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General Equilibrium with Competition in Schedules and Input-output Networks: an Existence Result

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Abstract

This paper presents a model of general equilibrium with firm-to-firm trade using double auctions, or competition in supply and demand functions relating quantities to prices. An equilibrium exists for general non-parametric technology, provided the best replies are convex-valued, under suitable regularity and boundedness assumptions.

JEL Classification: L13, D43, D44, D57.

Keywords: Production networks, Oligopoly, Double Auction, Supply Function Equilibrium, General Equilibrium.

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Introduction

Production of goods in modern economies typically features long and interconnected supply chains.¹ Moreover, many authors find that market power is a sizable phenomenon, some even argue increasing,² and many firms are large relative to their sector or even the whole economy.³ How are prices formed in an input-output network of non-price-taking firms? How is surplus split? How efficient is the process?

This paper provides a strategic non-cooperative model of large firms interacting in an input-output network consisting of many specific supply-customer relationships. It does so introducing the technique of competition in schedules, or supply and demand functions, to the modeling of general equilibrium oligopoly. The main interest lies in the fact that such a technique allows to have a fully strategic model in which firms understand and take into account their position in the network, and have market power on both inputs and outputs markets simultaneously, in an endogenously determined way. These features are rarely both present in input-output models of the macroeconomy,⁴ but I argue that they are important to analyze market power in input-output networks. In particular, the fact that firms are fully strategic and take their position in the network into account can generate large differences in the magnitude of distortions due to imperfect competition. The fact that firms have market power on both input and output markets, as opposed to only outputs, can generate large differences in the ranking of market power across firms or sectors. Both effects are particularly stark, especially when supply chains are long.

A further interest of the competition in schedules framework is that it is a standard model for procurement auctions ([Holmberg et al., 2025](#); [Klemperer and Meyer, 1989](#); [Ausubel et al., 2014](#)), where the consumer is the auctioneer. The results developed can help shed light on price formation in procurement auctions where the bidders are simply the last stage of a potentially complex supply chain. The exploration of the implications of

¹Recently the focus of a large literature, see [Carvalho and Tahbaz-Salehi \(2018\)](#), [Bernard et al. \(2018\)](#).

²See [De Loecker et al. \(2020\)](#), [Berry et al. \(2019\)](#).

³Now known as *superstar firms* since [Autor et al. \(2020\)](#).

⁴With some exceptions, see the discussion in the literature section.

this for design are an interesting avenue for further research.

Formally, firms have each a set of input and output goods, some of which are in turn outputs or inputs of other firms,⁵ and these trade relationships, or *input-output links*, are exogenous. Firms play a simultaneous game in which the available actions are supply and demand schedules, relating quantities of the traded goods to prices: as in a double auction, the realized price on every trade relationship is the one where demand and supply cross.

The detailed contributions are the following. First, I show that an equilibrium exists under general regularity and boundedness conditions on the set of feasible schedules (Theorem 1), I provide necessary conditions for equilibrium in the form of a system of partial differential equations (Theorem 2), and a condition under which the equilibrium is ex-post, in the sense that firms would not change their decisions even after the realization of uncertainty (Corollary 2.1).

Theorem 1 presents an existence result under the assumption that the best reply correspondences are convex-valued (or, in particular, single-valued). Other assumptions include general regularity and boundedness assumptions on the technology, consumer demands, and the set of feasible schedules. As in the seminal Klemperer and Meyer (1989) paper on Supply function equilibrium, uncertainty in the realized prices is key to avoiding a huge multiplicity of best replies. In this paper, the uncertainty comes from stochastic parameters in the transformation function of firms, which can be seen as input (and output)-specific productivity shocks. This is sufficient to generate enough variation in the schedules so that the equilibrium prices span all the feasible set, and the best reply is not indeterminate. The result departs from other existing results in the literature in that it does not impose parametric functional forms,⁶ and the presence of firm-to-firm trade.⁷ The regularity and boundedness assumptions on the set of feasible schedules allow to use Banach spaces techniques and the Ky Fan fixed

⁵For the purpose of the model, two different instances of the same good, that are traded between different producers at possibly different prices are labeled as different “goods”

⁶As Malamud and Rostek (2017)

⁷Contrary to Wilson (2008) or Holmberg and Philpott (2018) that study an oligopoly where a transmission network affects the demand.

point theorem to show existence, rather than looking for an equilibrium as a solution to a system of differential equations.⁸ The existence result provided can be of interest also to the modeling of financial markets where traders have price impacts, departing from the standard CARA - gaussian setting.

Then I explore what can be said about uniqueness of the best replies. Theorem 2 indeed expresses the necessary conditions for an equilibrium in the form of a system of partial differential equations, and clarifies that the equilibrium in this model is not ex-post (as in Klemperer and Meyer (1989)), due to network effects. Corollary 2.1 illustrates that the equilibrium is ex-post under a measurability condition, stating that the residual demand and supply depend on a number of uncertain parameters equal to the degree of each firm in the network. Under this condition, it is possible to prove that best replies are single-valued, thus complementing the result of Theorem 1 for existence. The measurability condition says that the degrees of freedom of each firm are as many as the independent sources of uncertainty. This condition is satisfied if the residual schedules are linear (the case to which the parametric model in the companion paper Bizzarri (2025)), or if the network is a sequence of sectors linearly connected, a network I label *regular layered supply chain*. This shows that the ex-post or ex-ante nature of the equilibrium depends on an interaction of the schedules' functional form and the network structure.⁹

Related literature

This paper contributes to three lines of literature: the literature on competition in supply and demand functions, the literature on production networks or networked markets, and the literature on general equilibrium

⁸As Klemperer and Meyer (1989), Glebkin et al. (2020).

⁹Rostek and Yoon (2021a) show that even in some cases when the equilibrium is not ex-post, if schedules are linear the optimization can still be expressed as a *pointwise* optimization over quantities given price impacts. This is not true anymore in the model of the present paper for nonlinear schedules: the crucial problem is that the way prices respond to a change in schedules (or quantities) is itself uncertain, because it depends on the realization of the stochastic parameters in the other markets. As clarified by Theorem 2, the optimization over quantities alone would miss this effect, even conditioning on the price impact. The consequence is that the optimization cannot be performed on prices anymore, but has to be done directly on schedules.

oligopoly. The closest paper is the companion paper [Bizzarri \(2025\)](#), where it is developed the case of the linear equilibrium.

My contribution to the literature on competition on supply and demand functions is to introduce the technique to the modeling of general equilibrium oligopoly, in particular with firm-to-firm trade, and providing a general existence result. Some papers have applied the concept to macroeconomic modeling, without considering the input-output dimension: [Vuong et al. \(2015\)](#) show how to non-parametrically estimate a supply function equilibrium oligopoly on international trade data; [Bornstein and Peter \(2022\)](#) study the effect of nonlinear pricing on misallocation. The literature has studied the situation where the demand firms receive comes from a network structure with a large dimension of uncertainty, in [Wilson \(2008\)](#), [Holmberg and Philpott \(2018\)](#), [Ruddell et al. \(2017\)](#), but their firms only supply to a node in the network, do not trade among themselves. Firm-to-firm trade is studied in a bilateral setting in [Weretka \(2011\)](#) and [Hendricks and McAfee \(2010\)](#), always constraining the schedules to a parametric functional form. In the finance literature the model is used to study simultaneous demand and supply of heterogeneous assets: [Malamud and Rostek \(2017\)](#), as well as [Rostek and Yoon \(2021a\)](#), [Rostek and Yoon \(2021b\)](#) and [Rostek and Weretka \(2012\)](#) analyze a parametric model yielding an equilibrium in linear strategies; [Glebkin et al. \(2020\)](#) and [Du and Zhu \(2017\)](#) study general functional forms, but in a centralized market (corresponding to a trivial network). [Ausubel et al. \(2014\)](#) and [Woodward \(2021\)](#) study general functional forms in the context of centralized auctions. [Vives \(2011\)](#) studies market power arising from asymmetric information, rather than network position.

My contribution to the production networks literature is to provide a model of competition in an input-output network in which all firms have market power on both input and output markets, and are fully strategic internalizing their position in the supply chain. Many models explicitly assume that firms have power to decide/affect prices only on one side of the market. To this class belong the workhorse sequential oligopoly games in [Spengler \(1950\)](#), [Salinger \(1988\)](#), [Ordover et al. \(1990\)](#), [Hart et al. \(1990\)](#).¹⁰ and the recent [Carvalho et al. \(2020\)](#). These models all feature sequential

¹⁰And used in classic textbook treatments, such as [Tirole \(1988\)](#).

moves in which downstream firms take input prices as given and, hence, one-sided market power. In another class of models authors assume that output prices are equal to the marginal cost times a markup. The concept of the marginal cost itself implicitly implies price-taking in the input market: indeed, it arises from the price-taking cost minimization problem of the firm. Hence, it is implicitly assuming unilateral market power. To this category belong [Grassi \(2017\)](#), [Bernard et al. \(2022\)](#), [Baqae \(2018\)](#), [Baqae and Farhi \(2019\)](#), [Baqae and Farhi \(2020\)](#), [Huremovic and Vega-Redondo \(2016\)](#), [Magerman et al. \(2020\)](#), [Dhyne et al. \(2019\)](#), [Huneus et al. \(2021\)](#), [Arkolakis et al. \(2021\)](#), [Pasten et al. \(2020\)](#), [Pellegrino \(2019\)](#). In [Galeotti et al. \(2021\)](#) only primary producers charge a markup, while the intermediate firms behave competitively, thus abstracting from the balance of market power among firms that trade with each other. The exception is [Acemoglu and Tahbaz-Salehi \(2020\)](#), that follows a mixed approach: input prices are taken as given when firms decide their input mix, but are then determined in equilibrium through a link-level alternating offers game, relying on exogenously specified bargaining weights. My results complement theirs, providing a model that does not rely on the choice of exogenously specified bargaining weights.

Except for [Acemoglu and Tahbaz-Salehi \(2020\)](#), all these papers feature also the implicit or explicit assumption that firms do not internalize the effect of their decisions on sectors/firms further downstream beside the direct customers. Sometimes this is a consequence of the assumption of a continuum of firms in each sector (and so sector-level aggregates are taken as given by every individual firm),¹¹ other times it is explicitly assumed.¹²

I contribute to the literature on general equilibrium with market power by providing a fully strategic model of the production side with endogenous market power and firm-to-firm trade; furthermore, the game does not depend on price normalization, and can incorporate general assumptions on owner’s preferences as in [Azar and Vives \(2021\)](#). In the recent literature on “general oligopolistic competition” ([Azar and Vives \(2021\)](#), [Azar and Vives \(2018\)](#) and [Ederer and Pellegrino \(2022\)](#)) do not consider firm-to-firm trade. An older related paper is [Nikaido \(2015\)](#), who also uses the

¹¹This is the case in, e.g. [Baqae \(2018\)](#) and various others listed in the literature.

¹²E.g., in [Grassi \(2017\)](#), [Kikkawa et al. \(2019\)](#).

market clearing conditions to back up quantities as functions of prices, but his method is limited to Leontief technology. In the literature on general equilibrium matching [Fleiner et al. \(2019\)](#) study firm-to-firm trade with distortions that are exogenous wedges rather than the outcome of a strategic setting as in the present paper.

1 The Model

In this section I introduce the primitives of the model, that is the firms and their technology, the input-output network, and the utility of the consumer. Firms play a game in which the strategies are supply and demand schedules. Finally, I introduce the technical assumptions needed for the subsequent results.

