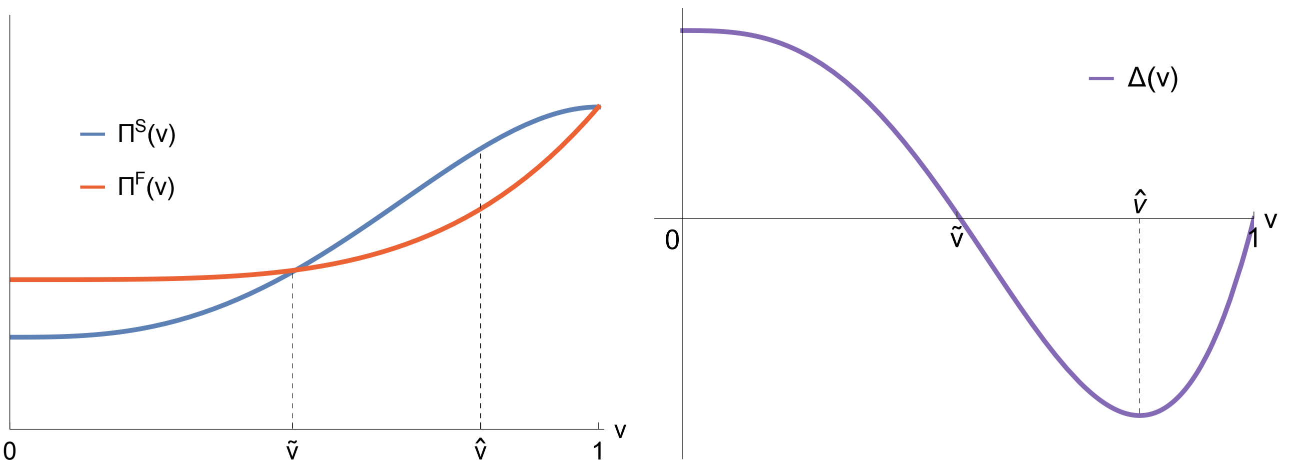

Figure 3.2: Interim revenue functions in the FPA and SPA (left) and interim revenue difference $\Delta(v)$ (right) with $n = 4$ and $v_i \sim \mathcal{U}[0, 1]$.

The first term is the expected highest bid in the FPA when v loses, while the second term is the expected second-highest bid in the SPA — which is the expected second-highest value — when v loses.

In the Appendix, we show the second term is equal to $\int_v^1 \psi(x)\,\mathrm{d}F^{n-1}(x)$. Therefore,

$$\Delta(v) = \int_v^1 \left(b^F(x) - \psi(x)\right) \mathrm{d}F^{n-1}(x), \tag{3.6}$$

and

$$\Delta'(v) = -\left(b^F(v) - \psi(v)\right) \mathrm{d}F(v)^{n-1}.$$

Critical points of $\Delta(v)$ are either 0 or such that $b^F(v) = \psi(v)$. In the Appendix, we show that $\Delta(v)$ has a maximum at $v = 0$ and $\Delta(0) > 0$ (see Lemma 14). Moreover, it is straightforward that $\Delta(1) = 0$, $\Delta'(1) \propto 1 - b^F(1) > 0$ and, by the RET, $\mathbb{E}[\Delta(v)] = 0$.

It follows that $\Delta(v)$ must cross the horizontal axis (from above) before 1 and must have minimum at $\hat{v}$ such that $b^F(\hat{v}) = \psi(\hat{v})$. Under our assumption that there is a unique v such that $b^F(v) = \psi(v)$, there are exactly two critical points of $\Delta(v)$: (i) the maximum at 0 is unique and (ii) the minimum at $\hat{v}$ is unique. Hence, $\Delta(v)$ cannot cross the horizontal axis more than once and so there is a unique $\tilde{v} \neq 1$ such that: (i) $\Delta(\tilde{v}) = 0$, (ii) $\mathit{\Pi}^F(v) > \mathit{\Pi}^S(v)$ if $v < \tilde{v}$, and (iii) $\mathit{\Pi}^F(v) < \mathit{\Pi}^S(v)$ if $\tilde{v} < v < 1$. □

To shed light on equation (3.6), we notice that, perhaps surprisingly, the second term in equation (3.5) is the ex-ante expected revenue in *any* auction with *reserve price* v *and* $n - 1$ *bidders*. The reason is the following. First, the probability that in an auction with n bidders a bidder with value v loses is exactly equal to the probability that, in an auction with $n - 1$ bidders and reserve price v, there is at least one bidder with value higher than the reserve price. Hence, $\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} \neq v\right)$ represents the probability that the seller's revenue is different from 0 in any such auction. Second, $\mathbb{E}_{\boldsymbol{v}|v}\left[v_{(2)}|v_{(1)} \neq v\right]$ — i.e., the expected second-highest value when one bidder among n has value v and this is not the highest value — is exactly equivalent to the expected seller's revenue in a SPA with $n - 1$ bidders and reserve price v, when the highest bidder has a value higher than the reserve price. In fact, the seller's revenue in this auction is the reserve price v when only one bidder has a value higher than v and is the second-highest value otherwise.

Intuitively, when a bidder with value v loses in a SPA with n bidder she acts exactly as a reserve price in a SPA with $n-1$ bidders, since his bid represents a lower bound to the price that the remaining bidders have pay.

Finally, the SPA with $n-1$ bidders and reserve v is revenue equivalent to any auction with $n-1$ bidders and reserve v, and yields an expected revenue equal to the expected virtual value of the highest bidder in the auction, conditional on this bidder having a value higher than the reserve price.

Theorem 2 states that the IRFs of the FPA and SPA cross only once for interior valuations, so that the seller obtains a higher interim revenue in the FPA if $v < \tilde{v}$, while she obtain a higher interim revenue in the SPA if $v > \tilde{v}$. Single crossing is relevant because it allows the seller to determine her preferred auction format not only when she knows the exact valuation of the special bidders, but also with different and arguably more realistic information structures. For example, in many application the seller may only know that the special bidder's value is either above or below an *arbitrary* threshold, possibly different from $\tilde{v}$. Because of single crossing, the seller prefers the FPA (resp. SPA) when she knows that the special bidder is below (resp. above) *any* threshold, since this is the auction format that yields the higher *expected* interim revenue conditional on this information. See Appendix 6.1 for a formalization of this argument.

3.3 Reserve Price

Our main result on the comparison between the FPA and SPA extends to a setting with a common reserve price R. Let $\Pi^a(v,R)$ be the IRF in auction a with reserve price R, and let the interim revenue difference between the FPA and SPA be

$$\Delta(v,R) \equiv \Pi^F(v,R) - \Pi^S(v,R).$$

Proposition 3. *There is a unique $\tilde{v} > R$ such that*

$$\begin{cases} \Delta(v,R) > 0 & \text{if } v \in [0,\tilde{v}) \\ \Delta(v,R) < 0 & \text{if } v \in (\tilde{v},1) \\ \Delta(v,R) = 0 & \text{if } v \in \{\tilde{v};1\} \end{cases}$$

Moreover, $\Delta(v,R)$ is maximized at all $v \leq R$ and has a global minimum at $\hat{v}$ such that $b^F(\hat{v},R) = \psi(\hat{v})$.

Figure 3.3 compares the IRFs of the FPA and SPA. Notice that, when the special bidder's value is lower R, the interim revenue is constant in both auction formats, but it is higher in the FPA than in the SPA. This is obvious when there are only two bidders: if the seller knows that one bidder has a value lower than R, then her revenue is positive if and only if the other bidder's value is higher than R. In this case, however, the seller's revenue is equal to the reserve price in the SPA, while it is equal to the expectation of the other bidder's bid in the FPA (which is higher than R because the bidder has a value higher than the reserve price). Similarly, with multiple bidders, when the highest bidder's value is higher than R (and the special bidder's value is lower than R): in the SPA the seller's revenue is the maximum between R and the second-highest among $n-1$ bidders, while in the FPA it is equal to the highest bidder's bid, which is the expectation of the maximum between R and the second-highest *among n bidders*.

Moreover, in both auction formats the interim revenue is discontinuous when the special bidder's value is exactly equal to the reserve price. The reason is that, if the seller knows that at least one bidder meets the reserve price, then her revenue is always strictly positive. In both auction formats, the discontinuous

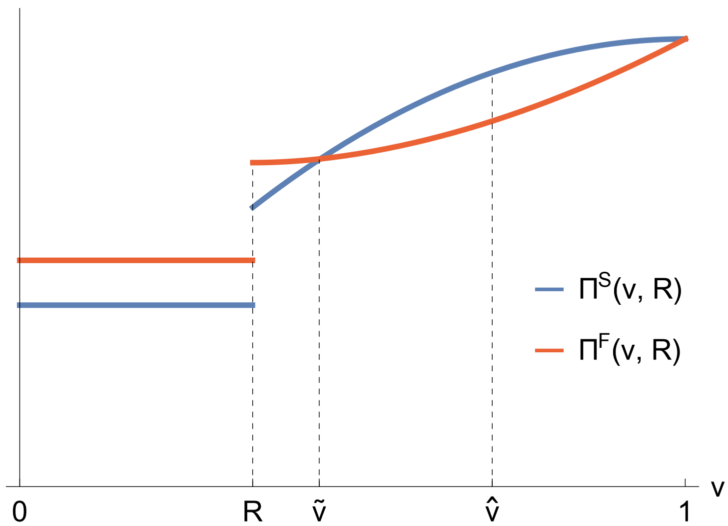

Figure 3.3: Interim revenue functions in the FPA and SPA with $n = 2$, $R = 0.35$ and $v_i \sim \mathcal{U}\,[0, 1]$.

increase in the interim revenue when $v = R$ is equal to the probability that all other bidders have a value lower than the reserve price, times the reserve price.[5] Finally, the intuition for the single crossing of the IRFs and for the equivalence of the two auction formats at $v = 1$ is analogous to the case without a reserve price.

Since Proposition 3 holds for any reserve price, our comparison between the FPA and SPA also holds when the seller sets the optimal (ex-ante) reserve prices, which are the same in both auctions. Therefore, while the (ex-ante) expected revenues in the optimal FPA and SPA are equal, the optimal FPA yields a higher (lower) interim revenue than the optimal SPA if v is low (high), exactly as with FPA and SPA without reserve prices.

4 Interim Revenue in Standard Auctions

We now turn to the analysis of a broad class of sealed-bid auction formats, referred to as *standard auctions*. Section 4.1 defines this class of auctions and shows that differences in their IRFs only depend on the interim transfer of a generic (non-special) bidder. We then consider special cases of standard auctions: the winner-pay auctions in Section 4.2 and the pay-as-bid auctions in Section 4.3. Finally, Section 4.4 characterizes the auction formats that maximize the interim seller's revenue when the special bidder's value is close to the extrema.

4.1 Standard Auctions

In all standard auctions, each bidder submits a bid which is a non-negative real number; bids are ranked and the object is assigned to the highest bid.[6] Standard auctions differ in how transfers are determined.

Let $[n] := \{1, \ldots n\}$. An element $i \in [n]$ is interpreted as the i^{th}-order statistic of submitted bids.

[5] In both auction formats, when all other bidders have a value lower than R, the seller's revenue is 0 when the special bidder's value is also lower than R, and is R when the special bidder's value is equal to R.

[6] Throughout the exposition, the bid identifies the bidder who submits it. Hence "the highest bid" is shorthand for "the bidder who submits the highest bid".

Definition 4. A standard auction a is characterized by a non-empty set $\mathcal{P}_a \subseteq [n]$ and by a function $T_a : \mathcal{P}_a \to [n]$ such that $T_a(j) \geq j$ for all $j \in \mathcal{P}_a$.

The set $\mathcal{P}_a$ specifies the order statistics that pay. The function T_a associates to each payer the the order statistic of the bid that he pays, with the constraint that a bidder cannot pay a bid higher than his own. $\mathcal{P}_a$ and T_a identify the bid-transfer map $\tilde{t}^a : \mathbb{R}^n \to \mathbb{R}^n$, where $\tilde{t}^a_i(\boldsymbol{b})$ denotes the payment that bidder i makes if the profile of bids is $\boldsymbol{b}$.

We can represent a standard auction concisely as a partial function from $[n]$ to itself. A partial function specifies the domain over which the function is defined, i.e. $\mathcal{P}_a \subseteq [n]$, and a function $(T_a : \mathcal{P}_a \to [n])$ over this domain.[7] Figure 4.1 gives the standard auction representation of common formats (FPA, SPA, APA), as well as uncommon ones (APL, LPA).

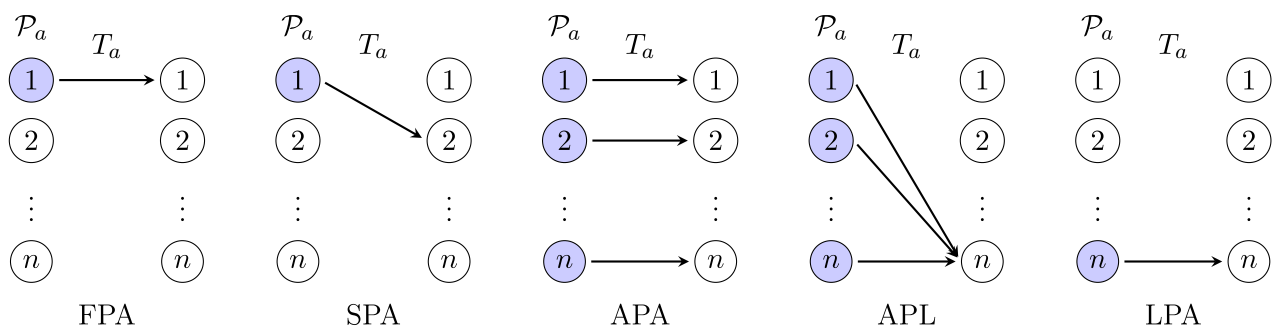

Figure 4.1: Examples of Standard Auctions. Filled circles represent the set of payers $\mathcal{P}_a$, arrows represent the transfer function T_a

Standard auctions encompass the following familiar formats.

Definition 5. A Winner-Pay Auction (WPA) is a standard auction with $\mathcal{P}_a = \{1\}$. A Pay-as-Bid-Auction (PBA) is a standard auction with $T_a(i) = i$, $\forall i \in \mathcal{P}_a$.

WPAs are auctions where only the bidder who obtains the object pays. The FPA and SPA are WPAs with $T_a(1) = 1$ and $T_a(1) = 2$, respectively. In general, WPAs are characterized by the order of the bid that determines the transfer: we refer to the WPA with $T_a(1) = k$ as the k^{th}-price auction (kPA).

PBAs are auctions where all payers pay their own bids. Notice that the FPA is a PBA while any other WPA is not; indeed to each set of payers is associated a unique PBA. The All-Pay Auction (APA) is the PBA with $\mathcal{P} = [n]$. The Last-Pay Auction (LPA, fifth panel in Figure 4.1) is the PBA with $\mathcal{P} = \{n\}$ — i.e., the auction where *only the lowest bidder* pays (and pays his bid). The All-Pay-Last Auction (APL, fourth panel) is the is the auction with $\mathcal{P} = [n]$ and $T(i) = n$, $\forall i$ — i.e., the auction where *everyone* pays the last bid.

