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WORKING PAPER NO. 760

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September 2025

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ISSN: 2240-9696

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Vessel Sharing Agreements under Non-Linear Costs

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Abstract
We examine a game-theoretic model of vessel sharing agreements in industries endowed with a general class of price functions and with classes of convex cost functions. We study the equilibrium structure thereof—in particular, the existence of a unique equilibrium aggregate and the existence of a unique equilibrium—and we provide a comparative statics analysis of consumer welfare with respect to an ordinal measure of concentration of the industry. We show that the a "high degree" of convexity of the cost functions can generate anti-competitive effects. In the presence of linear costs, the model satisfies a weak aggregative form in the sense of aggregative games. By allowing for the non-linearity of variable cost functions, we further weaken the aggregative nature of the games considered. Here we provide a specific new technique for treating these games in which both the equilibrium structure and the comparative statics analysis are based on the comparison of the equilibrium conditions of the players who positively vary their strategies within the groups that positively vary the group's equilibrium aggregate from an equilibrium with a smaller global aggregate associated with a less concentrated industry to an equilibrium with a larger global aggregate associated with a more concentrated industry.

Keywords: Vessel sharing agreements, Nash equilibrium uniqueness, Generalized concavity, Comparative statics, Welfare analysis.

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1 Introduction

The key aspect of a vessel sharing agreement is the joint production, by its members, of a liner transportation service of containerized cargo. The peculiarity of this joint production method lies in the absence of compensation for the costs incurred by each member of the agreement in producing a transportation service that is partly commercially managed by the other members of the agreement, who are the beneficiary of the revenues resulting from the operation of that part of the service. The fact that a vast majority (more than the 80%) of world trade takes place via maritime transportation and that agreements of this kind have become increasingly frequent[1] gives a measure of the dimension of the phenomenon studied here.

The importance of analyzing agreements that involve the sharing of independently owned means of production is widely recognized in economics (see, e.g., [5] and [20] as well as the subsequent literature). Beyond a purely theoretical interest, a real reason to examine these agreements is that they can give rise to anti-competitive effects and, in fact, some antitrust laws prohibit them, at least in principle. For instance, vessel sharing agreements are in principle prohibited by Article 101(1) of the Treaty on the Functioning of the European Union and they can be run only by virtue of the "emending" Article 101(3) of that Treaty, which allows for pro-competitive agreements.[2] Our understanding of the effects of modes of production that call for the joint use of independently owned means of production can thus have potential implications for the economy. Such understanding cannot ignore the particular mechanism used by companies to organize joint production, which is, in fact, at the origin of a possible change in the strategic behavior of the agents involved and of the repercussions on society.

To make the point of our discussion as clear as possible from the beginning, it is worth to immediately clarify the production mechanism envisaged by a vessel sharing agreement. Denoting by B a finite and nonempty set of carriers of a vessel sharing agreement where each member $n \in B$ contributes a maximum production capacity $\kappa_n \in \mathbb{R}_{++}$, the typical contract underlying that agreement stipulates that each member $n \in B$ dedicates a fraction

$$\frac{\kappa_l}{\sum_{i \in B} \kappa_i}$$

of its capacity κ_n to the transportation of containers commercially managed by the member l, for every member $l \in B$ of the agreement: the member l can thus *exclusively* use member n's production capacity up to the above specified portion and the remaining capacity cannot be used by any other member. The cost incurred by carrier $n \in B$ for the transportation of the containers commercially managed by a member $l \in B \backslash \{n\}$ are not compensated by l and each member $n \in B$ enjoys exclusive rights to the revenues associated with the transportation of the containers commercially managed by n. As is clear from this brief description, such peculiar production mechanism can be extended in principle to any other modes of transportation that calls for the use of production capacities, if not even—with due abstraction—to industries other than transportation.

[1] See, e.g., [19, p. 139].

[2] During the so-called CBER era—until 25th April 2024—the members of large prospective vessel sharing agreements had to self-assess the pro-competitive nature of the agreement; with the end of that "era", common European antitrust rules and procedures apply.

A model of a system of vessel sharing agreements in which a game of oligopolistic competition is associated with each possible configuration of vessel sharing agreements has been first proposed in [17]. The mentioned article makes the point that, since the transportation of containers can be understood as a homogeneous good (more correctly, as a homogeneous service) and since firms' legal identities are kept separate, carriers' revenues should be modelled as in the usual Cournot oligopoly: the revenue of each carrier thus depends on the sum of the quantities of transportation service supplied to the market *by all carriers of the industry*. Also, and importantly, that same article makes the point that the cost function of each carrier depends on a fraction of the sum of the quantities of transportation service supplied to the market *by all carriers of a vessel sharing agreement*.[3] As a result, in the case of a degenerate configuration of vessel sharing agreements (namely, the vessel sharing agreements formed by only one carrier) the associated game of oligopolistic competition is the usual Cournot game, but in all other cases the associated game is structurally dissimilar from it. For the games generated by a system of vessel sharing agreements, [17] has examined the equilibrium structure and provided a comparative statics analysis unfolding the pro-competitive effects of the formation and expansion of vessel sharing agreements. More precisely, under the assumption of strict concavity of the revenue function and of the linearity of variable cost functions, the mentioned article has proved that every game generated by a system has a unique Nash equilibrium and that the enlargement of vessel sharing agreements yields a decrease in the unique price equilibrium and, consequently, an increase in consumer welfare.

In the present contribution we generalize the investigation in [17] both on the demand side, by admitting a general class of price functions (which properly subsumes that considered in the mentioned article), and on the supply side, by admitting strictly increasing and convex (but possibly nonlinear) variable cost functions. Dealing with such generalization provides a new contribution, from both a game-theoretic and mathematical viewpoint and from an economic viewpoint, by not only responding to the need to carry an analysis based on hypotheses on the most general primitives but also, especially on the supply side, to remove assumptions that are not necessarily plausible. In particular, the linearity of costs may conflict with the heterogeneity of the productive efficiency of the various ships of the fleet of a carrier. Even when we assume that each vessel operates under linear and strictly increasing variable costs, in the presence of a fleet composed of multiple vessels with different linear variable costs, a simple microeconomic optimization exercise leads to variable costs for the carrier that are piecewise linear, convex, and strictly increasing. It must be conceded that the heterogeneity of the productive efficiency of the various ships of the fleet of a carrier is commonplace in the container shipping industry and hence that the assumption of linearity of variable cost functions is in fact restrictive and unrealistic.

From a game-theoretic and mathematical viewpoint, the relaxation of the cost linearity as-

[3] A point already clarified in [17] is that the formation of a vessel sharing agreement *does not alter the variable cost function* of its members (even though the mechanism *alters the argument of that function*): the simple reason for this is that such agreements do not alter a member's fleet. To the contrary, the fixed costs of providing a liner service are altered by the formation of vessel sharing agreements and a good part of the incentives for its members to sign the underlying contract might in fact arise from a decrease in fixed costs. Examining the members' incentives to form a vessel sharing agreement is not the object of this paper and hence, for the present analysis, the changes in fixed costs are just immaterial. For this reason we do not need to (and we will not) make any assumption about fixed costs.

sumption drastically changes the structure of the games studied. It is not difficult to check—if needed, see Remark 3—that the games considered here, as well as those in [17], are not typically aggregative games in the general sense of [1, Definition 1] as well as in the sense of [6] or of [7]. The main structural difference between the games examined in [17] and those considered here is that, by virtue of cost linearity, in the games examined in [17] the partial derivative of a player's utility function with respect to the player's strategic variable is equivalent to that of a player of a Cournot game and hence of an aggregative game. The analysis in [17] relies on the observation that, under certain assumptions on the (pseudo-)concavity of players' utility function in their own strategic variables, only the partial derivatives are what really matters and what actually determines the structure of the set of equilibria. *Lato sensu*, the games of vessel sharing agreements with linear costs considered in [17] are "almost smooth aggregative games" in the sense of [10, Definition 1] although, *stricto sensu*, a comparison is not possible because of some assumptions introduced by those authors in the definition of an almost smooth aggregative game.[4] The above observation is no longer valid when the linearity of variable cost functions is relaxed in that the structure of the resulting games cannot be assimilated to that of an almost smooth aggregative game, not even *lato sensu*. One of the contributions of the present paper is in fact providing a technique to deal with equilibrium uniqueness and to provide comparative statics when the games of vessel sharing agreements cannot be reduced to (almost smooth) aggregative games. Also, the present research might provide a stimulus for considering more general notions of an aggregative game.[5]

From an economic viewpoint, the present work confirms and partly extends to the case of convex variable cost functions the main economic conclusions in [17], where the formation and expansion of vessel sharing agreements is proved to have pro-competitive effects under the assumption of cost linearity. More precisely, the present work shows that the mentioned conclusions can be generalized only up to a certain degree of convexity of variable cost functions and that, in industries with "highly" convex variable cost functions, the formation and expansion of vessel sharing agreements can actually harm consumers. This indicates that the heterogeneity of firms' production efficiency may constitute a structural cause of non-competitive effect.

The rest of the article is organized as follows. Section 2 formally presents the fundamental structures needed to model a system of vessel sharing agreements and of a game associated with it. Section 3 deals with various definitions derived from that of a price function and recalls a notion of generalized convexity. Relying on those definitions, Section 4 introduces the main object of our analysis—convex systems of vessel sharing agreements—and discusses other conditions that a system of vessel sharing agreements might or might not satisfy. Section 5 contains the core findings of the paper: it investigates the uniqueness of the equilibrium aggregate, the equilibrium uniqueness, the positivity of equilibria in convex VSA-systems and examines the implications of the increase of concentration of a VSA-configuration on the variation in consumer welfare. The

[4] A similar nomenclature is used also in [15] and in other papers, but with a more traditional and restrictive meaning.

[5] We can't prove or disprove that the games examined here are *generalized quasi-aggregative games* in the sense of [11, Definition 2]: see also Observation II in [18] for a simple characterization thereof. Probably, to encompass the class of games considered here one needs at least two distinct interaction systems in the sense of the last-mentioned article and a more general definition of an aggregative game is in fact needed.

analysis is supplemented by examples illustrating the impossibility to drop specific assumptions. Section 6 concludes. A final Appendix 7 discusses a general condition on price functions (Appendix 7.1) and contains the fundamental theorems underlying the main results of this paper (Appendix 7.2).

2 Vessel sharing agreements

2.1 Container shipping industries

A **container shipping industry**, henceforth abbreviated CSI for short, is modelled as an oligopoly in which a set of carriers compete in the offer of a liner service for the transport of containers, here understood as a single homogeneous commodity. Each carrier has a maximum capacity (i.e., a cap to the volume of containers that such an operator can transport) and faces both a fixed cost of operating a liner service (i.e., the pure cost of operating a liner service independently of the quantity of containers transported) and a variable cost (which instead depends on the quantity of containers transported only). A container shipping industry is formally defined as follows.

Definition 1 *A **CSI** is a triple $I = (N, p, \{S_n, \phi_n, v_n\}_{n\in N})$ with*

- *a set $N = \{1, \ldots, \bar{n}\}$ is of $\bar{n} \geq 2$ **carriers**,*
- *a **price function** $p : \mathbb{R}_+ \to \mathbb{R}$*
- *a **capacity** $S_n = [0, \kappa_n] \subseteq \mathbb{R}$ with a **cap** $\kappa_n > 0$ for each $n \in N$;*
- *a **variable cost function** $v_n : S_n \to \mathbb{R}$ for each $n \in N$;*
- *a **stand-alone fixed cost** $\phi_n \in \mathbb{R}$ for each $n \in N$.*

2.2 VSA-configurations

A VSA-configuration is a structure of vessel sharing agreements within a CSI. As vessel sharing agreements usually contain some exclusivity clauses, it is natural to model a VSA-configuration as a partitions of the set N of carriers and to interpret a block of that partition as a set of carriers that have jointly signed a vessel sharing agreement. A block of a VSA-configuration will be also alternatively called a VSA (namely, a vessel sharing agreement). A singleton in a VSA-configuration is viewed as a degenerate VSA.

Definition 2 *Let $N = \{1, \ldots, \bar{n}\}$ be a set of carriers. The **set of all VSA-configurations for** N is set*

$$\mathcal{N}$$

*of all partitions of N. A partition C of N is called a **VSA-configuration** and a block in C is called a **VSA**: when C is a singleton we speak of a **degenerate VSA** (otherwise, we speak of a **non-degenerate VSA**) and when C is neither a singleton nor the entire set of carriers we speak of* *a* ***proper VSA*** *(otherwise, we speak of a* ***non-proper VSA****). For each VSA-configuration* $C \in \mathcal{N}$ *we denote by*

$$B_n^C$$

the VSA—i.e., the block in C*—to which carrier* n *belongs under* C*. For each carrier* $n \in N$ *and for each VSA-configuration* $C \in \mathcal{N}$*, the* ***measure of block-internal weight of*** n ***relative to*** C *is the real number* μ_n^C *defined by*

$$\mu_n^C = \frac{\kappa_n}{\sum_{l \in B_n^C} \kappa_l}. \tag{1}$$

Example 1 clarifies the notation adopted by presenting a specific VSA-configuration and a specific VSA and by computing the measure of block-internal weight of a specific carrier of that VSA relative to that VSA-configuration.

Example 1 *Let* $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *be a set of carriers. The partition*

$$C = \{\{1, 9\}, \{2\}, \{3, 4, 6\}, \{5, 7\}, \{8\}\}$$

is one of the many possible VSA-configurations in $\mathcal{N}$*; the block* $\{3, 4, 6\}$ *is one of the five VSAs in* C*; the VSA to which carrier* 4 *belongs under* C *is the block* $B_4^C = \{3, 4, 6\}$*. Supposing that* $\kappa_3 = 15$*,* $\kappa_4 = 30$*,* $\kappa_6 = 55$*, then* $\mu_4^C = 3/10$ *as*

$$\mu_4^C = \frac{30}{15 + 30 + 55}.$$

2.3 A measure of concentration

Many measures of concentration used in industrial organization theory (like, e.g., the Herfindahl index) are real-valued functions that endow $\mathcal{N}$ with a total preorder. The "more concentrated than" used in [17] endow $\mathcal{N}$ with a *partial* order and allows for the pairwise incomparability of VSA-configurations.

Definition 3 *Let* $N = \{1, \ldots, \bar{n}\}$ *be a set of carriers and let* $(C^\circ, C^\bullet)$ *be a pair of VSA-configurations in* $\mathcal{N}$*. We say that* $C^\bullet$ ***is more concentrated than*** C° *iff each VSA in* C° *is contained in some VSA in* $C^\bullet$ *and we write* $C^\circ \sqsubseteq C^\bullet$*.*

The pair $(\mathcal{N}, \sqsubseteq)$ is known to be a bounded partial order relation on $\mathcal{N}$ and hence a reflexive, transitive and antisymmetric binary relation on $\mathcal{N}$ with a greatest element and a least element.

