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M. Beatrice Lignola[*] and Jacqueline Morgan[†]

Abstract
Problems associated to Multi-Leader-Multi-Follower games may fail to have solutions, particularly in the case of pessimistic behavior of the leaders. In this paper, we propose to obviate to this deficiency by introducing approximate and viscosity solutions constructed with the aid of suitable approximate Nash equilibria of the second stage game. More specifically, the parametric followers Nash equilibrium problem is approximated by using Minty variational inequalities or classical variational inequalities and it is shown, among others, that approximate and viscosity solutions of the pessimistic Multi-Leader-Multi-Follower problem exist under not restrictive assumptions when we use Minty's variational inequalities.

Keywords: Multi-Leader-Multi-Follower game - Pessimistic behavior - Variational inequality - Minty variational inequality - Approximate solution - Viscosity solution.

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1 Introduction

In this paper, we consider Multi-Leader-Multi-Follower games ([29], [14], [10], [11], [7], [23],[2]...), MF for short, aimed to extend the investigation in [22], where we were concerned with Multi-Leader-Common-Follower games, CF for short, i.e. two-stage games with k players (leaders) at the first stage and one player (follower) at the second stage that replies with a common response to all leaders. The known example given by Fukushima and Pang in [29] shows that, even under good conditions, solutions of problems associated to CF games may fail to exist, due to the lack of *continuity* and *concavity* [9] properties of the follower's solutions map. Thus, in [22], we introduced, among others, suitable approximate solution concepts whose existence is guaranteed for the class of *weighted potential* CF games. More specifically, we approximated the solution map of the single follower, which amounts to be the optimal points map in this case, by considering the map of ϵ-optimal points and then we defined the concept of *approximate* and *viscosity solution* for the problem associated to weighted potential CF's.
Here, we consider several approximations of the Nash equilibria map $\mathcal{N}$ at the second stage and we obtain, under usual assumptions, appropriate continuity properties by considering also a class of differentiable payoffs.
In detail, we first consider classical ε-Nash equilibria maps, as defined in [5], [26], [27] and in [20], and we present previous and new results related to these approximations also showing their deficiencies. Then, to overcome these criticities, we approximate the second stage Nash equilibrium problem via two types of parametric variational inequalities associated to it. In [17] and [18], we already used approximation linked to classical variational inequalities ((VI) for short) [3], for well-posedness purposes in the setting of One-Leader-One-Follower and One-Leader-Multi-Follower games; in [15] and in [19], we also considered approximation of Minty variational inequalities ((MVI) for short) [24] for existence and stability purposes in Bilevel Optimization settings. Surprisingly, here we find that approximating (MVI)'s allows to get all conditions required for a satisfactory concept of approximate and viscosity solution for the pessimistic problem associated to weighted potential MF's. Indeed, we prove that, under not restrictive assumptions, the approximating (MVI)'s map is lower semicontinuous and asymptotically approaches the Nash equilibrium map of the second stage, two basic conditions for the existence of these approximate and viscosity solutions. On the contrary, using approximation by classical (VI)'s leads to obtain results only for a more restricted class of data.
Regarding the first stage game, we assume that all leaders have a cautious approach, so we focus our attention only on MF games with pessimistic leaders and on weighted potential MF games, as defined in Section 2, where are recalled definitions and results used in the paper. Approximate solutions to variational inequalities and to Nash equilibria are respectively analized in Section 3 and in Section 4. Finally, in Section 5 we define two types of approximate and viscosity solutions for the pessimistic problem associated to weighted potential MF's also giving related existence results.

2 Setting, Preliminaries and Basic Results

First, we point out that for all definitions concerned with set valued maps, as well with set convergences, we address to [1].

Let X_i be a nonempty closed subset of $\mathbb{R}^{l_i}$, for $i = 1, ..., k$, and let Y_j be a nonempty closed convex subset of $\mathbb{R}^{h_j}$, for $j = 1, ..., m$, where $k > 1$ and $m > 1$ are integers.

We set $l_1 + .. + l_k = l$, $h_1 + .. + h_m = h$, so that $X = \prod_{i=1,..,k} X_i \subseteq \mathbb{R}^l$, $Y = \prod_{j=1,..,m} Y_j \subseteq \mathbb{R}^h$.

For any $\bar{\boldsymbol{x}} = (\bar{x}_1, ..., \bar{x}_k) \in X$ and $\bar{\boldsymbol{y}} = (\bar{y}_1, ..., \bar{y}_h) \in Y$, we denote respectively by $\bar{\boldsymbol{x}}_{-i}$ and $\bar{\boldsymbol{y}}_{-j}$ the points $(\bar{x}_1, ..., \bar{x}_{i-1}, \bar{x}_{i+1}, ..., \bar{x}_k) \in X_{-i} = \prod_{p \neq i} X_p$ and $(\bar{y}_1, ..., \bar{y}_{j-1}, \bar{y}_{j+1}, ..., \bar{y}_m) \in Y_{-j} = \prod_{q \neq j} Y_q$, whereas $(x_i, \bar{\boldsymbol{x}}_{-i})$ and $(y_j, \bar{\boldsymbol{y}}_{-j})$ indicate the points $(\bar{x}_1, , .., \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, ..., \bar{x}_k) \in X$ and $(\bar{y}_1, , .., \bar{y}_{j-1}, y_j, \bar{y}_{j+1}, ..., \bar{y}_m) \in Y$.