Equilibrium and Revenue Equivalence Each standard auction defines a Bayesian game. We assume that this Bayesian game admits an efficient equilibrium, namely that there exists an increasing function $b^a : [0,1] \to \mathbb{R}$ with $b^a(0) = 0$ such that for each v, $b^a(v)$ maximizes the expected utility, computed under

[7] We keep the notation separate since it will be convenient to focus on the random set of payers, i.e. the domain specification of the partial function.

the rules of a, of a bidder of type v when his $n-1$ competitors use b^a,[8]

$$b^a(v) \in \arg\max_b v\mathbb{P}_{\boldsymbol{v}}\left(b > \max b^a(\boldsymbol{v}_{-i})\right) - \mathbb{E}_{\boldsymbol{v}}\left(\tilde{t}_i^a\left(b, b^a(\boldsymbol{v}_{-i})\right)\right).$$

From now on, we identify each auction with its efficient equilibrium and simplify our notation by not explicitly including the bidding function,[9] denoting $\tilde{t}^a(b^a(\boldsymbol{v}))$ as $\tilde{t}^a(\boldsymbol{v})$ the equilibrium transfer vector.

Let $\Pi^a : [0,1]^n \to \mathbb{R}$ denote the seller's revenue as a function of the bidders' valuations in the efficient equilibrium of auction format a, given by $\Pi^a(\boldsymbol{v}) = \sum_{i\in[n]} \tilde{t}_i^a(\boldsymbol{v})$. The RET implies that the transfer of the special bidder is independent of the auction format a, i.e.[10]

$$t(v) = \mathbb{E}_{\boldsymbol{v}|v}\left[\tilde{t}_1^a(\boldsymbol{v})\right] = \int_0^v x \mathrm{d}F^{n-1}(x) \tag{4.1}$$

and, therefore, $\mathbb{E}_{\boldsymbol{v}}[\Pi^a] = \bar{\Pi} = n\mathbb{E}[t(v)]$. The expected transfer of a bidder given that one competitor has valuation of v, referred to as *others' transfer*, is

$$t^a(v) \equiv \mathbb{E}_{\boldsymbol{v}|v}\left[\tilde{t}_i^a(\boldsymbol{v})\right] \text{ for } i \neq 1. \tag{4.2}$$

Notice the distinction between $t(v)$ and $t^a(v)$: the former represents the expected transfer of a bidder conditional on his own valuation being equal to v, while the latter is conditional on a competitor's valuation being equal to v. This transfer is not pinned down by the RET and depends on the auction format.

Finally, we define

$$t^a(x,v) = \mathbb{E}_{\boldsymbol{v}|v}\left[\tilde{t}_i^a(\boldsymbol{v})\,|v_i = x\right] \text{ for } i \neq 1. \tag{4.3}$$

the expected transfer in auction a of a bidder given that i) his valuation is x and ii) the valuation of a competitor is v. By construction, $t(x) = \mathbb{E}_v[t^a(x,v)]$ and $t^a(v) = \mathbb{E}_x[t^a(x,v)]$.[11]

Lemma 6. *For a standard auction a, the IRF satisfies*

$$\Pi^a(v) = t(v) + (n-1)\,t^a(v). \tag{4.4}$$

Moreover,

- $\mathbb{E}[\Pi^a(v)] = \bar{\Pi}$ *and* $\mathbb{E}[t^a(v)] = \mathbb{E}[t(v)]$
- $\Pi^a(v)$ *is continuous, increasing and bounded below by* $t(v)$

Both equations (4.4) and (3.3) provide an additive decomposition of the IRF in $t(v)$ and a term that is auction-specific. However, the approach leading to the two decompositions is conceptually different. Equation (4.4) is obtained by writing the revenue as the sum of the transfer made by each bidder, without

[8]Since all standard auction assign the object to the highest bidder, efficiency coincides with monotonicity of the bidding function. With slight abuse of notation, for a vector $\boldsymbol{v}$ we denote $b^a(\boldsymbol{v}) = [b^a(v_1), \dots, b^a(v_n)]$.

[9]Equation (6.19) in the Appendix characterizes this bidding function as the solution to a functional equation. We derive it analytically for uncommon standard auctions with uniform bidders.

[10]Recall that we have assumed that the special bidder "sits" in slot 1, so $\mathbb{E}_{\boldsymbol{v}|v}\left[\tilde{t}_1^a(\boldsymbol{v})\right]$

[11]Notice that t^a has two arguments — as in (4.3) — when we are conditioning both on the individual (first argument) and the competitor's (second argument) valuations, while it has a unique argument — as in (4.2) — when we are conditioning only on the competitor's valuation.

conditioning on his order: $t(v)$ is the expected transfer of a bidder with valuation v which is auction-independent by the RET; the "residual" part is what others pay, which is conveniently summarized by the $t^a(v)$ function. For Lemma 1, instead, we computed the revenues in the events the special bidder is the first-order statistic or any other order statistic. In the former case, the special bidder wins and, according to the rules of the auctions under consideration (the FPA and the SPA) is the only one to make a positive transfer: $t(v)$ in (3.3) results as the product of the probability that v wins and the revenues in the auction conditional on v winning which, in those formats, coincide with his transfer; the "residual" part is the expected revenue conditional on the event that a loser has valuation v. This construction remains valid for the other WPAs (which are still interim-equivalent conditional on the valuation of the winner, see Proposition 8), but not for other standard auctions where bidders other than the winner make positive transfers, i.e. where $\mathcal{P}_a \neq \{1\}$.

Equation (4.4) highlights that the auction format does not affect the interim revenue through the special bidder's transfer, pinned down to $t(v)$ by the RET, but only through the transfers of the other (interim symmetric) bidders. Indeed, as an immediate corollary we obtain that $\mathit{\Pi}^a(v) > \mathit{\Pi}^{a'}(v)$ if and only if $t^a(v) > t^{a'}(v)$, namely the interim revenues are higher in format a than in format a' if and only if a bidder makes a higher transfer in the efficient equilibrium of a than in the efficient equilibrium of a' when a *competitor* has valuation v.

For the applied interpretation of our analysis, a seller who chooses the auction format based on interim dominance exploits her information about the valuation of the special bidder by maximizing the surplus extracted from the *other* bidders: the seller has an information advantage over these bidders because the valuation of a competitor affects their equilibrium transfer, but their bids are submitted without accounting for that (unsophistication). The way the valuation of a competitor affects each bidder's equilibrium transfer differs across auction formats so, through this channel, the seller can exploit her information by manipulating the format.

The Lemma also gives some properties of the t^a and therefore $\mathit{\Pi}^a$ functions. Taking expectations under v, we obtain

$$\mathbb{E}\left[\mathit{\Pi}^a(v)\right] = \mathbb{E}\left[\mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^a\right]\right] = \mathbb{E}_{\boldsymbol{v}}\left[\Pi^a\right] = \bar{\Pi}.$$

Indeed, by iterated expectations (second equality), the expected interim revenue is the ex-ante expected revenue, and therefore it is the same in any standard auction. Using (4.4) and the RET yields

$$\bar{\Pi} = n\mathbb{E}\left[t(v)\right] = \mathbb{E}\left[t(v)\right] + (n-1)\,\mathbb{E}\left[t^a(v)\right],$$

which implies that $\mathbb{E}[t(v)] = \mathbb{E}[t^a(v)]$. This is intuitive: the two sides are equal to the ex-ante expected transfer because they denote the expected transfer of a bidder prior to drawing, respectively, his valuation and the valuation of one competitor.

Hence, $t^a(v)$ is *i*) a positive function, implying that $t(v)$ is a lower bound for all IRFs, and *ii*) it integrates to $\mathbb{E}[t(v)]$. Two natural candidates for t^a are the constant function $\mathbb{E}[t(v)]$ and the (increasing) function $t(v)$ itself. The standard auction such that $t(v) = \mathbb{E}[t(v)]$ is the All-Pay Auction, as in that format the *realized* transfer of each bidder is independent of a competitor's valuation. If $t(v) = t^a(v)$, then for each v a bidder makes the same expected transfer if v represent his own or a competitor's valuation. A sufficient condition is that for every realization of $\boldsymbol{v}$ all bidders make the same transfer. In a standard

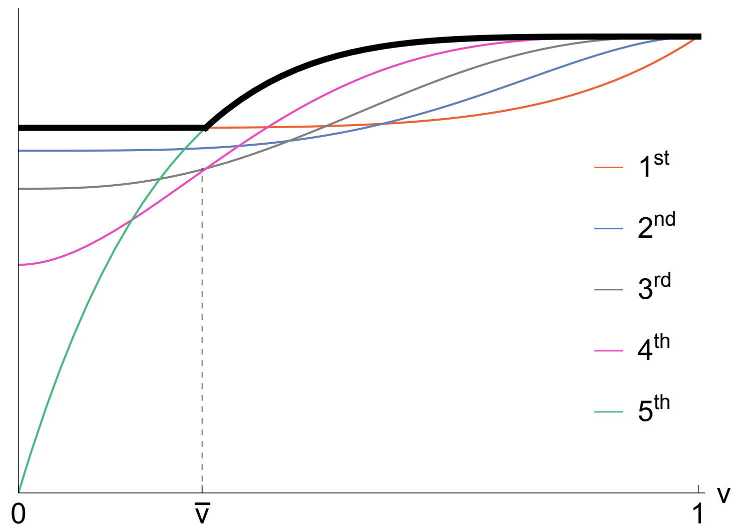

Figure 4.2: Interim revenue functions in kPA with $k = 1, ..., 5$, $n = 5$ and $v_i \sim \mathcal{U}[0,1]$.

auction happens only if everyone pays the last bid, i.e. in the APL.

Remark 7. The interim transfers of the formats in Figure 4.1 are: $t^{APA}(v) = \mathbb{E}[t(v)]$ constant in v; $t^{APL}(v) = t(v)$ monotonically increasing and bounded; $t^{LPA}(v) = \int_0^v \frac{t(x)}{1-F(x)} \mathrm{d}F(x)$ monotonically increasing and unbounded; $t^{FPA}(v) = \int_v^1 t(x)\,\mathrm{d}\log F(x)$ monotonically decreasing. If $n = 2$, then $t^{SPA}(v) = v(1 - F(v))$ hump-shaped.

These derivations highlight that t^a is not necessarily monotonic; indeed to prove that Π^a is increasing in any format we have to rely on a constructive argument. Using the monotonicity properties alone we can infer single crossing of the IRFs in many formats, e.g. the APA and the FPA, or the FPA and the APL.

4.2 Winner-Pay Auctions

Each WPA is a kPA, an auction where the highest bidder wins and pays the k^{th}-highest bid. We denote by Π^k, b^k and t^k the IRF, equilibrium bidding function and transfer of the kPA.[12]

Proposition 8. *$\Pi^k(0)$ is decreasing in k. $\Pi^k(1) = t(1)$ for every k, and there is a threshold $\hat{v}$ such that $\Pi^k(v)$ is increasing in k for $v \in (\hat{v}, 1)$.*

Proposition 8 generalizes Theorem 2 for extreme values of v, establishing that if the special bidder has very low (resp. high) valuation, the seller prefers the kPA with lower (resp. higher) k (see Fig.4.2). Therefore, among all WPAs: (i) the FPA — i.e., the kPA with $k = 1$ — yields the highest interim revenue when $v = 0$; (ii) the nPA — i.e., the auction where the winner pays the lowest bid — yields the highest interim revenue for $v \approx 1$.

We prove the result at $v = 0$ by showing that, for all x, $t^k(x, 0)$ defined in (4.3) is decreasing in k, which implies that $t^k(0) = \mathbb{E}_x\left[t^k(x, 0)\right]$ is also decreasing in k. In the FPA a type x makes a positive payment if and only if he is the highest bidder, an event that given a competitor is 0 has probability F^{n-2},

[12]To maintain consistency within this section we use the superscripts 1,2 (rather than F, S) to denote the FPA and SPA.

and conditional on that event he transfers his bid,

$$t^1(x,0) = F^{n-1}(x)\, b^1(x) = F^{n-1}(x)\, \mathbb{E}_{\boldsymbol{v}}\left[b^k\left(v_{(k)}(\boldsymbol{v})\right)|v_{(1)} = x\right]$$

where the last equality holds for all $k > 1$ (by the RET). The interim transfer in WPAs of higher order is instead

$$t^k(x,0) = F^{n-1}(x)\, \mathbb{E}_{\boldsymbol{v}}\left[b^k\left(v_{(k)}(\boldsymbol{v})\right)|v_{(1)} = x,\ v_{(n)} = 0\right].$$

Since the condition $v_{(n)} = 0$ amounts to removing one draw and thus reduces the expectation of any increasing function, we conclude that $t^1(x,0) > t^k(x,0)$ for all $k > 1$.

More in general, in the proof we show that the RET implies that for any k,[13]

$$b^k(x) = \mathbb{E}_{\boldsymbol{v}}\left[b^{k+1}\left(v_{(k+1)}(\boldsymbol{v})\right)|v_{(k)} = x\right] > \mathbb{E}_{\boldsymbol{v}}\left[b^{k+1}\left(v_{(k+1)}(\boldsymbol{v})\right)|v_{(k)} = x,\ v_{(n)} = 0\right]$$

which we can leverage to conclude that $t^k(x,0) > t^{k+1}(x,0)$ and hence that $\mathit{\Pi}^k(0)$ is decreasing in k.

When $v = 1$, all WPAs yield the same interim revenue; this is because, for all k, $t^k(1) = 0$: when a competitor has the highest possible valuation, a bidder never wins and therefore never pays in a WPA. Moreover, the IRFs achieve the lower bound $t(1)$. To show the existence of the threshold $\hat{v}$ in Proposition 8, we first leverage the RET to establish that[14]

$$\begin{aligned}\mathit{\Pi}^{k+1}(v) - \mathit{\Pi}^k(v) &= \sum_{i=k+1}^{n} \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)\left(\mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^{k+1}|v_{(i)} = v\right] - \mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^k|v_{(i)} = v\right]\right) \\ &> \sum_{i=k+1}^{n} \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)\left(b^{k+1}(v) - b^k(1)\right)\end{aligned}$$

where the inequality uses that, if $i \geq k$, $b^k(v)$ constitutes a lower bound for Π^k conditional on $v_{(i)} = v$, and so

$$b^k(v) < \mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^k|v_{(i)} = v\right] < b^k(1).$$

Therefore, if v is above $\hat{v}_k$ is such that $b^{k+1}(\hat{v}) = b^k(1)$ — which always exists by continuity and the fact that $b^{k+1}(1) > b^k(1)$— then $\mathit{\Pi}^{k+1}(v) \geq \mathit{\Pi}^k(v)$, strictly whenever $\sum_{i=k+1}^{n} \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right) > 0$, i.e. for $v < 1$. Then $\hat{v}$ in the Proposition is $\max_k \hat{v}_k$.[15]

[13] The first equality implies that, for all $j \geq k$,

$$\mathbb{E}_{\boldsymbol{v}}\left[\Pi^j|\Pi^k\right] = \Pi^k, \tag{4.5}$$

i.e. if $j > k$ then Π^j is a mean preserving spread of Π^k. Therefore, a risk-averse seller (ex-ante) prefers WPAs where lower orders pay. This is a generalization of Waehrer *et al.* (1998), who show that the FPA is the preferred auction format for a risk-averse seller.