Definition 4 *Let* $N = \{1, \ldots, \bar{n}\}$ *be a set of carriers. The VSA-configuration*

$$\{\{1, \ldots, \bar{n}\}\}$$

is the greatest element of the partially ordered set $(\mathcal{N}, \sqsubseteq)$*. The VSA-configuration*

$$\{\{1\}, \ldots, \{\bar{n}\}\}$$

is the least element of the partially ordered set $(\mathcal{N}, \sqsubseteq)$ *and is henceforth called the* ***Cournot VSA-configuration****.*

In general, the binary relation $\sqsubseteq$ is not a total order and two VSA-configurations might well be incomparable through $\sqsubseteq$. Example 2 illustrates the point.

Example 2 *Let* $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *be a set of carriers and consider the VSA-configurations*

$$C^{\circ} = \{\{1,4\},\{2,6,7\},\{3\},\{5,8\},\{9\}\}, \quad C^{\bullet} = \{\{1,4,9\},\{3\},\{2,5,6,7,8\}\}$$

and

$$C^{*} = \{\{1,3,4,9\},\{2,5,6,7\},\{8\}\}$$

in $\mathcal{N}$*. The VSA-configuration* $C^{\bullet}$ *is more concentrated than* C°*. However, neither* $C^{\bullet}$ *is more concentrated than* C^{*} *nor* C^{*} *is more concentrated than* $C^{\bullet}$*.*

2.4 VSA-system

Given a CSI, we can get a formal description of a VSA-system by specifying how a carrier's fixed and variable costs depend on all possible VSA-configurations.

Definition 5 *A* ***VSA-system*** Σ *is a pair* (I, Φ) *where* I *is a CSI specified as in Definition 1 and where* Φ *is a set*

$$\{\phi_n^C\}_{(n,C)\in N\times\mathcal{N}}$$

of ***configuration-dependent fixed costs*** *such that for all* $n \in N$*:*

- $\phi_n^C \in \mathbb{R}$ *for all* $C \in \mathcal{N}$*;*
- $\phi_n^{\{\{1\},\ldots,\{\bar{n}\}\}} = \phi_n$*.*

Remark 1 *The hypothesis that* $\phi_n^{\{\{1\},\ldots,\{\bar{n}\}\}} = \phi_n$ *for all* $n \in N$ *is just a consistency condition but is immaterial for the validity of the results of this paper.*

2.5 Operation and profits in a VSA-system

Consider a VSA-system $\Sigma = (I, \Phi)$ and suppose that its carriers commercially produce a vector $(s_1, \ldots, s_{\bar{n}})$ of quantities of transportation service. The VSA-configuration of the CSI-industry I of that system is immaterial as to the revenue of a carrier, say n, in N: such a revenue is

$$p(\textstyle\sum_{l\in N} s_l) \cdot s_n$$

no matter what fraction of s_n commercially managed (namely, commercially produced) by a carrier n is operated (namely, transported) by other carriers of the industry. VSA-configurations—and the way they discipline how quantities are operated by carriers—are instead important in the specification of costs. About fixed costs we have merely assumed that they are real numbers and we only have imposed a reasonable consistency condition: other reasonable assumptions might be reasonably imposed but they are just immaterial for the analysis conducted here. The specification of variable costs—which is instead crucial to the understanding of how vessel sharing agreements work—needs a longer explanation.[6] Now, suppose a VSA-configuration C in $\mathcal{N}$ is formed. Assuming a homogeneous distribution of production over time,[7] the vessel sharing agreement signed by carrier n implies that the *aggregate* quantity of transport service

$$\textstyle\sum_{l\in B_n^C} s_l$$

commercially managed by the members of B_n^C is operated by each of them proportionally to the quantity of capacity contributed to that VSA (namely, proportionally to B_n^C): such a proportion is the real number μ_n^C specified in (1). Therefore, the quantity operated by carrier n is

$$\mu_n^C \cdot \textstyle\sum_{l\in B_n^C} s_l \tag{2}$$

and hence the associated *variable* cost of carrier n is

$$v_n(\mu_n^C \cdot \textstyle\sum_{l\in B_n^C} s_l).$$

It is noted that vessel sharing agreements do not involve a change of property of the ships owned by n and hence they do not entail a change in the technology—and hence in the variable cost function—of carrier n, which continues to face a variable cost function v_n. Carrier n's *total* cost is thus

$$\phi_n^C + v_n(\mu_n^C \cdot \textstyle\sum_{l\in B_n^C} s_l)$$

and, consequently, carrier n's configuration-dependent profit is

$$p(\textstyle\sum_{l\in N} s_l) \cdot s_n - \phi_n^C - v_n(\mu_n^C \cdot \textstyle\sum_{l\in B_n^C} s_l). \tag{3}$$

Remark 2 *Consider a VSA-system $\Sigma = (I, \Phi)$. It is explicitly remarked that, for all $n \in N$, the quantity operated by carrier n specified in (2) satisfies the inequalities $0 \leq \mu_n^C \cdot \sum_{l\in B_n^C} s_l \leq \kappa_n$ and that each carrier $i \in B_n^C$ operates the fraction*

$$\frac{\kappa_i}{\sum_{l\in B_n^C} \kappa_l}$$

of quantity s_n commercially managed by the carrier $n \in B_n^C$ and hence that

$$0 \leq \frac{\kappa_i}{\sum_{l\in B_n^C} \kappa_l} \cdot s_n \leq \kappa_i$$

and

$$\textstyle\sum_{i\in B_n^C} \dfrac{\kappa_i}{\sum_{l\in B_n^C} \kappa_l} \cdot s_n = s_n.$$

This remark clarifies that the accounting of the quantities operated and commercially managed by carriers is consistent. It is observed, also, that when C is the Cournot configuration—put differently, when $B_l^C = \{l\}$ for all $l \in N$—carrier n's profit can be expressed by $p(\sum_{l\in N} s_l) \cdot s_n - \phi_n - v_n(s_n)$ in that $\mu_n^C = 1$ and carrier n's profit coincides with the usual specification of the profit of a firm of a Cournot oligopoly when C is the Cournot configuration. A VSA-system is thus a proper generalization of a Cournot oligopoly.

[6] See also [17] for a discussion on variable costs.

[7] See again, Sections 2 and 4.2 in [17] for a discussion of this assumption.

Example 3 illustrates, numerically, the difference between the quantity of liner service commercially managed by a carrier in a VSA and that operated by a carrier in a VSA.

Example 3 *Consider a VSA-system* (I, Φ) *with* $N = \{1, 2, 3, 4, 5, 6\}$ *and*

$$S_n = [0, 10 \cdot n]$$

for all $n \in N$ *and consider a VSA-configuration*

$$C = \{\{1, 2, 3\}, \{4, 6\}, \{5\}\}$$

for Σ. *Then*

$$(\mu_1^C, \mu_2^C, \mu_3^C) = (1/6, 1/3, 1/2).$$

Assume that each carrier n *commercially produces a vector* $s = (s_1, \ldots, s_6)$ *of quantities of transportation service with* $s_n = 2 \cdot n + 6$ *for all* $n \in N$. *Then*

$$(s_1, s_2, s_3) = (8, 10, 12)$$

and hence the quantity of liner service commercially managed by carrier 1 *(respectively,* 2 *and* 3*) is* 8 *(respectively,* 10 *and* 12*). The generic entry* A_{ij} *of the* 3×3 *square matrix* A *specified by*

$$A = \begin{bmatrix} \frac{1}{6} \cdot 8 & \frac{1}{6} \cdot 10 & \frac{1}{6} \cdot 12 \\ \frac{1}{3} \cdot 8 & \frac{1}{3} \cdot 10 & \frac{1}{3} \cdot 12 \\ \frac{1}{2} \cdot 8 & \frac{1}{2} \cdot 10 & \frac{1}{2} \cdot 12 \end{bmatrix}$$

is the quantity operated—namely, transported—by carrier i *of liner service commercially managed—namely, commercially produced—by carrier* j. *It is readily verified that*

$$\textstyle\sum_{j=1}^{3} A_{ij} = \mu_i^C \cdot \sum_{l \in B_i^C} s_l \text{ for all } i \in \{1, 2, 3\}$$

and so $\sum_{j=1}^{3} A_{ij}$ *is the quantity* $\mu_i^C \cdot \sum_{l \in B_i^C} s_l$ *operated—namely, transported—by carrier* $i \in \{1, 2, 3\}$ *according to the vessel sharing agreement. Likewise, it is readily verified that*

$$\textstyle\sum_{i=1}^{3} A_{ij} = s_j \text{ for all } j \in \{1, 2, 3\}$$

and so $\sum_{i=1}^{3} A_{ij}$ *is the quantity* s_j *commercially managed—namely, commercially produced—by carrier* $j \in \{1, 2, 3\}$.

2.6 VSA-games and equilibria

A game G is a triple $(N, \{S_n\}_{n \in N}, \{u_n\}_{n \in N})$ where: $N = \{1, \ldots, \bar{n}\}$ is a finite set with $\bar{n} \geq 2$ players; S_n is a nonempty set of player n's strategies; player n's utility function u_n is a real-valued function on the joint strategy set

$$S = \textstyle\prod_{l \in N} S_l.$$

Given a game G and a pair $(n, s) \in N \times S$, we put $s_{-n} = (s_l)_{l \in N \setminus \{n\}}$ and

$$S_{-n} = \textstyle\prod_{l \in N \setminus \{n\}} S_l,$$

we denote the joint strategy s by (s_n, s_{-n}) and we call player n's conditional utility function on s_{-n} the function $u_n(\cdot, s_{-n}) : S_n \to \mathbb{R}$ specified by

$$u_n(\cdot, s_{-n})(s_n) = u_n(s_n, s_{-n}).$$

A **Nash equilibrium** for a game G is a joint strategy $e \in S$ satisfying the implication $s_n \in S_n \Rightarrow u_n(e) \geq u_n(s_n, e_{-n})$ for all $n \in N$. When strategy sets are subsets of $\mathbb{R}$, the sum $e_1 + \ldots + e_{\bar{n}}$ of all components of a Nash equilibrium e for G is sometimes called an equilibrium aggregate. Recalled the basic definitions of a game and of a Nash equilibrium, we associate a game of oligopolistic competition to each possible VSA-configuration. The associated game—where utility functions are specified by the configuration-dependent profit obtained in (3)—will be called a VSA-game.

Definition 6 *Let Σ be a VSA-system specified as in Definition 5. For each VSA-configuration C in $\mathcal{N}$, the* ***VSA-game associated to C under*** *Σ is the game*

$$(N, \{S_n\}_{n\in N}, \{u_n^C\}_{n\in N})$$

where N is the set of carriers and, for all $n \in N$, player n's strategy set S_n is carrier n's capacity and player n's utility function u_n^C is the real-valued function on the joint strategy set S specified by

$$u_n^C(s) = p(\textstyle\sum_{l\in N} s_l) \cdot s_n - \phi_n^C - v_n(\mu_n^C \cdot \sum_{l\in B_n^C} s_l).$$

The utility function u_n^C is also called ***carrier n's configuration-dependent profit function****.*

Remark 3 clarifies the connection with [1, Definition 1]'s aggregative games.

Remark 3 *Consider a VSA-system and a VSA-configuration C with a proper VSA containing a carrier n. Consider the profit function u_n^C specified as in Definition 6. Assume that v_n strictly increasing and pick arbitrary $i \in B_n^C \backslash \{n\}$ and $l \in N \backslash B_n^C$. Denote by ω the zero vector of $\mathbb{R}^{\bar{n}}$ and by $\omega^{(i)}$ (by $\omega^{(l)}$) the vector in $\mathbb{R}^{\bar{n}}$ whose i-th (whose l-th) component equals $\max S_i$ (equals $\max S_l$) and where all other components are zero. Suppose the existence of a pair (g, Π_n) of functions with $g : S \to \mathbb{R}$ and $\Pi_n : S_n \times g[S] \to \mathbb{R}$ such that g is continuous on the Cartesian product S and strictly increasing in each of the $\bar{n}$ arguments. It is not difficult to see that the equality $u_n^C(s) = \Pi_n(s_n, g(s))$ cannot hold for all $s \in S$ and hence that the game VSA-game associated to C under Σ cannot be an aggregative game in the sense of [1, Definition 1]. To see this, note that the assumption that g is strictly increasing in every argument ensures that $\min\{g(\omega^{(i)}), g(\omega^{(l)})\} > g(\omega)$ and—by basic topological reasons—the continuity of g in turn implies the existence of a pair $(s^\circ, s^\bullet) \in S \times S$ such that*

$$s_i^\circ > 0 = s_j^\circ \text{ for all } j \in N\backslash\{i\} \text{ and } s_l^\bullet > 0 = s_j^\bullet \text{ for all } j \in N\backslash\{l\}$$

and that

$$g(s^\circ) = g(s^\bullet).$$

Clearly, $s_n^\circ = s_n^\bullet = 0$. If the equality $u_n^C(s) = \Pi_n(s_n, g(s))$ were true for all $s \in S$, then we should have

$$u_n^C(s^\circ) = \Pi_n(s_n^\circ, g(s^\circ)) = \Pi_n(0, g(s^\circ)) = \Pi_n(0, g(s^\bullet)) = \Pi_n(s_n^\bullet, g(s^\bullet)) = u_n^C(s^\bullet)$$

in contradiction with the fact that the strict increasingness of v_n implies

$$u_n^C(s^\circ) = -\phi_n^C - v_n(\mu_n^C \cdot s_i^\circ) < -\phi_n^C - v_n(0) = u_n^C(s^\bullet).$$

3 Price functions, derived notions and conditions thereon

The notion of price function has already been used in the context of a CSI. A general definition thereof—independent of that of an industry—is here given.

Definition 7 *A price function is a function from $\mathbb{R}_+$ to $\mathbb{R}$.*

The remainder of Sect. 3 provides the definitions of some notions associated with that of a price function and recalls a definition of generalized convexity that will be often imposed on one of them in the rest of the paper.

3.1 Derived notions

The domain of the revenue function specified in Definition 8 is—like in the case of a price function—the entire $\mathbb{R}_+$ while the domain of the three functions specified in Definition 9 is $\mathbb{R}_{++}$: the reason of this choice is essentially technical.

Definition 8 *Let $p : \mathbb{R}_+ \to \mathbb{R}$ be a price function. The **revenue function associated to** p is the function $r : \mathbb{R}_+ \to \mathbb{R}$ specified by*

$$r(x) = p(x) \cdot x.$$

It is observed here, that one can always handle standard specifications of a price function (e.g., the specification $p(x) = 1/\sqrt{x}$) that are well-defined only on $\mathbb{R}_{++}$ by assigning an arbitrary real value of p at 0 (e.g., by putting $p(0) = 1$ when $p(x) = 1/\sqrt{x}$ for all positive x): this arbitrary imposition makes r vanishing at 0, which is clearly a reasonable condition (continuing with the previous parenthetical example, whatever the specification of p at 0, the revenue function r is well-specified on the entire $\mathbb{R}_+$ by $r(x) = \sqrt{x}$). Before introducing additional notions associated to that of a price function, it is explicitly observed that in this paper we put

$$\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$$

and hence that $\mathbb{Z}_+$ is the set of all nonnegative integers.