1.1 Setting

Firms and Production Network There are N firms and M goods: their sets are respectively denoted $\mathcal{N}$ and $\mathcal{M}$. Each good might be produced by more firms, and each firm may produce more than one good. I write $i \rightarrow g$ if firm i produces good g , and $g \rightarrow i$ if firm i needs good g for production. Each firm produces using labor, and a set of inputs produced by other firms, which I denote as $\mathcal{N}_i^{in}$. The set of goods produced by firm i is $\mathcal{N}_i^{out}$. The consumers' utility depends directly on a subset of goods, denoted $\mathcal{C} \subseteq \mathcal{M}$. Firms, goods and the connections defined above define a directed bipartite graph $\mathcal{G} = (\mathcal{N}, \mathcal{M}, E)$, where $E = \{(i, g) \mid i \rightarrow g \text{ or } g \rightarrow i\}$ is the set of existing connections. I refer to $\mathcal{G}$ as the *input output network* of this economy. Note that in this setting, a *good* is identified by the fact that is exchanged by a number of firms for a specified price. That is, the framework can accommodate for a firm selling the same physical good to different groups of customers for different prices: simply, this case would show up in the model as two distinct goods. I denote $d_i^{out} = |\mathcal{N}_i^{out}|$ the *out-degree* (number of outputs) of i , and $d_i^{in} = |\mathcal{N}_i^{in}|$ the *in-degree* (number of inputs) of firm i , excluding labor.

Remark 1.1. In the case in which each firm produces only one good, and the goods are all distinct, we can identify the sets of firms and goods and

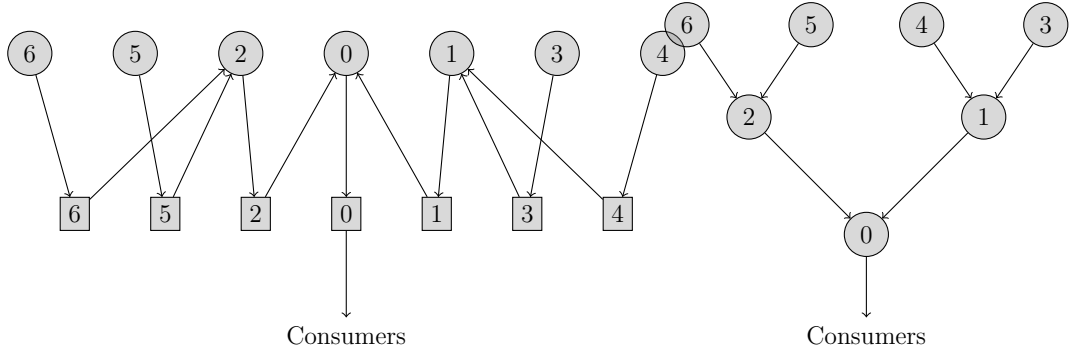


Figure 1: Left: bipartite representation of the production network: the circles are the firms, the squares are goods. An arrow from a good to a firm means the firm buys the good, an arrow from a firm to a good means that the firm sells the good. Right: classic representation of the network, where nodes are firms and links represent the flow of goods. In this example, in which each firm has one distinct output good, the two are equivalent. In general this representation is ambiguous, because it does not allow to see whether, e.g. 5 and 6 output is the same good or two distinct goods.

say that two firms are connected if one is a customer of the other. This is the more standard approach in the literature. Figure 1 illustrates the standard (Left) and the bipartite (Right) representation followed here, in the example of a tree network.

The production possibilities available to firm i are described by a *transformation function* Φ_i . This is a function of the input and output quantities, and also on a vector of stochastic parameters $\boldsymbol{\varepsilon}_i = (\varepsilon_{ig})_{g \in \mathcal{N}_i}$, one for each good traded by i . These can be thought of as technological shocks, increasing or decreasing the input quantity needed to achieve a certain level of output. As in Mas-Colell et al. (1995), input quantities are negative, while output quantities are positive. The production possibility set of firm i is thus $\{(q_{gi})_{i \rightarrow g}, (q_{ig})_{g \rightarrow i}, \ell_i \mid \Phi_i((q_{gi})_g, (-q_{ig})_g, \ell_i, \boldsymbol{\varepsilon}_i) \leq 0\}$. The reason to describe the technology as a transformation function is, besides generality, to treat symmetrically inputs and outputs: goods are allowed to be both, depending on what is more convenient given the market conditions and the implied prices. This a standard approach taken also in Mas-Colell et al. (1995). In our context, it allows a considerable technical simplification, allowing to abstract from corner solutions: negative quantities are allowed,

they simply mean trade in the opposite direction.

The price of good g is denoted p_g , so that for a firm buying and producing quantities $(q_{gi})_{i \rightarrow g}, (q_{ig})_{g \rightarrow i}, \ell_i$, the nominal profit is:

$$\Pi_i = \sum_{g, i \rightarrow g} p_g q_{gi} - \sum_{g, g \rightarrow i} p_g q_{ig} - w \ell_i$$

Consumers There is a continuum of identical consumers or, equivalently, a representative consumer. She gets utility $U((c_g)_{g \in \mathcal{C}}, L, \varepsilon_{i,c})$ from a subset of goods $\mathcal{C} \subseteq \mathcal{M}$, and disutility from labor L ; similarly to the firms, I am going to assume that the utility also depends on a vector of stochastic parameters $\varepsilon_c = (\varepsilon_{g,c})_{g \in \mathcal{C}}$, one for each good consumed. Denote the demand for good i derived by U as $D_{i,c}$, and the labor supply as L . The profits of the firms are rebated to the representative consumer, so that the total income is $wL + Pro$, where $Pro = \sum_i Pro_i$ is the aggregate profit. Welfare in this economy is the utility of the consumers in equilibrium: $U(c^*, L^*)$, where c^* and L^* are the equilibrium values of consumption and labor: since the profits are rebated, such welfare also includes the producers surplus.

Notation Bold symbols are used to denote vectors of prices and stochastic parameters: $\mathbf{p}$ is the vector of all prices, except the wage w , $\mathbf{p}_i^{in} = ((p_g)_{g \in \mathcal{N}_i^{in}})$ are the prices of all input goods of firm i , and similarly $\mathbf{p}_i^{out}$ is the price vector for the outputs, so that $\mathbf{p}'_i = (\mathbf{p}_i^{out}, \mathbf{p}_i^{in})'$. Similarly, $\mathbf{p}_c = (p_g)_{g \in \mathcal{C}}$ is the vector of prices of goods consumed by the consumer. The analogous notations hold for stochastic parameters, so that, e.g., ε is the vector that stacks all the stochastic parameters of all firms.

When A is a function of many variables, $\nabla A = (\partial_1 A, \dots, \partial_n A)'$ denotes the (column) vector of partial derivatives (the gradient). HA denotes the matrix of second derivatives, that is the Hessian of A . When A is a vector function of x , $\partial_x A$ denotes the square matrix with on each row the gradient of A_i with respect to x (the Jacobian matrix).

If B is a matrix, B_{-i} denotes the same matrix to which row and column i have been removed. If $\mathbf{b}$ is a vector, $\mathbf{b}_{-i}$ denotes the same vector to which element i has been removed. $B \geq C$ denotes the fact that $B - C$ is positive semidefinite (even when they are not symmetric).

The Game I: players and actions The competition among firms takes the form of a game in which firms compete choosing in supply and demand functions. This means that the players of the game are the firms, and the actions available to each firm i are a family of functions defined over a set $\mathcal{F}_i$ of tuples of input-output prices, wage, and a set of firm-specific stochastic parameters $\mathcal{E}_i$: $\mathcal{S}_i : (w, \mathbf{p}_i, \boldsymbol{\varepsilon}_i) \in \mathcal{F}_i \times \mathcal{E}_i \rightarrow \mathbb{R}^{d_i+1}$, where $\mathcal{F}_i \times \mathcal{E}_i \subset \mathbb{R}^{2(d_i+1)}$. Such functions are called *schedules*, and $\mathcal{S}_i = (S_i, -D_i, -\ell_i)$, composed by profiles of supply functions for outputs $S_i = (S_{gi})_{i \rightarrow g}$, demand functions for intermediate inputs $D_i = (D_{ig})_{g \rightarrow i}$, and for labor ℓ_i .¹³ The set of feasible supply and demand schedules for firm i (defined below) is denoted $\mathcal{A}_i$, and $\mathcal{A} = \prod_{i \in \mathcal{N}} \mathcal{A}_i$.

In the general model of this section we are not restricting traded quantities to be positive. This is a matter of interpretation: since trade has a direction, negative quantities can simply be interpreted as trade flowing in the opposite direction.¹⁴ This approach simplifies the analysis because rules out corner solutions in which firms decide not to buy some inputs (or sell some outputs) at all.

The Game II: prices and payoffs To complete the definition of the game, we have to define the payoffs. These are, in short, the expected profits calculated in the prices that satisfy the market clearing conditions. The market clearing conditions are:

$$\begin{aligned} \sum_{j, g \rightarrow j} D_{jg}(\mathbf{p}_j, w, \boldsymbol{\varepsilon}_j) &= \sum_{k, k \rightarrow g} S_{gk}(\mathbf{p}_k, w, \boldsymbol{\varepsilon}_k) \quad \text{if } g \in \mathcal{M} \\ D_{cg}(\mathbf{p}_c, w, \boldsymbol{\varepsilon}_c) &= \sum_{k, k \rightarrow g} S_{gk}(\mathbf{p}_k, w, \boldsymbol{\varepsilon}_k) \quad \text{if } g \in \mathcal{C} \\ \sum_i \ell_i(\mathbf{p}_i, w, \boldsymbol{\varepsilon}_i) &= L(w, \mathbf{p}_c, \boldsymbol{\varepsilon}_c) \end{aligned}$$

¹³The sign convention makes formulas simpler allowing the derivative to be positive semidefinite.

¹⁴Indeed, this is the interpretation followed by classic treatments of production theory, such as [Mas-Colell et al. \(1995\)](#).

Define a function $MC : \mathbb{R}^M \times \mathcal{E} \rightarrow \mathbb{R}^M$ such that (normalizing the wage to 1):

$$\begin{aligned} MC_g &= \sum_{k, k \rightarrow g} S_{gk}(\mathbf{p}_k, w, \boldsymbol{\varepsilon}_k) - \sum_{j, g \rightarrow j} D_{jg}(\mathbf{p}_j, w, \boldsymbol{\varepsilon}_j) \quad \text{if } g \in \mathcal{M} \\ MC_g &= \sum_{k, k \rightarrow g} S_{gk}(\mathbf{p}_k, w, \boldsymbol{\varepsilon}_k) - D_{cg}(w, p_c, \boldsymbol{\varepsilon}_c) \quad \text{if } g \in \mathcal{C} \end{aligned} \quad (1)$$

Throughout the paper I am going to assume that Walras' law is specified and the schedules are homogeneous of degree zero. Hence the market clearing conditions can be stated as $MC(\mathbf{p}, w, \boldsymbol{\varepsilon}) = \mathbf{0}$. Formally, we have the following definition.

Definition 1.1 (Pricing function and payoffs). Call $\mathcal{E} = \times_{i \in \mathcal{N}} \mathcal{E}_i$. Define a *feasible pricing function* as a function $(\mathbf{p}^*, w^*) : \mathcal{E} \rightarrow \mathbb{R}^M$ such that $MC(\mathbf{p}^*(\boldsymbol{\varepsilon}), w^*(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}) = \mathbf{0}$ for all $\boldsymbol{\varepsilon} \in \mathcal{E}$.