[14] Using (4.5), for $j > k$

$$\mathit{\Pi}^k(v) - \mathit{\Pi}^j(v) = \sum_{i=k+1}^{n} \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)\left(\mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^k|v_{(i)} = v\right] - \mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^j|v_{(i)} = v\right]\right). \tag{4.6}$$

Hence, the interim revenue difference between any two WPAs is driven by the events in which the special bidder is an order statistic above k, where k is the lowest order in the comparison. If $k = 1$ — i.e., comparing the FPA with all other WPAs — then "only losers matter" (Lemma 1).

[15] Figure 4.2 shows that, with uniformly distributed values, the higher is k, the flatter is the interim revenue of the kPA around 1. Formally, in the Appendix we show that, at $v = 1$, all derivatives of $\mathit{\Pi}^k$ up to order $k - 1$ are equal to 0.

In sum, we can interpret the interim ranking between the kPA with the $(k+1)$PA as a race between opposing effects: the fact that the $(k+1)$PA collects the bid of a higher type than the $(k+1)$PA and the fact that in the $(k+1)$PA bids are higher than in the kPA. If $v=0$, the latter effect is second-order, as bids are 0 irrespective of k, while for $v \approx 1$ the former effect is second-order, as the space to get "better bidders" is squeezed between v and 1.

As displayed in Figure 4.2, with uniformly distributed valuations the IRFs of two kPAs cross only once, and there is a threshold $\overline{v}$ such that the FPA is interim dominant (among all WPAs) if $v<\overline{v}$, while the nPA is interim dominant if $v>\overline{v}$. There are no easy conditions on primitives, however, that ensure that these results hold for other distributions.

4.3 Pay-as-Bid Auctions

Denote by PB-$\mathcal{P}$ the PBA with set of payers $\mathcal{P}$. In the PB-$\mathcal{P}$, the transfer of a bidder with value x is equal to his bid times the ex-ante probability of belonging to $\mathcal{P}$, denoted by $\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P})$.[16] Therefore, using the RET, the bid function is

$$b^{\text{PB-}\mathcal{P}}(x)=\frac{t(x)}{\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P})}. \tag{4.7}$$

Moreover, the interim transfer of a generic bidder x is[17]

$$t^{\text{PB-}\mathcal{P}}(x, v)=b^{\text{PB-}\mathcal{P}}(x) \cdot \mathbb{P}_{\boldsymbol{v}|x, v}(x \in \mathcal{P})=t(x) \frac{\mathbb{P}_{\boldsymbol{v}|x, v}(x \in \mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P})}, \tag{4.8}$$

where the ratio $\frac{\mathbb{P}_{\boldsymbol{v}|x, v}(x \in \mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P})}$ measures how the probability that bidder x belongs to the set $\mathcal{P}$ changes with the information that a competitor has value v.

The key property of PBAs is that the transfer conditional on paying is not affected by the valuation of a competitor, who only affects the interim transfer by changing the likelihood of being a payer.

The IRF is obtained by substituting the interim transfer in (4.4),

$$\mathit{\Pi}^{\text{PB-}\mathcal{P}}(v)=t(v)+(n-1) \mathbb{E}\left[t(x) \frac{\mathbb{P}_{\boldsymbol{v}|x, v}(x \in \mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P})}\right]. \tag{4.9}$$

Notice that the FPA is a PB-$\mathcal{P}$ with $\mathcal{P}=\{1\}$. Looking at the FPA as a PBA, we obtain the interim transfer $t^{\text{PB-}\{1\}}(v)=\int_v^1 t(x) \,\mathrm{d} \log F(x)$ by integrating the interim transfer of a generic bidder x, $t^{\text{PB-}\{1\}}(x, v)= \mathbf{1}\{x>v\} \frac{t(x)}{F(x)}$. Since a higher competitor always reduces the probability of being the maximum, the interim transfer is decreasing in v in the FPA. In what follows, we discuss some other special PBAs, then establish the interim ranking.

All-Pay Auction The APA is the PB-$[n]$. Each bidder pays his own bid irrespectively of the other bidders' values: by construction, for each bidder, the realized transfer depends on his valuation alone, and

[16] $\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P})=\sum_{j \in \mathcal{P}} \mathbb{P}_{\boldsymbol{v}|x}\left(v_{(j)}=x\right)$, where $\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(j)}=x\right)=\begin{pmatrix} n-1 \\ j-1 \end{pmatrix} F^{n-j}(x)(1-F(x))^{j-1}$ is the probability that x is the j^{th}-order statistic in a sample of n i.i.d. draws from F.

[17] We denote $\mathbb{P}_{\boldsymbol{v}|v, x}$ as the probability conditional on the event that $v_1=v$ and $v_2=x$.

not on the valuation of a competitor. Formally, notice that $\mathbb{P}_{\boldsymbol{v}|x}(x \in [n]) = \mathbb{P}_{\boldsymbol{v}|x,v}(x \in [n]) = 1$, so for every x, v it holds

$$t(x) = b^{APA}(x) = t^{APA}(x, v),$$

which implies $t^{APA}(v) = \mathbb{E}[t(x)]$. Therefore, the IRF in the APA is

$$\Pi^{APA}(v) = t(v) + (n-1)\,\mathbb{E}[t(x)]. \tag{4.10}$$

Some remarks are due. First, when $v = 1$ the APA dominates all WPA. This observation is mathematically straightforward as $t^k(1) = 0 < \mathbb{E}[t(x)] = t^{APA}(1)$, however, the mechanics behind it is somehow counterintuitive. The APA yields higher revenue because, in addition to the transfer of the special bidder — who wins for sure and pays $t(1)$ both in the APA and in all WPAs — the seller collects bids from all other bidders. Hence, in the APA the seller exploits her information advantage to extract surplus from the other bidders, and to receive positive payments from bidders who (she knows) will lose the auction.

Second, the IRFs of the APA and of the nPA cross at least twice. Since $b^n(0) = t^n(0) = t^n(1) = 0$, t^n is below its expectation, which is the interim transfer for the APA, at both extrema; by continuity it must hit its expectation at least once from below and once from above. Since the APA dominates at $v = 1$ for all k, a sufficient condition for multiple crossing is $t^k(0) < \mathbb{E}[t(x)]$. This condition is not met by the FPA,[18] which indeed crosses the APA only once, as $t^{\text{PB-}\{1\}}(v)$ is monotonically decreasing. A sufficient (and necessary) condition for $t^k(0) < \mathbb{E}[t(x)]$ for all $k \geq 2$ is that the ex-ante transfer $\mathbb{E}[t(x)]$ is higher in an auction with n than $n-1$ bidders.[19]

Finally notice that, since $t^{APL}(v) = t(v)$, the All-Pay-Last Auction (a non-PBA with the same set of payers of the APA) crosses the APA only once and from below. This constitutes an analog to Theorem 2 (and its generalization at the extrema in Proposition 8): even with $\mathcal{P} = [n]$, the PBA interim dominates associated non-PBAs for $v = 0$ and is dominated around $v = 1$, the only difference being that now the strict ranking is preserved at $v = 1$.

Last-Pay Auction The LPA is the PB-$\{n\}$. By (4.7), the equilibrium bidding function in the LPA is

$$b^L(x) = \frac{t(x)}{[1 - F(x)]^{n-1}},$$

which is unbounded. To have an intuition of why the bid function is unbounded, suppose otherwise and let M be an upper bound to the equilibrium bids. Then bidding above M is a profitable deviation, as it allows to win at price 0. In general, unbounded bids are a necessary condition whenever the first order statistic (i.e. the winner) is not a payer.

Clearly, unbounded bids do not imply an unbounded ex-ante revenue (because of RET): although in the LPA high-value bidders bid arbitrarily high, as they want to win and do not expect to pay their bids, low-value bidders bid arbitrarily low because they expect to lose and pay their bids.

[18] Dominance of the FPA over the APA at $v = 0$ is as a special case of Proposition 9, but also follows from a direct comparison. If the special bidder has value 0, in the FPA the interim transfer of a generic bidder is $F^{n-2}(x) \cdot b^F(x) = \frac{t(x)}{F(x)}$, which is higher than $t(x)$, his interim expected transfer in the APA.

[19] This condition ensures that the APA yields a higher interim revenue than the SPA which, by Proposition 8, implies that the APA also dominates all other kPA with $k > 2$. The condition is satisfied if n is large; for example, with uniformly distributed bidders, it requires $n > 3$.

Moreover, unbounded bids do not imply unbounded interim revenue as $v \to 1$, because the winner does not pay his bid. However, we can get the result by looking at the interim transfer. Since $\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})} = \mathbf{1}\{x < v\}\dfrac{1}{1-F(x)}$, we obtain that $t^L(v) = \int_0^v \dfrac{t(x)}{1-F(x)}\mathrm{d}F(x)$, which is clearly increasing and, we show in the Appendix, unbounded.[20] Notice that there is no non-PBA with $\mathcal{P} = \{n\}$ and no comparison along the lines of Theorem 2 can be made in this case.

Nevertheless, the next section shows that the interim revenue is indeed unbounded in the LPA (for examples of this, see Figure 4.3 and the Appendix).

Interim Ranking of PBAs By (4.9), finding the interim optimal PBA for a given v amounts to solving

$$\max_{\mathcal{P}\subseteq[n]} \mathbb{E}\left[t(x)\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})}\right]$$

i.e. finding the subset of payers that maximizes the expected product of t and the likelihood ratio associated to v. The next Proposition builds on the observation that, for extreme values of v, there exists a $\mathcal{P}$ that maximizes the likelihood ratio $\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})}$ *for every* x.

Proposition 9. *For any* $\mathcal{P} \subseteq [n]$, $\mathit{\Pi}^{FPA}(0) > \mathit{\Pi}^{PB\text{-}\mathcal{P}}(0) > \mathit{\Pi}^{LPA}(0)$ *and* $\mathit{\Pi}^{LPA}(v) > \mathit{\Pi}^{PB\text{-}\mathcal{P}}(v) > \mathit{\Pi}^{FPA}(v)$ *for* $v \approx 1$. *Moreover,* $\lim_{v\to1} \mathit{\Pi}^{LPA}(v) - \mathit{\Pi}^{PB\text{-}\mathcal{P}}(v) = \infty$.

Therefore, when $v = 0$, the FPA yields the highest interim revenue among all PBAs. This is because, at $v = 0$, the special bidder is the minimum (n^{th}-order statistic) with probability 1. As a consequence, the likelihood that a generic bidder is *any other* order statistics increases, but the most significant increase occurs for the likelihood that the bidder is the maximum. In general, knowing that a bidder has value 0 changes the distribution of all order statistics, but more so for the higher order ones. Hence, the seller prefers to receive payments only from the first-order statistic.

When $v \approx 1$, the FPA yields the lowest interim revenue, while the LPA yields the highest. The intuition here is similar: in this case, the special bidder is the first-order statistic, and the highest increase in probability occurs for a generic bidder being the minimum. Thus, the seller prefers to receive payments only the n^{th}-order statistic.

Proposition 9 also provides a quantitative ranking of auction formats at $v \approx 1$, showing that the difference between the interim revenue of the LPA and any other PBA diverges. This, in turn, implies that the LPA has unbounded interim revenue. A straightforward extension the proof of this result shows that a stronger version holds, namely that for $v \approx 1$ the limit behavior of the interim revenue in PBAs is determined by the lowest payer: $\lim_{v\to1} \mathit{\Pi}^{\text{PB-}\mathcal{P}}(v) - \mathit{\Pi}^{\text{PB-}\mathcal{P}'}(v) = \infty$ if and only if $\min\mathcal{P} > \min\mathcal{P}'$. Therefore, increasing the minimum of $\mathcal{P}$ (e.g., removing the winner or the highest payer from the set of payers) induces an infinite increase in the interim revenue for $v \approx 1$.

4.4 Interim Dominant Auctions

In this section we complete our analysis of interim-dominant auctions by establishing that the FPA yields higher (resp., lower) interim revenue than any other standard auction when $v = 0$ (resp, $v \approx 1$). This

[20] More generally, unbounded bids and interim revenue arise in all standard auctions where the winner does not pay: $\lim_{v\to1} \mathit{\Pi}^a(v) = \infty$ if and only if $1 \notin \mathcal{P}_a$.

generalizes the comparisons between the FPA and the SPA (Theorem 2), WPAs (Proposition 8), and PBAs (Proposition 9).

Proposition 10. *The FPA interim dominates all other standard auctions at $v = 0$ and is interim dominated by all other standard auctions for $v \approx 1$.*

The proof of this result uses (4.4) and establishes that, when the special bidder has value 0, the interim transfer of a generic bidder, $t^{FPA}(x, 0) > t^a(x, 0)$, for any other standard auction a and valuation $x > 0$. In particular, in the Appendix we show that the difference $t^{FPA}(x, 0) - t^a(x, 0)$ is proportional to

$$\sum_{j \in \mathcal{P}_a} \mathbb{E}_{\boldsymbol{v}} \left[b^a \left(v_{(T_a(j))} (\boldsymbol{v}) \right) | v_{(j)} = x \right] - \frac{n-j}{n-1} \mathbb{E}_{\boldsymbol{v}} \left[b^a \left(v_{(T_a(j))} (\boldsymbol{v}) \right) | v_{(j)} = x, \ v_{(n)} = 0 \right], \qquad (4.11)$$

where b^a is the equilibrium bidding function in auction format a. The conditional expectations in (4.11) represent the expected payment of a bidder with value x, when he is the j^{th}-order statistic, this order static is a payer, and his payment is the $T_a(j)^{\text{th}}$ bid.

If auction a is not a PBA, then the difference (4.11) is positive because the first expectation is higher than the second one: both expectations represent the expected bid of the same order statistic, but the second one uses $n - 1$ draws (reflecting the fact that one bidder has value 0) rather than n. This effect drives the interim dominance of the FPA on the SPA (which is not a PBA) when $v = 0$, as in the SPA the transfer of the winner is lower because of the information that a competitor has value 0 (see Section 3).[21]

If auction a is a PBA, instead, then $T_a(j) = j$ and the two conditional expectations in (4.11) are both equal to $b^a(x)$. In this case, the difference (4.11) is positive because $\frac{n-j}{n-1} < 1$ for all $j > 1$. This effect reflects the change in the probability ratio of a generic bidder being a payer that occurs with the information that $v = 0$, which drives the interim dominance of the FPA among PBAs (see Section 4.3).