Definition 9 *Let $p : \mathbb{R}_+ \to \mathbb{R}$ be a price function that is continuous on $\mathbb{R}_{++}$ and let $(z, \tau) \in \mathbb{Z}_+ \times \mathbb{R}_+$.*

- *The **normal primitive price function associated to** p is the function $P : \mathbb{R}_{++} \to \mathbb{R}$ such that*

  $$P(1) = 0 \text{ and } DP(x) = p(x) \text{ for all } x \in \mathbb{R}_{++}.$$

- *The **augmented revenue function associated to** p **and** (z, τ) is the function $R^z_\tau : \mathbb{R}_{++} \to \mathbb{R}$ specified by*

  $$R^z_\tau(x) = p(x + \tau) \cdot x + P(x + \tau) \cdot z.$$

- *The* ***price elasticity associated to*** p *is the function* $E : \mathbb{R}_{++} \to \mathbb{R}$ *specified by*

$$E(x) = \frac{Dp(x)}{p(x)} \cdot x,$$

provided p *is nonvanishing on* $\mathbb{R}_{++}$ *and differentiable on* $\mathbb{R}_{++}$.

It is remarked that $R_0^0(x) = r(x)$ for all $x \in \mathbb{R}_{++}$ and hence that R_0^0 in fact coincides with the restriction of r to $\mathbb{R}_{++}$.

3.2 Semistrictly demiconcave augmented revenue

This Section 3.2 recalls a notion of generalized concavity introduced in [13]: see the Introduction of the mentioned article for antecedents. Before providing the definition of semistrict demiconcavity, it is worth to clarify that in this paper a real interval is said to be proper when it is infinite.

Definition 10 *A continuous real-valued function* f *on a proper real interval* L *is* ***semistrictly demiconcave*** *iff there exist two (possibly empty) real intervals* L_1 *and* L_2 *such that:* $L_1 \cup L_2 = L$ *and* $L_1 \cap L_2 = \emptyset$*;* $x \leq y$ *for every pair* $(x, y) \in L_1 \times L_2$*;* f *is strictly concave on* L_1 *and decreasing on* L_2.

Proposition 1 recalls some important facts concerning semistrict demiconcavity: the reader is referred to Section 2.2–3 in [13] for a proof of Proposition 1.

Proposition 1 *Let* f *and* g *be continuous real-valued functions on a proper real interval* L.

1. *If* f *is semistrictly demiconcave, then* f *is quasiconcave.*
2. *If* f *is semistrictly demiconcave and* g *is increasing and convex, then* $f - g$ *is semistrictly demiconcave.*
3. *Suppose* L *is open and* f *is differentiable. Then* f *is strictly demiconcave if and only if the implication*

$$Df(x) > 0 \Rightarrow Df(y) > Df(x)$$

holds true for every pair $(x, y) \in L \times L$ *such that* $y < x$.

The main result of this work imposes semistrict demiconcavity on each augmented revenue functions in the precise sense of Definition 11.

Definition 11 *Let* $p : \mathbb{R}_+ \to \mathbb{R}$ *be a price function that is continuous on* $\mathbb{R}_{++}$. *We say that* ***each augmented revenue function*** R_τ^z ***is semistrictly demiconcave*** *iff* R_τ^z *is semistrictly demiconcave for every pair* $(z, \tau) \in \mathbb{Z}_+ \times \mathbb{R}_+$.

Proposition 2 provides a convenient characterization of the strict demiconcavity of an augmented revenue function. Proposition 2 follows directly from part 3 of Proposition 1 and its obvious proof is thus omitted.

Proposition 2 *Let $p : \mathbb{R}_+ \to \mathbb{R}$ be a price function that is differentiable on $\mathbb{R}_{++}$ and let $(z, \tau) \in \mathbb{Z}_+ \times \mathbb{R}_+$. Assertions I and II are equivalent .*

I. R_τ^z is semistrictly demiconcave.

II. $DR_\tau^z(x) > 0 \Rightarrow DR_\tau^z(x - w) > DR_\tau^z(x)$ for every $(w, x) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$ such that $w < x$.

Remark 4 contains a useful observation.

Remark 4 *Let $(w, x, z, \tau) \in \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{Z}_+ \times \mathbb{R}_+$ with $w < x$ and put $y_x = x + \tau$. The derivatives $DR_\tau^z(x)$ and $DR_\tau^z(x - w)$ in Proposition 2 can be expressed by*

$$DR_\tau^z(x) = Dp(y_x) \cdot x + p(y_x) \cdot (z + 1)$$

and

$$DR_\tau^z(x - w) = Dp(y_x - w) \cdot (x - w) + p(y_x - w) \cdot (z + 1).$$

4 Main assumptions

In this Section 4 we introduce all the relevant definitions concerning VSA-systems and VSA-configurations that are used as assumptions in the analysis of the equilibrium structure and welfare properties conducted in Section 5. Henceforth, given a function $f : X \to \mathbb{R}$ on a proper real interval X, we denote the right upper Dini derivative of f at $x \in X \backslash \{\sup X\}$ by $D^+ f(x)$ and by $D^- f(x)$ the left upper Dini derivative of f at $x \in X \backslash \{\inf X\}$: when f is either convex or right differentiable at $x \in X \backslash \{\sup X\}$, the derivative $D^+ f(x)$ is the right derivative of f at x; when f is convex or left differentiable at $x \in X \backslash \{\inf X\}$, the extended real number $D^- f(x)$ is the left derivative of f at x. When f is differentiable at x in the interior of X, we write $Df(x)$ to denote the derivative of f at x.

4.1 Convex VSA-systems

The basic assumption employed in the analysis of this paper is the convexity of a VSA-system, in the precise sense of Definition 12. Appendix A contains a discussion on condition H4 and shows sufficient conditions for the validity of conditions H4. Furthermore, Appendix A contains several examples of price functions that satisfy conditions H1–4.

Definition 12 *Let Σ be a VSA-system. The VSA-system Σ is said to be a* ***convex VSA-system*** *iff:*

H1. the price function p is differentiable on $\mathbb{R}_{++}$;

H2. the inequality $Dp(x) < 0$ holds for all $x \in \mathbb{R}_{++}$;

H3. the revenue function r is continuous;

H4. each augmented revenue function R_τ^z is semistrictly demiconcave;

H5. each variable cost function v_n is continuous;

H6. each variable cost function v_n is convex and strictly increasing.

4.2 Active VSA-systems

The activeness of a VSA-system essentially requires that the inactivity of all carriers—namely, zero production of liner service by any carrier—does not constitute an equilibrium state when all vessel sharing agreements are degenerate.

Definition 13 *Let Σ be a VSA-system. The VSA-system Σ is said to be an* ***active VSA-system*** *iff conditions H1, H2, H6 hold and*

$$\lim_{x \to 0} \; p(x) > \min\{D^{+}v_1(0), \ldots, D^{+}v_{\bar{n}}(0)\}.$$

4.3 VSA-systems with almost linear costs

The following condition imposes an upper bound to the "degree of convexity" of variable costs functions by requiring that the left-hand derivative of each (convex and strictly increasing) variable cost function v_n at the cap κ_n is not greater than[8]

$$\frac{\kappa_1 + \cdots + \kappa_{\bar{n}}}{\kappa_1 + \cdots + \kappa_{\bar{n}} - \min\{\kappa_1, \ldots, \kappa_{\bar{n}}\}} \tag{4}$$

times the right-hand derivative of v_n at 0.

Definition 14 *Let Σ be a VSA-system. The VSA-system Σ is said to be a* ***VSA-system with almost linear costs*** *iff condition H6 holds and each variable cost function v_n satisfies the inequality*

$$D^{-}v_n(\kappa_n) \leq \frac{\kappa_1 + \cdots + \kappa_{\bar{n}}}{\kappa_1 + \cdots + \kappa_{\bar{n}} - \min\{\kappa_1, \ldots, \kappa_{\bar{n}}\}} \cdot D^{+}v_n(0).$$

4.4 VSA-systems with normal price functions

The condition of normality of price function postulates the continuity of a price function bounded from above.

Definition 15 *Let Σ be a VSA-system. The VSA-system Σ is said to be a* ***VSA-system with a normal price function*** *p iff conditions H1, H2 hold and $\lim_{x\to 0} p(x) = +\infty$ when p is not continuous at zero.*

4.5 Almost smooth configurations

The almost smoothness condition is a technical assumption that will be used to ensure the uniqueness of an equilibrium. We note here that such condition is satisfied by every Cournot configuration and that it is satisfied also when all carriers of a CSI except at most one have variable cost functions that are differentiable on the interior of their domain.

Definition 16 *Let Σ be a VSA-system. A VSA-configuration C in $\mathcal{N}$ is* ***almost smooth*** *iff in each block B in C all carriers in B except at most one have variable cost functions that are differentiable on the interior of their domain.*

[8]The real number in (4) is always strictly larger than 1.

4.6 Some remarks

Conditions H1 and H2 imply that p is strictly decreasing on $\mathbb{R}_{++}$ and hence the limit in Definitions 13 and 15 is a well-defined extended real: in particular, that limits exist in $\mathbb{R} \cup \{+\infty\}$. Condition H6 implies that $D^+v_n(0)$ exists in $\mathbb{R}_+$ and that $D^-v_n(\kappa_n)$ exists in $\mathbb{R}_{++} \cup \{+\infty\}$: as $\bar{n} \geq 2$ and each κ_n is positive by assumption, a moment's reflection shows that the almost linearity condition stipulated in Definition 14 implies that $D^+v_n(0)$ and $D^-v_n(\kappa_n)$ exists in $\mathbb{R}_{++}$.

5 Equilibrium analysis

Section 5 examines the equilibrium structure of a convex VSA-system and provides an ordinal comparative statics analysis of the effects of an increase in the concentration of a VSA-configuration.

5.1 Equilibrium structure

Theorem 1 proves the existence, uniqueness and positivity of an equilibrium aggregate in any convex VSA-system and provides sufficient conditions for the positivity of the unique equilibrium price and for the uniqueness of an equilibrium in the strict sense. Corollary 1—which follows directly from part 3 of Theorem 1 and whose proof is omitted—is a particular consequence of part 3 of Theorem 1 on the uniqueness of an equilibrium.

Theorem 1 *Let Σ be a convex VSA-system and let C be a VSA-configuration in $\mathcal{N}$.*

1. *There exists at least one Nash equilibrium e for the VSA-game associated to C under Σ.*

2. *There exists exactly one Nash equilibrium aggregate η for the VSA-game associated to C under Σ. Furthermore, the strict inequalities $\eta > 0$ and $p(\eta) > 0$ hold true if Σ is active.*[9]

3. *There exists exactly one Nash equilibrium e for the VSA-game associated to C under Σ if C is an almost smooth VSA-configuration.*

Proof. 1. By virtue of parts 1 and 2 of Theorem 3, a routinary application of a known Nash equilibrium existence result—use, e.g., [9, Theorem 7.4]—ensures the existence of a Nash equilibrium for the VSA-game associated to C under Σ.

2. By virtue of Theorem 6, we need to prove only the first sentence of part 2 of Theorem 1. Part 1 of Theorem 1 ensures the existence of at least one Nash equilibrium—and hence of at least one Nash equilibrium aggregate—for the VSA-game associated to C. Suppose $e^{\triangleright}$ is a Nash equilibrium for the VSA-game associated to C under Σ and suppose $e^{\triangleleft}$ is a Nash equilibrium for the VSA-game associated to C under Σ. Put $\eta^{\triangleright} = e_1^{\triangleright} + \cdots + e_{\bar{n}}^{\triangleright}$ and $\eta^{\triangleleft} = e_1^{\triangleleft} + \cdots + e_{\bar{n}}^{\triangleleft}$. Furthermore, put $C^{\triangleright} = C$ and $C^{\triangleleft} = C$. Obviously, $C^{\triangleright} \sqsubseteq C^{\triangleleft}$ and $C^{\triangleleft} \sqsubseteq C^{\triangleright}$. As $e^{\triangleright}$ is a Nash equilibrium for the VSA-game associated to $C^{\triangleright}$ under Σ and $e^{\triangleleft}$ is a Nash equilibrium for the VSA-game associated to $C^{\triangleleft}$ under

[9]Clearly, the weak inequality $\eta \geq 0$—but not the weak inequality $p(\eta) \geq 0$—is true whether or not the convex VSA-system Σ is active.

Σ, part 1 of Theorem 4 implies that $\eta^{\triangleright} \leq \eta^{\triangleleft}$ and $\eta^{\triangleleft} \leq \eta^{\triangleright}$. Consequently, $\eta^{\triangleright} = \eta^{\triangleleft}$ and hence there exists exactly one Nash equilibrium aggregate for the VSA-game associated to C under Σ.

3. Assume that C is an almost smooth VSA-configuration. Part 1 of Theorem 1 ensures the existence of at least one Nash equilibrium e° for the VSA-game associated to C under Σ and part 2 of Theorem 1 ensures that $e_1^\circ + \ldots + e_{\bar{n}}^\circ = e_1^\bullet + \ldots + e_{\bar{n}}^\bullet$ for any other Nash equilibrium $e^\bullet$ for the VSA-game associated to C under Σ. Consequently, there exists exactly one Nash equilibrium for the VSA-game associated to C under Σ by Theorem 5. ∎

Corollary 1 *Let Σ be a convex VSA-system and let C be a VSA-configuration in $\mathcal{N}$.*

1. *There exists exactly one Nash equilibrium e for the VSA-game associated to C under Σ provided C is a Cournot configuration.*

2. *There exists exactly one Nash equilibrium e for the VSA-game associated to C under Σ provided at most one carrier n in N has a (stand-alone) variable cost function v_n that is not differentiable on $(0, \kappa_n)$.*

5.2 Equilibrium welfare

Theorem 2 proves that in any convex VSA-system with almost linear costs an increase in the concentration of the industry generates beneficial effects for consumers: it increases the unique equilibrium aggregate (part 1 of Theorem 2) and, when either the price function is normal or the VSA-system is active, it decreases the unique equilibrium price (part 2 of Theorem 2). and yields an increase in consumer welfare (Corollary 2).

Theorem 2 *Let Σ be a convex VSA-system with almost linear costs and let $(C^\circ, C^\bullet)$ be a pair of VSA-configurations in $\mathcal{N}$ such that*

$$C^\circ \sqsubseteq C^\bullet.$$

There exists at least one Nash equilibrium e° for the VSA-game associated to C° and there exists at least one Nash equilibrium $e^\bullet$ for the VSA-game associated to $C^\bullet$. Put $\eta^\circ = e_1^\circ + \ldots + e_{\bar{n}}^\circ$ and $\eta^\bullet = e_1^\bullet + \ldots + e_{\bar{n}}^\bullet$.