Let $L_1, ..., L_k$, $F_1, ..., F_m$ be real-valued functions defined on $X \times Y$ and let $K_1, ..., K_m$ be set-valued maps from X to Y. Given $\boldsymbol{x} = (x_1, ..., x_k) \in X$, we assume that $\boldsymbol{x}$ corresponds to a strategy profile of k leaders playing first non-cooperatively, and that the followers play the non-cooperative normal form game

$$\Omega(\boldsymbol{x}) = \{m, K_1(\boldsymbol{x}), ..., K_m(\boldsymbol{x}), F_1(\boldsymbol{x}, \cdot), ..., F_m(\boldsymbol{x}, \cdot)\}.$$

As reaction to a strategy profile $\boldsymbol{x}$ chosen by the leaders, the followers are assumed to answer with a Nash equilibrium of the game $\Omega(\boldsymbol{x})$ [28], that is a point $\boldsymbol{y} \in K(\boldsymbol{x})$ such that

$$F_j(\boldsymbol{x}, \boldsymbol{y}) = \inf_{z_j \in K_j(\boldsymbol{x})} F_j(\boldsymbol{x}, z_j, \boldsymbol{y}_{-j}), \ \forall\, j = 1, .., m.$$

The best reply map

$$\mathcal{N} : \boldsymbol{x} \in X \rightrightarrows \mathcal{N}(\boldsymbol{x}) = \left\{ \boldsymbol{y} \in \prod_{j=1,..,m} K_j(\boldsymbol{x}) : F_j(\boldsymbol{x}, \boldsymbol{y}) = \inf_{z_j \in K_j(\boldsymbol{x})} F_j(\boldsymbol{x}, z_j, \boldsymbol{y}_{-j}), \ \forall\, j = 1, ., m \right\} \tag{1}$$

could be set-valued and we assume throughout the paper that it is nonempty-valued.

The case where, instead of minimizing, the followers maximize the payoffs $F_1, .., F_m$, can be described observing that

$$\sup_{z_j \in K_j(\boldsymbol{x})} F_j(\boldsymbol{x}, z_j, \boldsymbol{y}_{-j}) \ = \ - \inf_{z_j \in K_j(\boldsymbol{x})} -F_j(\boldsymbol{x}, z_j, \boldsymbol{y}_{-j}),$$

but, in this paper, we always consider the case where the followers minimize their payoffs.

The leaders, wishing to minimize their payoffs, can have a cautious approach by considering the function

$$\mathcal{P}_i : \boldsymbol{x} \in X \to \mathcal{P}_i(\boldsymbol{x}) = \sup_{\boldsymbol{y} \in \mathcal{N}(\boldsymbol{x})} L_i(\boldsymbol{x}, \boldsymbol{y}), \ \text{ for } i = 1, ..., k \tag{2}$$

and modelizing the Nash Equilibrium Problem associated to the normal form game defined by $\{k, X_1, .., X_k, \mathcal{P}_1, .., \mathcal{P}_k\}$

$$\text{find } \bar{\boldsymbol{x}} \in X \text{ such that } \ \mathcal{P}_i(\bar{\boldsymbol{x}}) \ = \ \inf_{x_i \in X_i} \mathcal{P}_i(x_i, \bar{\boldsymbol{x}}_{-i}) \ \text{ for } i = 1, .., k.$$

This choice leads to formulate the *Pessimistic Multi-Leader-Multi-Follower Problem*, (pessimistic) MF problem for short:

(PMF) find $\bar{\boldsymbol{x}} \in X$ such that

$$\sup_{\boldsymbol{y}\in\mathcal{N}(\bar{\boldsymbol{x}})} L_i(\bar{\boldsymbol{x}},\boldsymbol{y}) = \inf_{x_i\in X_i} \sup_{\boldsymbol{y}\in\mathcal{N}(x_i,\bar{\boldsymbol{x}}_{-i})} L_i(x_i,\bar{\boldsymbol{x}}_{-i},\boldsymbol{y}) \ \ \forall\, i = 1,..,k. \tag{3}$$

A solution to (PMF) is called a *pessimistic solution to the MF game.*
Such solutions may fail to exist: in [22, Example 2.1], we have shown that the functions $\mathcal{P}_i$ may not satisfy standard assumptions for the existence of Nash equilibria, even when the functions L_i's are linear, there is only one common follower and the map $\mathcal{N}$ is single-valued. So, as in [22], we restrict our analysis to the class of weighted potential games as below defined.

Definition 2.1 *A Multi-Leader-Multi-Follower game is said weighted potential if there exists a real-valued function π, defined on $X \times Y$ and called a weighted MF-potential of the game (weighted potential for short), and, for any $i = 1,...,k$, there exists $\beta_i \in \mathbb{R}_{++}$ such that, for all $(\boldsymbol{x},\boldsymbol{y}) \in X \times Y$, one has:*

$$L_i(x_i,\boldsymbol{x}_{-i},\boldsymbol{y}) - L_i(x_i',\boldsymbol{x}_{-i},\boldsymbol{y}') = \beta_i\,[\pi(x_i,\boldsymbol{x}_{-i},\boldsymbol{y}) - \pi(x_i',\boldsymbol{x}_{-i},\boldsymbol{y}')] \ \ \forall\,(x_i',\boldsymbol{y}') \in X_i \times Y. \tag{4}$$

The next characterization is in line with that given in [8, Theorem 2.1] for one stage potential games in the sense of Monderer and Shapley [25].

Proposition 2.1 *A Multi-Leader-Multi-Follower game is a weighted potential game if and only if there exists a real-valued function Π defined on $X \times Y$, and, for any $i = 1,...,k$, there exist $\beta_i \in \mathbb{R}_{++}$ and a real-valued function Ψ_i, defined on X_{-i}, such that, for all $(\boldsymbol{x},\boldsymbol{y}) \in X \times Y$, one has:*

$$L_i(\boldsymbol{x},\boldsymbol{y}) \ = \ \beta_i\Pi(\boldsymbol{x},\boldsymbol{y}) \ + \ \Psi_i(\boldsymbol{x}_{-i}). \tag{5}$$

Proof If MF is a weighted potential game, then, for any $i = 1,..,k$, from (4) we infer condition (5) by considering $\Pi = \pi$ and $\Psi_i = L_i - \beta_i\Pi$. Conversely, from (5) we get (4) by considering $\pi = \Pi$. $\square$

The above definition extends to MF games the definition given in [22, Definition 3.2] for CF games and those given in [12, Definition 2.2] and in [13, Definition 3.2] for MF games.
However, as shown in [22, Example 3.2], also pessimistic problems (PMF) associated to weighted potential MF games may fail to have exact equilibria even if there is only one common follower and the leaders payoffs are linear. Thus, we aim to approximate the followers equilibria map by appropriate maps which could help to obtain the existence of approximate and viscosity solutions for (PMF).
To this end, the following well known result (e.g. [6, Theorem 3.1]) is crucial.