The payoff of firm (player) i is the mapping from supply and demand schedules in $\mathcal{A}_i$ to real numbers defined by the profits, normalized by the wage:

$$\begin{aligned} \pi_i(S_i, D_i, \ell_i) &= \\ \mathbb{E}_F \left(\sum_{g, i \rightarrow g} p_g^*(\boldsymbol{\varepsilon}) S_{g,i}(\mathbf{p}_i^*(\boldsymbol{\varepsilon}), w^*(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}_i) - \sum_{g, g \rightarrow i} p_g^*(\boldsymbol{\varepsilon}) D_{ig}(\mathbf{p}_i^*(\boldsymbol{\varepsilon}), w^*(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}_i) \right. \\ &\quad \left. - w^*(\boldsymbol{\varepsilon}) \ell_i(\mathbf{p}_i^*(\boldsymbol{\varepsilon}), w^*(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}_i) \right) / w \end{aligned}$$

In summary, we defined a game: $G = (\mathcal{N}, (\mathcal{A}_i)_{i \in \mathcal{N}}, (\pi_i)_{i \in \mathcal{N}})$. Proposition 1 below shows that the pricing function exists, so the payoffs are well defined, and moreover that the equilibrium does not depend on the normalization of prices. I call a Nash equilibrium of this game a Supply and Demand function equilibrium. Notice that the equilibrium defines a probability distribution over all the endogenous objects; prices, quantities, hence welfare.

Generalizations: objectives of the firm While the *price* normalization is inconsequential, the *uniform* normalization of profits is. Hence, in the appendix (Section 2) Theorem 1 and 2 are proven under the more gen-

eral assumption that there are two distinct types of agents, workers and owners, and *firms optimize the indirect utility of shareholders*, following Azar and Vives (2021). The construction in the main text corresponds to the case in which owners are identical and only value a good produced independently from the network, whose price is in fixed proportion with the wage. This is equivalent to the approach followed in Ederer and Pellegrino (2022), and the polar opposite of the assumption maintained in Azar and Vives (2021), in which owners have the same utility as consumers. Both are evidently abstractions: in the main text I follow the former for two reasons: first, our focus is on the effect of endogenous market power on firm-to-firm trade, rather than the interactions of market power and owner's incentives, that are instead the focus of Azar and Vives (2021).

1.2 Assumptions

In this paragraph I collect all the assumptions needed for Theorems 1 and 2.

Assumption 1 - Demand Consumers have aggregate demand D_c that has negative semidefinite jacobian with corank 1 with respect to both prices $\mathbf{p}_c, w$. It is positive definite with respect to stochastic parameters ε_c ; D_c is positive and differentiable. Finally, if for a converging sequence of price vectors $(\mathbf{p}^n, w^n) \rightarrow (\mathbf{p}^*, w^*)$ we have that the limit satisfies $p_g^* = 0$ for some good g , then the demand for that good diverges: $\lim_{(\mathbf{p}^n, w^n) \rightarrow (\mathbf{p}^*, w^*)} D_{cg}(\mathbf{p}^n, \varepsilon_c) = \infty$. Moreover, consumer preferences give rise to a labor supply function ℓ^S , nonnegative and differentiable. If for a converging sequence of price vectors $(\mathbf{p}^n, w^n) \rightarrow (\mathbf{p}^*, w^*)$ we have that the limit satisfies $w^* = 0$, then the consumer stops working: $\lim_{(\mathbf{p}^n, w^n) \rightarrow (\mathbf{p}^*, w^*)} \ell^S(\mathbf{p}^n, \varepsilon_c) = 0$.

Assumption 2 - Technology The transformation function Φ_i is differentiable, convex and increasing in the quantities $\mathbf{q}_i = (\mathbf{q}^{out}, -\mathbf{q}^{in})$: $\nabla \Phi_i >> 0$. It satisfies the Inada condition that $\lim_{q_j \rightarrow 0} \partial_{q_j} \Phi_i = +\infty$. The joint support of the distribution F of all stochastic parameters $\varepsilon = ((\varepsilon_i)_{i \in \mathcal{N}}, \varepsilon_c)$, call it $\mathcal{E}$, is the closure of an open set, bounded in

norm by K_e (hence compact), and the distribution admits a differentiable density f .

Assumption 3 - Feasible schedules Define $\mathcal{A}_i$ as the set of schedules such that:

- a) *Homogeneity* each schedule $\mathcal{S}_i$ is homogeneous of degree 0 in $\mathbf{p}_i, w$;
- b) *Feasibility* each schedule $\mathcal{S}_i$ satisfies the **technology constraint**, that is, for any possible $(\mathbf{p}_i, w, \boldsymbol{\varepsilon}_i)$, it must be:

$$\Phi_i(\mathcal{S}_i(\mathbf{p}_i, w, \boldsymbol{\varepsilon}_i), \boldsymbol{\varepsilon}_i) \leq 0 \quad (2)$$

- c) *Regularity* the schedules $\mathcal{S}_i$ are infinitely differentiable and have Jacobian derivative with respect to prices $\partial_{\mathbf{p}_i, w} \mathcal{S}_i$ that is positive semidefinite with rank $d_i - 1$ ¹⁵; the derivative with respect to stochastic parameters $\partial_{\boldsymbol{\varepsilon}_i} \mathcal{S}_i$ is positive definite.

- d) *Bounds* The feasible schedules are uniformly bounded in the sup norm: there is a K_S such that $\|\mathcal{S}_i(\mathbf{p}_i, w, \boldsymbol{\varepsilon}_i)\|_\infty = \sup_{\mathbf{p}_i, w, \boldsymbol{\varepsilon}_i} \|\mathcal{S}_i(\mathbf{p}_i, w, \boldsymbol{\varepsilon}_i)\|_2 \leq K_S$. Moreover, the slopes are bounded too. That is, there exist constants k and K such that for all $\mathbf{p}_i, w, \boldsymbol{\varepsilon}_i$ $\|\partial_{\boldsymbol{\varepsilon}_i} \mathcal{S}_i\|_2 \leq K$ and $\partial_{\mathbf{p}_i, w} \mathcal{S}_i \geq kI_i$, where $\|\cdot\|_2$ is the spectral matrix norm, and I_i is the identity matrix of appropriate dimension. Moreover, if for a converging sequence of price vectors $(\mathbf{p}^n, w^n) \rightarrow (\mathbf{p}^*, w^*)$ we have $w^* = 0$ then the labor demands become larger than all other demands: $\lim_{(\mathbf{p}^n, w^n) \rightarrow (\mathbf{p}^*, w^*)} -\mathcal{S}_{i, \ell}(\mathbf{p}^n, w^n, \boldsymbol{\varepsilon}_i) \geq \max_{g \in \mathcal{N}_i} \{\mathcal{S}_{ig}\}$.

Denote $\mathcal{A} = \prod_i \mathcal{A}_i$.

Most of these assumptions are technical in nature: in particular, the boundedness and regularity assumptions are crucial in establishing compactness of the feasible set. The assumption on the limiting behavior says that the schedules are such that for extreme values of prices, at least on demand or supply diverges: this is used to show existence of a positive market clearing price vector. As part of the proof of Theorem 1 I am going to show that there is a bounded set of prices, bounded away from zero, where we can focus without loss of generality.

¹⁵The rank cannot be maximum because of homogeneity in prices.

For a given vector of parameters ε_i , the assumptions on the transformation function are quite standard: if the firm has a single output y produced with a strictly concave increasing production function f_i (for example a CES with decreasing returns to scale), then $\Phi_i(y, q_1, \dots, q_n) = y - f_i(-q_1, \dots, -q_n, \varepsilon_i)$ (remembering that negative quantities represent inputs) is indeed convex and increasing in the q variables. The assumptions on stochastic parameters guarantee that they represent productivity parameters, each of which has an independent effect.

The regularity and boundedness assumptions 3c) – d) guarantee that the demand and schedules are well behaved, enough to solve the market clearing system. The various boundedness assumptions are useful for various technical steps, and ultimately to guarantee compactness of the set of schedules, that is necessary to use the Schauder fixed point theorem in Theorem 1.

Example 1. Standard Supply Function Equilibrium

The model by [Klemperer and Meyer \(1989\)](#) can be seen as a special case of this setting, in which there is only one sector, the network $\mathcal{G}$ is empty, the only uncertainty is on the consumers, and the labor market is competitive. Their setting is a “partial” equilibrium one, in which the consumers do not supply labor to firms but appear only through a demand function $D(\cdot)$, and firms have a cost function for production $C(\cdot)$, that does not explicitly represent payments to anyone. Nonetheless, the game played by the firms is precisely the same: if the transformation function is $\Phi(q_i, -\ell_i) = q_i - C(\ell_i)$, and the consumer utility gives rise to a demand of the form $D_c + \varepsilon_c$, the game G played by firms is precisely the same as in [Klemperer and Meyer \(1989\)](#).

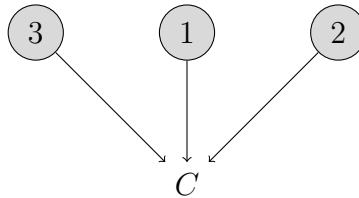


Figure 2: The (degenerate) production network of Example 1: there is only 1 Sector whose firm sell to the consumer.

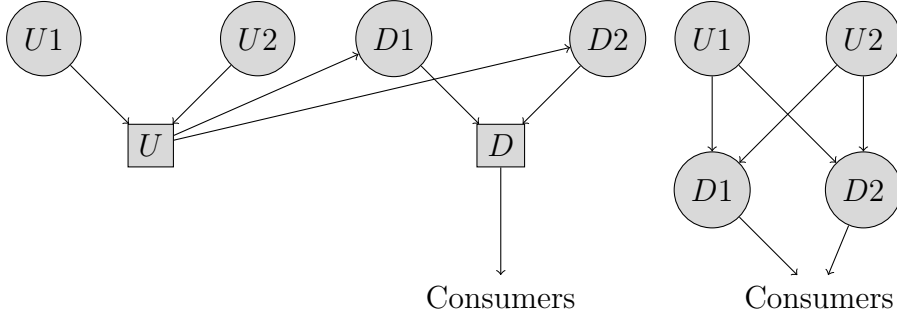


Figure 3: A layered supply chain. Left: bipartite representation, the squares represent goods, the circles firms. Right: firm-only representation.

Example 2 (Regular layered supply chain). A regular layered supply chain is a production structure in which firms are divided in m layers, as in Figure 3. There are m goods, each produced by all the firms in a layer; there are n firms per layer. Firms in layer $i + 1$ sell to firms in layer i , firm 0 sells its output to the consumer, and firms in layer m are the only ones to use labor.

The following example illustrates the simplest assumption on the behavior of owners, that will also be useful in the parametric model.

Example 3. Owners consuming an outside good.

If the owners' utility only depends on one good, o , and, moreover, such a good is produced from a continuum of firms that use only labor as input (hence are isolated from the network), with constant marginal cost, then two things happen. First, the indirect utility of the owners of group i is simply $\pi_i = \frac{\pi_i}{p_o}$, that is the profit divided by p_o , that is the same across owners: $\mathcal{P}_i = \mathcal{P}_j = p_o$. Second, that the price of such a good is equal to marginal cost $p_o = C_o$, hence it is itself a constant. It follows that in this case it is without loss of generality to assume that managers optimize the profits of the firms.

2 Results

In this section I present Theorems 1, 2 and Corollary 2.1. First, I prove as a preliminary result that a pricing function exists and is unique, hence the

payoffs above are well-defined (Proposition 1 below), and moreover the set of feasible prices is bounded, that is going to be important for the argument of Theorem 1.

2.1 The game is well-defined

We show that under our Assumptions, the game is well-defined and independent of the price normalization.