1. *$\eta^\circ \leq \eta^\bullet$.*

2. *$p(\eta^\bullet) \leq p(\eta^\circ)$ if Σ is either a VSA-system with a normal price function or an active VSA-system.*

Proof. Part 1 of Theorem 1 ensures the existence of at least one Nash equilibrium e° for the VSA-game associated to C° and of at least one Nash equilibrium $e^\bullet$ for the VSA-game associated to $C^\bullet$. Part 2 of Theorem 4 ensures the validity of the inequality $\eta^\circ \leq \eta^\bullet$. Clearly, $0 \leq \eta^\circ$ as pointed out in fn. 9. This proves part 1 of Theorem 2. Henceforth assume that Σ is either a VSA-system with a normal price function or an active VSA-system. If p is continuous on $\mathbb{R}_+$, then the validity of conditions H1 and H2 entails the strict decreasingness of p on $\mathbb{R}_+$ and hence that $p(\eta^\bullet) \leq p(\eta^\circ)$ by part 1 of Theorem 2. If p is not continuous, then the normality of p implies that

(p is not continuous at 0 by virtue of assumption H1 and hence that) the limit $\lim_{x\to 0} p(x) = +\infty$: the validity of such limit in turn implies that $\lim_{x\to 0} p(x) > \min\{D^+ v_1(0), \ldots, D^+ v_{\bar{n}}(0)\}$ and hence that Σ is active. Consequently, when p is not continuous, Theorem 6 ensures that $0 < \eta^\circ \leq \eta^\bullet$ and from the last two inequalities we infer that $p(\eta^\bullet) \leq p(\eta^\circ)$ by virtue of conditions H1 and H2. ■

Corollary 2 proves that in any active convex VSA-system with almost linear costs an increase in the concentration of the industry generates beneficial effects to consumers by yielding a nonnegative consumer surplus variation, which is defined as follows. Suppose for a moment that $p : \mathbb{R}_+ \to \mathbb{R}$ is a price function of a VSA-system that is decreasing on the interior of its domain and pick an arbitrary pair $(e^\circ, e^\bullet)$ of vectors in $[0, \kappa_1] \times \cdots \times [0, \kappa_{\bar{n}}]$ such that $\eta^\circ \leq \eta^\bullet$ and that $0 \leq p(\eta^\bullet)$ with η° and $\eta^\bullet$ defined by $\eta^\circ = e_1^\circ + \ldots + e_{\bar{n}}^\circ$ and $\eta^\bullet = e_1^\bullet + \ldots + e_{\bar{n}}^\bullet$: we henceforth refer to the real number $\Delta_{SP}(e^\circ, e^\bullet)$ specified by

$$\Delta_{SP}(e^\circ, e^\bullet) = r(\eta^\circ) - r(\eta^\bullet) + \int_{\eta^\circ}^{\eta^\bullet} p(x)dx$$

as to the **consumer surplus variation from e° to $e^\bullet$.**

Corollary 2 *Let Σ be an active convex VSA-system with almost linear costs and let $(C^\circ, C^\bullet)$ be a pair of VSA-configurations in $\mathcal{N}$ such that*

$$C^\circ \sqsubseteq C^\bullet.$$

Pick an arbitrary Nash equilibrium e° for the VSA-games associated to C° under Σ and by $e^\bullet$ an arbitrary Nash equilibrium for the VSA-games associated to $C^\bullet$ under Σ. Then

$$0 \leq \Delta_{SP}(e^\circ, e^\bullet)$$

(where p is the price function of the VSA-system Σ).

Proof. Put $\eta^\circ = e_1^\circ + \ldots + e_{\bar{n}}^\circ$ and $\eta^\bullet = e_1^\bullet + \ldots + e_{\bar{n}}^\bullet$. Then $0 < \eta^\circ \leq \eta^\bullet$ and $0 < p(\eta^\bullet) \leq p(\eta^\circ)$ by Theorems 2 and 6. Clearly, p is strictly decreasing (and hence positive) on the possibly degenerate real interval $[\eta^\circ, \eta^\bullet]$ by virtue of conditions H1 and H2. Consequently,

$$p(x) - p(\eta^\bullet) \geq 0 \text{ for all } x \in [\eta^\circ, \eta^\bullet]. \tag{5}$$

As $r(\eta^\circ) = p(\eta^\circ) \cdot \eta^\circ$ and

$$r(\eta^\bullet) = p(\eta^\bullet) \cdot \eta^\bullet = p(\eta^\bullet) \cdot \eta^\circ + \int_{\eta^\circ}^{\eta^\bullet} p(\eta^\bullet)dx,$$

we can express $\Delta_{SP}(e^\circ, e^\bullet)$ as the sum of the nonnegative real number $(p(\eta^\circ) - p(\eta^\bullet)) \cdot \eta^\circ$ and of the nonnegative real number $\int_{\eta^\circ}^{\eta^\bullet} (p(x) - p(\eta^\bullet))dx$. Needless to say, the previous conclusions follow from the assertion in (5) and from the inequalities inferred at the beginning of this proof. ■

5.3 On almost smoothness and almost linearity

Example 4 shows that, in part 3 of Theorem 1, the almost smoothness condition cannot be simply dropped.

Example 4 *Putting $N = \{1, 2\}$, consider the CSI*

$$I = (N, p, \{S_n\}_{n\in N}, \{\phi_n\}_{n\in N}, \{v_n\}_{n\in N})$$

with a price function specified by

$$p(x) = 40 - 4x$$

where $S_n = [0, 5]$ and

$$v_n(x) = \begin{cases} 30x & \text{if } x \leq 1 \\ 59x - 29 & \text{if } x > 1 \end{cases}$$

for all $n \in N$. Suppose, for instance, that $\phi_n = \frac{5}{2}$ for all $n \in N$ and consider then the VSA-system $\Sigma = (I, \Phi)$ where, for instance,

$$\phi_n^C = \frac{5}{2} \cdot \mu_n^C$$

for all $n \in N$ and all $C \in \mathcal{N}$. The VSA-system Σ is an active convex VSA-system with a normal price function and with almost linear costs. Having observed this, put

$$\bar{C} = \{\{1, 2\}\}$$

and note that the VSA-configuration $\bar{C}$ for the VSA-system Σ is not an almost smooth VSA-configuration. It is readily checked that the vector e^* and e^{**} specified by

$$e^* = (81/80, 79/80)$$

and

$$e^{**} = (79/80, 81/80)$$

are distinct Nash equilibria for the VSA-game associated to $\bar{C}$ under Σ and hence that there exists a multiplicity of equilibria.

By making use of Example 4, Remark 5 shows the difficulty for our analysis of the equilibrium structure to make use of the theory of potential games started with [12].

Remark 5 *Letting $\pi : [0, 5] \to \mathbb{R}$ be the function specified by*

$$\pi(x) = \begin{cases} 30x & \text{if } x \leq 1 \\ 59x - 29 & \text{if } x > 1 \end{cases}$$

reasoning as in Proposition 2 in [4], one can readily prove that the VSA-game associated to $\bar{C}$ under Σ in Example 4 is an (exact) potential game with a potential $\Pi : S \to \mathbb{R}$ specified by

$$\Pi(s) = 40(s_1 + s_2) - 4(s_1^2 + s_2^2 + s_1 \cdot s_2) - \frac{5}{4} - \pi(s_1/2 + s_2/2).$$

Therefore, the game in Example 4 admits a potential: even more, such a potential is strictly concave. Noting that $\arg\max \Pi = \{(1,1)\}$*, it should be clear from Example 4 that the maximization of the potential does not provide a description of the entire set of equilibria. This observation shows that, even by imposing more restrictive assumptions on the price and cost functions, the attempt to convey the class of games considered here into the class of games with an exact potential (or into some generalization thereof such as [8]' pseudo-potential) does not simplify the analysis of the structure of equilibria in that the maximization of the potential might disregard some equilibria even when price functions are linear and cost functions are piecewise linear.*

5.4 On almost linearity

Example 5 shows that, in Theorem 2 and Corollary 2, the condition of almost linearity of cost functions cannot be simply dropped.

Example 5 *Putting* $N = \{1, \ldots, 31\}$*, consider the CSI*

$$I = (N, p, \{S_n\}_{n\in N}, \{\phi_n\}_{n\in N}, \{v_n\}_{n\in N})$$

with a price function specified by

$$p(x) = 100 - x$$

where $S_n = [0, 1]$ *and* $v_n(x) = x$ *for all* $n \in \{1, \ldots, 30\}$ *and where* $S_{31} = [0, 30]$ *and*

$$v_{31}(x) = \begin{cases} 20 \cdot x & \text{if } x \leq 26, \\ 44 \cdot x - 624 & \text{if } x > 26. \end{cases}$$

Suppose, for instance, that $\phi_n = 10$ *for all* $n \in \{1, \ldots, 30\}$ *and that* $\phi_{31} = 300$ *and then consider the VSA-system* $\Sigma = (I, \Phi)$ *where*

$$\phi_n^C = 10 \cdot \kappa_n \cdot \mu_n^C$$

for all $n \in N$ *and all* $C \in \mathcal{N}$ *(where* κ_n *is the maximum of* S_n *for all* $n \in N$*). The VSA-system* Σ *is an active convex VSA-system whose price function is normal and any VSA-configuration for* Σ *is an almost smooth VSA-configuration. However, the VSA-system* Σ *is not VSA-system with almost linear costs. Having observed this, put* $C^\circ = \{\{1\}, \ldots, \{n\}\}$ *and* $C^\bullet = \{\{1, \ldots, n\}\}$ *and note that*

$$C^\circ \sqsubseteq C^\bullet.$$

By part 2 of Corollary 1 there exists exactly one Nash equilibrium e° *for the VSA-game associated to* C° *under* Σ*: this equilibrium is specified by* $e_n^\circ = 1$ *for all* $n \in \{1, \ldots, 30\}$ *and by*

$$e_{31}^\circ = 25.$$

By part 2 of Corollary 1 there exists exactly one Nash equilibrium $e^\bullet$ *for the VSA-game associated to* $C^\bullet$ *under* Σ*: this equilibrium is specified by* $e_n^\bullet = 1$ *for all* $n \in \{1, \ldots, 30\}$ *and by*

$$e_{31}^\bullet = 24.$$

Putting $\eta^\circ = e_1^\circ + \ldots + e_{31}^\circ$ *and* $\eta^\bullet = e_1^\bullet + \ldots + e_{31}^\bullet$, *it is readily checked that*

$$54 = \eta^\bullet < \eta^\circ = 55$$

and that

$$45 = p(\eta^\circ) < p(\eta^\bullet) = 46.$$

Finally, it is readily checked that

$$-54.5 = \Delta_{SP}(e^\circ, e^\bullet) < 0.$$

6 Final discussion

We have presented a model of competition between the carriers of the container shipping industry under general hypotheses on the demand and supply side that relax the assumptions of a previous model in [17] with strictly concave revenue functions and linear cost functions. We have examined the equilibrium structure of this model and derived a comparative statics analysis from it. The relaxation of linearity assumptions on costs has drastically changed the structure of strategic interaction between the carriers and has required a non-obvious reappraisal of the proof technique needed to handle equilibrium problems that cannot be reduced to 1-dimensional ones.

Confirming and generalizing previous results on the structure of equilibria, we have shown how the nonlinearity of convex costs can have anti-competitive implications on consumer welfare, which were absent in the linear case. From our analysis it thus emerges that, in the presence of "highly" convex variable costs (due, for example, to the presence of carriers with highly heterogeneous fleets in terms of transportation efficiency), the formation of vessel sharing agreements can generate price increases. The last example of the paper has shown that this can occur even in the presence of elementary linear price functions and all but one firms with linear costs. Our present results thus confirm and generalize the previous analysis in [17] but at the same time shows that the relaxation of cost linearity beyond the condition that we have called "almost linearity" is compatible with anti-competitive effects.

From a mathematical point of view, the main novelty of this work was to offer a proof technique—based on a particular dichotomy described first in the Abstract—to prove equilibrium uniqueness and to perform comparative static analysis in games of competition among carriers that satisfy a weak form of aggregativity but that cannot be bear classified as aggregative games (at least, in the usual sense). Theorem 4 uses this technique and all the main results of this work are based on that result. A less sophisticated version of it, which relies on the simple dichotomy of the set of all players in a group consisting of those who increase their equilibrium strategy in the transition between two equilibria with different equilibrium aggregates and the group of all remaining players, has already been used in the context of certain types of aggregative games: see, e.g., [13] and the references therein. This less sophisticated technique cannot be used in the present context. This paper has shown that the more refined dichotomy formed by the set Z defined in the proof of Theorem 4 and by its complement to the set of all players can instead be fruitfully employed in the present context. It would be interesting to examine the applicability of the new technique to other classes of games that—though non-aggregative in the usual sense—satisfy weak forms of aggregativity.

7 Appendix

7.1 Appendix A: On condition H4

It is here shown that the class of price functions considered in this work subsumes and expands the class used in [17], where attention is restricted to price functions with an associated strictly concave revenue function. Even though Propositions 3 and 4 are essentially known, it is convenient to have clear and simple statements that show the generality of the class of price functions considered here, as well as some illustrating examples. Proposition 3 is in fact a variant of part 1 of Proposition 4.3 in [13] that dispenses with unnecessary assumptions on the positivity of p imposed in that article: it is observed that a strictly decreasing linear function cannot be positive everywhere and hence the relaxation of that positivity assumption is important if, in our analysis, we want admit also the elementary class of all strictly decreasing linear price functions.

Proposition 3 *Suppose $p : \mathbb{R}_+ \to \mathbb{R}$ is a price function satisfying condition H1. If the revenue function r is strictly concave, then conditions H2 and H4 are satisfied.*

Proof. Assume that the function $r : \mathbb{R}_+ \to \mathbb{R}$ is strictly concave. Lemma 1 in [17] ensures that condition H2 is satisfied and hence that p is strictly decreasing on $\mathbb{R}_{++}$. Consequently, P is strictly concave. Fix an arbitrary pair $(z, \tau) \in \mathbb{Z}_+ \times \mathbb{R}_+$. The function $g : \mathbb{R}_{++} \to \mathbb{R}$ specified by $g(x) = z \cdot P(x + \tau)$ is concave because P is strictly so and because z is a nonnegative integer. As $R^0_\tau(x) = p(x + \tau) \cdot x$ for all x in the domain $\mathbb{R}_{++}$ of R^0_τ, Lemma 1 in [17] implies that R^0_τ is strictly concave when $\tau > 0$. When $\tau = 0$, the strict concavity of R^0_τ is an immediate consequence of the assumption that r is strictly concave in that $R^0_0(x) = r(x)$ for all x in the domain $\mathbb{R}_{++}$ of R^0_0. As the sum of a strictly concave function and a concave function is strictly concave, the observation that $R^z_\tau = R^0_\tau + g$ implies that R^z_τ is strictly concave and we are in a position to conclude that condition H4 is satisfied. ■

Remark 6 recalls a connection between the concavity of a strictly decreasing price function and the strict concavity of the revenue function. In the literature on Cournot equilibrium, the condition of strict concavity of the revenue function has been used in the [16].