Proposition 2.2 *Assume that for every $\boldsymbol{x} \in X$ and $j = 1,..,m$ the function $F_j(\boldsymbol{x},\cdot)$ is differentiable with respect to y_j on Y. Then, any Nash equilibrium for the game $\Omega(\boldsymbol{x})$ is a solution to the variational inequality*

$$\textit{find}\ \ \boldsymbol{y} \in \prod_{j=1,..,m} K_j(\boldsymbol{x}) \ \textit{such that}\ \langle A(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{y}-\boldsymbol{v}\rangle \le 0 \quad \forall \boldsymbol{v} \in \prod_{j=1,..,m} K_j(\boldsymbol{x}), \tag{6}$$

where the function $A(\boldsymbol{x}, \cdot)$ associates to every $\boldsymbol{y} \in Y$ the element $A(\boldsymbol{x}, \boldsymbol{y})$ such that

$$\langle A(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{v} \rangle \;=\; \sum_{j=1}^{m} \left\langle \frac{\partial F_j}{\partial y_j}(\boldsymbol{x}, \boldsymbol{y}), v_j \right\rangle. \tag{7}$$

If, moreover, for every $j = 1, .., m$ and $(\boldsymbol{x}, \boldsymbol{y}) \in X \times Y$, the function $F_j(\boldsymbol{x}, \cdot, \boldsymbol{y}_{-j})$ is pseudo-convex on Y_j, then the converse holds.

Now, we outline definitions and properties that will be used in this paper

- The lower limit of a sequence of sets $(S_n)_n$, $S_n \subseteq \mathbb{R}^h$, denoted by $\liminf_n S_n$, is the set of the points $\boldsymbol{y} \in \mathbb{R}^h$ such that $\boldsymbol{y} = \lim_k \boldsymbol{y}_n$ where $\boldsymbol{y}_n \in S_n$ for n sufficiently large.
- The upper limit of a sequence of sets $(S_n)_n$, $S_n \subseteq \mathbb{R}^h$, denoted by $\limsup_n S_n$, is the set of the points $\boldsymbol{y} \in \mathbb{R}^h$ such that $\boldsymbol{y} = \lim_k \boldsymbol{y}_{n_k}$ where $\boldsymbol{y}_{n_k} \in S_{n_k}$ for a subsequence $(n_k)_k$.
- A set-valued map T from $X \subseteq \mathbb{R}^l$ to $Y \subseteq \mathbb{R}^h$ is *lower semicontinuous* over X if for every $\boldsymbol{x} \in X$, every sequence $(\boldsymbol{x}_n)_n$ converging to $\boldsymbol{x}$ in X and every $y \in T(\boldsymbol{x})$ there exists a sequence $(y_n)_n$ converging to y such that $y_n \in T(\boldsymbol{x}_n)$ for n sufficiently large, i.e.
$$T(\boldsymbol{x}) \subseteq \liminf_n T_n(\boldsymbol{x}_n);$$
- A set-valued map T from $X \subseteq \mathbb{R}^l$ to $Y \subseteq \mathbb{R}^h$ is *closed* over X if for any $\boldsymbol{x} \in X$ and any $(\boldsymbol{x}_n)_n$ converging to $\boldsymbol{x}$ in X, if $(y_k)_k$ converges to y in Y and, for a subsequence $(n_k)_k$, $y_k \in T(\boldsymbol{x}_{n_k})$ for any $k \in \mathbb{N}$, then we have that $y \in T(\boldsymbol{x})$, i.e.
$$\limsup_n T_n(\boldsymbol{x}_n) \subseteq T(\boldsymbol{x});$$
- A set-valued map T from $X \subseteq \mathbb{R}^l$ to $Y \subseteq \mathbb{R}^h$ is subcontinuous over X if for any $\boldsymbol{x} \in X$ and every $(\boldsymbol{x}_n)_n$ converging to $\boldsymbol{x}$ in X, any $(\boldsymbol{y}_n)_n$ such that $\boldsymbol{y}_n \in T(\boldsymbol{x}_n)$, for any $n \in \mathbb{N}$, has a convergent subsequence;
- A function $G : Y \subseteq \mathbb{R}^h \to \mathbb{R}^h$ is monotone over Y if for any $\boldsymbol{y} \in Y$ and any $\boldsymbol{v} \in Y$
$$\langle G(\boldsymbol{y}) - G(\boldsymbol{v}), \boldsymbol{y} - \boldsymbol{v} \rangle \geq 0;$$
- A function $G : Y \subseteq \mathbb{R}^h \to \mathbb{R}^h$ is pseudomonotone over Y if for any $\boldsymbol{y} \in Y$ and any $\boldsymbol{v} \in Y$
$$\langle G(\boldsymbol{y}), \boldsymbol{y} - \boldsymbol{v} \rangle \leq 0 \Rightarrow \langle G(\boldsymbol{v}), \boldsymbol{y} - \boldsymbol{v} \rangle \leq 0;$$
- A function $G : Y \subseteq \mathbb{R}^h \to \mathbb{R}^h$ is hemicontinuous over Y if it is continuous over the segments of line contained in Y.

From now on, we denote by K the set-valued map

$$K: \boldsymbol{x} \in X \rightrightarrows K(\boldsymbol{x}) = \prod_{j=1,..,m} K_j(\boldsymbol{x})$$

and we recall that K is lower semicontinuous, closed and convex-valued over X if K_j is lower semicontinuous, closed and convex-valued over X for any $j = 1, .., m$.
As mentioned in the Introduction, in next section we will investigate the lower semicontinuity property for various types of approximate solutions to parametric (VI)'s as well their asymptotic behaviour.