Proposition 1 (Feasible pricing and price normalization). *1. There exist a feasible pricing function $(\mathbf{p}, w) : \mathcal{E} \times \mathcal{A} \rightarrow \mathbb{R}_+^M$, and is unique up to normalization. Moreover, the payoffs are independent of price normalization.*

2. Normalizing the wage to 1, the image of the pricing function $\mathcal{P} = \mathbf{p}(\mathcal{E} \times \mathcal{A})$ is bounded, that is there is a $k_p > 0$ such that for any $\mathbf{p} \in \mathcal{P}$ $\|\mathbf{p}\|_2 < k_p$.

The proof relies on the regularity assumptions 3c) and the bounds in 3d) to show that the pricing function exists thanks to a global form of the implicit function theorem. The uniqueness up to normalization follows from homogeneity of the schedules, that translates into homogeneity of the excess supply MC . The second part follows from the bounds in Assumption 3d) and an application of the mean value theorem.

Thanks to the normalization by the wage, the profits depend only on price ratios, and so the game does not depend on the specific price normalization. For this reason, from now on, I am going to focus on homogenized schedules obtained normalizing the wage to 1, writing, with a slight abuse of notation, $\mathcal{S}_i(\mathbf{p}_i, \boldsymbol{\varepsilon}_i)$ for $\mathcal{S}_i(\mathbf{p}_i, 1, \boldsymbol{\varepsilon}_i)$. Moreover, since the technology constraint is binding, from now on we focus on $\mathcal{S}_{i,-\ell} = (S_i, -D_i)$, that is the profile of schedules for input and output goods *excluding* labor. Because of the above assumptions $\partial_{\mathbf{p}_i} \mathcal{S}_i$ is positive definite.

2.2 Existence

The main argument is a fixed point theorem. The main obstacle is establishing compactness of the set of feasible schedules. In order to do this it

is crucial first to limit the domains of the schedules to a compact set. In general, for a compact domain D , define $\mathcal{A}(D)$ as the set of schedules in $\mathcal{A}$ that are restricted to $D \times \mathcal{E}$. To be precise, $\mathcal{S}_i$ is restricted to the projection of D on the space of input and output prices of i , call it D_i . Second, it is necessary to consider the *closure* of $\mathcal{A}(D)$, denoted $\overline{\mathcal{A}}_i$, with respect to the $\|\cdot\|_\infty$ -norm on the set of schedules: $\|\mathcal{S}_i\|_\infty = \max_{D \times \mathcal{E}} |\mathcal{S}_i(\mathbf{p}_i, w, \varepsilon_i)|$, which is well defined thanks to the compactness of $D \times \mathcal{E}$. Lemma A.2 in the Appendix shows that the pricing function is Lipschitz, and so can be extended without problems to $\overline{\mathcal{A}}(D)$. To obtain compactness, thanks to the Ascoli-Arzelà theorem, the last piece we need is to choose an upper bound K to the norm of the price derivatives $\|\partial_{\mathbf{p}_i} \mathcal{S}_i\|_2 < K$: denote $\overline{\mathcal{A}}(D)^K$ the set of schedules that satisfy this bound. The formal statement of the theorem is as follows.

Theorem 1.

If the best reply correspondences are convex-valued, there exists a compact domain $\tilde{\mathcal{P}} \subseteq \overline{\mathcal{P}}$ such that the game G has a pure strategy Nash equilibrium in $\overline{\mathcal{A}(\tilde{\mathcal{P}})}^K$.

Furthermore, all prices in $\tilde{\mathcal{P}}$ can arise for some value of ε , and $\tilde{\mathcal{P}}$ is the closure of an open set (in particular, it has positive measure).

The second part of the statement guarantees that, thanks to our assumptions on stochastic parameters, the equilibrium spans a set of prices that is “large” enough, in particular in which derivatives are meaningful.

The proof of the first part applies the Ky Fan fixed point theorem to $\overline{\mathcal{A}(\tilde{\mathcal{P}})}^K$. Via a standard argument the differentiability and boundedness assumptions on the schedules in $\mathcal{A}(\tilde{\mathcal{P}})^K$ are enough to guarantee equicontinuity, and applying the Ascoli-Arzelà theorem we obtain that the closure is compact. Assumption 2 on the technology is also sufficient to show that $\overline{\mathcal{A}(\tilde{\mathcal{P}})}^K$ is convex. Hence, if the best reply is convex-valued, there exist a fixed point by Ky Fan’s fixed point theorem. For the second part, thanks to the assumptions of positive definiteness of $\partial_{\varepsilon_i} \mathcal{S}_i$ we can show that the pricing function is locally (right-)invertible, and this allows to conclude that the set of feasible prices is the closure of an open set. The formal proof is in Appendix A.2.

2.3 Necessary conditions for equilibrium

In this section I derive necessary conditions for best replies and describe the insights that emerge on the structure of the equilibrium.

The necessary conditions are best expressed in terms of the *residual schedule*, the schedule that collects the residual demands and supplies that the firm faces on all its input-output connections. It can be formally constructed as follows.

Definition 2.1 (Conditional pricing function and residual schedule). Given a profile of schedules $(\mathcal{S}_i)_{i \in \mathcal{N}}$, the *pricing function conditional on i* is the function $\mathbf{p}_{-i}(\cdot \mid i)$, defined on $\mathbf{p}_i, w, \varepsilon$ that satisfies the market clearing conditions 1, excluding those relative to the input and output prices of i :

$$MC_g(\mathbf{p}_{-i}(\mathbf{p}_i, w, \varepsilon \mid i), \mathbf{p}_i, \varepsilon) = 0 \quad \forall g \notin \mathcal{N}_i$$

The *residual schedule* of firm i is:

$$\mathcal{S}^r(\mathbf{p}_i, w, \varepsilon) = - \sum_{j \neq i} \mathcal{S}_j(\mathbf{p}_j(\mathbf{p}_i, w, \varepsilon \mid i), \varepsilon_j)$$

The next lemma sums up some properties of the residual schedules that are going to be useful.

Lemma 2.1. Under Assumptions 1,2 and 3, the residual schedule is homogeneous of degree zero in $\mathbf{p}_i, w$, differentiable, has positive semidefinite derivative $\partial_{\mathbf{p}_i, w} \mathcal{S}_i^r$ of corank 1 (i.e. has maximum rank minus 1).

Example 4. (Regular layered supply chain)

The easiest setting in which to understand the mechanics of the residual demand is the layered production chain illustrated in 3 and defined below: respectively, firm $U1$ and $U2$, and $D1$ and $D2$ produce perfect substitutes, and have the same technology.

When other firms play a profile of schedules $\mathcal{S}_{-i}$, the demand curve faced by $U1$ is:

$$\underbrace{D_{D1}(p_2^*, p_1, \varepsilon_D) + D_{D2}(p_D^*, p_U, \varepsilon_D)}_{\text{Direct demand from sector 2}} - \underbrace{S_{U1}(p_U, \varepsilon_U)}_{\text{Supply of competitor}}$$

for different choices of a supply function S_{U1} , different prices p_1 realize, as functions of the realizations of ε_2 . But naturally, given profile $\mathcal{S}_{-i}$ also p_D^* is endogenous, and a rational $U1$ takes this into account when optimizing. In particular the price p_D is determined by the market clearing conditions for good D :

$$S_{U1}(p_D, p_U, \varepsilon_U) + S_{U2}(p_D, p_U, \varepsilon_U) = D(p_D) + \varepsilon_c$$

as a function of p_U and the stochastic parameters. If we assume that all other players are using *linear* supply and demand schedules $S_{1U}(p_U, \varepsilon_U) = B_U(p_U - \varepsilon_U)$, $D_{2D}(p_D, p_U, \varepsilon_D) = B_{2D}(p_D - p_U - \varepsilon_D)$, using the downstream market clearing condition to solve for p_D we get the residual demand $U1$ faces as function of p_U and the ε s alone:

$$D_U^r = \frac{2B_D}{B_c + 2B_D} (A_c + \varepsilon_c - B_c p_U) - B_U(p_U - \varepsilon_U) \quad (3)$$

which clarifies how, even if each firms acts "locally" choosing its own input and output prices, actually the problem depends from the parameters of the whole economy.

The first lemma guarantees that under the imposed assumption, the pricing function spans all the possible prices in $\mathcal{P}$.

Theorem 2.

Remember that $\mathcal{S}_{i,-\ell}$ denotes the schedule played by firm i excluding labor demand (and similarly for $\mathcal{S}_{i,-\ell}^r$). Assume a schedule profile $\mathcal{S}_i \in \mathcal{A}$ is twice differentiable, the spectral norm $\|\cdot\|_2$ of the schedules is differentiable, and the boundary of $\tilde{\mathcal{P}}$ is differentiable. $\mathcal{S}_i$ is an interior best reply to the profile $\mathcal{S}_{-i}$ only if satisfies the following partial differential equation for all $(\mathbf{p}_i, \varepsilon_i) \in \mathcal{P} \times \mathcal{E}$:

$$\mathbb{E} \left[([\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}] + [\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}^r])^{-1} (-\mathcal{S}_{i,-\ell} + \partial_{\mathbf{p}_i, w} \mathcal{S}_i^r((\mathbf{p}_i, 1)' - \lambda \nabla \Phi_i)) \mid \mathbf{p}_i, \varepsilon_i \right] = 0 \quad (4)$$

and the technology constraint: $\Phi_i(\mathcal{S}_i, \varepsilon_i) = 0$.

The proof in is Appendix A.3. The first-order condition can be understood as follows. The term $-\mathcal{S}_{i,-\ell} + \partial_{\mathbf{p}_i} \mathcal{S}_i^r((\mathbf{p}_i, 1)' - \lambda \nabla \Phi_i)$ represents the

sensitivity of the profit to a variation in the prices. In this context the “marginal cost” of producing an additional unit of output is an ill-suited concept: indeed, the standard marginal cost is intimately connected with the assumption of taking input prices as given, being the multiplier in the standard cost minimization problem. In our setting, where firms have some market power on all input and output markets, the relevant generalization is the marginal value of relaxing the technology constraint, which is exactly the multiplier λ_i , times $\nabla\Phi_i$, that represents the marginal product of each input/output. Hence the vector $(\mathbf{p}'_i, 1)' - \lambda_i \nabla\Phi_i$ can be thought as the vector of markups (for outputs) and markdowns (for inputs). The reason why the schedule without labor demand $\mathcal{S}_{i,-\ell}$ appears in the expression is because we normalized the wage to 1: this is inconsequential, as we showed that price normalization does not affect the payoffs nor the schedules. Then, we can see that this term of the FOC has a very similar intuition to the standard Lerner equation: the higher the responsiveness of demand/supply to prices, the smaller the markups/markdowns that can be charged.

The term $([\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}] + [\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}^r])^{-1}$ represents the sensitivity of the prices to a variation in the schedules. Again, the schedules without labor demand appear because of the normalization of the wage. The key difference from [Klemperer and Meyer \(1989\)](#) is the presence of the expectation in the expression. The reason is somewhat different from [Holmberg and Philpott \(2018\)](#) and [Wilson \(2008\)](#), in which the equilibrium is not ex-post because of the possibility of binding transmission capacities in an otherwise linear transmission network. To understand why this is the case, consider [Figure 4](#). A seller faces a residual demand of the form $\varepsilon_D D_U^r + \varepsilon_c$, where ε_D and ε_c are two distinct sources of uncertainty. Computing first the optimal prices for given ε_D , and varying ε_c , we find the red curve (Left panel). This is what happens computing the best reply in a standard supply function competition. But now note that ε_D changes the slope of the residual demand, so is also affecting the optimal price, and in such a way that *the optimal price realizes a different demand quantity*. Hence if we represent on the same graph (Right panel) the optimal price quantity pairs varying ε_D , they do not lie on the red line, they form another curve. Hence, no single supply function can touch all the ex-post optimal points, but has to

trade-off between them, depending on the relative probability. This is the reason why the expectation appears in the necessary conditions. Moreover, in general the optimization is also not *pointwise*, in the sense described by [Rostek and Yoon \(2021a\)](#). Namely, the optimization in schedules is not equivalent to a pointwise optimization in quantities traded, taking the price impacts as given. It would be only in case the price sensitivity term $([\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}] + [\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}^r])^{-1}$ drops from equation 4, which happens only when is measurable with respect to $\mathbf{p}_i, \boldsymbol{\varepsilon}_i$, as discussed in the following section. In general such a sensitivity might depend on the realization of the residual uncertain parameters in a way that correlates with the slope of the residual demand, modifying the marginal impact of changing the schedule, and hence the optimal choice.