Remark 6 *Suppose $p : \mathbb{R}_+ \to \mathbb{R}$ is a strictly decreasing price function. If p is concave, then the revenue function r is strictly concave.*[10] *As is clear from the first specification in Example 6, the converse of the previous implication is generally false.*

[10] To see why, suppose p is concave and strictly decreasing and, by way of contradiction, suppose r is not strictly concave. Then there exists $(\lambda, x^\circ, x^\bullet) \in (0, 1) \times \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$r(\lambda \cdot x^\circ + (1 - \lambda) \cdot x^\bullet) \leq \lambda \cdot r(x^\circ) + (1 - \lambda) \cdot r(x^\bullet) \tag{6}$$

and $x^\circ \neq x^\bullet$. Without loss of generality, suppose $x^\circ < x^\bullet$. It is readily seen that

$$0 < \lambda \cdot (1 - \lambda) \cdot (x^\bullet - x^\circ) \cdot (p(x^\circ) - p(x^\bullet)) \tag{7}$$

Example 6 provides some instances of price functions whose associated revenue function is strictly concave. In Example 6, the specification of $p(0)$ is immaterial and can be arbitrarily chosen by the reader.

Example 6 *Let (α, β, γ) be a triple of real numbers with β and γ positive. Any price function $p : \mathbb{R}_+ \to \mathbb{R}$ specified at all $x > 0$ by*

$$p(x) = \alpha + \beta \cdot x^{-\frac{\gamma}{\gamma+1}} \tag{9}$$

as well as by

$$p(x) = \alpha - \beta \cdot x^{\gamma} \tag{10}$$

or by

$$p(x) = \alpha + \frac{\beta}{x + \gamma}. \tag{11}$$

possesses an associated revenue functions that is strictly concave. In particular, each of the previous specifications makes p and r satisfy all conditions H1–4. It is observed that any price function p specified on $\mathbb{R}_{++}$ as in (9) is not continuous—even though the associated revenue function is continuous—and that, when $\alpha > 0$, the associated price elasticity is not decreasing: this last observation proves that the strict concavity of r does not imply the decreasingness of E. Finally, it is observed that the class of price functions specified in (10) includes the class of all strictly decreasing linear price functions discussed above.

Proposition 4 follows, essentially, from part 3 of Proposition 4.5 in [13]. Proposition 4 is explicitly proved here for the sake of completeness and readability. In the literature on Cournot equilibrium, the condition of decreasingness of the price elasticity has been used in the [14].

Proposition 4 *Suppose $p : \mathbb{R}_+ \to \mathbb{R}$ is a price function satisfying condition H1 and H2 that is positive on $\mathbb{R}_{++}$. If the price elasticity E is decreasing, then condition H4 is satisfied.*

Proof. Assume that the price elasticity E is decreasing and, by way of contradiction, suppose condition H4 is not satisfied. Proposition 2 ensures the existence of a quadruple $(w, x, z, \tau) \in \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{Z}_+ \times \mathbb{R}_+$ such that $w < x$, that $DR^z_\tau(x) > 0$ and $DR^z_\tau(x - w) \leq DR^z_\tau(x)$. Put $y_x = x + \tau$. It is clear from Remark 4 that the inequality $DR^z_\tau(x) > 0$ can be equivalently rewritten as

$$Dp(y_x) \cdot x + p(y_x) \cdot (z + 1) > 0 \tag{12}$$

and that the inequality $DR^z_\tau(x - w) \leq DR^z_\tau(x)$ can be equivalently rewritten as

$$Dp(y_x - w) \cdot (x - w) + p(y_x - w) \cdot (z + 1) \leq Dp(y_x) \cdot x + p(y_x) \cdot (z + 1). \tag{13}$$

by the strict decreasingness of p. Put $\delta = \lambda \cdot x^\circ + (1 - \lambda) \cdot x^\bullet$. Clearly, $\delta > 0$ as $\lambda \in (0, 1)$ and $0 \leq x^\circ < x^\bullet$. Observe that

$$\delta \cdot p(\lambda \cdot x^\circ + (1 - \lambda) \cdot x^\bullet) < \delta \cdot (\lambda \cdot p(x^\circ) + (1 - \lambda) \cdot p(x^\bullet)) \tag{8}$$

as the left-hand (right-hand) side of the inequality in (8) is the sum of the left-hand (right-hand) sides of the inequalities in (6) and (7). Consequently, $p(\lambda \cdot x^\circ + (1 - \lambda) \cdot x^\bullet) < \lambda \cdot p(x^\circ) + (1 - \lambda) \cdot p(x^\bullet)$ in contradiction with the concavity of p.

As $0 < w < y_x$, the positivity and the strict decreasingness of p on $\mathbb{R}_{++}$ imply $p(y_x - w) > p(y_x) > 0$: from (12) and (13) we then infer that

$$\frac{Dp(y_x - w) \cdot (x - w) + p(y_x - w) \cdot (z + 1)}{p(y_x - w)} < \frac{Dp(y_x) \cdot x + p(y_x) \cdot (z + 1)}{p(y_x)}$$

and hence that

$$E(y_x - w) \cdot \frac{x - w}{y_x - w} < E(y_x) \cdot \frac{x}{y_x}. \tag{14}$$

Put $\xi = \frac{x}{x-w} \cdot \frac{y_x - w}{y_x}$ and note that $\xi \geq 1$. The inequality in (14) implies that $E(y_x - w) < E(y_x) \cdot \xi$ and hence that $E(y_x - w) < E(y_x)$ because $E(y_x) < 0$ and $\xi \geq 1$. But the inequality $E(y_x - w) < E(y_x)$ is in contradiction with the assumption that E is decreasingness. ■

Remark 7 recalls a connection between the decreasingness of the price elasticity and the [2]'s log-concavity of a price function.

Remark 7 *Suppose $p : \mathbb{R}_+ \to \mathbb{R}$ is a positive and decreasing price function satisfying conditions H1. If p is log-concave (namely, if $\ln p$ is concave), then the price elasticity E is decreasing.*[11] *As is clear from the first specification in Example 7, the converse of the previous implication is generally false.*

Example 7 provides some instances of price functions whose associated price elasticity is decreasing. Also in Example 7, the specification of $p(0)$ is immaterial and can be arbitrarily chosen by the reader.

Example 7 *Let (α, β, γ) be a triple of positive real numbers. Any price function $p : \mathbb{R}_+ \to \mathbb{R}$ specified at all $x > 0$ by*

$$p(x) = \alpha \cdot \frac{1}{\exp(x^\beta)} \tag{15}$$

as well as by

$$p(x) = \alpha \cdot \frac{1}{(x + \gamma)^\beta} \tag{16}$$

or by

$$p(x) = \alpha \cdot \frac{1}{x^\beta + \gamma} \tag{17}$$

possess an associated decreasing price elasticities. In particular, each of the previous specifications makes p and r satisfy all conditions H1–4. It is observed that any price function p specified on $\mathbb{R}_{++}$ as in (15) has an associated revenue function that is not strictly concave (in fact, not even concave): this last observation proves that the decreasingness of E does not imply the strict concavity of r. Finally, it is observed that no price function specified on $\mathbb{R}_{++}$ as in (16) is log-concave.

[11] To see why, pick arbitrary x° and $x^\bullet$ in $\mathbb{R}_{++}$ such that $x^\circ < x^\bullet$. As p is positive and decreasing on its domain $\mathbb{R}_+$ and differentiable on $\mathbb{R}_{++}$, the function $D \ln p$ is well-defined on $\mathbb{R}_{++}$ and nonpositive on $\mathbb{R}_{++}$. The concavity of $\ln p$ on $\mathbb{R}_{++}$ implies the decreasingness of $D \ln p$ on $\mathbb{R}_{++}$. As $D \ln p$ is decreasing on $\mathbb{R}_{++}$ and nonpositive on $\mathbb{R}_{++}$, we have that $D \ln p(x^\bullet) \leq D \ln p(x^\circ) \leq 0$ and hence that $E(x^\bullet) = x^\bullet \cdot D \ln p(x^\bullet) \leq x^\circ \cdot D \ln p(x^\circ) = E(x^\circ)$ as $0 < x^\circ < x^\bullet$. We thus conclude that E is decreasing on its domain $\mathbb{R}_{++}$.

To conclude, it is explicitly pointed out that the class of price functions satisfying condition H4 is by no means limited to the union of the classes of price functions that satisfy the conditions in Propositions 3 and 4: Proposition 4.6 in [13] provides evidence of this claim.

7.2 Appendix B: Fundamental theorems

Theorem 3 *Let Σ be a convex VSA-system and let C be a VSA-configuration in $\mathcal{N}$. Consider the VSA-game associated to C under Σ and suppose $n \in N$.*

1. *The function u_n^C is continuous on S.*
2. *The function $u_n^C(\cdot, s_{-n})$ is quasiconcave on S_n for all $s_{-n} \in S_{-n}$.*
3. *The function $u_n^C(\cdot, s_{-n})$ possesses exactly one maximizer on S_n for all $s_{-n} \in S_{-n}$.*

Proof. Denote by ω the zero vector $(0, \ldots, 0)$ of $\mathbb{R}^{\bar{n}}$.
1. Observe that

$$u_n^C(s) = r(s_1 + \ldots + s_{\bar{n}}) \cdot \frac{s_n}{s_1 + \cdots + s_{\bar{n}}} - \phi_n^C - v_n(\mu_n^C \cdot \textstyle\sum_{l \in B_n^C} s_l)$$

for all $s \in S \backslash \{\omega\}$ and that

$$u_n^C(\omega) = -\phi_n^C - v_n(0).$$

As r and v_n are continuous, the first initial observation is readily seen to imply the continuity of u_n^C at all $s \in S \backslash \{\omega\}$. As v_n and r are continuous at 0 and r is even vanishing at 0, the validity of the inequalities

$$0 \leq \frac{s_n}{s_1 + \ldots + s_{\bar{n}}} \leq 1$$

for all $s \in S \backslash \{\omega\}$ and the two initial observations imply the continuity of u_n^C at ω by virtue of the Police Theorem and by other basic facts concerning limits. Therefore, u_n^C is continuous on S.
2. By assumption, R_τ^0 is semistrictly demiconcave on its domain $\mathbb{R}_{++}$ for all $\tau \in \mathbb{R}_+$: therefore, R_τ^0 is semistrictly demiconcave on $S_n \backslash \{0\}$ for all $\tau \in \mathbb{R}_+$ in the precise sense that the restriction of R_τ^0 to $S_n \backslash \{0\}$ is semistrictly demiconcave on its domain $S_n \backslash \{0\}$ for all $\tau \in \mathbb{R}_+$. Observe that, for all $s \in S \backslash \{\omega\}$, the value of the function $u_n^C(\cdot, s_{-n})$ at s_n is expressed by

$$u_n^C(\cdot, s_{-n})(s_n) = R_\tau^0(s_n) - \phi_n^C - v_n(\mu_n^C \cdot \textstyle\sum_{l \in B_n^C} s_l) \text{ with } \tau = \textstyle\sum_{l \in N \backslash \{n\}} s_l.$$

Given this observation[12], we infer that $u_n^C(\cdot, s_{-n})$ is semistrictly demiconcave on $S_n \backslash \{0\}$ for all $s \in S$ by virtue of part 2 of Proposition 1 and hence that $u_n^C(\cdot, s_{-n})$ is quasiconcave on $S_n \backslash \{0\}$ for all $s \in S$ by virtue of part 1 of Proposition 1. The continuity of u_n^C implies the continuity of $u_n^C(\cdot, s_{-n})$ on its domain S_n for all $s \in S$ and hence—see, e.g., Theorem 2.2.12 in [3]—$u_n^C(\cdot, s_{-n})$ is quasiconcave on its domain S_n for all $s \in S$.

3. Since S_n is a nonempty compact subset of $\mathbb{R}_+$ and since part 1 of Theorem 3 ensures the continuity of $u_n^C(\cdot, s_{-n})$ on S_n for all $s_{-n} \in S_{-n}$, the function $u_n^C(\cdot, s_{-n})$ possesses at least one

[12] Observe also that the function $g : S_n \backslash \{0\} \to \mathbb{R}$ specified by $g(s_n) = \phi_n^C + v_n(\mu_n^C \cdot \sum_{l \in B_n^C} s_l)$ is continuous, convex and strictly increasing in that so is v_n by assumption.

maximizer on S_n by the Weierstrass Theorem. Now, by way of contradiction, suppose x^* and x^{**} are distinct maximizers of $u_n^C(\cdot, s_{-n})$ on S_n for some $s_{-n} \in S_{-n}$. Without loss of generality, suppose $x^* < x^{**}$. The real interval (x^*, x^{**}) is a nonempty open convex subset of $S_n \backslash \{0\}$ and $u_n^C(\cdot, s_{-n})$ is constant on (x^*, x^{**}) because $u_n^C(\cdot, s_{-n})$ is quasiconcave on S_n by part 2 of Theorem 3 and because x^* and x^{**} are maximizers thereof. We have already observed in the proof of part 2 of Theorem 3 that

$$u_n^C(\cdot, s_{-n})(s_n) = R_\tau^0(s_n) - \phi_n^C - v_n(\mu_n^C \cdot \textstyle\sum_{l \in B_n^C} s_l) \text{ with } \tau = \sum_{l \in N \backslash \{n\}} s_l$$

for all $s \in S \backslash \{\omega\}$ and that R_τ^0 is semistrictly demiconcave on $S_n \backslash \{0\}$. By the strict demiconcavity of R_τ^0 on $S_n \backslash \{0\}$, there exist real intervals L_1 and L_2 such that: $L_1 \cup L_2 = S_n \backslash \{0\}$ and $L_1 \cap L_2 = \emptyset$; $x \leq y$ for every pair $(x, y) \in L_1 \times L_2$; R_τ^0 is strictly concave on L_1 and decreasing on L_2. So, as v_n is strictly increasing and convex by assumption (see again fn. 12), the function $u_n^C(\cdot, s_{-n})$ is strictly concave on L_1 and *strictly* decreasing on L_2: as the real interval (x^*, x^{**}) is a nonempty open convex subset of $S_n \backslash \{0\}$, it is readily seen that we get a contradiction with the fact that $u_n^C(\cdot, s_{-n})$ is constant on (x^*, x^{**}). ■

It is noted here, that both the aggregate $\sum_{l \in B} e_l^\bullet$ and the aggregate $\sum_{l \in B} e_l^\circ$ in the definition of $\mathcal{Z}$ given in Lemma 1 are relative to the blocks in the VSA-configuration C°. Also, it is noted here that the sets $\mathcal{Z}$ and Z defined in Lemma 1 might well be empty: however, this fact is immaterial as to the validity of Lemma 1 and Theorem 4. Finally, it is noted here that in the statement of Lemma 1 the vectors e° and $e^\bullet$ are mere joint strategies and not necessarily Nash equilibria: we denote those vectors by e° and $e^\bullet$ only so that the subsequent application of Lemma 1 is even more immediate.