3 Approximate solutions to parametric variational inequalities

In this section we consider $X \subseteq \mathbb{R}^l$ and $Y \subseteq \mathbb{R}^h$ and, for $\boldsymbol{x} \in X$, the constrained parametric variational inequality defined by A and K (see, for example, [3]), where $A : X \times Y \to Y$ is a function and $K : X \rightrightarrows Y$ is a set-valued map, which consists in finding

$$\text{(VI)}(\boldsymbol{x}) \qquad \boldsymbol{y} \in K(\boldsymbol{x}) \ \text{ s.t. } \ \langle A(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{y} - \boldsymbol{v} \rangle \leq 0 \ \ \forall\, \boldsymbol{v} \in K(\boldsymbol{x}) \tag{8}$$

and we denote by $\mathcal{S}$ the solution map of (VI)($\boldsymbol{x}$)

$$\mathcal{S} : \boldsymbol{x} \in X \rightrightarrows \{\boldsymbol{y} \in K(\boldsymbol{x}) \text{ s.t. } \langle A(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{y} - \boldsymbol{v} \rangle \ \leq 0 \ \ \forall\, \boldsymbol{v} \in K(\boldsymbol{x})\}\,.$$

For two nonnegative real numbers ε_1, ε_2, consider $\varepsilon = (\varepsilon_1, \varepsilon_2)$ and define the maps

$$\mathcal{S}^{\epsilon} \ : \ \boldsymbol{x} \in X \rightrightarrows \mathcal{S}^{\epsilon}(\boldsymbol{x}) \ = \ \{\boldsymbol{y} \in Y \ : \ d(\boldsymbol{y}, K(\boldsymbol{x})) \leq \varepsilon_2 \text{ and } \langle A(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{y} - \boldsymbol{v} \rangle \leq \varepsilon_1 \ \ \forall\, \boldsymbol{v} \in K(\boldsymbol{x})\}\,,$$

$$\widetilde{\mathcal{S}}^{\epsilon} \ : \ \boldsymbol{x} \in X \rightrightarrows \widetilde{\mathcal{S}}^{\epsilon}(\boldsymbol{x}) \ = \ \{\boldsymbol{y} \in Y \ : \ d(\boldsymbol{y}, K(\boldsymbol{x})) < \varepsilon_2 \text{ and } \langle A(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{y} - \boldsymbol{v} \rangle < \varepsilon_1 \ \ \forall\, \boldsymbol{v} \in K(\boldsymbol{x})\}\,,$$

$$\mathcal{M}^{\varepsilon} \ : \ \boldsymbol{x} \in X \rightrightarrows \mathcal{M}^{\varepsilon}(\boldsymbol{x}) \ = \ \{\boldsymbol{y} \in Y \ : \ d(\boldsymbol{y}, K(\boldsymbol{x})) \leq \varepsilon_2 \text{ and } \langle A(\boldsymbol{x}, \boldsymbol{v}), \boldsymbol{y} - \boldsymbol{v} \rangle \leq \varepsilon_1 \ \ \forall\, \boldsymbol{v} \in K(\boldsymbol{x})\}\,,$$

$$\widetilde{\mathcal{M}}^{\varepsilon} \ : \ \boldsymbol{x} \in X \rightrightarrows \widetilde{\mathcal{M}}^{\varepsilon}(\boldsymbol{x}) \ = \ \{\boldsymbol{y} \in Y \ : \ d(\boldsymbol{y}, K(\boldsymbol{x})) < \varepsilon_2 \text{ and } \langle A(\boldsymbol{x}, \boldsymbol{v}), \boldsymbol{y} - \boldsymbol{v} \rangle < \varepsilon_1 \ \ \forall\, \boldsymbol{v} \in K(\boldsymbol{x})\}\,,$$

that are the *large* and *strict* versions of suitable approximate solutions to (VI)($\boldsymbol{x}$) and to the associated Minty variational inequality [24]

$$\text{(MVI)}(\boldsymbol{x}) \quad \text{find } \ \boldsymbol{y} \in \ K(\boldsymbol{x}) \text{ such that } \langle A(\boldsymbol{x}, \boldsymbol{v}), \boldsymbol{y} - \boldsymbol{v} \rangle \leq 0 \ \ \forall\, \boldsymbol{v} \in K(\boldsymbol{x}) \tag{9}$$

whose solutions map is

$$\mathcal{M} : \boldsymbol{x} \in X \rightrightarrows \{\boldsymbol{y} \in K(\boldsymbol{x}) \text{ s.t. } \langle A(\boldsymbol{x}, \boldsymbol{v}), \boldsymbol{y} - \boldsymbol{v} \rangle \leq 0 \ \ \forall\, \boldsymbol{v} \in K(\boldsymbol{x})\}\,.$$

Here, and throughout the paper, we assume that the maps $\mathcal{S}$ and $\mathcal{M}$ are nonempty-valued.

It is known [3] that:

- if the function $A(\boldsymbol{x}, \cdot)$ is pseudomonotone over Y, then $\mathcal{S}(\boldsymbol{x}) \subseteq \mathcal{M}(\boldsymbol{x})$ for any $\boldsymbol{x} \in X$;

- if the function $A(\boldsymbol{x},\cdot)$ is hemicontinuous over Y, then $\mathcal{M}(\boldsymbol{x}) \subseteq \mathcal{S}(\boldsymbol{x})$ for any $\boldsymbol{x} \in X$;
- if the function $A(\boldsymbol{x},\cdot)$ is hemicontinuous and pseudomonotone over Y, then $\mathcal{M}(\boldsymbol{x}) = \mathcal{S}(\boldsymbol{x})$, i.e. (VI)($\boldsymbol{x}$) and (MVI)($\boldsymbol{x}$) are equivalent problems, for any $\boldsymbol{x} \in X$.

However, even if the inclusion $\widetilde{\mathcal{S}}^{\epsilon}(\boldsymbol{x}) \subseteq \widetilde{\mathcal{M}}^{\varepsilon}(\boldsymbol{x})$ holds whenever the function $A(\boldsymbol{x},\cdot)$ is pseudomonotone over Y, the maps $\widetilde{\mathcal{M}}^{\varepsilon}$ and $\widetilde{\mathcal{S}}^{\epsilon}$ do not coincide in general even if $A(\boldsymbol{x},\cdot)$ is continuous and monotone, as shown by the following simple example in which the function $A(\boldsymbol{x},\cdot)$ is even linear.