Remark 2.1 (Optimization is not pointwise unless residual schedule is linear). The FOC makes it clear that the equilibrium is not in general *ex-post*, since the uncertainty about the prices (and stochastic parameters) in other markets crucially enters the equation. It is useful to note that the optimization is also not *pointwise*, in the sense described by [Rostek and Yoon \(2021a\)](#), unless the residual schedule is linear in the stochastic parameters. By pointwise here is meant that the optimization in schedules is equivalent to a pointwise optimization in quantities traded, taking the price impacts as given. Such an optimization would yield as a FOC:

$$\mathbb{E} \left[\left(-\mathcal{S}'_{i,-\ell} + (\mathbf{p}'_i - \lambda \nabla \Phi'_i) \partial_{\mathbf{p},-1} \mathcal{S}_i^r - \frac{\mathbf{p}'_i \mathcal{S}_i}{P_i} \nabla_{\mathbf{p}_i,-1} P_i \right) \frac{1}{P_i} \mid \mathbf{p}_i, \boldsymbol{\varepsilon}_i \right] = 0$$

We can see that only if the residual schedule is linear the jacobian derivative $\partial_{\mathbf{p}} \mathcal{S}^r$ is deterministic, and so the term $([\partial_{\mathbf{p}} \mathcal{S}_i]_{-1} + [\partial_{\mathbf{p}} \mathcal{S}_i^r]_{-1})^{-1}$ drops from the equation 4. For example, if we perform a similar exercise as in [Rostek and Yoon \(2021a\)](#), where firms commit to “unconditional supply schedules”, when the residual schedule is linear we find a similar result: optimization is pointwise but not *ex-post*. When the residual schedule is not linear the term $([\partial_{\mathbf{p}} \mathcal{S}_i]_{-1} + [\partial_{\mathbf{p}} \mathcal{S}_i^r]_{-1})^{-1}$ is present and so the optimization is not pointwise.

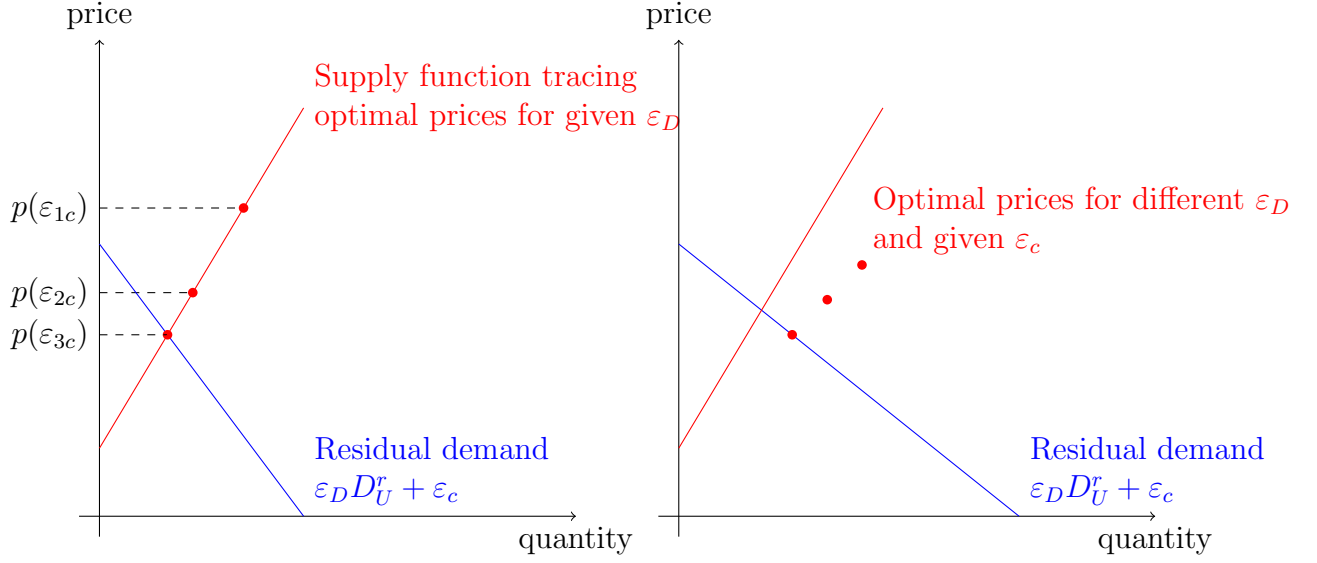


Figure 4: A supply function is not equivalent to ex-post price setting when uncertainty has enough dimensions.

We can rewrite the equation as:

$$\mathcal{S}'_i = \mathbb{E}_{\mathbf{p}_i} \left[\left((\mathbf{p}'_i - \lambda \nabla \Phi'_i) \partial_p \mathcal{S}_i^r - \frac{\mathbf{p}'_i \mathcal{S}_i}{P_i} \nabla_{\mathbf{p}_i} P_i \right) (\partial_p \mathcal{S}_i + \partial_p \mathcal{S}_i^r)^{-1} \mid \mathbf{p}_i, \boldsymbol{\varepsilon}_i \right] \times \\ \mathbb{E}_{\mathbf{p}_i} \left[\frac{1}{P_i} ((\partial_p \mathcal{S}_i + \partial_p \mathcal{S}_i^r)^{-1} \mid \mathbf{p}_i, \boldsymbol{\varepsilon}_i) \right]^{-1}$$

In case the owners consume an external good (Example 3), the equation is reduced to:

$$\mathcal{S}'_i = (\mathbf{p}'_i - \lambda \nabla \Phi'_i) \mathbb{E}_{\mathbf{p}_i} [\partial_p \mathcal{S}_i^r (\partial_p \mathcal{S}_i + \partial_p \mathcal{S}_i^r)^{-1} \mid \mathbf{p}_i, \boldsymbol{\varepsilon}_i] \mathbb{E}_{\mathbf{p}_i} [((\partial_p \mathcal{S}_i + \partial_p \mathcal{S}_i^r)^{-1} \mid \mathbf{p}_i, \boldsymbol{\varepsilon}_i)]^{-1}$$

that clarifies the role of uncertainty: the markups do not only depend on the responsiveness of the residual demand, but on the *covariance* of the responsiveness of the residual demand and the excess demand. If the two are independent, only the expectation of the slope of the residual demand matters. In general no: because when adjusting the schedule firms take into account both the variation in purchases from neighbors, and the variation in the expected prices: if the responsiveness of the residual demand is high whenever the responsiveness of the excess demand is, then markups are smaller: the intuition is that a high responsiveness of the excess de-

mand moves the prices more than under independence, hence making the responsiveness of the residual demand more stark.

The proof proceeds computing the Gateaux derivative along a direction, then imposing that all Gateaux derivatives are zero: since this is true for any direction η_i , this allows to conclude that the expression in the Theorem is zero. Details are in the Appendix.

2.4 Unique best reply

In case the degrees of freedom of the firms are exactly the same as the uncertain parameters they face we can prove that best replies are single-valued. In this case the equilibrium is *ex-post*, and the partial differential equation 4 boils down to an implicit equation. The key assumption needed for this is the following:

Assumption 4-Measurability for each firm i , there exist a function f_i such that the residual demand is measurable with respect to $(\mathbf{p}_i, \boldsymbol{\varepsilon}_i)$, that is, it satisfies $\partial_{\mathbf{p}} \mathcal{S}_i^r(\mathbf{p}_i, \boldsymbol{\varepsilon}) = f_i(\mathbf{p}_i, \boldsymbol{\varepsilon}_i)$.

The immediate consequence of this assumption is that the residual schedule is completely known once we know $\mathbf{p}_i$ and $\boldsymbol{\varepsilon}_i$, hence there is no residual uncertainty and hence the expectation in 4 is trivial. So, for an interior solution for which the positive definite constraints are not binding, the FOC reduces to:

$$(-\mathcal{S}_{i,-\ell} + \partial_{\mathbf{p}_i} \mathcal{S}_i^r(\mathbf{p}_i' - \lambda \nabla \Phi_i(-\mathcal{S}_i^r, \boldsymbol{\varepsilon}_i))) ([\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}] + [\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}^r])^{-1} = 0$$

where now the term $\frac{1}{P_i}([\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}] + [\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}^r])^{-1}$ simplifies away, and we are left with:

$$\mathcal{S}_{i,-\ell} = \partial_{\mathbf{p}_i} \mathcal{S}_i^r((\mathbf{p}_i, 1) - \lambda_i \nabla \Phi_i(-\mathcal{S}_i^r, \boldsymbol{\varepsilon}_i)) \quad (5)$$

This is an equation that directly defines the best reply schedule $\mathcal{S}_{i,-\ell}$ as a function of prices and schedules played by competitors. Hence it is immediate to conclude that in this context the best reply is unique. Moreover we recover both the pointwise optimization and the ex-post equilibrium as in [Klemperer and Meyer \(1989\)](#). We can summarize the above discussion as follows.

Corollary 2.1. Under Assumptions 1, 2, 3 and 4, if the constraints are not binding ($\mathcal{I}_i = \mathcal{J}_i = 0$), the best reply is single valued in the interior of $\mathcal{A}(\tilde{\mathcal{P}})$.

The measurability assumption is not vacuous. An example that satisfies it for any network is when the profile of schedules played is linear, case to which is devoted the companion paper [Bizzarri \(2025\)](#). In this case the function f_i is actually a *constant*, independent of ε and $\mathbf{p}_i$. The following is another example, where it is not the functional form, but the structure of the network that determines the measurability.

Example 5 (Regular layered supply chain). In the context of a regular layered supply chain each firm has 1 degree of freedom, because it has to decide a schedule for inputs and outputs, constrained by the technology. Hence, it is sufficient one stochastic parameter to generate enough variation in the realized prices to span the whole feasible set. Assume that the only stochastic parameter is the one of consumer demand ε_c , while the transformation functions of firms, and the schedules, are all deterministic.¹⁶ In this case the measurability assumption is satisfied, because, under the assumptions above, realizations of the stochastic parameter ε_c are one to one with price variation, for any firm. Details are in Online Appendix [B.6](#).

Conclusion

This paper provides a way to model oligopoly in general equilibrium as a game in which firms fully internalize their position in the supply chain and have market power both over inputs and outputs, in an endogenously determined way. I show that such features are desirable in a input-output model with market power: if absent, both the aggregate and the relative ranking of distortions due to imperfect competitions is crucially affected. This suggests that, when modeling complex networks of large firms with market power, simplifying assumptions might affect in a sizable way the results. The parametric functional form introduced is suitable for quantitative work, and the strategic complementarity structure of the equilibrium

¹⁶Or, equivalently, the distribution of ε_i is a Dirac for all i .

makes it computationally tractable: the exploration of the quantitative implications of the supply and demand function equilibrium for the analysis of market power is an interesting avenue for future research.

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Appendix

A Proofs

As anticipated in the text, the proofs of this section are done under a more general assumption for the payoffs, consistent with the literature on general equilibrium oligopoly (in particular [Azar and Vives \(2021\)](#)), namely, that firms optimize the indirect utility of their owners. The details are as follows.