Lemma 1 *Let Σ be a VSA-system satisfying condition H6 and let $(C^\circ, C^\bullet)$ be a pair of VSA-configurations in $\mathcal{N}$ such that $C^\circ \sqsubseteq C^\bullet$. Assume that e° is a joint strategy for the VSA-game associated to C° under Σ and that $e^\bullet$ is a joint strategy for the VSA-game associated to $C^\bullet$ under Σ. Put*

$$\mathcal{Z} = \{B \in C^\circ : \textstyle\sum_{l \in B} e_l^\bullet < \sum_{l \in B} e_l^\circ\}$$

and suppose $n \in \bigcup_{B \in \mathcal{Z}} B$. Then

$$0 < \mu_n^{C^\circ} \cdot \textstyle\sum_{l \in B_n^{C^\circ}} e_l^\circ \leq \kappa_n \tag{18}$$

and

$$0 \leq \mu_n^{C^\bullet} \cdot \textstyle\sum_{l \in B_n^{C^\bullet}} e_l^\bullet < \kappa_n. \tag{19}$$

Furthermore,

$$\mu_n^{C^\bullet} \cdot D^+ v_n(\mu_n^{C^\bullet} \cdot \textstyle\sum_{l \in B_n^{C^\bullet}} e_l^\bullet) \leq \mu_n^{C^\circ} \cdot D^- v_n(\mu_n^{C^\circ} \cdot \sum_{l \in B_n^{C^\circ}} e_l^\circ) \tag{20}$$

if at least one of the following additional conditions holds:

1. $\mu_n^{C^\circ} = \mu_n^{C^\bullet}$;

2. *Σ is a VSA-system with almost linear costs.*

Proof. Clearly, $\emptyset \neq B_n^{C^\circ} \subseteq B_n^{C^\bullet}$ in that $n \in B_n^{C^\circ}$ by the definition of $B_n^{C^\circ}$ and $C^\circ \sqsubseteq C^\bullet$ by assumption. As $\kappa_l > 0$ for all $l \in N$, the inclusion $B_n^{C^\circ} \subseteq B_n^{C^\bullet}$ and the definitions of $\mu_n^{C^\circ}$ and $\mu_n^{C^\bullet}$ imply

$$0 < \mu_n^{C^\bullet} \leq \mu_n^{C^\circ} \leq 1. \tag{21}$$

As e° and $e^\bullet$ are elements of $[0, \kappa_1] \times \cdots \times [0, \kappa_{\bar{n}}]$ and $\emptyset \neq B_n^{C^\circ} \subseteq B_n^{C^\bullet}$, from the assumption that $n \in \bigcup_{B \in \mathcal{Z}} B$ and the definition of $\mathcal{Z}$ it follows that

$$0 \leq \textstyle\sum_{l \in B_n^{C^\circ}} e_l^\bullet < \sum_{l \in B_n^{C^\circ}} e_l^\circ \leq \sum_{l \in B_n^{C^\circ}} \kappa_l \tag{22}$$

and

$$0 \leq \textstyle\sum_{l \in B_n^{C^\bullet} \setminus B_n^{C^\circ}} e_l^\bullet \leq \sum_{l \in B_n^{C^\bullet} \setminus B_n^{C^\circ}} \kappa_l. \tag{23}$$

The definition of $\mu_n^{C^\circ}$ and the inequalities in (21) and (22) imply $0 < \mu_n^{C^\circ} \cdot \sum_{l \in B_n^{C^\circ}} e_l^\circ \leq \mu_n^{C^\circ} \cdot \sum_{l \in B_n^{C^\circ}} \kappa_l = \kappa_n$. This proves the validity of (18). The inequalities in (22) and (23) imply

$$0 \leq \textstyle\sum_{l \in B_n^{C^\bullet}} e_l^\bullet < \sum_{l \in B_n^{C^\bullet}} \kappa_l. \tag{24}$$

The definition of $\mu_n^{C^\bullet}$ and the inequalities in (21) and (24) imply $0 \leq \mu_n^{C^\bullet} \cdot \sum_{l \in B_n^{C^\bullet}} e_l^\bullet < \mu_n^{C^\bullet} \cdot \sum_{l \in B_n^{C^\bullet}} \kappa_l = \kappa_n$. This proves the validity of (19).
1. Assume for a moment that $\mu_n^{C^\circ} = \mu_n^{C^\bullet}$. As $\emptyset \neq B_n^{C^\circ} \subseteq B_n^{C^\bullet}$ and $\kappa_l > 0$ for all $l \in N$, the assumption that $\mu_n^{C^\circ} = \mu_n^{C^\bullet}$ and the definition of $\mu_n^{C^\circ}$ and $\mu_n^{C^\bullet}$ imply $B_n^{C^\circ} = B_n^{C^\bullet}$: from (22) we then infer that

$$0 \leq \textstyle\sum_{l \in B_n^{C^\bullet}} e_l^\bullet < \sum_{l \in B_n^{C^\circ}} e_l^\circ \leq \sum_{l \in B_n^{C^\circ}} \kappa_l. \tag{25}$$

The definition of $\mu_n^{C^\circ}$ and the inequalities in (21) and (25) imply

$$0 \leq \mu_n^{C^\bullet} \cdot \textstyle\sum_{l \in B_n^{C^\bullet}} e_l^\bullet < \mu_n^{C^\circ} \cdot \sum_{l \in B_n^{C^\circ}} e_l^\circ \leq \mu_n^{C^\circ} \cdot \sum_{l \in B_n^{C^\circ}} \kappa_l = \kappa_n. \tag{26}$$

Given the validity of the inequalities in (26), condition H6 implies $0 \leq D^+ v_n(\mu_n^{C^\bullet} \cdot \sum_{l \in B_n^{C^\bullet}} e_l^\bullet) \leq D^- v_n(\mu_n^{C^\circ} \cdot \sum_{l \in B_n^{C^\circ}} e_l^\circ)$ and the inequalities in (21) in turn imply the validity of (20).

2. Assume now that Σ is a VSA-system Σ with almost linear costs. When $\mu_n^{C^\circ} = \mu_n^{C^\bullet}$, the validity of part 2 of Lemma 1 follows from the validity of part 1 of Lemma 1. So, henceforth assume that

$$\mu_n^{C^\circ} \neq \mu_n^{C^\bullet}. \tag{27}$$

From (21) and (27) we infer that

$$0 < \mu_n^{C^\bullet} < \mu_n^{C^\circ} \leq 1. \tag{28}$$

As $\kappa_l > 0$ for all $l \in N$, the initial observation that $\emptyset \neq B_n^{C^\circ} \subseteq B_n^{C^\bullet}$ and the second inequality in (28) imply

$$\emptyset \neq B_n^{C^\circ} \subset B_n^{C^\bullet} \tag{29}$$

by the definitions of $\mu_n^{C^\circ}$ and $\mu_n^{C^\bullet}$. Note that (29) entails the existence of at least two distinct elements in $B_n^{C^\bullet}$: as $B_n^{C^\bullet}$ contains at least two distinct elements and $\kappa_l > 0$ for all $l \in N$, from the first inequality in (28) and the definition of $\mu_n^{C^\bullet}$ we infer that

$$0 < \mu_n^{C^\bullet} = \frac{\kappa_n}{\sum_{l \in B_n^{C^\bullet}} \kappa_l} < \frac{\kappa_n}{-\min\{\kappa_l : l \in B_n^{C^\bullet}\} + \sum_{l \in B_n^{C^\bullet}} \kappa_l}$$

and from the strict inclusion in (29) and the definition of $\mu_n^{C^\circ}$ that

$$\frac{\kappa_n}{-\min\{\kappa_l : l \in B_n^{C^\bullet}\} + \sum_{l\in B_n^{C^\bullet}} \kappa_l} \leq \frac{\kappa_n}{\sum_{l\in B_n^{C^\circ}} \kappa_l} = \mu_n^{C^\circ}.$$

Therefore,

$$0 < \mu_n^{C^\bullet} = \frac{\kappa_n}{\sum_{l\in B_n^{C^\bullet}} \kappa_l} < \frac{\kappa_n}{-\min\{\kappa_l : l \in B_n^{C^\bullet}\} + \sum_{l\in B_n^{C^\bullet}} \kappa_l} \leq \mu_n^{C^\circ} \tag{30}$$

and hence[13]

$$\frac{\mu_n^{C^\bullet}}{\mu_n^{C^\circ}} \leq \frac{-\min\{\kappa_l : l \in B_n^{C^\bullet}\} + \sum_{l\in B_n^{C^\bullet}} \kappa_l}{\sum_{l\in B_n^{C^\bullet}} \kappa_l} = 1 - \frac{\min\{\kappa_l : l \in B_n^{C^\bullet}\}}{\sum_{l\in B_n^{C^\bullet}} \kappa_l}. \tag{31}$$

Given the validity of the inequalities in (18) and (19), condition H6 implies $D^+v_n(0) \leq D^-v_n(\mu_n^{C^\circ} \cdot \sum_{l\in B_n^{C^\circ}} e_l^\circ)$ and $D^+v_n(\mu_n^{C^\bullet} \cdot \sum_{l\in B_n^{C^\bullet}} e_l^\bullet) \leq D^-v_n(\kappa_n)$: from (28) we then infer that

$$\mu_n^{C^\circ} \cdot D^+v_n(0) \leq \mu_n^{C^\circ} \cdot D^-v_n(\mu_n^{C^\circ} \cdot \textstyle\sum_{l\in B_n^{C^\circ}} e_l^\circ) \tag{32}$$

and

$$\mu_n^{C^\bullet} \cdot D^+v_n(\mu_n^{C^\bullet} \cdot \textstyle\sum_{l\in B_n^{C^\bullet}} e_l^\bullet) \leq \mu_n^{C^\bullet} \cdot D^-v_n(\kappa_n). \tag{33}$$

As $B_n^{C^\bullet}$ is a nonempty subset of N by (29) and $\kappa_l > 0$ for all $l \in N$, we have that $\min\{\kappa_l : l \in B_n^{C^\bullet}\} \geq \min\{\kappa_1, \ldots, \kappa_{\bar{n}}\} > 0$ and $\kappa_1 + \cdots + \kappa_{\bar{n}} \geq \sum_{l\in B_n^{C^\bullet}} \kappa_l > 0$. Consequently,

$$1 - \frac{\min\{\kappa_l : l \in B_n^{C^\bullet}\}}{\sum_{l\in B_n^{C^\bullet}} \kappa_l} \leq 1 - \frac{\min\{\kappa_1, \ldots, \kappa_{\bar{n}}\}}{\kappa_1 + \cdots + \kappa_{\bar{n}}}. \tag{34}$$

Recall that

$$D^-v_n(\kappa_n) \leq \frac{\kappa_1 + \cdots + \kappa_{\bar{n}}}{\kappa_1 + \cdots + \kappa_{\bar{n}} - \min\{\kappa_1, \ldots, \kappa_{\bar{n}}\}} \cdot D^+v_n(0) \tag{35}$$

by the assumption that Σ is a VSA-system with almost linear costs. Note that—because of condition H6—the validity of the inequality in (35) implies that $D^-v_n(\kappa_n)$ is a well-defined positive real number and note that the first factor in the right-hand side of the inequality in (35) is a positive real number because $\bar{n} \geq 2$ and because all caps are positive. Having observed this, from (35) we infer that

$$1 - \frac{\min\{\kappa_1, \ldots, \kappa_{\bar{n}}\}}{\kappa_1 + \cdots + \kappa_{\bar{n}}} \leq \frac{D^+v_n(0)}{D^-v_n(\kappa_n)}. \tag{36}$$

From (31), (34), (36) we conclude that

$$\frac{\mu_n^{C^\bullet}}{\mu_n^{C^\circ}} \leq \frac{D^+v_n(0)}{D^-v_n(\kappa_n)}$$

and hence—as $\mu_n^{C^\circ}$ and $D^-v_n(\kappa_n)$ are positive real numbers—that

$$\mu_n^{C^\bullet} \cdot D^-v_n(\kappa_n) \leq \mu_n^{C^\circ} \cdot D^+v_n(0). \tag{37}$$

The validity of (20) follows from (32), (33) and (37). ■

[13] Note that the right-hand side of the inequality in (31) is the quotient of the left- and the right-hand side of the second inequality in (30).