Proposition 10, however, does not yield a revenue ranking for all auction formats. For $v \approx 1$, for example, it only implies that both the SPA and the LPA interim dominate the FPA. While the revenue difference between the SPA and the FPA vanishes for $v \to 1$, however, the revenue difference between the LPA and the FPA diverges, implying that the LPA dominates the SPA. This suggests that changing the set of payers has a stronger effect than only changing the bidders who determine the transfer. Indeed, raising the *minimum* order statistic in the set of payers yields an unbounded revenue increase in PBAs for $v \to 1$.[22]

Figure 4.3 gives a graphical representation of our results for standard auctions with 3 bidders and uniformly distributed valuations (in the Appendix we derive the IRFs analytically for all displayed formats). The bottom-left panel displays the behavior of standard auctions at $v \approx 0$, where the FPA (red line) is best and the LPA (yellow line) is worst. Notice that each PBA (continuous line) lies above the corresponding non-PBA with the same set of payers (dashed line of the same color). The bottom-right panel displays the behavior of standard auctions at $v \approx 1$, where the LPA is best and the FPA is worst. In this case, each non-PBA (dashed line) lies above the corresponding PBA (continuous line of the same color). Finally,

[21] We conjecture that, when $v = 0$, among all auctions with the same set of payers, the PBA yields the highest interim revenue. For example, the difference in the transfers conditional on the realization of the order statistic of a payer drives the interim dominance of the PBA vs. non-PBAs when $v = 0$ for single-payer auctions (i.e., when $|\mathcal{P}_a| = 1$).

[22] Quantifying the gain from changing the bidders who determine the transfer, given an arbitrary set of payers, is challenging because there is no closed form for the equilibrium bid. In the Appendix we show that, when $\mathcal{P} = \{n - 1\}$, the limit revenue difference between the (unique) non-PBA and the PBA is *strictly positive but finite* (see Figure 4.3).

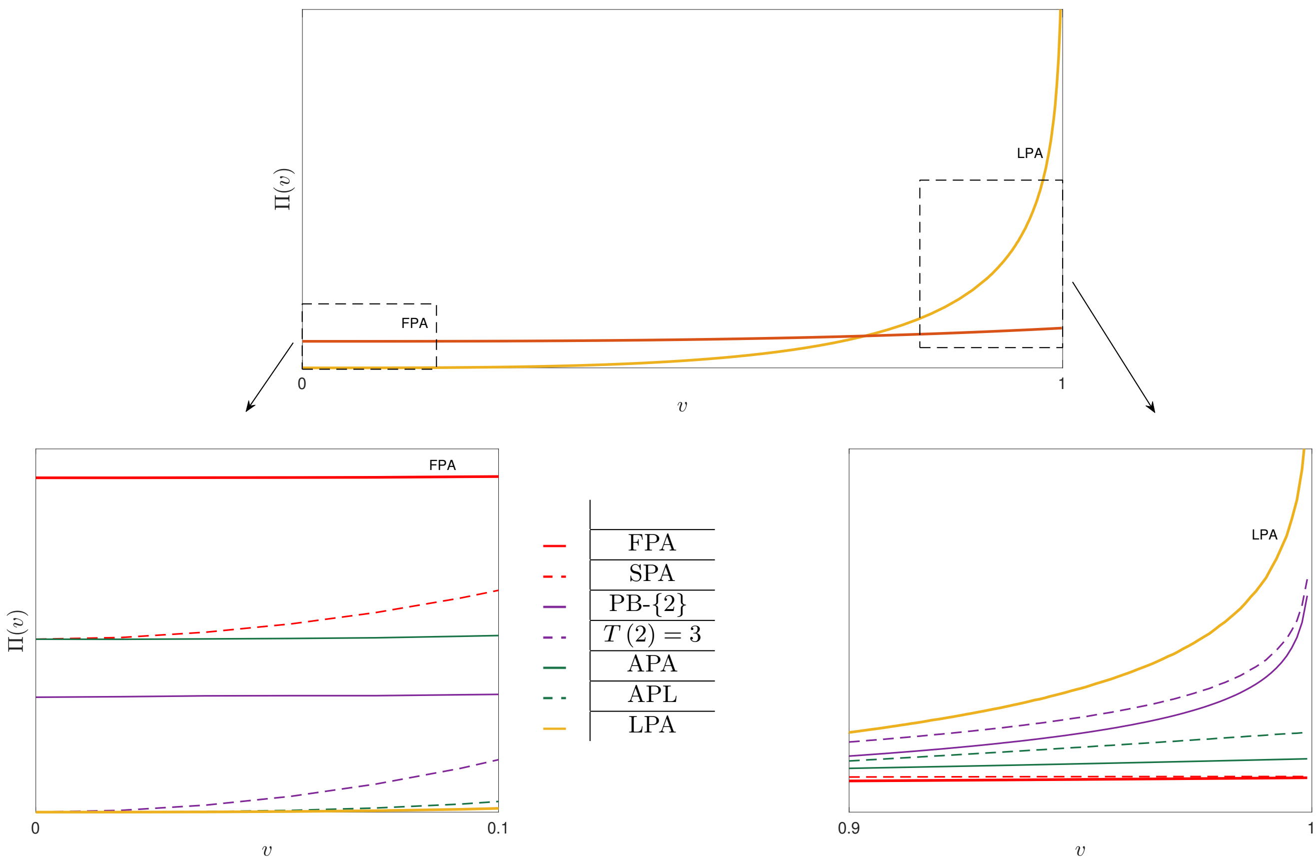

Figure 4.3: Interim revenue functions in standard auctions with $n = 3$ and $v_i \sim \mathcal{U}[0,1]$. Column $T(\cdot)$ reports the image of the corresponding element of $\mathcal{P}$. The bottom panels zoom around $v \approx 0$ and $v \approx 1$.

for $v \to 1$ all standard auctions with $1 \notin \mathcal{P}$ diverge, but the LPA does it faster. Moreover, the difference between the LPA and the PB- $\{2\}$ (purple solid line) is infinite (Proposition 9) while the difference between the PB- $\{2\}$ and the non-PBA with $\mathcal{P} = \{2\}$ (purple dashed line) remains finite.

Finally, Proposition 10 and monotonicity of the IRF imply that

Corollary. *For any standard auction a,* $Im\left(\Pi^{FPA}\right) \subset Im\left(\Pi^{a}\right) \subseteq Im\left(\Pi^{LPA}\right)$.

This results suggests that, although a generic standard auction is not a mean-preserving spread of the FPA (in contrast to what happens with WPAs — see (4.5)), the FPA can still be considered the less risky standard auction format for the seller. First, the FPA has the smallest range of possible interim revenues, and hence the lowest variability among standard auctions. Second, the FPA maximizes the lowest possible interim seller's revenue, so it provides the best insurance for the event that $v = 0$.

5 Conclusions

We have studied how the revenue in the efficient equilibrium of a broad class of sealed-bid auction formats is influenced by the information that one bidder has valuation v. Our analysis identifies how the marginal contribution of a single bidder varies across formats. Indeed, the difference between the interim revenue evaluated at v and the ex-ante revenue reflects the amount a seller would be willing to pay to substitute a generic bidder with one of valuation v. This amount is not merely equal to the expected transfer from v, but must also consider how the presence of v impacts the expected transfer from other bidders. Since the transfer from v is independent of the auction format, only the indirect effect determines the interim differences.

This has a natural interpretation in terms of how a seller can tailor the auction format to leverage information about a bidder. The RET implies that she cannot influence the transfer received from that specific bidder, but she can still utilize her informational advantage over other bidders. For example, when aware of a high bidder, she prefers the all-pay auction over formats where only the winner pays, as it enables her to secure transfers from bidders with no chance of winning the object.

This interpretation highlights a potential limitation in applying our results to scenarios where the seller selects the auction format based on information about a bidder's valuation.[23] We assume that bidders play the efficient equilibrium of the auction with symmetric competitors. This assumption presupposes that (i) the seller's information is private — i.e., other bidders do not know the value of the special bidder — and (ii) bidders do not infer anything about their competitors from the seller's choice of auction format.

A preliminary analysis in Appendix 6.1 suggests that bidders' sophistication limits the ability of the seller to exploit her information, and derives an equilibrium where the seller primarily (i.e. for almost all values of the special bidder) uses the SPA. With this caveat in mind, our results further identify the type of information that is most beneficial to the seller.

Due to the single crossing property, the seller can determine whether her expected revenue is higher in the FPA or SPA by simply knowing whether the bidder's valuation is below than *any* threshold. In practical terms, the seller would only need to know whether a bidder has made a purchase from a competitor, even if she is unaware of the competitor's pricing strategy.

Finally, while our analysis focused on the interim expectation, it would be interesting to explore how different moments of the conditional distributions vary across formats depending on v. For instance, although the FPA and the SPA are interim equivalent at $v = 1$, their revenues are not the same conditional random variable. Specifically, the FPA is degenerate at $t(1)$, while the SPA is not.

[23]Absent this concern, we can see our analysis as a contribution to the literature exploring why certain auction formats are prevalent. Other studies focus on risk preferences of the seller (Waehrer et al. (1998)) or bidders (Maskin and Riley (1984)), and the seller's ability to manipulate bids (Akbarpour and Li (2020)). Our explanation is instead based on the seller's information advantage.

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6 Appendix

6.1 Extensions

Savvy Bidders

Consider a setting where bidders are aware that the seller observes the valuation v of the special bidder before choosing the auction format. The identity (but not the valuation) of this special bidder, as well as the fact that the seller selects the auction format based on his valuation, is common knowledge.

Define the function $\mathcal{E} : [0,1] \to \mathcal{A}$, where $\mathcal{A}$ represents the set of standard auctions from which the seller can choose. $\mathcal{E}$ partitions the set of possible special bidder's valuation, depending on the associated auction format. By choosing the function $\mathcal{E}(v)$, the seller chooses an auction format for each special bidder's valuation. Let $\mathcal{E}^{-1}(a)$ denote the subset of special bidder's values to which the function $\mathcal{E}$ associates the auction format a, i.e. $\mathcal{E}^{-1}(a) = \{w \in [0,1] : \mathcal{E}(w) = a\}$. Then, $\mathcal{E}^{-1}(\mathcal{E}(v))$ denotes the atom of the partition to which v belongs (i.e., all bidders' valuations that induce the same auction format as v).

For a standard auction $a \in \mathcal{A}$ and a set $V \subset [0,1]$, construct the function $\Pi_V^a : [0,1]^n \to \mathbb{R}$ as follows.

1. *Equilibrium Bids.* Compute an equilibrium of auction a with asymmetric bidders in $V \times [0,1]^{n-1}$. The equilibrium specifies the bidding functions $b_V^{a,S} : V \to \mathbb{R}$ and $b_V^{a,N} : [0,1] \to \mathbb{R}$, which are mutual best responses for the special and normal bidders, respectively.

2. *Equilibrium Bids Extension.* Extend the equilibrium to the full space $[0,1]^n$, by computing, for each $v \notin V$, the best response to $n-1$ bidders playing $b_V^{a,N}$. This means that types of the special bidders for which the seller should not choose format a, play a best response to the equilibrium in auction a. Define this function $\tilde{b}_V^{a,S} : [0,1] \to \mathbb{R}$, noting that $\tilde{b}_V^{a,S}$ extends $b_V^{a,S}$ on $[0,1] \setminus V$.

3. *Interim Revenue Function.* Let $\Pi_V^a : [0,1]^n \to \mathbb{R}$ denote the revenue in such an auction as a function of the entire profile of bidders' valuations. Define the interim counterpart $\mathit{\Pi}_V^a(v) := \mathbb{E}[\Pi_V^a | v_1 = v]$, which is also defined for $v \notin V$.

In words, Π_V^a represents the seller's revenue in auction format a assuming that: *i*) Auction format a is chosen only if the special bidder's valuation is in V; *ii*) Non-special bidders always play according to the equilibrium strategy for an asymmetric auction where one competitor is in the set V; *iii*) The special bidder responds optimally to the equilibrium strategy, even when his valuation is not in V.[24]

Definition 11. The function $\mathcal{E}(v)$ is a *savvy-bidder equilibrium* if the following conditions are satisfied.
1. $\Pi_{\Omega_a}^a$ is well-definite $\forall a \in \mathcal{A}$, i.e. there exist bid functions as defined in Step 1 and Step 2
2. For all $v \in [0,1]$ and $a \in \mathcal{A}$, $\mathit{\Pi}_{\mathcal{E}^{-1}(\mathcal{E}(v))}^{\mathcal{E}(v)}(v) \geq \mathit{\Pi}_{\mathcal{E}^{-1}(a)}^a(v)$.

In the SPA, knowing that a competitor belongs to the set $V \subset [0,1]$ does not affect bids. Therefore, for all sets V, $\mathit{\Pi}_V^{SPA}(v)$ exists and is equal to $\mathit{\Pi}^{SPA}(v)$, the interim revenue studied in this paper. For any other auction format a, computing $\Pi_{\mathcal{E}^{-1}(a)}^a$ requires the full characterization of the bid functions

[24]The rationale behind this selection criterion, as formalized in Step 2, is that only the special bidder can detect any deviation by the seller, as he is the only one aware of his own valuation.

with asymmetric bidders (Step 1), a problem that presents notorious difficulties.[25] However, savvy-bidder equilibria that partition $[0,1]$ into $\{0\}$ and $(0,1]$ are immune to this problem, as the equilibrium played when there is common knowledge that a bidder has value 0 coincides with the equilibrium of the auction with $n-1$ bidders. In the FPA, with uniform bidders, we can compute the best response under deviation (Step 2) and analytically characterize the interim revenue function (Step 3).

Proposition 12. *Suppose F is the uniform CDF and $\mathcal{A} = \{FPA, SPA\}$. Then, for each n*

$$\mathcal{E}(v) = \begin{cases} FPA & v = 0 \\ SPA & v > 0 \end{cases}$$

constitutes a savvy-bidder equilibrium in which

$$b_0^{FPA,N}(x) = \frac{n-2}{n-1}x, \quad \tilde{b}_0^{FPA,S}(x) = \max\left\{\frac{n-1}{n}x, \frac{n-2}{n-1}\right\}, \quad b_{(0,1]}^{SPA,N}(x) = \tilde{b}_{(0,1]}^{SPA,S}(x) = x$$

Proof. See Online Appendix. □

Proposition 12 characterizes an equilibrium in which the seller chooses the FPA if and only if the special bidder's valuation is zero, and chooses the SPA otherwise. In this scenario, the equilibrium bids of non-special bidders in the FPA coincide with the equilibrium bids in a standard FPA with $n-1$ bidders. This is because, by participating in an FPA, bidders learn that the special bidder's valuation is 0, which is equivalent to competing with one fewer bidder (Step 1). The deviation strategy of the special bidder in the event the FPA is chosen when he has value $v > 0$ is the best response to these bids (Step 2). With uniform distribution, this best response, $\tilde{b}_0^{FPA,S}(x)$, can be derived in closed form.[26] Given the strategies on and off equilibrium, we show that $\Pi_0^F(v)$ equals $\Pi^S(v)$ at $v = 0$, while it is lower than it for all $v > 0$. At $v = 0$, the interim revenue in the FPA coincides with the interim revenue in the SPA, as non-special bidders bid as in an auction with one fewer competitor in both auction formats. Local to $v = 0$, gains are of order n in the FPA, lower than gains of order $n-1$ in the SPA, and the FPA never catches up.