Workers and Owners As in [Azar and Vives \(2021\)](#), there are two types of agent: workers, and owners. There is a continuum of identical workers or, equivalently, there is a representative worker, whose utility is $U((c_g)_{g \in \mathcal{C}}, L, \varepsilon_{i,c})$. The workers have aggregate demand D_c^w that has negative semidefinite jacobian with maximum rank (which is $|\mathcal{C}| - 1$) with respect to both prices $\mathbf{p}_c$ and stochastic parameters ε_c .

The owners, instead, do not work, but own the firms. They are a continuum, partitioned in N groups, and owners in group i collectively own all the shares of firm i . They have utility functions homogeneous of degree 1, generating aggregate indirect utilities $V_i = \frac{\Pi_i}{P_i}$, where Π_i is the profit of firm i , P_i is a function of prices, homogeneous of degree 1 (the *price index* relative to owners of group i) and differentiable. These assumptions are enough to generate an aggregate demand that is differentiable and has negative definite jacobian as in the main text.

As anticipated, firms optimize the indirect utility of shareholders. Hence the payoff of firm i is:

$$\pi_i(\mathcal{S}) = \mathbb{E} \frac{\Pi_i}{P_i} = \mathbb{E} \left(\sum_{i \rightarrow g} \frac{p_g}{P_i} S_{g,i}(\mathbf{p}_i, \varepsilon_i) - \sum_{g \rightarrow i} \frac{p_g}{P_i} D_{ig}(\mathbf{p}_i, \varepsilon_i) - \frac{w}{P_i} \ell_i(\mathbf{p}_i, \varepsilon_i) \right)$$

Note that this depends only on ratios p_g/P_i , hence not on price normalization. The assumption followed in the main body, of firms maximizing profits Π_i , can be recovered as a special case of this setting assuming that the owners' utility only depends on only one good, o , and, moreover, such a good is produced from a continuum of firms that use only labor as input (hence are isolated from the network). Hence in this case the price indices are all $P_i = w$, and we recover the main text formulation.

A.1 Proof of Proposition 1

We are going to need the following Lemmas, proved in the online Appendix.

Lemma A.1. Under Assumptions 3c, d) the map MC has positive definite jacobian derivative $\partial_{\mathbf{p}}MC$. Moreover, there are $\underline{k}$ and $\overline{K}$ such that $\|\partial_{\mathbf{p}}MC\|_2 \leq \overline{K}$ and $\|\partial_{\mathbf{p}}MC^{-1}\|_2 \leq \underline{k}^{-1}$.

Lemma A.2. There is a constant K_p such that the derivatives of the pricing function with respect to the stochastic parameters, and the (Fréchet) derivatives with respect to the schedules are bounded above: $\|\partial_{\boldsymbol{\varepsilon}}\mathbf{p}\| \leq K_p$ and $\|\partial_{\mathcal{S}_i}\mathbf{p}\|_2^{op} \leq K_p$. Here $\|\cdot\|^{op}$ denotes the operator norm: relative to the $\|\cdot\|_{\infty}$ norm in the domain: if A is a linear operator $\mathcal{A}_i \rightarrow \mathbb{R}^M$, $\|A\|^{op} = \max\{\|A\mathcal{S}_i\|_2 \mid \|\mathcal{S}_i\|_{\infty} = 1\}$.

Part 1 First, focus on schedules in $\mathcal{A}$. We want to show that for every $\boldsymbol{\varepsilon}$ there is a price vector satisfying the market clearing conditions. For every given $\boldsymbol{\varepsilon}$ the map $z(\mathbf{p}) = -MC(\mathbf{p}, \boldsymbol{\varepsilon})$ satisfies all the properties of the excess demand in Proposition 17.C.1 in Mas-Colell et al. (1995), except property (v) that has to be replaced with:

$$(v') \quad \text{if } \lim_{(\mathbf{p}^n, w^n) \rightarrow (\mathbf{p}^*, w^*)} \text{ with } p_g^* = 0 \text{ or } w^* = 0 \\ \text{then } \lim_{(\mathbf{p}^n, w^n) \rightarrow (\mathbf{p}^*, w^*)} z(\mathbf{p}^n, w^n) \leq \max_{g \in \mathcal{N}_i} \{z_{ig}\}$$

this follows from the assumption 1 that the demand diverges when some good prices go to zero, and the assumption 3d) that labor demand grows more than other demands when the wage tends to zero.

Hence, an equilibrium price vector exists with $\mathbf{p}, w \gg 0$.

Moreover, the Lemma A.1 guarantees that the jacobian of z is negative definite, so that we can use a global inversion theorem (Theorem 3.1.8 in Ambrosetti and Prodi (1995)) to conclude that the equilibrium is unique. Now, using the implicit function theorem applied to the map $(\mathbf{p}, \boldsymbol{\varepsilon}) \mapsto (MC(\mathbf{p}, \boldsymbol{\varepsilon}), \boldsymbol{\varepsilon})$, we can conclude that $\mathbf{p}(\boldsymbol{\varepsilon})$ is differentiable on $\mathring{\mathcal{E}}$, the interior of $\mathcal{E}$. The Lemma A.2 guarantees that it is Lipschitz, so it can be extended uniquely to the whole of $\mathcal{E}$.

Now let us consider schedules in the closure of $\mathcal{A}$. Lemma A.2 guarantees that the map $\mathbf{p} : \mathcal{E} \times \mathcal{A} \rightarrow \mathbb{R}^M$ is Lipschitz, hence it can be extended

in a unique way to the closure of the domain.

So far, we produced a unique function $\mathbf{p}(w, \boldsymbol{\varepsilon})$ for each fixed w . Now consider two functions such that $MC(\mathbf{p}(w, \boldsymbol{\varepsilon}), w, \boldsymbol{\varepsilon}) = 0$ and $MC(\mathbf{p}'(w', \boldsymbol{\varepsilon}), w', \boldsymbol{\varepsilon}) = 0$. Since MC is homogeneous of degree zero:

$$MC(\mathbf{p}'(w', \boldsymbol{\varepsilon}), w', \boldsymbol{\varepsilon}) = MC(\mathbf{p}'(w', \boldsymbol{\varepsilon})w/w', w, \boldsymbol{\varepsilon}) = 0,$$

and so $\mathbf{p}'(w', \boldsymbol{\varepsilon})w/w' = \mathbf{p}(w, \boldsymbol{\varepsilon})$, that is, the functions are the same up to a positive normalization. Since the payoffs only depend on price ratios, they are independent of the normalization chosen.

Part 2 Fix a schedule $\bar{\mathcal{S}}_i \in \mathcal{A}_i$ and a value $\boldsymbol{\varepsilon}$, and call $\mathcal{S}_t = t\mathcal{S} + (1-t)\bar{\mathcal{S}}$ and $\boldsymbol{\varepsilon}_t = t\boldsymbol{\varepsilon}' + (1-t)\boldsymbol{\varepsilon}$. By the mean value theorem in Banach spaces (e.g., Proposition 7.2 in [Luenberger \(1997\)](#)):

$$\begin{aligned} \|\mathbf{p}(\mathcal{S}, \boldsymbol{\varepsilon}') - \mathbf{p}(\bar{\mathcal{S}}, \boldsymbol{\varepsilon})\|_2 &\leq \\ \sup_{t \in [0,1]} \|\partial_{\mathcal{S}} \mathbf{p}(\boldsymbol{\varepsilon}_t, \mathcal{S}_t)\|^{op} \|\mathcal{S} - \bar{\mathcal{S}}\|_{\infty} &+ \sup_{t \in [0,1]} \|\partial_{\boldsymbol{\varepsilon}} \mathbf{p}(\boldsymbol{\varepsilon}_t, \mathcal{S}_t)\|_2 \|\boldsymbol{\varepsilon}' - \boldsymbol{\varepsilon}\|_2 \end{aligned}$$

Now by the Lemma [A.2](#) and Assumption 3d, such a norm is bounded above by $k_p = 2K_p K_S + 2K_p K_e$, and in particular the image of $\mathcal{E} \times \mathcal{A}$ via $\mathbf{p}$ is bounded by k_p . $\square$

A.2 Proof of Theorem 1

Thanks to Proposition [1](#) the set $\bar{\mathcal{P}}$ is bounded; since it is closed by definition, it is compact, hence all schedules and their derivatives have upper bounds on it.

The set of all differentiable schedules that are bounded, with bounded derivatives, and compact domain $\mathcal{P} \times \mathcal{E}$ is equicontinuous (Theorem 14.2 in [Treves \(2016\)](#)), hence, by the Ascoli-Arzelà theorem, its closure is compact in the sup-norm. The set $\overline{\mathcal{A}(\mathcal{P})}^K$ is a subset of such a compact set. Moreover, it is closed by definition, being the closure of $\mathcal{A}(\mathcal{P})^K$. Hence it is a closed subspace of a compact set, and so is compact. Since the pricing function is Lipschitz, it can be extended uniquely to such closure: hence the game is well defined also on $\overline{\mathcal{A}(\mathcal{P})}^K$.

Since the profit function is continuous, the best reply problem has a solution. Moreover, by the maximum theorem the solution correspondence is upper-hemicontinuous (in particular, if single valued, is a continuous function).

It remains to prove that $\mathcal{A}(\mathcal{P})^K$ is convex. Consider $\mathcal{S}_i$ and $\mathcal{S}'_i$ in $\mathcal{A}_i(\mathcal{P})$. All the regularity assumptions are inherited by any convex combination, and it has the same domain by definition. The bounds are also inherited:

$$kI_i \leq \alpha \partial_{\mathbf{p}_i} \mathcal{S}_i + (1 - \alpha) \partial_{\mathbf{p}_i} \mathcal{S}'_i \text{ and } \|\alpha \partial_{\mathbf{p}_i, w} \mathcal{S}_i + (1 - \alpha) \partial_{\mathbf{p}_i, w} \mathcal{S}'_i\|_2 \leq K$$

and similarly for $\partial_{\boldsymbol{\varepsilon}_i} \mathcal{S}_i$. By convexity of Φ , the technology constraint is also satisfied:

$$\Phi_i(\alpha \mathcal{S}_i + (1 - \alpha) \mathcal{S}'_i, \boldsymbol{\varepsilon}_i) \leq \alpha \Phi_i(\mathcal{S}_i, \boldsymbol{\varepsilon}_i) + (1 - \alpha) \Phi_i(\mathcal{S}'_i, \boldsymbol{\varepsilon}_i) \leq 0$$

which is what we wanted to show.

So, if best replies are convex-valued (or in particular single valued), the best reply map is continuous on a set $\overline{A(\mathcal{P})}^K$ that is compact and convex, hence by the Ky Fan fixed point theorem the game has an equilibrium.

Denote the equilibrium profile as $\mathcal{S}^*$. Now, it is possible to further restrict the domain of each schedule to $\tilde{\mathcal{P}}_i = \mathbf{p}_i(\mathcal{E}, \mathcal{S}^*)$, that is the image of $\mathcal{E}$ via the equilibrium profile. This in general might be smaller than $\mathcal{P}$. Nevertheless, the profile $\mathcal{S}^*$ remains an equilibrium. Indeed, all the price values in $\mathcal{P} \setminus \mathbf{p}_i(\mathcal{E}, \mathcal{S}^*)$ have probability zero, so they do not affect the payoffs. Hence we can restrict each schedule to $\tilde{\mathcal{P}}_i = \mathbf{p}_i(\mathcal{E}, \mathcal{S}^*)$, to have an equilibrium in which the whole domain is spanned. Finally, the following Lemma (proven in the Online Appendix) uses the positive definiteness of $\partial_{\boldsymbol{\varepsilon}_i} \mathcal{S}_i$ to guarantee that each $\tilde{\mathcal{P}}_i$ is the closure of an open set. $\square$

Lemma A.3. Under Assumptions 1-3, $\mathcal{P}_i$ is the closure of an open set.