Theorem 4 *Let Σ be a convex VSA-system and let $(C^\circ, C^\bullet)$ be a pair of VSA-configurations in $\mathcal{N}$ such that $C^\circ \sqsubseteq C^\bullet$. Assume that e° is a Nash equilibrium for the VSA-game associated to C° under Σ and that $e^\bullet$ is a Nash equilibrium for the VSA-game associated to $C^\bullet$ under Σ. Let η° and $\eta^\bullet$ be the equilibrium aggregates defined by $\eta^\circ = e_1^\circ + \ldots + e_{\bar{n}}^\circ$ and $\eta^\bullet = e_1^\bullet + \ldots + e_{\bar{n}}^\bullet$. Then*

$$\eta^\circ \leq \eta^\bullet$$

if at least one of the following additional conditions holds:

1. *$C^\circ = C^\bullet$;*

2. *Σ is a VSA-system with almost linear costs.*

Proof. Assume that either $C^\circ = C^\bullet$ or Σ is a VSA-system with almost linear costs. Denote by ω the zero vector of $\mathbb{R}^{\bar{n}}$. Clearly, the equality $C^\circ = C^\bullet$ implies that $\mu_n^{C^\circ} = \mu_n^{C^\bullet}$ for all $n \in N$. By way of contradiction, suppose $\eta^\bullet < \eta^\circ$. Then

$$0 < \eta^\circ - \eta^\bullet. \tag{38}$$

Put

$$\mathcal{Z} = \{B \in C^\circ : \textstyle\sum_{l \in B} e_l^\bullet < \sum_{l \in B} e_l^\circ\}$$

and

$$Z = \{l \in \textstyle\bigcup_{B \in \mathcal{Z}} B : e_l^\bullet < e_l^\circ\}.$$

Furthermore, put

$$z = |Z|.$$

A moment's reflection shows[14] the validity of the **Fundamental Membership**

$$z \in \mathbb{Z}_+ \backslash \{0\}. \tag{39}$$

Lemma 1 ensures that

$$0 < \mu_n^{C^\circ} \cdot \textstyle\sum_{l \in B_n^{C^\circ}} e_l^\circ \leq \kappa_n \tag{40}$$

for all $n \in Z$, that

$$0 \leq \mu_n^{C^\bullet} \cdot \textstyle\sum_{l \in B_n^{C^\bullet}} e_l^\bullet < \kappa_n \tag{41}$$

for all $n \in Z$ and that

$$\mu_n^{C^\bullet} \cdot D^+ v_n(\mu_n^{C^\bullet} \cdot \textstyle\sum_{l \in B_n^{C^\bullet}} e_l^\bullet) \leq \mu_n^{C^\circ} \cdot D^- v_n(\mu_n^{C^\circ} \cdot \sum_{l \in B_n^{C^\circ}} e_l^\circ) \tag{42}$$

[14] Clearly, z is a nonnegative integer. To check the Fundamental Membership, observe that

$$\eta^\circ - \eta^\bullet = \textstyle\sum_{B \in \mathcal{Z}} \sum_{n \in B} (e_n^\circ - e_n^\bullet) + \sum_{B \in C^\circ \backslash \mathcal{Z}} \sum_{n \in B} (e_n^\circ - e_n^\bullet).$$

The very definition of $\mathcal{Z}$ entails that $\sum_{B \in C^\circ \backslash \mathcal{Z}} \sum_{n \in B} (e_n^\circ - e_n^\bullet) \leq 0$ and hence we have that $\eta^\circ - \eta^\bullet \leq \sum_{B \in \mathcal{Z}} \sum_{n \in B} (e_n^\circ - e_n^\bullet)$: the inequality in (38) in turn implies

$$0 < \textstyle\sum_{B \in \mathcal{Z}} \sum_{n \in B} (e_n^\circ - e_n^\bullet).$$

The very definition of Z entails that $\sum_{B \in \mathcal{Z}} \sum_{n \in B} (e_n^\circ - e_n^\bullet) \leq \sum_{n \in Z} (e_n^\circ - e_n^\bullet)$: we thus conclude that $0 < \sum_{n \in Z} (e_n^\circ - e_n^\bullet)$ and hence that $z \neq 0$.

for all $n \in Z$. As e° and $e^\bullet$ are elements of $\mathbb{R}_+^{\bar{n}}$ and $\eta^\bullet < \eta^\circ$, we have that

$$0 \leq \eta^\bullet < \eta^\circ \tag{43}$$

and

$$e^\circ \in \mathbb{R}_+^{\bar{n}} \backslash \{\omega\}. \tag{44}$$

As $\kappa_l > 0$ for all $l \in N$, the definition of $\mu_n^{C^\circ}$ and $\mu_n^{C^\bullet}$ implies

$$0 < \mu_n^{C^\bullet} \leq \mu_n^{C^\circ} \tag{45}$$

for all $n \in N$. We briefly show that also

$$e^\bullet \in \mathbb{R}_+^{\bar{n}} \backslash \{\omega\}. \tag{46}$$

Indeed, suppose for a moment that $e^\bullet = \omega$ and pick an arbitrary $i \in N$ such that $e_i^\circ > 0$: the existence of such i is implied by (44). As $u_i^{C^\bullet}(\cdot, e_{-i}^\bullet)$ has exactly one maximizer by part 3 of Theorem 3 and by the assumption that Σ is a convex VSA-system, the inequality $u_i^{C^\bullet}(e_i^\circ, e_{-i}^\bullet) - u_i^{C^\bullet}(e_i^\bullet, e_{-i}^\bullet) < 0$ is true in that $e_i^\bullet$ is a maximizer of $u_i^{C^\bullet}(\cdot, e_{-j}^\bullet)$ and $e_i^\bullet = 0 \neq e_i^\circ$. The inequality $u_i^{C^\bullet}(e_i^\circ, e_{-i}^\bullet) - u_i^{C^\bullet}(e_i^\bullet, e_{-i}^\bullet) < 0$ can be equivalently rewritten as

$$p(e_i^\circ) \cdot e_i^\circ + v_i(0) - v_i(\mu_i^{C^\bullet} \cdot e_i^\circ) < 0. \tag{47}$$

As $i \in B_i^{C^\circ}$ by the definition of $B_i^{C^\circ}$ and $e^\circ \in \mathbb{R}_+^{\bar{n}} \backslash \{\omega\}$ by (44), a moment's reflection shows that (47) implies

$$p(\eta^\circ) \cdot e_i^\circ + v_i(-\mu_i^{C^\circ} \cdot e_i^\circ + \mu_i^{C^\circ} \cdot \textstyle\sum_{l \in B_i^{C^\circ}} e_l^\circ) - v_i(\mu_i^{C^\circ} \cdot \textstyle\sum_{l \in B_i^{C^\circ}} e_l^\circ) < 0 \tag{48}$$

in that p is strictly decreasing on $\mathbb{R}_{++}$ by conditions H1 and H2 and $0 < e_i^\circ \leq \eta^\circ$ by the choice of i and the definition of η° and in that v_i is convex by condition H6 and $0 < \mu_i^{C^\bullet} \leq \mu_i^{C^\circ}$ by the validity of (45) for all $n \in N$.[15] By virtue of the momentary assumption that $e^\bullet = \omega$, the inequality in (48) can be equivalently rewritten as $u_i^{C^\circ}(e_i^\circ, e_{-i}^\circ) - u_i^{C^\circ}(e_i^\bullet, e_{-i}^\circ) < 0$ and this suffices to infer a contradiction with the assumption that e° is a Nash equilibrium. This concludes the proof of the validity of (46). From (43) and (46) we also infer that

$$0 < \eta^\bullet < \eta^\circ. \tag{49}$$

Put $\sigma^\circ = \sum_{n \in Z} e_n^\circ$ and $\sigma^\bullet = \sum_{n \in Z} e_n^\bullet$. Clearly,

$$0 \leq \min\{\eta^\bullet - \sigma^\bullet, \eta^\circ - \sigma^\circ\}. \tag{50}$$

A moment's reflection shows[16] the validity of the **Fundamental Inequality**

$$\eta^\circ - \eta^\bullet \leq \sigma^\circ - \sigma^\bullet. \tag{51}$$

[15] As v_j is convex, $v_j(a) - v_j(b) \geq v_j(c) - v_j(d)$ when $d - c \geq b - a$ and $c \geq a$.

[16] Put $T = \bigcup_{B \in \mathcal{Z}} B$. To check the Fundamental Inequality, observe that

$$\sigma^\circ - \sigma^\bullet = \textstyle\sum_{n \in Z}(e_n^\circ - e_n^\bullet)$$

and that

$$\eta^\circ - \eta^\bullet = \textstyle\sum_{n \in Z}(e_n^\circ - e_n^\bullet) + \sum_{n \in T \backslash Z}(e_n^\circ - e_n^\bullet) + \sum_{B \in C^\circ \backslash \mathcal{Z}} \sum_{n \in B}(e_n^\circ - e_n^\bullet).$$

As $\sum_{B \in C^\circ \backslash \mathcal{Z}} \sum_{n \in B}(e_n^\circ - e_n^\bullet) \leq 0$ by the very definition of $\mathcal{Z}$ and $\sum_{n \in T \backslash Z}(e_n^\circ - e_n^\bullet) \leq 0$ by that of T and Z, we get $\eta^\circ - \eta^\bullet \leq \sum_{n \in Z}(e_n^\circ - e_n^\bullet)$ and hence $\eta^\circ - \eta^\bullet \leq \sigma^\circ - \sigma^\bullet$.

The definition of $\sigma^\bullet$ implies $\sigma^\bullet \geq 0$ in that $e^\bullet$ is an element of $\mathbb{R}_+^{\bar{n}}$: the inequalities in (49) and (51) in turn imply $0 \leq \sigma^\bullet < \sigma^\circ$. By making use of conditions H1 and H2, from the inequalities in (49) we infer that

$$0 < p(\eta^\bullet) - p(\eta^\circ) \tag{52}$$

and that $Dp(\eta^\bullet)$ and $Dp(\eta^\circ)$ are negative real numbers. Condition H6 and the validity of the inequalities in (40) for all $n \in Z$ imply the existence of $D^- v_n(\mu_n^{C^\circ} \cdot \sum_{l \in B_n^{C^\circ}} e_l^\circ)$ in $\mathbb{R}_{++} \cup \{+\infty\}$ for all $n \in Z$ while condition H6 and the validity of the inequalities in (41) for all $n \in Z$ imply the existence of $D^+ v_n(\mu_n^{C^\bullet} \cdot \sum_{l \in B_n^{C^\bullet}} e_l^\bullet)$ in $\mathbb{R}_+$ for all $n \in Z$ and hence that

$$D^+ v_n(\mu_n^{C^\bullet} \cdot \textstyle\sum_{l \in B_n^{C^\bullet}} e_l^\bullet) \in \mathbb{R}_+ \tag{53}$$

for all $n \in Z$. We briefly show that $D^- v_n(\mu_n^{C^\circ} \cdot \sum_{l \in B_n^{C^\circ}} e_l^\circ) \neq +\infty$ for all $n \in Z$ and hence that

$$D^- v_n(\mu_n^{C^\circ} \cdot \textstyle\sum_{l \in B_n^{C^\circ}} e_l^\circ) \in \mathbb{R}_{++} \tag{54}$$

for all $n \in Z$. Indeed, if the equality $D^- v_m(\mu_m^{C^\circ} \cdot \sum_{l \in B_m^{C^\circ}} e_l^\circ) = +\infty$ were true for some $m \in Z$, then $\mu_m^{C^\circ} \cdot D^- v_m(\mu_m^{C^\circ} \cdot \sum_{l \in B_m^{C^\circ}} e_l^\circ) = +\infty$ by (45) and hence, equivalently,

$$\lim_{x \to 0^-} \frac{v_m(\mu_m^{C^\circ} \cdot (x + \sum_{l \in B_m^{C^\circ}} e_l^\circ)) - v_m(\mu_m^{C^\circ} \cdot \sum_{l \in B_m^{C^\circ}} e_l^\circ)}{x} = +\infty;$$

however, the equilibrium conditions and the validity of the inequalities in (41) for all $n \in Z$ imply

$$\liminf_{x \to 0^-} \frac{u_m^{C^\circ}(\cdot, e_{-m}^\circ)(e_m^\circ + x) - u_m^{C^\circ}(\cdot, e_{-m}^\circ)(e_m^\circ)}{x} \in \mathbb{R}_+ \cup \{+\infty\}$$

in that $m \in Z$ and hence—writing extensively the sum of the two previous limits as the limit of their sum—we would obtain

$$\liminf_{x \to 0^-} \frac{p(\eta^\circ + x) \cdot (e_m^\circ + x) - p(\eta^\circ) \cdot (e_m^\circ)}{x} = +\infty$$

in contradiction with the fact that $Dp(\eta^\circ) e_m^\circ + p(\eta^\circ)$ is a well-defined real number. This completes the proof of the validity of (54) for all $n \in Z$: by the validity of (45) for all $n \in N$, we are in a position to conclude that

$$\mu_n^{C^\circ} \cdot D^- v_n(\mu_n^{C^\circ} \cdot \textstyle\sum_{l \in B_n^{C^\circ}} e_l^\circ) \in \mathbb{R}_{++} \tag{55}$$

for all $n \in Z$. Clearly, the validity of (45) and (53) for all $n \in Z$ implies

$$\mu_n^{C^\bullet} \cdot D^+ v_n(\mu_n^{C^\bullet} \cdot \textstyle\sum_{l \in B_n^{C^\bullet}} e_l^\bullet) \in \mathbb{R}_+$$

for all $n \in Z$. We are now in a position to conclude that, for all $n \in Z$, the left derivative $D^- u_n^{C^\circ}(\cdot, e_{-n}^\circ)(e_n^\circ)$ of $u_n^{C^\circ}(\cdot, e_{-n}^\circ)$ at e_n° is a well-defined real number specified by

$$Dp(\eta^\circ) \cdot e_n^\circ + p(\eta^\circ) - \mu_n^{C^\circ} \cdot D^- v_n(\mu_n^{C^\circ} \cdot \textstyle\sum_{l \in B_n^{C^\circ}} e_l^\circ) \tag{56}$$

and the right derivative $D^+ u_n^{C^\bullet}(\cdot, e_{-n}^\bullet)(e_n^\bullet)$ of $u_n^{C^\bullet}(\cdot, e_{-n}^\bullet)$ at $e_n^\bullet$ is a well-defined real-number specified by

$$Dp(\eta^\bullet) \cdot e_n^\bullet + p(\eta^\bullet) - \mu_n^{C^\bullet} \cdot D^+ v_n(\mu_n^{C^\bullet} \cdot \textstyle\sum_{l \in B_n^{C^\bullet}} e_l^\bullet). \tag{57}$$

Having clarified these points, we continue the proof observing that the equilibrium conditions imply that

$$D^{+}u_n^{C^\bullet}(\cdot, e^\bullet_{-n})(e^\bullet_n) \leq 0 \leq D^{-}u_n^{C^\circ}(\cdot, e^\circ_{-n})(e^\circ_n) \tag{58}$$

for all $n \in Z$ and hence that

$$\textstyle\sum_{n\in Z} D^{+}u_n^{C^\bullet}(\cdot, e^\bullet_{-n})(e^\bullet_n) \leq 0 \leq \sum_{n\in Z} D^{-}u_n^{C^\circ}(\cdot, e^\circ_{-n})(e^\circ_n). \tag{59}$$

Given the specifications of $D^{-}u_n^{C^\circ}(\cdot, e^\circ_{-n})(e^\circ_n)$ and $D^{+}u_n^{C^\bullet}(\cdot, e^\bullet_{-n})(e^\bullet_n)$ in (56) and (57) for all $n \in Z$, it should be clear that the membership in (39), the validity of the membership in (55) for all $n \in Z$ and the second inequality in (59) entail that

$$0 < Dp(\eta^\circ) \cdot \sigma^\circ + p(\eta^\circ) \cdot z \tag{60}$$

and it should be clear that the validity of the inequality in (42) for all $n \in Z$, the membership in (39) and the inequalities in (59) entail that

$$Dp(\eta^\bullet) \cdot \sigma^\bullet + p(\eta^\bullet) \cdot z \leq Dp(\eta^\circ) \cdot \sigma^\circ + p(\eta^\circ) \cdot z. \tag{61}$$

The inequality in (51) implies that $\sigma^\bullet \leq \sigma^\circ + \eta^\bullet - \eta^\circ$: from the already observed negativity of $Dp(\eta^\bullet)$ and the inequality in (61) we then infer that

$$Dp(\eta^\bullet) \cdot (\sigma^\circ + \eta^\bullet - \eta^\circ) + p(\eta^\bullet) \cdot z \leq Dp(\eta^\circ) \cdot \sigma^\circ + p(\eta^\circ) \cdot z. \tag{62}$$