Intuitively, when bidders are unaware that the seller's choice of auction format reflects her information about the value of the special bidder (as in our main analysis), the FPA maximizes the seller's revenue for low values of v. However, when bidders are aware that the seller is using information about a competitor, observing that they play in a FPA indicates that they are competing against a bidder with a low value. Consequently, they adjust their bids accordingly, and the equilibrium tends to favor the auction format where interim information does not influence bidding behavior.

Comparing Proposition 12 with Theorem 2 we see that in the presence of savvy bidders, the seller cannot exploit the information about a bidder, as the others will respond and adjust their bids leading

[25] For the FPA, Maskin and Riley (2000) compute the equilibrium in asymmetric auction with two asymmetric bidders, supposing that one of the valuation is drawn from a set of the form $[0, v_u]$. Kaplan and Zamir (2012) extend the result considering an asymmetric auction with two bidders having uniform distributions defined on sets of the form $[\underline{v}_i, \overline{v}_i]$. Olszewski et al. (2023) provide conditions for the existence of equilibria in first-price auctions with asymmetric bidders; in particular they provide sufficient conditions for the existence of equilibria when one bidder is drawn from a set V which can be of arbitrary form, but do not provide characterization of the equilibrium.

[26] Notice that the responsive part of the special bidder's best response is the equilibrium bidding function in the FPA with n symmetric bidders. This is also a bidders' best response when $n-1$ competitors adopt *any* linear bidding strategy (a property that relies on the uniform assumption). Of course, the special bidder never bids more than $b^F(1, n-1) = \frac{n-2}{n-1}$, the highest bid by non-special bidders.

to an unraveling process that ultimately favors the choice of an auction format where bidding behavior is unaffected by information about competitors.

Notice that within the class of standard auctions only the SPA is immune to manipulations given information about a competitor. A natural question is, then, whether *only* this format can be played in a savvy-bidder equilibrium, i.e. whether the equilibrium of Proposition 12 is unique. Guided by Theorem 2, we look in the class of cutoff strategies, that is where the seller selects a cutoff $\bar{v} \in [0,1]$ and chooses the FPA if and only if $v \leq \bar{v}$ and the SPA otherwise. Notice that the equilibrium in Proposition 12 belongs to this family, but is degenerate as it sets $\bar{v} = 0$. However, proving uniqueness (even in this family) has proven elusive as a direct unraveling argument is not possible and equilibria in the FPA with asymmetric bidders (Step 1 in the derivation of $\Pi^F_{[0,\bar{v}]}$ for $\bar{v} > 0$) do not have a closed-form representation, except in special cases (uniform with two bidders, see Kaplan and Zamir (2012)).

General Information and Implications of Single Crossing

We have analyzed the auction format preferred by the seller when she knows that one bidder has valuation v. A natural extension of our analysis is to assume instead that the seller has some, though not perfect, information about v. Let $\tilde{F}$ be a distribution over $[0,1]$ that represents such information. The ex-ante case corresponds to $\tilde{F} = F$, while the analysis of this paper corresponds to $\tilde{F} = \delta_v$ for some $v \in [0,1]$.[27] Computing the interim revenue in this setting is a straightforward extension of our analysis since

$$\Pi^a\left(\tilde{F}\right) := \mathbb{E}\left[\Pi^a | v_1 \sim \tilde{F}\right] = \mathbb{E}_{\tilde{F}}\left[\Pi^a(v)\right]$$

Some qualitative implications of Theorem 2 do not extend to this stochastic setting. In particular, although Theorem 2 implies that

$$\Pi^S(v) - \Pi^F(v) > 0 \quad \Rightarrow \quad \Pi^S\left(v'\right) - \Pi^F\left(v'\right) > 0$$

whenever $v' > v$, improving (in FOSD sense) the distribution of the special bidder does not necessarily make the SPA more desirable. In fact, it is possible that $\tilde{F}_2$ first-order stochastically dominates $\tilde{F}_1$ but[28]

$$\mathbb{E}_{\tilde{F}_1}\left[\Pi^S(v) - \Pi^F(v)\right] > 0 \text{ and } \mathbb{E}_{\tilde{F}_2}\left[\Pi^S(v) - \Pi^F(v)\right] < 0.$$

So the implication that "a better special bidder preserves SPA dominance" is not valid in general. There are, however, signal structures where single crossing is sufficient to determine unambiguously the seller's preference. For example, suppose that the seller observes whether v is above or below a random threshold, but is ignorant of the precise threshold and even of the distribution from which it is drawn.This type of information emerges naturally in settings where a seller is only able to observe whether one of his

[27] We still work at an interim stage and not model explicitly where $\tilde{F}$ comes from, though it is natural to interpret $\tilde{F}$ as the posterior resulting from observing some signal about the special bidder's valuation.

[28] This observation follows immediately from the fact that the interim revenue difference is non-monotonic (see, e.g. Figure 3.2). Thus, shifting mass above $\hat{v}$ shrinks the advantage of the SPA. As an example of $\tilde{F}_1, \tilde{F}_2$ that yield the desired contradition, take $\tilde{F}_1 = \begin{cases} 0 & p \\ \hat{v} & 1-p \end{cases}$ where p is chosen such that $\mathbb{E}_{\tilde{F}_1}\left[\Pi^S(v) - \Pi^F(v)\right] = \epsilon$ for ϵ positive but small. Then, $\tilde{F}_2 = \begin{cases} 0 & p \\ 1 & 1-p \end{cases}$ dominates $\tilde{F}_1$ but is such that $\mathbb{E}_{\tilde{F}_2}\left[\Pi^S(v) - \Pi^F(v)\right] < 0$.

bidders has previously completed a purchase or not from a competitor, without observing the price or the competitor's pricing strategy. If the interim revenue functions of two formats intersect only once, this information is sufficient to determine which format the seller prefers.[29]

Proposition 13. *Suppose the seller observes* $s = \mathbb{I}\{v < P\}$ *for some random variable* P *with support in* $[0,1]$*, independent of* v *but of unknown distribution. If* $s = 1$*, the interim revenue is higher in the FPA. If* $s = 0$ *the interim revenue is higher in the SPA.*

Proof. See Online Appendix. □

Information that one bidder is relatively weak —i.e. below a random threshold— makes the seller prefer the FPA.

6.2 Proofs

Proof of Lemma 1

The statement is a special case of (4.6) with $k = 1$ and $j = 2$.

Proof of Theorem 2

We complete the proof in the main text by proving (3.6) and Lemma 14. Notice that

$$\begin{aligned}\int_v^1 \int_0^x \max\{v, y\} \frac{\mathrm{d}F^{n-2}(y)}{F^{n-2}(x)} \mathrm{d}F^{n-1}(x) &= \int_v^1 \left[v \frac{F^{n-2}(v)}{F^{n-2}(x)} + \int_v^x y \frac{\mathrm{d}F^{n-2}(y)}{F^{n-2}(x)} \right] \mathrm{d}F^{n-1}(x) \\ &= \int_v^1 \left[x - \int_v^x \frac{F^{n-2}(y)}{F^{n-2}(x)} \mathrm{d}y \right] \mathrm{d}F^{n-1}(x),\end{aligned}$$

where the second equality uses integration by parts. Now let

$$g(v) := \int_v^1 \left[x - \int_v^x \frac{F^{n-2}(y)}{F^{n-2}(x)} \mathrm{d}y \right] \mathrm{d}F^{n-1}(x).$$

Notice that $g(1) = 0$ and

$$\begin{aligned} g'(v) &= -v \mathrm{d}F^{n-1}(v) + \int_v^1 \frac{F^{n-2}(v)}{F^{n-2}(x)} \mathrm{d}F^{n-1}(x) \\ &= -\mathrm{d}F^{n-1}(v) \left[v - \frac{1 - F(v)}{f(v)} \right] = -\mathrm{d}F^{n-1}(v)\, \psi(v). \end{aligned}$$

Finally, by the fundamental theorem of calculus:

$$\begin{aligned} g(v) &= g(1) - \int_v^1 g'(w)\, \mathrm{d}w \\ &= \int_v^1 \psi(w)\, \mathrm{d}F^{n-1}(w). \end{aligned}$$

[29] We state the result for FPA vs. SPA, though it is immediate to extend it to any pair of auction formats (e.g., the FPA and APA) whose interim revenue functions cross only once.

Lemma 14. *$\Delta(v)$ has a maximum at $v = 0$ and $\Delta(0) > 0$.*

Proof. We denote by $b^F(x, R, n)$ be the equilibrium bidding function in the FPA with n bidders and reserve price R. Integration by parts yields

$$\int_v^1 b^F(x, v, n)\,\mathrm{d}F^n(x) = \int_v^1 \psi(x)\,\mathrm{d}F^n(x),$$

for every n. Therefore, using (3.6),

$$\Delta(v) = \int_v^1 \left[b^F(x, 0, n) - b^F(x, v, n-1)\right] \mathrm{d}F^{n-1}(x).$$

Since $b^F(x, 0, n) - b^F(x, 0, n-1) > 0$, this implies that $\Delta(0) > 0$.

To show that $v = 0$ is a local maximum, write $\Delta'(v) = -\Delta_1(v) \cdot \Delta_2(v)$ where $\Delta_1(v) = b^F(v) - \psi(v)$ and $\Delta_2(v) = \mathrm{d}F(v)^{n-1}$. By the Leibniz rule, for $k \geq 1$,

$$\Delta^{(k-1)}(v) = \sum_{j=0}^{k-1} \binom{j}{k} \Delta_1^{(k-j)}(v)\,\Delta_2^{(j)}(v).$$

Notice that, for all $j < n-1$, $\Delta_2^{(j)}(0) \propto F(0) = 0$. Then, $\Delta^{(k)}(0) = 0$ for all $k < n-1$ while

$$\Delta^{(n-1)}(0) = \binom{n-2}{n-1} \Delta_1(0)\,\Delta_2^{(n-2)}(0) \propto -\Delta_1(0) = \psi(0) < 0,$$

proving that 0 is a local maximum. Since the other critical point $b^F(v) = \psi(v)$ must be a minimum, 0 is also the global maximum. □

Proof of Proposition 3

Suppose that $v < R$. Then,

$$\mathit{\Pi}^F(v) = \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} > v\right) \cdot \mathbb{E}_{\boldsymbol{v}|v}\left[b^F\left(v_{(1)}, R\right) | v_{(1)} > R\right]$$

and

$$\mathit{\Pi}^S(v) = \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} > R\right) \cdot \mathbb{E}_{\boldsymbol{v}|v}\left[\max\left\{v_{(2)}; R\right\} | v_{(1)} > R\right].$$

Therefore, when $v < R$, the IRFs of the FPA and SPA do not depend on v and the IRF of the FPA is strictly higher than the SPA. At $v = R$ the IRFs of both formats have an upward jump equal to $R \cdot \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} < R\right)$.

For $v > R$, Lemma 1 yields

$$\Delta(v, R) = \int_v^1 b^F(x, R) - \psi(x)\,\mathrm{d}F^{n-1}(x),$$

and the arguments in the proof of Theorem (2) without reserve directly extend, giving a unique minimum at $b^F(x, R) = \psi(x)$ and hence a unique $\widetilde{v} \neq 1$ such that $\Delta(\widetilde{v}, R) = 0$.

Standard Auctions (Section (4))

We first fill some gaps that are left in the development of standard auction and lead to Lemma 6, which is an immediate consequence of our construction.

We characterize the transfer map on the space of bids $\tilde{t}^a : \mathbb{R}^n \to \mathbb{R}^n$ from the defining objects of the standard auction a, $\mathcal{P}_a, T_a$, as follows. Let $s : \mathbb{R}^n \to \mathbb{N}^n$ denote the ordering map, that associates each vector with the vector of order statistics.[30] For example, $s([12, 31, 18]) = [3, 1, 2]$. Then,

$$\tilde{t}_i^a(\boldsymbol{b}) = \begin{cases} b_{j:s_j(\boldsymbol{b})=T_a(s_i(\boldsymbol{b}))} & \text{if } s_i(\boldsymbol{b}) \in \mathcal{P}_a \\ 0 & \text{if } s_i(\boldsymbol{b}) \notin \mathcal{P}_a \end{cases} \tag{6.1}$$

notice that $b_{j:s_j(\boldsymbol{b})=T_a(s_i(\boldsymbol{b}))}$ represents $T_a(s_i(\boldsymbol{b}))^{th}$-order statistic of $\boldsymbol{b}$. For example,

$$\tilde{t}_i^{FPA}(\boldsymbol{b}) = \begin{cases} b_{\underbrace{j : s_j(\boldsymbol{b}) = 1}_{=i}} & \text{if } s_i(\boldsymbol{b}) \in \{1\} \\ 0 & \text{if } s_i(\boldsymbol{b}) \notin \{1\} \end{cases}$$

and, since in general $\{j : s_j(\boldsymbol{b}) = s_i(\boldsymbol{b})\} = i$, we obtain for PBAs

$$\tilde{t}_i^{\text{PB-}\mathcal{P}}(\boldsymbol{b}) = \begin{cases} b_i & \text{if } s_i(\boldsymbol{b}) \in \mathcal{P} \\ 0 & \text{if } s_i(\boldsymbol{b}) \notin \mathcal{P} \end{cases}.$$

which becomes $\tilde{t}_i^{APA}(\boldsymbol{b}) = b_i$. For the non-PBAs discussed in the text, we have

$$\tilde{t}_i^k(\boldsymbol{b}) = \begin{cases} b_{j:s_j(\boldsymbol{b})=k} & \text{if } s_i(\boldsymbol{b}) \in \{1\} \\ 0 & \text{if } s_i(\boldsymbol{b}) \notin \{1\} \end{cases}$$

and

$$\tilde{t}_i^{APL}(b) = b_{j:s_j(\boldsymbol{b})=n}.$$

We can now characterize the equilibrium bidding function. We now characterize the equilibrium bidding function in standard auction a. Using the structure of the auction, the expected transfer of a bidder with value v is

$$\mathbb{E}_{\boldsymbol{v}|v}\left[\tilde{t}_1^a\right] = \sum_{j\in\mathcal{P}_a} \mathbb{P}_{\boldsymbol{v}|v}\left[v_{(j)} = v\right] \mathbb{E}_{\boldsymbol{v}}\left[b^a\left(v_{(T_a(j))}(\boldsymbol{v})\right) | v_{(j)} = v\right], \tag{6.2}$$

because v pays only if he is in the set of payers $\mathcal{P}_a$, and conditional on being the j^{th}-order statistic he pays the $T_a(j)^{th}$-highest bid. By the RET the left-hand-side is known and auction independent for every v; therefore (6.2) gives a functional equation for b^a. We can obtain a tighter characterization of (6.2) using the following following result in probability theory (see, e.g. Casella and Berger, Ex. 5.27.a).