Example 3.1 Let $m = l = 1$, $X = [0,+\infty[$, $Y = [0,1]$, $A(x,u) = xu$ and $K(x) = [0,1]$ for any x. Then, for any $\varepsilon < 1/4$, one checks that $\mathcal{M}^{\varepsilon}(x) = \widetilde{\mathcal{M}}^{\varepsilon}(x) = [0,1]$ if $x \in [0,4\varepsilon]$, $\mathcal{M}^{\varepsilon}(x) = [0, 2\sqrt{\varepsilon/x}]$ if $x \in]4\varepsilon, 1]$, $\widetilde{\mathcal{M}}^{\varepsilon}(x) = [0, 2\sqrt{\varepsilon/x}[$ if $x \in]4\varepsilon, 1]$, $\mathcal{S}^{\epsilon}(x) = \widetilde{\mathcal{S}}^{\epsilon}(x) = [0,1]$ if $x \in [0,\varepsilon]$, $\mathcal{S}^{\epsilon}(x) = [0, \sqrt{\varepsilon/x}]$ if $x \in]\varepsilon, 1]$, $\widetilde{\mathcal{S}}^{\epsilon}(x) = [0, \sqrt{\varepsilon/x}[$ if $x \in]\varepsilon, 1]$.

We start by analyzing the behavior of the above maps with respect to lower semicontinuity property.

Proposition 3.1 *If the function A is continuous over $X \times Y$, the set-valued map K is subcontinuous, lower semicontinuous and closed over X, then the map $\widetilde{\mathcal{M}}^{\varepsilon}$ is lower semicontinuous over X.*
If, in addition, the set Y is convex and the map K is also convex-valued over X, then also the map $\mathcal{M}^{\varepsilon}$ is lower semicontinous over X and both $\widetilde{\mathcal{M}}^{\varepsilon}$ and $\mathcal{M}^{\varepsilon}$ are convex-valued.

Proof Let $\bar{\boldsymbol{x}} \in X$ and $(\bar{\boldsymbol{x}}_n)_n$ be a sequence converging to $\bar{\boldsymbol{x}}$ in X. Consider $\bar{\boldsymbol{y}} \in \widetilde{\mathcal{M}}^{\varepsilon}(\bar{\boldsymbol{x}})$, i.e.

$$d(\bar{\boldsymbol{y}}, K(\bar{\boldsymbol{x}})) < \varepsilon_2 \text{ and } \langle A(\bar{\boldsymbol{x}}, \boldsymbol{v}), \bar{\boldsymbol{y}} - \boldsymbol{v}\rangle < \varepsilon_1 \ \forall \boldsymbol{v} \in K(\bar{\boldsymbol{x}}).$$

Let $\boldsymbol{v} \in K(\bar{\boldsymbol{x}})$ such that $||\bar{\boldsymbol{y}} - \boldsymbol{v}|| < \varepsilon_2$ and let $(\boldsymbol{v}_n)_n$ be a sequence, existing since K is lower semicontinuous over X, which converges to $\boldsymbol{v}$ such that $\boldsymbol{v}_n \in K(\bar{\boldsymbol{x}}_n)$ for n sufficiently large. Then, the sequence $(\boldsymbol{y}_n)_n$, defined by $\boldsymbol{y}_n = \bar{\boldsymbol{y}}$ for any n, converges to $\bar{\boldsymbol{y}}$ and $\boldsymbol{y}_n \in \widetilde{\mathcal{M}}^{\varepsilon}(\bar{\boldsymbol{x}}_n)$ for n large since:

- $\limsup_n d(\boldsymbol{y}_n, K(\bar{\boldsymbol{x}}_n)) \leq d(\bar{\boldsymbol{y}}, K(\bar{\boldsymbol{x}})) \leq ||\bar{\boldsymbol{y}} - \boldsymbol{v}|| < \varepsilon_2$;
- if there exists an increasing sequence of integers $(n_h)_h$ and a sequence $(\boldsymbol{v}_h)_h$ in Y such that $\boldsymbol{v}_h \in K(\bar{\boldsymbol{x}}_{n_h})$ and $\langle A(\bar{\boldsymbol{x}}_{n_h}, \boldsymbol{v}_h), \boldsymbol{y}_{n_h} - \boldsymbol{v}_h\rangle \geq \varepsilon_1 \ \forall h \in N$, subcontinuity and closedness assumptions imply that a subsequence of $(v_h)_h$ should converge to a point $\bar{\boldsymbol{v}} \in K(\bar{\boldsymbol{x}})$, so, by continuity of A we infer $\langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{v}}), \bar{\boldsymbol{y}} - \bar{\boldsymbol{v}}\rangle \geq \varepsilon_1$ and we get a contradiction.

Now, we show that $\mathcal{M}^{\varepsilon}$ is lower semicontinuous over X if we also assume that Y is convex and K is convex-valued.
Let $\bar{\boldsymbol{x}} \in X$ and $(\bar{\boldsymbol{x}}_n)_n$ be a sequence converging to $\bar{\boldsymbol{x}}$ in X. From the proof above, we have

$$cl\,\widetilde{\mathcal{M}}^{\varepsilon}(\bar{\boldsymbol{x}}) \subseteq cl\, \liminf_n \widetilde{\mathcal{M}}^{\varepsilon}(\bar{\boldsymbol{x}}_n) = \liminf_n \widetilde{\mathcal{M}}^{\varepsilon}(\bar{\boldsymbol{x}}_n) \subseteq \liminf_n \mathcal{M}^{\varepsilon}(\bar{\boldsymbol{x}}_n),$$

so, in order to get the result, it will be sufficient to prove that $\mathcal{M}^{\varepsilon}(\boldsymbol{x}) \subseteq cl\,\widetilde{\mathcal{M}}^{\varepsilon}(\boldsymbol{x})$ for any $\boldsymbol{x} \in X$.