If the equality $\sigma^\bullet = 0$ were true, then the inequality in (61) would imply that $(p(\eta^\bullet) - p(\eta^\circ)) \cdot z \leq Dp(\eta^\circ) \cdot \sigma^\circ$: a contradiction with the fact that the left-hand side of the previous inequality is positive by (39) and (52) while its right-hand side is negative by the already observed negativity of $Dp(\eta^\circ)$ and by the already observed positivity of σ°. So $\sigma^\bullet > 0$ and from (38) and (51) we infer that

$$0 < \eta^\circ - \eta^\bullet < \sigma^\circ. \tag{63}$$

Put $\tau = \eta^\circ - \sigma^\circ$ and $w = \eta^\circ - \eta^\bullet$. Furthermore, put $x = \sigma^\circ$. It is readily observed that the right-hand sides of (60) and (62) can be equivalently rewritten as $DR_\tau^{z-1}(x)$ while the left-hand side of (62) as $DR_\tau^{z-1}(x - w)$: noting that $(\tau, z) \in \mathbb{R}_+ \times \mathbb{Z}_+$ with $z \geq 1$ by (39) and (50) and that $(w, x) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$ with $w < x$ by the inequalities in (63), we are in a position to conclude that the inequalities in (60) and (62) are in contradiction with Proposition 2 and conditions H1 and H4. ■

Theorem 5 *Let Σ be a VSA-system satisfying conditions H1, H2, H6 and let C be an almost smooth VSA-configuration in $\mathcal{N}$. Assume that e° is a Nash equilibrium for the VSA-game associated to C under Σ and that $e^\bullet$ is a Nash equilibrium for the VSA-games associated to C under Σ. Let η° and $\eta^\bullet$ be the equilibrium aggregates defined by $\eta^\circ = e^\circ_1 + \ldots + e^\circ_{\bar{n}}$ and $\eta^\bullet = e^\bullet_1 + \ldots + e^\bullet_{\bar{n}}$. If $\eta^\circ = \eta^\bullet$, then $e^\circ = e^\bullet$.*

Proof. Assume that $\eta^\circ = \eta^\bullet$ and, by way of contradiction, suppose $e^\circ \neq e^\bullet$. Put $\eta = \eta^\circ$. Then $\eta = \eta^\bullet$ and

$$0 < \eta \tag{64}$$

in that e° and $e^\bullet$ are distinct elements of the Cartesian product $[0,\kappa_1]\times\cdots\times[0,\kappa_{\bar{n}}]$ such that $\eta^\circ = \eta^\bullet$. Therefore, $Dp(\eta)$ is a well-defined negative real number by conditions H1 and H2. Recalling that

$$0 < \mu_n^C = \frac{\kappa_n}{\sum_{l\in B_i^C}\kappa_l} \leq 1 \tag{65}$$

for all $n \in N$, we continue the proof by distinguishing two exhaustive cases.

Case $\sum_{l\in B_n^C} e_l^\circ = \sum_{l\in B_n^C} e_l^\bullet$ **for all** $n \in N$. Suppose for a moment that $\sum_{l\in B_n^C} e_l^\circ = \sum_{l\in B_n^C} e_l^\bullet$ for all $n \in N$. The inequality $e^\circ \neq e^\bullet$ then implies the existence of $m \in N$ and of i and j in B_m^C such that

$$e_i^\circ < e_i^\bullet \tag{66}$$

and

$$e_j^\bullet < e_j^\circ.$$

As i and j belongs to B_m^C, from the definition of B_i^C and B_j^C we infer that $B_m^C = B_i^C = B_j^C$ and from the last two inequalities we conclude that $i \neq j$. As C is an almost smooth VSA-configuration, either v_i or v_j is differentiable on the interior of its domain. Without loss of generality, suppose v_i is differentiable on the interior of its domain. Keeping in mind the validity of (65) for all $n \in N$, note that

$$0 < \mu_i^C \cdot \textstyle\sum_{l\in B_i^C} e_l^\circ < \kappa_i \tag{67}$$

because $e_l^\circ \in [0,\kappa_l]$ for all $l \in B_i^C$ and because i and j are elements of B_i^C such that $e_i^\circ < e_i^\bullet$ and $e_j^\bullet < e_j^\circ$. Clearly,

$$0 < \mu_i^C \cdot \textstyle\sum_{l\in B_i^C} e_l^\bullet < \kappa_i \tag{68}$$

in that $\sum_{l\in B_i^C} e_l^\circ = \sum_{l\in B_i^C} e_l^\bullet$ by the momentary assumption. Note that

$$0 \leq e_i^\circ < e_i^\bullet \leq \kappa_i \tag{69}$$

by (66) and by the fact that e_i° and $e_i^\bullet$ are elements of $[0,\kappa_i]$: recalling that $Dp(\eta)$ is a well-defined real number and that v_i is differentiable on the interior of its domain $[0,\kappa_i]$, it is readily observed that the inequalities in (67) and (68) imply that the right-derivative $D^+u_i^C(\cdot, e_{-i}^\circ)(e_i^\circ)$ of $u_i^C(\cdot, e_{-i}^\circ)$ at e_i° and the left-derivative $D^-u_i^C(\cdot, e_{-i}^\bullet)(e_i^\bullet)$ of $u_i^C(\cdot, e_{-i}^\bullet)$ at $e_i^\bullet$ exist in $\mathbb{R}$ and are respectively specified by

$$Dp(\eta)\cdot e_i^\circ + p(\eta) - \mu_i^C \cdot Dv_i(\mu_i^C \cdot \textstyle\sum_{l\in B_i^C} e_l^\circ) \tag{70}$$

and

$$Dp(\eta)\cdot e_i^\bullet + p(\eta) - \mu_i^C \cdot Dv_i(\mu_i^C \cdot \textstyle\sum_{l\in B_i^C} e_l^\bullet). \tag{71}$$

The Nash equilibrium conditions then imply

$$D^+u_i^C(\cdot, e_{-i}^\circ)(e_i^\circ) \leq 0 \leq D^-u_i^C(\cdot, e_{-i}^\bullet)(e_i^\bullet) \tag{72}$$

by virtue of (69). As $\sum_{l\in B_i^C} e_l^\circ = \sum_{l\in B_i^C} e_l^\bullet$, from the specifications of the sided derivatives in (70) and (71) and from the inequalities in (72) we conclude that $Dp(\eta)\cdot e_i^\circ \le Dp(\eta)\cdot e_i^\bullet$: a contradiction with the inequality in (66) and the already observed fact that $Dp(\eta)$ is a negative real number.

Case $\sum_{l\in B_n^C} e_l^\circ \neq \sum_{l\in B_n^C} e_l^\bullet$ for some $n\in N$. Suppose now that $\sum_{l\in B_n^C} e_l^\circ \neq \sum_{l\in B_n^C} e_l^\bullet$ for some $n \in N$. In particular, but without loss of generality, suppose

$$\textstyle\sum_{l\in B_n^C} e_l^\circ < \sum_{l\in B_n^C} e_l^\bullet. \tag{73}$$

Then there exists $i \in B_n^C$ such that

$$e_i^\circ < e_i^\bullet. \tag{74}$$

Recalling that $e_l^\bullet \in [0,\kappa_l]$ for all $l \in B_i^C$ and keeping in mind the validity of (65) and (73), note that

$$0 \le \mu_i^C \cdot \textstyle\sum_{l\in B_i^C} e_l^\circ < \mu_i^C \cdot \sum_{l\in B_i^C} e_l^\bullet \le \kappa_i. \tag{75}$$

It is now remarked that if $\mu_i^C \cdot \sum_{l\in B_i^C} e_l^\bullet = \kappa_i$ and $D^-v_i(\kappa_i) = +\infty$, then $e^\bullet$ cannot be a Nash equilibrium since in that case the left derivative of $u_i^C(\cdot, e_{-i}^\bullet)$ at $e_i^\bullet$ is $-\infty$: to see this, note that such left derivative is the limit

$$\lim_{x\to 0^-} \frac{p(\eta+x)\cdot(e_i^\bullet + x) - v_i(\mu_i^C\cdot(x+\kappa_i)) - p(\eta)\cdot e_i^\bullet + v_i(\mu_i^C\cdot\kappa_i)}{x}$$

and observe that the limit

$$\lim_{x\to 0^-} \frac{p(\eta+x)\cdot(e_i^\bullet + x) - p(\eta)e_i^\bullet}{x}$$

exists in $\mathbb{R}$ by the differentiability of p at η and that the limit

$$\lim_{x\to 0^-} \frac{v_i(\mu_i^C\cdot(x+\kappa_i)) - v_i(\mu_i^C\cdot\kappa_i)}{x}$$

equals $+\infty$ since it can be expressed as the product $\mu_i^C \cdot D^-v_i(\kappa_i)$ of the positive real μ_i^C and the positively infinite left derivative $D^-v_i(\kappa_i)$. Having remarked this, from (75) we are in a position to conclude that the right derivative at $\mu_i^C\cdot\sum_{l\in B_i^C} e_l^\circ$ and the left derivative at $\mu_i^C\cdot\sum_{l\in B_i^C} e_l^\bullet$ of the convex and strictly increasing function v_i exist in $\mathbb{R}$. Recalling that $Dp(\eta)$ and μ_i^C are real numbers, we then readily infer that the right derivative of $u_i^C(\cdot, e_{-i}^\circ)$ at e_i° and the left derivative of $u_i^C(\cdot, e_{-i}^\bullet)$ at $e_i^\bullet$ exist in $\mathbb{R}$ and are respectively specified by

$$Dp(\eta)\cdot e_i^\circ + p(\eta) - \mu_i^C\cdot D^+v_i(\mu_i^C\cdot \textstyle\sum_{l\in B_i^C} e_l^\circ) \tag{76}$$

and

$$Dp(\eta)\cdot e_i^\bullet + p(\eta) - \mu_i^C\cdot D^-v_i(\mu_i^C\cdot \textstyle\sum_{l\in B_i^C} e_l^\bullet). \tag{77}$$

Note that

$$0 \le e_i^\circ < e_i^\bullet \le \kappa_i \tag{78}$$

by (74) and by the fact that e_i° and $e_i^\bullet$ are elements of $[0,\kappa_i]$. The equilibrium conditions then imply

$$D^+u_i^C(\cdot, e_{-i}^\circ)(e_i^\circ) \le 0 \le D^-u_i^C(\cdot, e_{-i}^\bullet)(e_i^\bullet) \tag{79}$$

by virtue of (78). As $\sum_{l\in B_i^C} e_l^\circ < \sum_{l\in B_i^C} e_l^\bullet$, the convexity of v_i implies

$$D^+v_i(\mu_i^C \cdot \textstyle\sum_{l\in B_i^C} e_l^\circ) \leq D^-v_i(\mu_i^C \cdot \sum_{l\in B_i^C} e_l^\bullet)$$

and from the specifications of the sided derivatives in (76) and (77) and from the inequalities in (79) we conclude that $Dp(\eta) \cdot e_i^\circ \leq Dp(\eta) \cdot e_i^\bullet$: a contradiction with the inequality in (74) and the already observed fact that $Dp(\eta)$ is a negative real number. ■

Theorem 6 *Let Σ be an active VSA-system and let C be a VSA-configuration in $\mathcal{N}$. Assume that e is a Nash equilibrium for the VSA-game associated to C under Σ and put $\eta = e_1 + \cdots + e_{\bar{n}}$. Then $\eta > 0$ and $p(\eta) > 0$.*

Proof. Pick $n \in \{l \in N : D^+v_l(0) = \min\{D^+v_1(0), \ldots, D^+v_{\bar{n}}(0)\}\}$ and put $\omega = (0, \ldots, 0) \in \mathbb{R}^{\bar{n}}$. The right derivative $D^+u_n^C(\cdot, \omega_{-n})(\omega_n)$ of $u_n^C(\cdot, \omega_{-n})$ at ω_n exists in $\mathbb{R}_{++} \cup \{+\infty\}$ and is

$$\lim_{x\to 0^+} \frac{p(x) \cdot x - v_n(\mu_n^C \cdot x) + v_n(0)}{x}.$$

It is obvious that such right derivative—if it exists—is the limit expressed above. It is less obvious that the limit expressed above exists in $\mathbb{R}_{++} \cup \{+\infty\}$. To see this, note that

$$\lim_{x\to 0^+} \frac{p(x) \cdot x - v_n(\mu_n^C \cdot x) + v_n(0)}{x}$$

exists in the extended reals and coincides with

$$\lim_{x\to 0^+} p(x) - \lim_{x\to 0^+} \frac{v_n(\mu_n^C \cdot x) - v_n(0)}{x} \tag{80}$$

by the fact that the first limit in (80) exists in $\mathbb{R} \cup \{+\infty\}$ because conditions H1 and H2 imply the decreasingness of p on $\mathbb{R}_{++}$ and by the fact that the second limit in (80) exists in $\mathbb{R}_+$ as

$$\lim_{x\to 0^+} \frac{v_n(\mu_n^C \cdot x) - v_n(0)}{x} = \mu_n^C \cdot D^+v_n(0)$$

with $0 < \mu_n^C \leq 1$ by the definition of μ_n^C and with $D^+v_n(0) \in \mathbb{R}_+$ by condition H6. The last two observed facts imply also $\mu_n^C \cdot D^+v_n(0) \leq D^+v_n(0)$ and hence

$$\lim_{x\to 0^+} \frac{p(x) \cdot x - v_n(\mu_n^C \cdot x) + v_n(0)}{x} \geq \lim_{x\to 0^+} p(h) - D^+v_n(0).$$

Noting that $\lim_{x\to 0^+} p(x) - D^+v_n(0) > 0$ by the assumption that Σ is active, we are in a position to get the desired conclusion that $D^+u_n^C(\cdot, \omega_{-n})(\omega_n)$ exists in $\mathbb{R}_{++} \cup \{+\infty\}$. Therefore, ω cannot be a Nash equilibrium and hence $e \in \mathbb{R}_+^{\bar{n}} \backslash \{\omega\}$ as $e_l \in [0, \kappa_l]$ for all $l \in N$. This proves that $\eta > 0$ and implies the existence of i in N such that $e_i > 0$. If $p(\eta) \leq 0$, then

$$p(\eta - e_i) \cdot 0 - \phi_i^C - v_i(\mu_i^C \cdot (-e_i + \textstyle\sum_{l\in B_i^C} e_l)) > p(\eta) \cdot e_i - \phi_i^C - v_i(\mu_i^C \cdot \sum_{l\in B_i^C} e_l)$$

because e_i and μ_i^C are positive real numbers and because v_i is strictly increasing: the observation that the left-hand side of the last equality is $u_i^C(\omega_i, e_{-i})$ and that the right-hand side of the last equality is $u_i^C(e_i, e_{-i})$ yields a contradiction with the assumption that e is a Nash equilibrium. So, $p(\eta) > 0$. ■

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