[30] The operator s is well definite only for vectors that have no equal entries. We can assume an arbitrary tie-breaking rule (say, the slot of the bidder) as equal entries arise in equilibrium with zero probability.

Fact 15. *The pdf of the $(k+j)^{th}$-order statistic, conditioned on the realization of the k^{th}-order statistic, x, is the pdf of the j^{th}-order statistic from a sample of $(n-k)$ draws from the distribution truncated at x.*

Denote $F^{v}_{(j,m)}:[0,1]\to[0,1]$ the CDF of the j^{th} order statistic from m draws from distribution F truncated at v (implying $F^{v}_{(j,m)}(x)=1$ for all $x\geq v$), with the following conventions:[31]

- For all v,n, $F^{v}_{(0,n)}(w)=\mathbb{I}(w\geq v)$
- For all v,j, $F^{v}_{(j,0)}(w)=\mathbb{I}(w\geq 0)$

Using Fact and the RET, (6.2) can be written as

$$t(v)=\sum_{j\in\mathcal{P}_a}\mathbb{P}_{\boldsymbol{v}|v}\left[v_{(j)}=v\right]\int_0^v b^a(x)\,\mathrm{d}F^{v}_{(T_a(j)-j,n-j)}(x) \tag{6.3}$$

which has to hold for all $v\in(0,1)$. Notice that the only unknown in (6.3) is the bidding function b^a. If (6.3) admits a monotone solution, with initial condition $b^a(0)=0$, then such solution constitutes an equilibrium of the standard auction a.

We follow the main text and, abusing notation, denote $\tilde{t}^a(\boldsymbol{v}):[0,1]^n\to\mathbb{R}^n$ as the composition of the $\tilde{t}^a(\boldsymbol{b})$ function defined by (6.1) on the space of bids (that characterizes the rule of the standard auction) with the equilibrium bidding function b^a. We can now write the revenue in auction a as a function of the vector of bidders' valuations $\boldsymbol{v}$ in the following two equivalent ways

$$\Pi^a(\boldsymbol{v})=\sum_{i=1}^{n}\tilde{t}^a_i(\boldsymbol{v})=\sum_{j\in\mathcal{P}_a}b^a_{s^{-1}_{T_a(j)}(\boldsymbol{v})}(\boldsymbol{v}). \tag{6.4}$$

where $s^{-1}_{T_a(j)}(\boldsymbol{v})=\{i:s_i(\boldsymbol{v})=T_a(j)\}$ is the index of the bidder that submitted the $T_a(j)^{\text{th}}$-highest bid. The first expression obtains revenues summing the transfers from each bidder; the second is instead more "constructive" as it reflects the rules of the standard auction: for each order statistic in the set of payers it finds the bid that determines his transfer and adds it to the seller's revenue.[32]

[31] The conventions are just used so to make (6.3) hold in general. In case $T_a(j)-j=0$, i.e. for PBAs, then $\int_0^v b^a(x)\,\mathrm{d}F^{v}_{(T_a(j)-j,n-j)}(x)=b^a(v)$.

[32] If T_a were surjective then, letting $\mathcal{T}_a$ be the image of $\mathcal{P}_a$ through T_a, we could write revenues as $\Pi^a(\boldsymbol{v})=\sum_{j\in\mathcal{T}_a}b^a_{s^{-1}(j)(\boldsymbol{v})}(\boldsymbol{v})$ — i.e., as the sum of bids that determine the transfer (e.g., in the SPA only the second bid, in the APA all bids). In that case, the transfer vector $\tilde{t}^a(\boldsymbol{v})$ contains entries that are either zero or equal to one of the bids that is . As $T_a^{-1}(j)$ might be not singleton (for example, in the APL) the general expression requires to associate payer-wise the bid that determines transfer and account for repetitions.

Proof of Lemma 6

To obtain decomposition (4.4) we use the first expression in (6.4). Using only definitions (4.1)-(4.2) in the main text and linearity of expectations we get

$$\begin{aligned}
\mathit{\Pi}^a(v) &= \mathbb{E}_{\boldsymbol{v}|v}\left(\Pi^a(\boldsymbol{v})\right) = \mathbb{E}_{\boldsymbol{v}|v}\left(\sum_{i=1}^{n}\tilde{t}_i^a(\boldsymbol{v})\right) \\
&= \mathbb{E}_{\boldsymbol{v}|v}\left(\tilde{t}_1^a(\boldsymbol{v})\right) + \mathbb{E}_{\boldsymbol{v}|v}\left(\sum_{i=2}^{n}\tilde{t}_i^a(\boldsymbol{v})\right) \\
&= t(v) + (n-1)\,t^a(v)
\end{aligned}$$

That, for all a, $\mathbb{E}\left[\mathit{\Pi}^a(v)\right] = \bar{\Pi}$ and $\mathbb{E}\left[t^a(v)\right] = \mathbb{E}\left[t(v)\right]$ is proved in the main text.

To prove that the IRF of each standard auction a is continuous and increasing in the special bidder's valuation we instead rely on the second expression in (6.4). Fix an arbitrary vector $v_{-1} \in [0,1]^{n-1}$ representing the valuations of all bidders excluding the special one, and notice that $\sum_{j\in\mathcal{P}_a} b^a_{s^{-1}_{T_a(j)}(v,v_{-1})}(v, v_{-1})$ is continuous and increasing in the special bidders value v because each order statistic of the bids is continuously increasing in v.[33] The result follows because the IRF is the expectation of $\sum_{j\in\mathcal{P}_a} b^a_{s^{-1}_{T_a(j)}(v,v_{-1})}(v, v_{-1})$ with respect to v_{-1} (under the $n-1$ product measure $\times_1^{n-1}F$) and inherits its continuity and monotonicity.

Proof of Proposition 8

Let

$$\Delta^k(v) \equiv \mathit{\Pi}^{k+1}(v) - \mathit{\Pi}^k(v).$$

To prove the statement, we show that, for all k, $\Delta^k(v) > 0$ at $v = 0$, $\Delta^k(v) < 0$ for $v \approx 1$ and $\Delta^k(1) = 0$. We start by establishing the following result.

Lemma 16. *For all k, j with $k \geq j$, $\mathbb{E}\left[\Pi^k | \Pi^j\right] = \Pi^j$. Moreover,*

$$\Delta^k(v) = \sum_{i=k+1}^{n} \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)\left[\mathbb{E}\left(\Pi^{k+1}|v_{(i)} = v\right) - \mathbb{E}\left(\Pi^k|v_{(i)} = v\right)\right]. \tag{6.5}$$

Proof. Since $\Pi^k = b^k\left(x_{(k)}\right)$ and $b^k(\cdot)$ is a strictly monotonic function, conditioning on Π^k is equivalent to conditioning on the realization of the k^{th}-order statistic — i.e.

$$\mathbb{E}\left[\Pi^{k+1}|\Pi^k\right] = \mathbb{E}\left[\Pi^{k+1}|x_{(k)} = \left(b^k\right)^{-1}\left(\Pi^k\right)\right].$$

We first prove that $\mathbb{E}\left[\Pi^k|\Pi^{k-1}\right] = \Pi^{k-1}$ by showing that

$$\mathbb{E}[\underbrace{b^{k+1}\left(x_{(k+1)}\right)}_{\Pi^{k+1}}|x_{(k)}] = x = \underbrace{b^k(x)}_{\Pi^k},$$

using the following result in probability theory (see, e.g. Casella and Berger, Ex. 5.27.a).

[33] Notice that we need the second expression of the revenue, as for almost all v_{-1}, $\tilde{t}_i^a(v, v_{-1})$ is not continuous in v.

Fact 17. *The pdf of the $(k+j)^{th}$-order statistic, conditioned on the realization of the k^{th}-order statistic, x, is the pdf of the j^{th}-order statistic from a sample of $(n-k)$ draws from the distribution truncated at x.*

Let $G^{y}_{(j,n)}(x)$ denote the CDF of the j^{th}-order statistic among n draws below the threshold y. By the PET,

$$\int_0^v b^k(x)\,\mathrm{d}G^v_{(k-1,n-1)}(x) = \int_0^v b^{k+1}(y)\,\mathrm{d}G^v_{(k,n-1)}(y), \tag{6.6}$$

where the RHS (LHS) is the expected payment conditional on winning of a generic bidder with valuation v in the $(k+1)^{\text{th}}(k^{\text{th}})$ -price auction. Applying the LIE,

$$\int_0^v b^{k+1}(y)\,\mathrm{d}G^v_{(k,n-1)}(y) = \int_0^v \mathbb{E}\left[b^{k+1}(y)\,|x\right]\mathrm{d}G^v_{(k-1,n-1)}(x). \tag{6.7}$$

Combining equations (6.6) and (6.7) and differentiating with respect to v yields

$$b^k(v) = \mathbb{E}\left[b^{k+1}(y)\,|v\right],$$

and the result follows from Fact 17. The first statement of Lemma 16 follows from repeatedly applying the LIE:

$$\mathbb{E}[\Pi^k|\Pi^j] = \mathbb{E}[\ldots\overbrace{\mathbb{E}[\underbrace{\mathbb{E}[\Pi^k|\Pi^{k-1}]}_{\Pi^{k-1}}|\Pi^{k-2}]}^{\Pi^{k-2}}\ldots|\Pi^j] = \Pi^j.$$

We now prove (6.5). Using the Law of Total Probability,

$$\Delta^k(v) = \sum_{i=1}^{n}\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)}=v\right)\left[\mathbb{E}\left(\Pi^{k+1}|v_{(i)}=v\right) - \mathbb{E}\left(\Pi^k|v_{(i)}=v\right)\right].$$

However, all terms in the summation up to $i=k$ are equal to 0. The reason is that, by strict monotonicity of the bidding function, $\mathbb{E}\left[\Pi^k|v_{(i)}=v\right] = \mathbb{E}\left[\Pi^k|\Pi^i=b^i(v)\right]$ and, as we have shown,

$$\mathbb{E}\left[\Pi^{k+1}|v_{(i)}=v\right] = \Pi^i = \mathbb{E}\left[\Pi^k|v_{(i)}=v\right] \qquad \forall i\le k.$$

□

Consider first $v=0$.

$$\begin{aligned}\Delta^k(0) &= \mathbb{E}_{\boldsymbol{v}|0}\left[\Pi^k - \Pi^{k+1}\right]\\ &= \mathbb{E}_{\boldsymbol{v}|0}\left[\mathbb{E}_{\boldsymbol{v}|0}\left[\Pi^k - \Pi^{k+1}|\Pi^k\right]\right]\\ &> \mathbb{E}_{\boldsymbol{v}|0}\left[\Pi^k - \mathbb{E}_{\boldsymbol{v}}\left[\Pi^{k+1}|\Pi^k\right]\right]\\ &= \mathbb{E}_{\boldsymbol{v}|0}\left[\Pi^k - \Pi^k\right] = 0.\end{aligned}$$

The first equality is a definition, the second equality is the LIE, the inequality uses $\boldsymbol{v} > \boldsymbol{v}|0$, and the last line uses Lemma 16.

Consider now $v \approx 1$. We establish that, for all k, there is a neighborhood of 1 where $\Delta^k(v) < 0$. By a Taylor expansion of order N around 1,

$$\Delta^k(v) = \sum_{j=0}^{N} \Delta^k(1)^{(j)} \frac{(v-1)^j}{j!} + o\left((v-1)^N\right),$$

where $\Delta^k(v)^{(j)}$ denotes the j^{th}-order derivative of $\Delta^k(v)$. Since $\Delta^k(1) = 0$, the sign of the the function Δ^k in a neighborhood of 1 is determined by the sign of its first non-zero derivative at 1. By Leibniz's rule,

$$\Delta^k(v)^{(j)} = \sum_{i=k+1}^{n} \sum_{s=0}^{j} \binom{j}{s} \mathbb{P}\left(v_{(i)} = v\right)^{(s)} \Delta_i^k(v)^{(j-s)}, \tag{6.8}$$

where $\Delta_i^k(v) \equiv \mathbb{E}\left(\Pi^k | v_{(i)} = v\right) - \mathbb{E}\left(\Pi^{k+1} | v_{(i)} = v\right)$. Since $\Delta_i^k(v)$ has finite derivatives, the corresponding term in (6.8) is equal to 0 if $\mathbb{P}\left(v_{(i)} = v\right)^{(s)} = 0$.

Lemma 18. *For all i,*

$$\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)^{(s)}\big|_{v=1} = 0 \qquad \forall s < i-1,$$

and

$$(-1)^{i-1} \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)^{(i-1)}\big|_{v=1} > 0.$$

Proof. For all k and i, $\mathbb{E}\left[\Pi^k | v_{(i)} = v\right]$ is n times continuously differentiable with finite derivatives. Using

$$\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right) = (1 - F(v))^{i-1} F(v)^{n-i} \binom{n-1}{i-1}$$

and Leibniz's rule, we have

$$\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)^{(j)} = \binom{n-1}{i-1} \sum_{s=0}^{j} \binom{j}{s} \left[(1-F(v))^{i-1}\right]^{(s)} \left[F(v)^{n-i}\right]^{(j-s)}.$$

Therefore, for all $s < i-1$, $\left[(1-F(v))^{i-1}\right]^{(s)}\big|_{v=1} = 0$ because $1 - F(1) = 0$, and $\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)^{(s)}\big|_{v=1} = 0$ because it is the sum of addenda that are all zero. By contrast,

$$\left[(1-F(v))^{i-1}\right]^{(i-1)}\big|_{v=1} = (i-1)!\,(-1)^{i-1} f(1)^{i-1},$$

and hence $\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)^{(i-1)}\big|_{v=1} \propto (-1)^{i-1}$. □

Lemma 18 implies that, for all $j < k$, $\Delta^k(1)^{(j)} = 0$ since all addenda in (6.8) are equal to 0. Moreover, for $j = k$ the only non-zero addendum in (6.8) at $v = 1$ is

$$\mathbb{P}\left(v_{(k+1)} = v\right)^{(k)} \Delta_{k+1}^k(v)\Big|_{v=1}.$$

Notice that $\Delta_{k+1}^k(v) = \mathbb{E}\left[b^k(v') \,|\, v' > v\right] - b^{k+1}(v)$ and converges to $b^k(1) - b^{k+1}(1) < 0$ as $v \to 1$.