A.3 Proof of Theorem 2

We derive necessary conditions for an interior solution in $\mathcal{A}$. The necessary conditions for optimization are the usual Lagrange multiplier equations

(Luenberger (1997)). The Lagrangian is:

$$\mathcal{L}_i(\mathcal{S}_i) = \mathbb{E} [\mathbf{p}'_{i,-\ell} \mathcal{S}_{i,-\ell} + \mathcal{S}_1 - \lambda_i \Phi_i(\mathcal{S}_i, \varepsilon_i)]$$

where $\mathbf{p}_i(\mathcal{S}, \varepsilon)$ is the unique pricing function (from Proposition 1) such that the wage is 1.

We have to show that this is Fréchet differentiable, and the necessary condition is setting the Fréchet differential to 0. To do so, in the following Lemma (proven in the Online Appendix) we compute the Gateaux differential in the direction η_i . Under the assumption we made on η_i , it is always possible to choose h small enough such that $\mathcal{S}_i + h\eta_i \in \mathcal{A}$.

Lemma A.4. Assume that $\partial\tilde{\mathcal{P}}$ has differentiable boundary. The Gateaux differential of the Lagrangian in a direction η_i , satisfying the above assumptions, is:¹⁷ $\mathbb{E} [\eta'_{i,-\ell} G_i]$, where:

$$G_i = ([\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}] + [\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}^r])^{-1} \left(\mathcal{S}_i^r + \partial_{\mathbf{p}_{i,-\ell}} \mathcal{S}_{i,-\ell}^r((\mathbf{p}_i, 1) - \lambda_i \nabla \Phi_i) - \frac{\mathbf{p}'_i \mathcal{S}_i}{P_i} \nabla_{\mathbf{p}_{i,-1}} P_i \right)$$

The assumption of differentiable boundary is necessary to apply the divergence theorem, and integrate by parts the derivative of the constraints, eliminating the derivatives of η_i from the expression.

Now by the law of iterated expectations we can rewrite the expectation as $\mathbb{E}[\eta'_{i,-\ell}(\mathbf{p}_i, \varepsilon_i) \mathbb{E}[G_i \mid \mathbf{p}_i, \varepsilon_i]]$, and by the arbitrariness of η_i the FOC is equivalent to $\mathbb{E}[G_i \mid \mathbf{p}_i, \varepsilon_i] = 0$. Using Lemma A.3 to conclude that $\mathcal{S}_i^r = -\mathcal{S}_i$ for all the possible prices, we obtain the expression in the main text, noting that in that case $\nabla_{\mathbf{p}_{i,-\ell}} P_i = 0$. $\square$

¹⁷Note that the component of η relative to labor does not directly enter the equation, but this is not strange because it is implicitly determined by the technology constraint.

B Additional proofs of section 2

B.1 Proof of Lemma A.1

Part I: Positive definite

By the *lifting* procedure as in Malamud and Rostek (2017), we can consider every supply function as defined on the set of all prices instead then the prices of the neighboring goods, and similarly having values in tuples of all the goods: $\hat{\mathcal{S}}_i : \mathbb{R}_+^M \rightarrow \mathbb{R}^M$. The consistency required is, of course,

$$\hat{\mathcal{S}}_i(\mathbf{p}_i, \mathbf{p}_{-i}, \varepsilon) = \mathcal{S}_i(\mathbf{p}_i, \varepsilon) \forall g \in \mathcal{N}_i \quad \hat{\mathcal{S}}_{gi}(\mathbf{p}_i, \mathbf{p}_{-i}, \varepsilon) = 0 \forall g \notin \mathcal{N}_i$$

With this notation, we can write the excess supply function as:

$$MC_g = \sum_i \hat{\mathcal{S}}_{gi} - \hat{D}_c + \hat{\ell}_c$$

Denote $\mathcal{S}_{i,-\ell}$ the schedule of i excluding (if present) labor demand. Moreover, MC is homogeneous of degree zero, hence naturally we cannot invert it as a full function of prices. For convenience we consider it a function of $\mathbf{p}_{-w}$, the vector of prices excluding the wage.

The Jacobian derivative is:

$$\partial_{\mathbf{p}} MC = \sum_i \partial_{\mathbf{p}} \hat{\mathcal{S}}_{i,-\ell} - \partial_{\mathbf{p}} \hat{D}_c$$

This is symmetric if all the derivatives are symmetric. We are going to prove that, once we normalize by a price, this is also positive definite. By Theorem 6 in Gale and Nikaido (1965), this implies that the realized prices are well defined on any convex domain.

Considering any vector $\mathbf{x} \in \mathbb{R}^M \setminus \{0\}$, we have

$$\mathbf{x}' \partial_{\mathbf{p}} MC \mathbf{x} = \sum_i \mathbf{x}' (\partial_{\mathbf{p}_i} \hat{\mathcal{S}}_{i,-\ell} - \partial_{\mathbf{p}} \hat{D}) \mathbf{x} = \sum_i \mathbf{x}'_i \partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell} \mathbf{x}_i + \mathbf{x}'_c (-\partial_{\mathbf{p}} D_c) \mathbf{x}_c$$

where, as for the prices, we denote $\mathbf{x}_i = (x_g)_{g \in \mathcal{N}_i}$. Now if there is a λ_i such that $\mathbf{p}_i = \lambda_i \nabla \Phi_i$ for each i , then $\mathcal{S}_{i,-\ell}$ is positive definite, because the original schedules have co-rank 1. In this case, it follows that $\partial_{\mathbf{p}} MC$ is

positive definite. If not, $\mathcal{S}_{i,-\ell}$ has co-rank 1, and satisfies:

$$\sum_g u_{ig}[\mathcal{S}_{i,-\ell}]_{hg} = 0 \forall h \quad u_{ig} = \left(\frac{1}{p_h} p_g - \frac{1}{\partial_{p_h} \Phi_i} \nabla_p \Phi_i \right) \text{ for some } h \in \mathcal{N}_i$$

So, if there is a nonzero vector $\mathbf{x}$ such that $\mathbf{x}' \partial_{\mathbf{p}_{-w}} MC \mathbf{x} = 0$, it must be $\mathbf{x}_i = \mathbf{u}_i$ for some i , and $\mathbf{x}_i = 0$ otherwise. Where, since $\mathbf{u}_i$ is nonzero, and the sum is null, at least two entries of the vector $\mathbf{u}_i$ are nonzero, corresponding to, say, good g and h . Then $x_g = u_{ig} \neq 0$, and x_g is also an element of $\mathbf{x}_j$, so also $\mathbf{x}_j = \mathbf{u}_j$. Repeating the reasoning, we can go on until we reach a firm k such that the good g such that $u_{kg} \neq 0$, and g is a good consumed by the consumer: in that case x_g cannot be zero, and we reach a contradiction. Hence the quantity $\mathbf{x}' \partial_{\mathbf{p}} MC \mathbf{x}$ is positive, and the jacobian $\partial_{\mathbf{p}} MC$ is positive definite.

Part II: bounds

For the lower bound, by Assumption 3d) we have:

$$k \sum_i \hat{I}_i \leq \partial_{\mathbf{p}} MC = \sum_i \partial_{\mathbf{p}_i} \hat{\mathcal{S}}_{i,-\ell} - \partial_{\mathbf{p}} \hat{D}_c$$

where $\hat{I}_i$ is the lifting of the identity matrix relative to i , having a 1 on the diagonal whenever g, h are both traded by firm i , and zero otherwise. The sum of such matrices is still diagonal. In particular, the entry in position g, h is $n_g k$, where n_g is the sum of firms that trade good g , plus (eventually) the consumer. Anyhow this is larger than $2k$, so the matrix is bounded below, and so it can be found a $\underline{k}$ such that $\underline{k}I \leq K \sum_i I_i$. Now by definition this is the same as $\underline{k}I \leq H(\partial_{\mathbf{p}} MC)$, where $H(A) = (A + A')/2$ denotes the symmetric part of a matrix. For a property of the positive semidefinite ordering, it follows that $\underline{k}^{-1}I \geq H(\partial_{\mathbf{p}} MC)^{-1}$, that implies $\underline{k}^{-1} \geq \|H(\partial_{\mathbf{p}_i} MC)^{-1}\|_2$. By Lemma 2.1 in Mathias (1992) it follows that $\|\partial_{\mathbf{p}_i} MC^{-1}\|_2 \leq \|H(\partial_{\mathbf{p}} MC)^{-1}\|_2 \leq \underline{k}^{-1}$.

Concerning the upper bound, it is sufficient to apply subadditivity of the norm and again Assumption 3d)

$$\left\| \sum_i \partial_{\mathbf{p}_i} \hat{\mathcal{S}}_{i,-\ell} - \partial_{\mathbf{p}} \hat{D}_c \right\|_2 \leq (N+1)K = \overline{K}$$

□

B.2 Proof of Lemma A.2

We have to prove that the Fréchet derivative of $\mathbf{p}$ with respect to the schedules is bounded. By the implicit function theorem is:

$$\partial_S \mathbf{p} = -(\partial_{\mathbf{p}} MC)^{-1} \partial_S MC$$

We have to compute the Gateaux derivatives in all the directions η that satisfy the constraints:

$$\partial_S MC(\eta) = \partial_h MC(\mathcal{S} + h\eta, \mathbf{p}, \boldsymbol{\varepsilon}) = MC(\eta, \mathbf{p}, \boldsymbol{\varepsilon})$$

and $MC(\eta, \mathbf{p}, \boldsymbol{\varepsilon}) = \sum_i \hat{\eta}_i(\mathbf{p}, \boldsymbol{\varepsilon})$ where $\hat{\eta}_i$ is the lifting of η , as in the proof of Proposition 1.

Hence $\|(\partial_{\mathbf{p}} MC)^{-1} \partial_S MC\|_2^{op} = \max_{\|\eta\|_\infty=1} \|(\partial_{\mathbf{p}} MC)^{-1} MC(\eta, \mathbf{p}, \boldsymbol{\varepsilon})\|_2 \leq \sum_i \|\eta_i\|_2 = NK_S$. Moreover, from Lemma A.1 follows $\|(\partial_{\mathbf{p}} MC)^{-1}\|_2 \leq \underline{k}^{-1}$. Hence, for any η $\|\partial_S MC(\eta)\|_2 \leq \underline{k}^{-1}N$. By definition of operator norm, the operator norm of $\|\partial_S \mathbf{p}\|$ is bounded above by the same constant.

Similarly,

$$\|\partial_{\boldsymbol{\varepsilon}} \mathbf{p}\|_2 = \|-(\partial_{\mathbf{p}} MC)^{-1} \partial_{\boldsymbol{\varepsilon}} MC\|_2 \leq \underline{k}^{-1}NK$$

and now define $K_p = \max\{\underline{k}^{-1}NK, \underline{k}^{-1}N\}$. □

B.3 Proof of Lemma 2.1

Consider the excess supply function, neglecting all g that are produced or used by firm i . We obtain a function:

$$MC_g^i : (\mathbf{p}, \boldsymbol{\varepsilon}) \mapsto MC_g(\mathbf{p}, \boldsymbol{\varepsilon}) \quad \forall g \notin \mathcal{N}_i$$

With a reasoning totally analogous, this is a function that can be inverted, expressing p_{-i} as a function of $\mathbf{p}_i$ (including labor). Moreover, this function is homogeneous of degree 1 in prices.