Assume that there exists a point $\bar{\boldsymbol{x}} \in X$ such that $\mathcal{M}^\varepsilon(\bar{\boldsymbol{x}}) \not\subseteq cl\, \widetilde{\mathcal{M}}^\varepsilon(\bar{\boldsymbol{x}})$ and let $\bar{\boldsymbol{y}}$ be a point in $\mathcal{M}^\varepsilon(\bar{\boldsymbol{x}})$ such that $\bar{\boldsymbol{y}} \notin cl\widetilde{\mathcal{M}}^\varepsilon(\bar{\boldsymbol{x}})$. Then, $d(\bar{\boldsymbol{y}}, K(\bar{\boldsymbol{x}})) \leq \varepsilon_2$ and $\langle A(\bar{\boldsymbol{x}}, \boldsymbol{v}), \bar{\boldsymbol{y}} - \boldsymbol{v}\rangle \leq \varepsilon_1\ \forall \boldsymbol{v} \in K(\bar{\boldsymbol{x}})$. Let $\bar{\boldsymbol{z}} \in \mathcal{M}(\bar{\boldsymbol{x}})$ be a solution to (MVI), so that $d(\bar{\boldsymbol{z}}, K(\bar{\boldsymbol{x}})) < \varepsilon_2$ and $\langle A(\bar{\boldsymbol{x}}, \boldsymbol{v}), \bar{\boldsymbol{z}} - \boldsymbol{v}\rangle < \varepsilon_1\ \forall \boldsymbol{v} \in K(\bar{\boldsymbol{x}})$. For a sequence of positive integers $(\alpha_n)_n$ decreasing to 0 in $]0,1[$, consider the sequence $(\bar{\boldsymbol{y}}_n)_n$ defined by $\bar{\boldsymbol{y}}_n = \alpha_n \bar{\boldsymbol{z}} + (1-\alpha_n)\bar{\boldsymbol{y}}$ which converges to $\bar{\boldsymbol{y}}$. If we prove that $\bar{\boldsymbol{y}}_n \in \widetilde{\mathcal{M}}^\varepsilon(\bar{\boldsymbol{x}})$, we will get a contradiction with $\bar{\boldsymbol{y}} \notin cl\, \widetilde{\mathcal{M}}^\varepsilon(\bar{\boldsymbol{x}})$.
Since $K(\bar{\boldsymbol{x}})$ is closed and convex, for any $n \in \mathbb{N}$ we have:

$$d(\bar{\boldsymbol{y}}_n, K(\bar{\boldsymbol{x}})) \leq \alpha_n d(\bar{\boldsymbol{z}}, K(\bar{\boldsymbol{x}})) + (1-\alpha_n) d(\bar{\boldsymbol{y}}, K(\bar{\boldsymbol{x}})) < \alpha_n \varepsilon_2 + (1-\alpha_n)\varepsilon_2 = \varepsilon_2$$

and

$$\langle A(\bar{\boldsymbol{x}}, \boldsymbol{v}), \bar{\boldsymbol{y}}_n - \boldsymbol{v}\rangle = \alpha_n \langle A(\bar{\boldsymbol{x}}, \boldsymbol{v}), \bar{\boldsymbol{z}} - \boldsymbol{v}\rangle + (1-\alpha_n)\langle A(\bar{\boldsymbol{x}}, \boldsymbol{v}), \bar{\boldsymbol{y}} - \boldsymbol{v}\rangle\ < \alpha_n \varepsilon_1 + (1-\alpha_n)\varepsilon_1 = \varepsilon_1\, \forall \boldsymbol{v} \in K(\bar{\boldsymbol{x}})$$

which imply that $\bar{\boldsymbol{y}}_n \in \widetilde{\mathcal{M}}^\varepsilon(\bar{\boldsymbol{x}})$ and a contradiction.
If $\boldsymbol{y}_1 \in \mathcal{M}^\varepsilon(\boldsymbol{x})$, $\boldsymbol{y}_2 \in \mathcal{M}^\varepsilon(\boldsymbol{x})$ and $\lambda \in [0,1]$ we have

$$d(\lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2, K(\boldsymbol{x})) \leq \lambda d(\boldsymbol{y}_1, K(\boldsymbol{x})) + (1-\lambda) d(\boldsymbol{y}_2, K(\boldsymbol{x})) \leq \lambda\varepsilon_2 + (1-\lambda)\varepsilon_2 = \varepsilon_2$$

$$\langle A(\bar{\boldsymbol{x}}, \boldsymbol{v}), \lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2 - \boldsymbol{v}\rangle \leq\ \lambda\varepsilon_1 + (1-\lambda)\varepsilon_1 = \varepsilon_1\ \forall \boldsymbol{v} \in K(\bar{\boldsymbol{x}}),$$

so $\mathcal{M}^\varepsilon$ and $\widetilde{\mathcal{M}}^\varepsilon$ are convex-valued. □

Concerning $\widetilde{\mathcal{S}}^\epsilon$ we have:

Proposition 3.2 *If the function A is continuous over $X \times Y$, the set valued map K is subcontinuous, lower semicontinuous and closed over X, then the map $\widetilde{\mathcal{S}}^\epsilon$ is lower semicontinuous over X.*

Proof The proof goes as in the first part of Proposition 3.1, so we omit it.

However, the map $\mathcal{S}^\epsilon$, as well the map $\mathcal{S}$, may fail to be lower semicontinuous and convex-valued even in the case of affine data, as shown in the example below.

Example 3.2 Let $l = h = 1$, $X = [0,1]$, $Y = [0,1]$, $A(x,u) = 1 + x - u$ and $K(x) = [0,1]$ for any x. Then, for any $\varepsilon < 1/4$, one can check that
$\mathcal{S}(x) = \{0,1\}$ if $x = 0$,
$\mathcal{S}(x) = \{0\}$ if $x \in]0,1]$,
$\mathcal{S}^\epsilon(x) = \left[0, (x+1)/2 - 1/2\sqrt{(x+1)^2 - 4\varepsilon}\right] \cup \left[(x+1)/2 + 1/2\sqrt{(x+1)^2 - 4\varepsilon}, 1\right]$ if $x \in\ [0,\varepsilon]$,
$\mathcal{S}^\epsilon(x) = \left[0, (x+1)/2 - 1/2\sqrt{(x+1)^2 - 4\varepsilon}\right]$ if $x \in]\varepsilon, 1]$,
$\widetilde{\mathcal{S}}^\epsilon(x) = \left[0, (x+1)/2 - 1/2\sqrt{(x+1)^2 - 4\varepsilon}\right[\cup \left](x+1)/2 + 1/2\sqrt{(x+1)^2 - 4\varepsilon}, 1\right]$ if $x \in\ [0,\varepsilon[$,
$\widetilde{\mathcal{S}}^\epsilon(x) = \left[0, (x+1)/2 - 1/2\sqrt{(x+1)^2 - 4\varepsilon}\right[$ if $x \in\ [\varepsilon, 1]$,
so, the map $\mathcal{S}$ is not semicontinuous at $x = 0$, the map $\mathcal{S}^\epsilon$ is not lower semicontinuous at $x = \varepsilon$, both of them are not convex-valued, but $\widetilde{\mathcal{S}}^\epsilon$ is lower semicontinuous over [0,1].