Therefore, by Fact 18,

$$\Delta^k (1)^{(k)} = \mathbb{P}\left(v_{(k+1)} = 1\right)^{(k)} \Delta^k_{k+1} (1) \propto (-1)^{k-1}.$$

Summing up, we have shown that

$$\Delta^k (1)^{(j)} (-1)^j = \begin{cases} 0 & \text{if } j < k \\ < 0 & \text{if } j = k \end{cases} \tag{6.9}$$

which, using (6.8), proves the statement that $\Delta^k (v) < 0$ for $v \approx 1$.

Proofs of the Statements in Section 4.3

The IRFs of the FPA and APA cross only once. To see this notice that, by (4.8), the IRF of the PB-$\mathcal{P}$ is

$$\begin{aligned} \mathit{\Pi}^{\text{PB-}\mathcal{P}} (v) &= t(v) + (n-1)\,\mathbb{E}\left[b^{\text{PB-}\mathcal{P}} (x)\, \mathbb{P}_{\boldsymbol{v}|x,v} (x \in \mathcal{P})\right] \\ &= t(v) + (n-1)\,\mathbb{E}\left[t(x) \frac{\mathbb{P}_{\boldsymbol{v}|x,v} (x \in \mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x} (x \in \mathcal{P})}\right]. \end{aligned} \tag{6.10}$$

Ex-interim, the seller obtains from the special bidder his transfer. From each other bidder, she obtains the transfer weighted by the ratio of the likelihood of belonging to the set of payers, conditional on v, relative to its ex-ante counterpart.

In the APA, $\mathcal{P} = [n]$ and $\frac{\mathbb{P}_{\boldsymbol{v}|v}(x \in \mathcal{P})}{\mathbb{P}_{\boldsymbol{v}}(x \in \mathcal{P})} = 1$ for all v and x. In the FPA, $\mathcal{P} = \{1\}$ and

$$\frac{\mathbb{P}_{\boldsymbol{v}|v} (x \in \mathcal{P})}{\mathbb{P}_{\boldsymbol{v}} (x \in \mathcal{P})} = \begin{cases} \frac{F^{n-2}(x)}{F^{n-1}(x)} = \frac{1}{F(x)} & x > v \\ 0 & x < v \end{cases}$$

Therefore,

$$\mathit{\Pi}^{FPA} (v) - \mathit{\Pi}^{APA} (v) = (n-1)\left[\mathbb{E}\left[\frac{t(x)}{F(x)} \mathbb{I}[x > v]\right] - \mathbb{E}[t(x)]\right].$$

Notice that $\mathit{\Pi}^{FPA} (0) - \mathit{\Pi}^{APA} (0) \propto \mathbb{E}\left[\frac{t(x)}{F(x)}\right] - \mathbb{E}[t(x)] > 0$ and $\mathit{\Pi}^{FPA} (1) - \mathit{\Pi}^{APA} (1) \propto -\mathbb{E}[t(x)] < 0$. Moreover, the difference is monotonically decreasing in v, proving single crossing.

Compare now the APA with other WPAs. By (6.10), $\mathit{\Pi}^{APA} (0) = (n-1)\,\mathbb{E}[t(x,n)]$. By contrast, since at $v = 0$ the SPA is equivalent to any efficient auction with $n-1$ bidders, $\mathit{\Pi}^{SPA} (0) = (n-1)\,\mathbb{E}[t(x,n-1)]$. Therefore,

$$\mathit{\Pi}^{APA} (0) - \mathit{\Pi}^{SPA} (0) \propto \mathbb{E}[t(x,n) - t(x,n-1)].$$

Proof of Proposition 9

By (6.10),

$$\mathit{\Pi}^{\text{PB-}\mathcal{P}} (v) > \mathit{\Pi}^{\text{PB-}\mathcal{P}'} (v) \qquad \Leftrightarrow \qquad \mathbb{E}\left[t(x) \frac{\mathbb{P}_{\boldsymbol{v}|x,v} (x \in \mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x} (x \in \mathcal{P})}\right] > \mathbb{E}\left[t(x) \frac{\mathbb{P}_{\boldsymbol{v}|x,v} (x \in \mathcal{P}')}{\mathbb{P}_{\boldsymbol{v}|x} (x \in \mathcal{P}')}\right]. \tag{6.11}$$

The following Lemma proves the statements by showing that, for every x, the ratio $\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})}$ is highest (lowest) in the FPA (LPA) than in any other PBA when $v=0$, while it is highest (lowest) in the LPA (FPA) when $v=1$. Since $t(x)$ is positive, this implies the result.

Lemma 19. *For all x and $\mathcal{P}$,*

$$\frac{\mathbb{P}_{\boldsymbol{v}|x,0}\left(x=v_{(1)}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(1)}\right)}>\frac{\mathbb{P}_{\boldsymbol{v}|x,0}\left(x\in\mathcal{P}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x\in\mathcal{P}\right)}>\frac{\mathbb{P}_{\boldsymbol{v}|x,0}\left(x=v_{(n)}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(n)}\right)} \tag{6.12}$$

and

$$\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(x=v_{(n)}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(n)}\right)}>\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(x\in\mathcal{P}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x\in\mathcal{P}\right)}>\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(x=v_{(1)}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(1)}\right)}. \tag{6.13}$$

Proof. Let $p_j^n(x)$ be the probability that x is the j^{th}-highest among n bidders — i.e., $p_j^n(x)=\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(j)}\right)$ — so that $\mathbb{P}_{\boldsymbol{v}|x,0}\left(x=v_{(j)}\right)=p_j^{n-1}(x)$ and $\mathbb{P}_{\boldsymbol{v}|x,1}\left(x=v_{(j)}\right)=p_{j-1}^{n-1}(x)$. Then,

$$\frac{\mathbb{P}_{\boldsymbol{v}|x,0}\left(x=v_{(1)}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(1)}\right)}=\frac{p_1^{n-1}(x)}{p_1^n(x)},\qquad\frac{\mathbb{P}_{\boldsymbol{v}|x,0}\left(x\in\mathcal{P}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x\in\mathcal{P}\right)}=\frac{\sum_{j\in\mathcal{P}}p_j^{n-1}(x)}{\sum_{j\in\mathcal{P}}p_j^n(x)}.$$

Notice that

$$\frac{p_j^{n-1}(x)}{p_j^n(x)}=\frac{(1-F(x))^{j-1}F(x)^{n-j-1}\binom{n-2}{j-1}}{(1-F(x))^{j-1}F(x)^{n-j}\binom{n-1}{j-1}}=\frac{n-j}{F(x)(n-1)}. \tag{6.14}$$

This ratio is decreasing in j, which proves (6.12).[34]

Similarly,

$$\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(x=v_{(n)}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(n)}\right)}=\frac{p_{n-1}^{n-1}(x)}{p_n^n(x)},\qquad\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(x\in\mathcal{P}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x\in\mathcal{P}\right)}=\frac{\sum_{j\in\mathcal{P}}p_{j-1}^{n-1}(x)}{\sum_{j\in\mathcal{P}}p_j^n(x)}$$

and

$$\frac{p_{j-1}^{n-1}(x)}{p_j^n(x)}=\frac{(1-F(x))^{j-2}F(x)^{n-j}\binom{n-2}{j-2}}{(1-F(x))^{j-1}F(x)^{n-j}\binom{n-1}{j-1}}=\frac{j-1}{(1-F(x))(n-1)}. \tag{6.15}$$

This ratio is increasing in j, which proves (6.13) (see footnote 34). ☐

Finally, we show that $\lim_{v\to1}\mathit{\Pi}^{\text{LPA}}(v)-\mathit{\Pi}^{\text{PB-}\mathcal{P}}(v)=\infty$. First, from the proof of Lemma (19), $\mathit{\Pi}^{\text{PB-max}\mathcal{P}}(1)>\mathit{\Pi}^{\text{PB-}\mathcal{P}}(1)$. Consider single-payer auctions, and compare the PB-$\{j\}$ and PB-$\{k\}$. Using

[34] We use the following algebraic fact (whose proof follows from a simple induction argument on the cardinality of M). Let $\{a_i\}_{i=1}^N$ and $\{b_i\}_{i=1}^N$ be sequences of positive numbers such that $\frac{a_i}{b_i}$ is (strictly) decreasing in i. Then for, any $M\subseteq\{1,\ldots,N\}$, $\frac{a_1}{b_1}\geq\frac{\sum_{i\in M}a_i}{\sum_{i\in M}b_i}\geq\frac{a_N}{b_N}$, with strict inequality if $M\neq\{1\}$. Moreover, if $\min\{i:i\in M\}\geq j$, then $\frac{a_j}{b_j}\geq\frac{\sum_{i\in M}a_i}{\sum_{i\in M}b_i}$, with strict inequality if $M\neq\{j\}$.

(6.11) and (6.15),

$$\Pi^{\text{PB-}\{j\}}(1) - \Pi^{\text{PB-}\{k\}}(1) = \mathbb{E}\left[t(x)\left(\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(v_{(j)}=x\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(j)}=x\right)} - \frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(v_{(k)}=x\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(k)}=x\right)}\right)\right]$$
$$= \mathbb{E}\left[t(x)\left(\frac{j-1}{(1-F(x))(n-1)} - \frac{k-1}{(1-F(x))(n-1)}\right)\right]$$
$$= \frac{j-k}{n-1}\mathbb{E}\left[\frac{t(x)}{1-F(x)}\right]. \tag{6.16}$$

Notice that, for generic $l \in (0, v)$,

$$\mathbb{E}\left[\frac{t(x)}{1-F(x)}\right] = \lim_{v\to 1}\int_0^v \frac{t(x)}{1-F(x)} f(x)\,\mathrm{d}x$$
$$> t(l)\lim_{v\to 1}\int_l^v \frac{f(x)}{1-F(x)}\mathrm{d}x$$
$$= -t(l)\lim_{v\to 1}\int_l^v \mathrm{d}\log(1-F(x))$$
$$\propto -t(l)\lim_{v\to 1}\log(1-F(v)) = \infty, \tag{6.17}$$

where the inequality holds because the function $\frac{t(x)}{1-F(x)} f(x)$ is bounded in the closed interval $[0, v]$ and $0 < t(l) \le t(x) \le t(1) < \infty$. Therefore, if $n \notin \mathcal{P}$ then $\Pi^{\text{PB-}\{n\}}(1) - \Pi^{\text{PB-max}\mathcal{P}}(1)$ diverges and $\Pi^{\text{PB-max}\mathcal{P}}(1) > \Pi^{\text{PB-}\mathcal{P}}(1)$.

For the case $n \in \mathcal{P}$, let $j = \max\mathcal{P} \setminus \{n\}$. Then, from the proof of Lemma (19), $\Pi^{\text{PB-}\{n,j\}}(1) > \Pi^{\text{PB-}\mathcal{P}}(1)$. Moreover,

$$\Pi^{\text{PB-}\{n\}}(1) - \Pi^{\text{PB-}\{n,j\}}(1)$$

$$= \mathbb{E}\left[t(x)\left(\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(v_{(n)}=x\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(n)}=x\right)} - \frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(v_{(j)}=x\right) + \mathbb{P}_{\boldsymbol{v}|x,1}\left(v_{(n)}=x\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(j)}=x\right) + \mathbb{P}_{\boldsymbol{v}|x}\left(v_{(n)}=x\right)}\right)\right]$$
$$= \mathbb{E}\left[t(x)\left(\frac{p^{n-1}_{n-1}(x)}{p^n_n(x)} - \frac{p^{n-1}_{j-1}(x) + p^{n-1}_{n-1}(x)}{p^n_j(x) + p^n_n(x)}\right)\right]$$
$$= \mathbb{E}\left[t(x)\left(\frac{F(x)^{n-j}\left[\binom{n-1}{j-1} - \binom{n-2}{j-2}\right]}{(1-F(x))\left(F(x)^{n-j}\binom{n-1}{j-1} + (1-F(x))^{n-j}\right)}\right)\right] \propto \mathbb{E}\left[\frac{t(x)}{1-F(x)}\right], \tag{6.18}$$

where the asymptotic comparison holds because $F(x)^{n-j}\binom{n-1}{j-1} + (1-F(x))^{n-j}$ is bounded away from 0. As the last term is infinite by (6.17), $\Pi^{\text{PB-}\{n\}}(1) - \Pi^{\text{PB-}\{n,j\}}(1)$ diverges and $\Pi^{\text{PB-}\{n,j\}}(1) > \Pi^{\text{PB-}\mathcal{P}}(1)$.

Proof of Proposition 10

Let the increasing function $b^a : [0, 1] \to \mathbb{R}$ denote the equilibrium bidding function in auction a. By PET and using Fact 17 (and the corresponding notation), for all v,

$$t(v) = \sum_{j\in\mathcal{P}_a} \mathbb{P}_{\boldsymbol{v}|v}\left[v_{(j)} = v\right] \mathbb{E}_{\boldsymbol{v}}\left[b^a\left(v_{(T_a(j))}(\boldsymbol{v})\right) | v_{(j)} = v\right] \tag{6.19}$$

because, by definition, in auction a a bidder with value v pays if and only if he is an order statistic $j \in \mathcal{P}_a$ and, in this case, pays the bid submitted by the (lower) bidder $T_a(j)$.[35]

Moreover, notice that we can write the interim transfer $t^a(x, v)$ as

$$t^a(x, v) = \sum_{j \in \mathcal{P}_a} \mathbb{P}_{\boldsymbol{v}|x,v}\left[v_{(j)} = x\right] \mathbb{E}_{\boldsymbol{v}|v}\left[b^a\left(v_{(T_a(j))}(\boldsymbol{v})\right) | v_{(j)} = x\right]$$

Therefore, when $v = 0$ and a bidder with value x is the j^{th}-order statistic and pays, his expected payment is the bid submitted by the lower bidder $T_a(j)$, taking into account that there are $n - j - 1$ bidders with unknown values lower than x (and one bidder with value 0).