Now, for $g \in \mathcal{N}$, the residual schedule is simply:

$$\mathcal{S}_g^r(\mathbf{p}_i, \boldsymbol{\varepsilon}) := MC_g(\mathbf{p}_{-i}(\mathbf{p}_i), \mathbf{p}_i, \boldsymbol{\varepsilon}) - \mathcal{S}_g(\mathbf{p}_i, \boldsymbol{\varepsilon}_i)$$

Homogeneity follows immediately. Hence, we normalize the wage to 1.

Define $\hat{MC}^i$ the function such that $\hat{MC}^i = MC_g(\mathbf{p}_{-i}(\mathbf{p}_i), \mathbf{p}_i, \boldsymbol{\varepsilon}) - \mathcal{S}_g(\mathbf{p}_i, \boldsymbol{\varepsilon}_i)$. Notice that by definition of the excess supply function this is actually independent of $\mathcal{S}$. Now, we can compute the derivative of the partially solved prices:

$$\partial_{\mathbf{p}_i} \mathbf{p}_{-i} = -(\partial \hat{MC}_{-i \mathbf{p}_{-i}})^{-1} \partial \hat{MC}_{\mathbf{p}_i}$$

and so define:

$$\partial_{\mathbf{p}_i} \mathcal{S}^r = \partial_{\mathbf{p}_i} \mathcal{S}^r = \partial \hat{MC}_{\mathbf{p}_i} - \partial \hat{MC}_{i \mathbf{p}_{-i}} (\partial \hat{MC}_{\mathbf{p}_{-i}})^{-1} \partial \hat{MC}_{\mathbf{p}_i}$$

that is the Schur complement of $\partial \hat{MC}_{-i \mathbf{p}_{-i}}$ in the jacobian $\partial \hat{MC}$, appropriately reordered to have all $g \in \mathcal{N}$ in the upper left corner, and all others in the rest:

$$\begin{aligned} \partial \hat{MC} &= \begin{pmatrix} \partial \hat{MC}_{i \mathbf{p}_i} & \partial \hat{MC}_{i \mathbf{p}_{-i}} \\ \partial \hat{MC}_{-i \mathbf{p}_i} & \partial \hat{MC}_{-i \mathbf{p}_{-i}} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} (\partial \hat{MC}_{i \mathbf{p}_i} - \partial \hat{MC}_{i \mathbf{p}_{-i}} (\partial \hat{MC}_{-i \mathbf{p}_{-i}})^{-1} \partial \hat{MC}_{\mathbf{p}_i})^{-1} & B \\ C & D \end{pmatrix} \end{aligned}$$

Hence we conclude that if all schedules have positive definite derivatives then $\partial_{\mathbf{p}_i} \mathcal{S}^r$ is positive definite because principal submatrices of positive definite matrices are still positive definite. $\square$

B.4 Proof of Lemma A.3

Fix $\mathcal{S}$. Since the stochastic parameters are $\sum_i d_i \geq M$, the map $\mathbf{p}$ is not invertible. We can consider a restriction such that it is. Namely, impose that the uncertain parameters relative to the same good are the same across firms: $\varepsilon_{gi} = \varepsilon_{gj}$ for all i, j and g . Let us denote the stochastic parameters remained independent as $\tilde{\boldsymbol{\varepsilon}}$, and their domain as $\tilde{\mathcal{E}} \subset \mathbb{R}^M$. This is a compact set, because it is a closed subset of a compact set. This way, the uncertain parameters behave formally exactly like prices, and with

analogous reasoning as in Proposition 1 we obtain that $\partial_{\tilde{\varepsilon}}MC$ is positive definite. Moreover, repeating the reasoning in the proof of Proposition 2.1, we obtain that if we consider constants the parameters relative to one firm i , this is equivalent to calculate the matrix $\partial_{\varepsilon/i}MC = \partial_{\tilde{\varepsilon}}MC - \partial_{\tilde{\varepsilon}}\hat{\mathcal{S}}_i$, and this is still positive definite, exactly as $\partial_{\mathbf{p}}MC - \partial_{\mathbf{p}}\hat{\mathcal{S}}_i$ is still positive definite. In particular, it is invertible.

Hence, in the interior of $\tilde{\mathcal{E}}$:

$$\partial_{\varepsilon/i}\mathbf{p} = -(\partial_{\mathbf{p}}MC)^{-1}\partial_{\varepsilon/i}MC$$

is invertible, and so the map $\mathbf{p} : \tilde{\mathcal{E}} \rightarrow \mathbb{R}^M$ is locally invertible: for any $\tilde{\varepsilon} \in \tilde{\mathcal{E}}$ there is an open $U_{\tilde{\varepsilon}}$ such that $\mathbf{p}|_{U_{\tilde{\varepsilon}}}$ is invertible. In particular, $\mathbf{p}(U_{\tilde{\varepsilon}})$ is open, and so $\mathbf{p}(\tilde{\mathcal{E}}) = \cup_{\tilde{\varepsilon}} U_{\tilde{\varepsilon}}$ is open too; hence $\mathbf{p}(\tilde{\mathcal{E}})$ is the closure of an open set. $\square$

B.5 Proof of Lemma A.4

Consider the perturbation in the direction of η : $\mathcal{S}_i + h\eta_i$. Write $\mathbf{p}_i(h)$ for $\mathbf{p}_i(\mathcal{S}_i + h\eta_i, \varepsilon_i)$. Define the functions:

$$N(h) = -\mathbb{E} \left[\frac{\mathbf{p}'_i(h)}{P_i(\mathbf{p}_i(h))} \mathcal{S}_i^r(\mathbf{p}_i(h), \varepsilon_i) \right] \quad (6)$$

$$M(h) = \Phi_i(\mathcal{S}_i^r(\mathbf{p}_i(h), \varepsilon_i)) \quad (7)$$

The Gateaux derivatives in direction η_i are $N'(0)$, $M'(0)$. Note that we can exchange derivatives and integrals since all the functions involved have bounded derivative (and the price space is supposed compact), hence dominated (because a probability space has finite measure) (see Billingsley (2008), Theorem 16.8). We have first to compute the derivative of $\mathbf{p}_i(\mathcal{S}_i + h\eta_i, \varepsilon)$ with respect to h , that by the chain rule is:

$$\partial_h \mathbf{p}_i(\mathcal{S}_{i,-\ell} + h\eta_{i,-\ell}, \varepsilon) = \partial_{\mathcal{S}_{i,-\ell}} \mathbf{p}_i \eta_{i,-\ell}$$

where $\partial_{\mathcal{S}_{i,-\ell}} \mathbf{p}_i$ is the Gateaux derivative of the prices as functions of the

schedules chosen, that can be computed via the implicit function theorem:¹⁸

$$[\partial_{\mathcal{S}_{i,-\ell}} \mathbf{p}_i(\eta_i)] = -([\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}] + [\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}^r])^{-1} \eta_{i,-\ell}$$

where I use the fact that the both the submatrices are positive semidefinite, and the residual demand is positive definite. Remember that $\mathcal{S}_{i,-\ell}$ denotes the components of the schedule $\mathcal{S}_i$ excluding the labor demand, and similarly for $\mathcal{S}_{i,-\ell}^r$ and $\eta_{i,-\ell}$. Hence, now:

$$\begin{aligned} N'(h) &= -\frac{\partial}{\partial h} \mathbb{E} \left[\frac{\mathbf{p}'_i(h)}{P_i(\mathbf{p}_i(h))} \mathcal{S}_i^r(\mathbf{p}_i(h), \varepsilon_i) \right] \\ &= -\mathbb{E} \left[\frac{\partial_h \mathbf{p}_i(h)' \mathcal{S}_i^r + \mathbf{p}'_i \partial_p \mathcal{S}_i^r \partial_h \mathbf{p}_i(h)}{P_i} - \frac{\mathbf{p}'_i(h) (\mathcal{S}_i^r(\mathbf{p}_i(h), \varepsilon_i))}{P_i^2} \nabla_{\mathbf{p}_i} P_i' \partial_h \mathbf{p}_i(h) \right] \\ &= \mathbb{E} \left[\left((\mathcal{S}_{i,-\ell}^r)' + \mathbf{p}'_i \partial_p \mathcal{S}_{i,-\ell}^r + \frac{(\mathcal{S}_i(\mathbf{p}_i(h), \varepsilon_i))}{P_i} \nabla_{\mathbf{p}_i} P_{i,-1} \right) \times \right. \\ &\quad \left. \frac{1}{P_i} ([\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}] + [\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}^r])^{-1} \eta_{i,-\ell} \right] \end{aligned}$$

for any direction η_i . Moreover:

$$\begin{aligned} M'(h) &= \Phi_i(\mathcal{S}_i^r(\mathbf{p}_i(h), \varepsilon_i)) \\ &= \nabla \Phi_i \partial_p \mathcal{S}_i^r \partial_h \mathbf{p}_i(h) \\ &= -\nabla \Phi_i \partial_p \mathcal{S}_{i,-\ell}^r ([\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}] + [\partial_{\mathbf{p}_i} \mathcal{S}_{i,-\ell}^r])^{-1} \eta_{i,-\ell} \end{aligned}$$

for any direction η_i . □

B.6 Details of Example 2

Write $D_i = \sum_{j \in i} D_{ij}$ for the aggregate demand function from firms in layer i , and similarly $S_i = \sum_{j \in i} S_{ij}$ for the supply. Hence $MC = (S_1 - D_2, \dots, S_n - D_c)$. Now consider the matrix:

$$diag(\mathbf{p})^{-1} \partial_{\mathbf{p}} MC diag(\mathbf{p}) =$$

¹⁸Alternatively, we can compute directly $\partial_h \mathbf{p}_i$ using the implicit function theorem, the procedures are identical.

$$\begin{pmatrix} \partial_{p_1} S_1 - \partial_{p_1} D_2, & -\partial_{p_2} D_2 \frac{p_2}{p_1}, & 0 & \dots & 0 \\ -\partial_{p_1} S_2 \frac{p_1}{p_2} & \partial_{p_2} S_2 - \partial_{p_2} D_3, & -\partial_{p_3} D_3 \frac{p_3}{p_2}, & \dots & 0 \\ 0 & \dots & 0 & \partial_{p_{n-1}} S_n \frac{p_{n-1}}{p_n} & \partial_{p_n} S_n - \partial_{p_n} D_c \end{pmatrix}$$

By homogeneity, $\frac{\partial_{p_{i-1}} D_i}{\partial_{p_i} D_i} = -\frac{p_i}{p_{i-1}}$, and $\frac{\partial_{p_{i-1}} S_i}{\partial_{p_i} S_i} = -\frac{p_i}{p_{i-1}}$, so on each row of this matrix the sum of the off-diagonal terms is equal to $\partial_{p_i} S_i + \partial_{p_i} D_{i+1}$, which is exactly equal to the diagonal element, but for row 1 and 2, in which one of the addenda is missing and so the diagonal element is larger. Hence the matrix is weakly chained diagonally dominant, so positive definite. So, by similarity, also $\partial_p MC$ is positive definite, and since it has negative off-diagonal elements, it is an M -matrix and $\partial_p MC^{-1}$ has all positive entries. Now $\partial_\varepsilon MC = (0, \dots, -\partial_\varepsilon D_c)$, and so $\partial_\varepsilon p_i > 0$ for all i . Moreover if $\varepsilon \rightarrow \infty$ $D_c \rightarrow \infty$ and $\varepsilon \rightarrow \infty$ $D_c \rightarrow 0$, so the whole price space is reached.

Finally, $\partial_\varepsilon p_i \neq 0$ implies $\partial_{p_i} \varepsilon \neq 0$, that is there exist g_i such that $\varepsilon = g_i(p_i)$, hence the measurability assumption is satisfied.