Nevertheless, we can obtain the lower semicontinuity of $\mathcal{S}^\epsilon$ if we consider a restricted class of data to which funtion A in Example 3.2 does not belong but funtion A in Example 3.1 does belong.

Proposition 3.3 *Under all assumptions of Proposition 3.1, if, for any $\boldsymbol{x} \in X$, the function $A(\boldsymbol{x}, \cdot)$ is affine and monotone over $K(\boldsymbol{x})$, then the map $\mathcal{S}^\epsilon$ is lower semicontinuous over X.*

Proof Let $\bar{\boldsymbol{x}} \in X$, $(\bar{\boldsymbol{x}}_n)_n$ be a sequence converging to $\bar{\boldsymbol{x}}$ in X. By Proposition 3.2 we have that, similarly as in Proposition 3.1:

$$cl\, \widetilde{\mathcal{S}}^\epsilon(\bar{\boldsymbol{x}}) \subseteq cl\, \liminf_n \widetilde{\mathcal{S}}^\epsilon(\bar{\boldsymbol{x}}_n) \; = \liminf_n \widetilde{\mathcal{S}}^\epsilon(\bar{\boldsymbol{x}}_n) \; \subseteq \; \liminf_n \mathcal{S}^\epsilon(\bar{\boldsymbol{x}}_n).$$

So, for the result, it is sufficient to prove that $\mathcal{S}^\epsilon(\bar{\boldsymbol{x}}) \subseteq cl\widetilde{\mathcal{S}}^\epsilon(\bar{\boldsymbol{x}})$.
To this end, consider $\bar{\boldsymbol{y}} \in \mathcal{S}^\epsilon(\bar{\boldsymbol{x}})$, a point $\bar{\boldsymbol{z}} \in \mathcal{S}(\bar{\boldsymbol{x}})$ (i.e. such that $\bar{\boldsymbol{z}} \in K(\bar{\boldsymbol{x}})$ and $\langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}), \bar{\boldsymbol{z}} - \boldsymbol{v}\rangle \leq 0 \; \forall \boldsymbol{v} \in K(\bar{\boldsymbol{x}})$) and the sequence $(\bar{\boldsymbol{y}}_n)_n$ defined as $\bar{\boldsymbol{y}}_n = \alpha_n \bar{\boldsymbol{z}} + (1-\alpha_n)\bar{\boldsymbol{y}}$, where $(\alpha_n)_n$ is a sequence of positive integers decreasing to 0 in $]0,1[$. The sequence $(\bar{\boldsymbol{y}}_n)_n$ converges to $\bar{\boldsymbol{y}}$ in Y and we are going to prove that $\bar{\boldsymbol{y}}_n \in \widetilde{\mathcal{S}}^\epsilon(\bar{\boldsymbol{x}})$ for any $n \in N$, which will complete the proof.
First, since $K(\bar{\boldsymbol{x}})$ is convex, we have $d(\bar{\boldsymbol{y}}_n, K(\bar{\boldsymbol{x}})) \leq \alpha_n d(\bar{\boldsymbol{z}}, K(\bar{\boldsymbol{x}})) + (1-\alpha_n) d(\bar{\boldsymbol{y}}, K(\bar{\boldsymbol{x}})) < \varepsilon_2$, and the monotonicity of $A(\bar{\boldsymbol{x}}, \cdot)$ over Y says that

$$\langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}) - A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}), \bar{\boldsymbol{z}} - \bar{\boldsymbol{y}}\rangle \; \geq 0. \tag{10}$$

Since $A(\bar{\boldsymbol{x}}, \cdot)$ is affine, for any $\boldsymbol{v} \in K(\bar{\boldsymbol{x}})$ and $n \in \mathbb{N}$ we have

$$\langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}_n), \bar{\boldsymbol{y}}_n - \boldsymbol{v}\rangle = \alpha_n \langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}), \bar{\boldsymbol{y}}_n - \boldsymbol{v}\rangle + (1-\alpha_n)\langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}), \bar{\boldsymbol{y}}_n - \boldsymbol{v}\rangle = \alpha_n^2 \langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}), \bar{\boldsymbol{z}} - \boldsymbol{v}\rangle$$
$$+\alpha_n(1-\alpha_n)\left[\langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}), \bar{\boldsymbol{y}} - \boldsymbol{v}\rangle + \langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}), \bar{\boldsymbol{z}} - \boldsymbol{v}\rangle\right] + (1-\alpha_n)^2 \langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}), \bar{\boldsymbol{y}} - \boldsymbol{v}\rangle.$$

Moreover, from (10) we get

$$\langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}), \bar{\boldsymbol{y}} - \boldsymbol{v}\rangle + \langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}), \bar{\boldsymbol{z}} - \boldsymbol{v}\rangle \leq \langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}), \bar{\boldsymbol{y}} - \boldsymbol{v}\rangle + \langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}), \bar{\boldsymbol{z}} - \boldsymbol{v}\rangle + \langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}) - A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}), \bar{\boldsymbol{z}} - \bar{\boldsymbol{y}}\rangle =$$
$$= \langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}), \bar{\boldsymbol{z}} - \boldsymbol{v}\rangle + \langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}), \bar{\boldsymbol{y}} - \boldsymbol{v}\rangle \leq \varepsilon_1 < 2\varepsilon_1,$$

and

$$\langle A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}_n), \bar{\boldsymbol{y}}_n - \boldsymbol{v}\rangle < \alpha_n^2 \varepsilon_1 + (1-\alpha_n)^2 \varepsilon_1 + \alpha_n(1-\alpha_n) 2\varepsilon_1 \; = \; \varepsilon_1,$$

and we can conclude that $\bar{\boldsymbol{y}}_n \in \widetilde{\mathcal{S}}^\epsilon(\bar{\boldsymbol{x}})$. □

We remark that, in light of Example 3.2, the above proposition does not longer hold if we weaken the assumptions by considering a pseudomonotone function $A(\boldsymbol{x}, \cdot)$.