The result at $v = 0$ follows because, for every standard auction a and $x \in (0, 1]$,[36]

$$t^{FPA}(x, 0) > t^a(x, 0) \quad \Leftrightarrow \quad \frac{t(x)}{F(x)} > \sum_{j \in \mathcal{P}_a} \mathbb{P}_{\boldsymbol{v}|x,0}\left[v_{(j)} = x\right] \mathbb{E}_{\boldsymbol{v}}\left[b^a\left(v_{(T_a(j))}(\boldsymbol{v})\right) | v_{(j)} = x,\ v_{(n)} = 0\right]$$

$$\Leftrightarrow \quad t(x) > \sum_{j \in \mathcal{P}_a} \mathbb{P}_{\boldsymbol{v}|x,0}\left[v_{(j)} = x\right] F(x) \mathbb{E}_{\boldsymbol{v}}\left[b^a\left(v_{(T_a(j))}(\boldsymbol{v})\right) | v_{(j)} = x,\ v_{(n)} = 0\right]$$

$$\Leftrightarrow \quad \sum_{j \in \mathcal{P}_a} \mathbb{P}_{\boldsymbol{v}|x}\left[v_{(j)} = x\right] \mathbb{E}_{\boldsymbol{v}}\left[b^a\left(v_{(T_a(j))}(\boldsymbol{v})\right) | v_{(j)} = x\right] >$$

$$> \sum_{j \in \mathcal{P}_a} \frac{n-j}{n-1} \mathbb{P}_{\boldsymbol{v}|x}\left[v_{(j)} = x\right] \mathbb{E}_{\boldsymbol{v}}\left[b^a\left(v_{(T_a(j))}(\boldsymbol{v})\right) | v_{(j)} = x,\ v_{(n)} = 0\right],$$

where we have used $t^{FPA}(x, 0) = \frac{t(x)}{F(x)}$, (6.19) and the fact that $\mathbb{P}_{\boldsymbol{v}|x,0}\left[v_{(j)} = x\right] F(x) = \frac{n-j}{n-1} \mathbb{P}_{\boldsymbol{v}|x}\left[v_{(j)} = x\right]$ (see (6.14)). The inequality holds because $\frac{n-j}{n-1} < 1$ for all $j > 1$ and, for all x, j and $k > j$,

$$\mathbb{E}_{\boldsymbol{v}}\left[b^a\left(v_{(k)}(\boldsymbol{v})\right) | v_{(j)} = x\right] \geq \mathbb{E}_{\boldsymbol{v}}\left[b^a\left(v_{(k)}(\boldsymbol{v})\right) | v_{(j)} = x,\ v_{(n)} = 0\right]$$

because b^a is increasing and the k^{th}-order statistic of n independent draws FOSD the k^{th}-order statistic of $n' < n$ independent draws from the same distribution.

Computations of the Interim Revenue Functions in Figure 4.3

Let $n = 3$ and $v \sim \mathcal{U}[0, 1]$, so that $t(v) = \frac{2}{3}v^3$. The bid and interim revenue functions in the FPA, SPA and APA are standard and omitted.

Consider first the PB-$\{2\}$ and PB-$\{3\}$. By (4.7), the bidding functions are

$$b^{\text{PB-}\{2\}} = \frac{t(v)}{\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(2)} = v\right)} = \frac{v^2}{3(1-v)},$$

and

$$b^{\text{PB-}\{3\}} = \frac{t(v)}{\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(3)} = v\right)} = \frac{2}{3} \frac{v^3}{(1-v)^2}.$$

[35] Denoting $\mathrm{d}F^v_{(j,n)}(w)$ the density of the j^{th} order statistic from n draws from distribution F truncated at v, we obtain a more explicit characterization of the bidding function $b^a : [0, 1] \to \mathbb{R}$ as $t(v) = \sum_{j \in \mathcal{P}_a} p^j_n(v) \int_0^v b^a(w) \mathrm{d}G^v_{(T_a(j)-j, n-j)}(w)$

[36] A type $x = 0$ transfers nothing in all standard auctions, so we can disregard this case and assume $F(x) > 0$.

We compute the interim revenue using (4.9). The likelihood rations are

$$\frac{\mathbb{P}_{\boldsymbol{v}|x,v}\left(v_{(2)}=x\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(2)}=x\right)}=\begin{cases}\frac{1}{2x} & \text{if } x>v\\ \frac{1}{2(1-x)} & \text{if } x<v\end{cases}$$

$$\frac{\mathbb{P}_{\boldsymbol{v}|x,v}\left(v_{(3)}=x\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(3)}=x\right)}=\begin{cases}0 & \text{if } x>v\\ \frac{1}{1-x} & \text{if } x<v\end{cases}$$

and the interim revenues are

$$\Pi^{\text{PB-}\{2\}}\left(v\right)=t\left(v\right)+\frac{4}{3}\left[\int_0^v\frac{x^3}{2\left(1-x\right)}\mathrm{d}x+\int_v^1\frac{x^2}{2}\mathrm{d}x\right]$$

$$\Pi^{\text{PB-}\{3\}}\left(v\right)=t\left(v\right)+\frac{4}{3}\left[\int_0^v\frac{x^3}{1-x}\mathrm{d}x\right].$$

It is immediate to check that

$$\Pi^{\text{PB-}\{1\}}\left(0\right)>\Pi^{\text{PB-}\{2\}}\left(0\right)>\Pi^{\text{PB-}\{3\}}\left(0\right)=0$$

and

$$\lim_{v\to1}\Pi^{\text{PB-}\{2\}}\left(v\right)=\lim_{v\to1}\Pi^{\text{PB-}\{3\}}\left(v\right)=\lim_{v\to1}\Pi^{\text{PB-}\{3\}}\left(v\right)-\Pi^{\text{PB-}\{2\}}\left(v\right)=\infty.$$

Consider now the standard auction with $\mathcal{P}=\{2\}$ and $T\left(2\right)=3$, which we denote compactly as the $2,3$ auction. By the PET,

$$b^{\text{PB-}\{2\}}\left(v\right)v=\int_0^v b^{2,3}\left(w\right)\mathrm{d}w,$$

and hence the bidding function is

$$b^{2,3}\left(v\right)=\frac{v^2\left(3-2v\right)}{3\left(1-v\right)^2}.$$

The interim transfer of a generic bidder against competitor v is

$$t^{2,3}\left(v\right)=\frac{1}{2}\left(1-v\right)^2b^{2,3}\left(v\right)+\frac{1}{2}\int_0^v 2\left(v-w\right)b^{2,3}\left(w\right)\mathrm{d}w,$$

as she pays $b^{2,3}\left(v\right)$ when v is third and she is second, and the bid of the other competitor when v is first and she is second (and nothing when v is second).[37] Using (4.4),

$$\Pi^{2,3}\left(v\right)=t\left(v\right)+\left(1-v\right)^2b^{2,3}\left(v\right)+\int_0^v 2\left(v-w\right)b^{2,3}\left(w\right)\mathrm{d}w$$

[37] Notice that conditional on the information that v is the j^{th} order statistic, then the two remaining bidders are any other order statistic with equal probability before drawing their type.

The interim revenue difference with the associated PBA is[38]

$$\Pi^{2,3}(v) - \Pi^{\text{PB-}\{2\}}(v) = \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(3)} = v\right)\left(b^{2,3}(v) - \int_v^1 b^{\text{PB-}\{2\}}(w)\,2(1-w)\,\mathrm{d}w\right)$$
$$= \frac{v^2(3-2v)}{3} - \int_v^1 \frac{2}{3}w^2\mathrm{d}w,$$

from which it follows that $\Pi^{2,3}(0) - \Pi^{\text{PB-}\{2\}}(0) < 0$ and $\Pi^{2,3}(1) - \Pi^{\text{PB-}\{2\}}(1) > 0$; again the PBA dominates at $v = 0$, is dominated at $v = 1$, and single crossing holds as $\frac{\mathrm{d}}{\mathrm{d}v}\left(\Pi^{2,3}(v) - \Pi^{\text{PB-}\{2\}}(v)\right) = \frac{2}{3}v(3-2v) > 0$.

Finally, consider the All-Pay-Last auction (APL), that has $\mathcal{P} = \{1,2,3\}$ and $T(1) = T(2) = T(3) = 3$.

$$b^{APA}(v) = \frac{2}{3}v^3 = \mathbb{E}_{\boldsymbol{v}|v}\left[b^{APL}\left(v_{(3)}(\boldsymbol{v})\right)\right]$$
$$= b^{APL}(v)(1-v)^2 + \int_0^v b^{APL}(w)\,2(1-w)\,\mathrm{d}w.$$

Differentiating and simplifying, we obtain the differential equation $\frac{\mathrm{d}}{\mathrm{d}v}b^{APL}(v) = \frac{2v^2}{(1-v)^2}$, which implies

$$b^{APL}(v) = \frac{2v(2-v)}{1-v} + 4\log(1-v),$$

a divergent bid. This is natural: although the winning bidder pays, he does not pay his own bid, and bounded bids unravel in any equilibrium. However, in this case the interim revenue remains finite because

$$t^{APL}(v) = \mathbb{E}_{\boldsymbol{v}|v}\left[b^{APL}\left(v_{(3)}(\boldsymbol{v})\right)\right] = t(v)$$

so the IRF is $\Pi^{APL}(v) = 3t(v)$. Recall that the IRF of the APA, the associated PBA, is $\Pi^{APA}(v) = t(v) + 2\mathbb{E}[t(x)]$. Hence, $\Pi^{APL}(v) - \Pi^{APA}(v) = 2(t(v) - \mathbb{E}[t(x)])$ from which we immediately get interim dominance of the PBA at 0 (at 1, strict) and single crossing of the IRFs.

Online Appendix

Proof of Proposition 12

To show that $\mathcal{E}$ is a savvy-bidder equilibrium we need

$$\Pi_0^F(0) \geq \Pi_{(0,1]}^S(0) \tag{6.20}$$

and

$$\Pi_0^F(v) \leq \Pi_{(0,1]}^S(v), \forall v > 0. \tag{6.21}$$

[38] As an extension of the, the $2,3$ and $PB-2$ auctions are interim equivalent conditional on v being the first, and the second. We can check this. The in

The argument made in the text establishes $b_{(0,1]}^{SPA,N}(x) = \tilde{b}_{(0,1]}^{SPA,S}(x) = x$ and that

$$\Pi_{(0,1]}^{S}(v) = \Pi^{S}(v) = \frac{n-2}{n} + v^{n-1} - \frac{n-1}{n}v^{n}. \tag{6.22}$$

We construct Π_0^F following the three steps in its definition.

Step 1. The FPA is chosen only when $v = 0$, so non-special bidders play $b_0^{FPA,N}(x)$, the equilibrium in the $n-1$ bidders auction, while $b_0^{FPA,S}$ is defined only on $\{0\}$.

Step 2. If the seller chose the FPA even when $v > 0$ the special bidder chooses

$$\begin{aligned}\tilde{b}_0^{FPA,S}(v) &= \arg\max_b (v-b)\,\mathbb{P}\left(b > \max b_0^{FPA,N}(x)\right) \\ &= \arg\max_b (v-b) \max\left\{\left(\frac{n-1}{n-2}b\right)^{n-1}, 1\right\}\end{aligned}$$

if $\left(\frac{n-1}{n-2}b\right)^{n-1} < 1$ then the problem is equivalent to maximizing $(v-b)\,b^{n-1}$, yielding the equilibrium bid with n bidders; bidding above the value $\frac{n-2}{n-1}$ that ensures to win is dominated so the equilibrium bid is capped at this level and we obtain $\tilde{b}_0^{FPA,S}(v) = \max\left\{\frac{n-1}{n}v, \frac{n-2}{n-1}\right\}$.

Step 3. By deviating to the FPA when $v > 0$, the seller obtains $\max\left\{\tilde{b}_0^{FPA,S}(v), \frac{n-2}{n-1}y\right\}$, where y denotes the maximum of the valuation of the non-special bidders.[39] That is

$$\Pi_0^F = \begin{cases} \tilde{b}_0^{FPA,S}(v) & \text{if } \frac{n-2}{n-1}y < \tilde{b}_0^{FPA,S}(v) \\ \frac{n-2}{n-1}y & \text{else} \end{cases}$$

which yields interim revenue

$$\begin{aligned}\Pi_0^F(v) &= \begin{cases} \frac{n-1}{n}v\left(\frac{(n-1)^2}{n(n-2)}v\right)^{n-1} + \int_{\frac{(n-1)^2}{n(n-2)}v}^{1} \frac{n-2}{n-1}y\,\mathrm{d}y^{n-1} & v < \frac{n(n-2)}{(n-1)^2} \\ \frac{n-2}{n-1} & \text{else} \end{cases} \\ &= \min\left\{\frac{n-2}{n} + \frac{(n-1)^{2n-1}}{(n-2)^{n-1}n^{n+1}}v^n, \frac{n-2}{n-1}\right\}.\end{aligned}$$

Comparing with (6.22) we obtain that $\Pi_{(0,1]}^S(0) = \Pi_0^F(0) = 0$, which establishes (6.20). Moreover, if $v \leq \frac{n(n-2)}{(n-1)^2}$ then

$$\Pi_{(0,1]}^S(v) - \Pi_0^F(v) \propto 1 - \left(\frac{n-1}{n} + \frac{(n-1)^{2n-1}}{(n-2)^{n-1}n^{n+1}}\right)v$$

which is positive for all $v \in \left[0, \frac{n(n-2)}{(n-1)^2}\right]$ since it is decreasing in v and positive at $v = \frac{n(n-2)}{(n-1)^2}$ for all $n \geq 2$. If $v > \frac{n(n-2)}{(n-1)^2}$, then $\Pi_{(0,1]}^S(v) > \Pi_{(0,1]}^S\left(\frac{n(n-2)}{(n-1)^2}\right) > \Pi_0^F\left(\frac{n(n-2)}{(n-1)^2}\right) = \Pi_0^F(v)$ as Π_0^F is flat in that region. Therefore, $\Pi_{(0,1]}^S(v) > \Pi_0^F(v)$ for all $v > 0$, which establishes (6.21) and completes the proof.

[39] Notice that the equilibrium is inefficient whenever $v < y < \frac{(n-1)^2}{n(n-2)}v$; by bidding more aggressively than his competitors, the special bidder wins too often.

Proof of Proposition 13

Using the LIE,

$$\mathbb{E}_{s=1}\left[\Pi^{F}(v)-\Pi^{S}(v)\right]=\mathbb{E}_{P}\left[\mathbb{E}\left[\Pi^{F}(v)-\Pi^{S}(v)\,|v<P\right]\right]$$

we show that this expectation is positive no matter the distribution of P by showing that for each realization p of P,

$$\mathbb{E}\left[\Pi^{F}(v)-\Pi^{S}(v)\,|v<p\right]>0.$$

from independence of p and v we get

$$\begin{aligned}\mathbb{E}\left[\Pi^{F}(v)-\Pi^{S}(v)\,|v<p\right]&=\frac{1}{F(p)}\int_{0}^{p}\Pi^{F}(v)-\Pi^{S}(v)\,\mathrm{d}F(v)\\&=-\frac{1}{F(p)}\int_{p}^{1}\Pi^{F}(v)-\Pi^{S}(v)\,\mathrm{d}F(v)\end{aligned}$$

where the last line uses $\int_0^1 \Pi^{F}(v)-\Pi^{S}(v)\,\mathrm{d}F(v)=0$. Let $\tilde{v}$ be the threshold from Theorem 2 such that $\text{sign}\left(\Pi^{F}(v)-\Pi^{S}(v)\right)=\text{sign}\left(\tilde{v}-v\right)$. If $p<\tilde{v}$, then $\Pi^{F}(v)-\Pi^{S}(v)>0$ for all $v<p$, implying that the integral in the first line is positive. If $p>\tilde{v}$ then $\Pi^{F}(v)-\Pi^{S}(v)\leq 0$ for all $v>p$ (and strictly for $v<1$),[40] implying that the integral in the second line is positive. The argument for $s=0$ is the specular and omitted.

[40]If $P\sim\delta_1$, then $v<P$ is uninformative and auctions are equivalent by the RET.