Now, consider a sequence $(\varepsilon_n)_n = (\varepsilon_n^1, \varepsilon_n^2)_n$, where $(\varepsilon_n^1)_n$ and $(\varepsilon_n^2)_n$ decrease to zero, in order to analize the behaviour of ε_n-*approximating* sequences.

Proposition 3.4 *If the function A is continuous over $X \times Y$, the set valued map K is sub-continuous, lower semicontinuous and closed over X, then for any $\boldsymbol{x} \in X$ and any sequence $(\boldsymbol{x}_n)_n$ converging to $\boldsymbol{x}$ in X, we have*

$$\limsup_n \widetilde{\mathcal{M}}^{\varepsilon_n}(\boldsymbol{x}_n) \; \subseteq \limsup_n \mathcal{M}^{\varepsilon_n}(\boldsymbol{x}_n) \; \subseteq \; \mathcal{M}(\boldsymbol{x}) \subseteq \mathcal{S}(\boldsymbol{x}) \tag{11}$$

and

$$\limsup_n \widetilde{\mathcal{S}}^{\varepsilon_n}(\boldsymbol{x}_n) \subseteq \limsup_n \mathcal{S}^{\varepsilon_n}(\boldsymbol{x}_n) \subseteq \mathcal{S}(\boldsymbol{x}) \tag{12}$$

Proof We only prove the first statement since (12) can be proved in exactly the same manner. Consider $\boldsymbol{y} \in \limsup_n \mathcal{M}^{\varepsilon_n}(\boldsymbol{x}_n)$, that is there exists a subsequence $(n_h)_h$ and a sequence $(\boldsymbol{y}_h)_h$ such that

$$\lim_h \boldsymbol{y}_h = \boldsymbol{y} \ \text{ and } \ \boldsymbol{y}_h \in \mathcal{M}^{\varepsilon_{n_h}}(\boldsymbol{x}_{n_h}) \ \forall\, h \in \mathbb{N}.$$

Then, $\boldsymbol{y} \in K(\boldsymbol{x})$ since $d(\boldsymbol{y}, K(\boldsymbol{x})) \le \liminf_h d(\boldsymbol{y}_h, K(\boldsymbol{x}_{n_h})) \le \lim_h \varepsilon^2_{n_h} = 0$.
In order to prove that $\langle A(\boldsymbol{x}, \boldsymbol{v}), \boldsymbol{y} - \boldsymbol{v}\rangle \le 0 \ \forall \boldsymbol{v} \in K(\boldsymbol{x})$, consider $\boldsymbol{v} \in K(\boldsymbol{x})$; the lower semicontinuity of K gives the existence of a sequence $(\boldsymbol{v}_h)_h$ such that

$$\lim_h \boldsymbol{v}_h = \boldsymbol{v} \ \text{ and } \ \boldsymbol{v}_h \in K(\boldsymbol{x}_{n_h})$$

for h sufficiently large.
So, being A continuous we get

$$\langle A(\boldsymbol{x}, \boldsymbol{v}), \boldsymbol{y} - \boldsymbol{v}\rangle = \lim_h \langle A(\boldsymbol{x}_{n_h}, \boldsymbol{v}_h), \boldsymbol{y}_h - \boldsymbol{v}_h\rangle \le \lim_h \varepsilon^1_{n_h} = 0,$$

so $\boldsymbol{y} \in \mathcal{M}(\boldsymbol{x})$ and, as observed at the beginning of this section, $\mathcal{M}(\boldsymbol{x}) \subseteq \mathcal{S}(\boldsymbol{x})$ due to continuity of A.
The left inclusion in (11) is obvious since $\widetilde{\mathcal{M}}^{\varepsilon}(\boldsymbol{x}) \subseteq \mathcal{M}^{\varepsilon}(\boldsymbol{x})$ for any $\boldsymbol{x} \in X$ and the proof is completed. $\square$

The above proposition says that both $\mathcal{M}^{\varepsilon}$ and $\mathcal{S}^{\epsilon}$ can be used to approximate $\mathcal{S}$ when ε tends to zero but, as observed in the Introduction, also the lower semicontinuity property will play a crucial rule in the approximating scheme for MF games that we will present in Section 5. In the next section we recall classical approximate equilibria for the second stage games as well new approximations via variational inequalities.

4 Approximate Solutions to Parametric Nash Equilibria

Following notations in Section 2, for any $\boldsymbol{x} \in X$, we consider the game

$$\Omega(\boldsymbol{x}) = \{m, K_1(\boldsymbol{x}), ..., K_m(\boldsymbol{x}), F_1(\boldsymbol{x}, \cdot), ..., F_m(\boldsymbol{x}, \cdot)\}$$

and the set of Nash equilibria $\mathcal{N}(\boldsymbol{x})$ which is assumed to be nonempty.
The map $\mathcal{N} : \boldsymbol{x} \in X \rightrightarrows \mathcal{N}(\boldsymbol{x})$ is closed if the payoffs are continuous and the constraint maps are lower semicontinuous and closed over X [16], but, as well the map of optimal points in an optimization problem, it may be not lower semicontinuous [26], [27], so, suitable approximate maps which could be closed and lower semicontinuous have been considered in [26], [27], [18], [20], and here, in line with the approximations considered in Section 3, we slightly enlarge them by setting:

$$\mathbb{N}^{\varepsilon} : \boldsymbol{x} \in X \rightrightarrows \left\{ \boldsymbol{y} \in Y : d(\boldsymbol{y}, K(\boldsymbol{x})) \le \varepsilon_2 \text{ and } F_j(\boldsymbol{x}, \boldsymbol{y}) \le \inf_{z_j \in K_j(\boldsymbol{x})} F_j(\boldsymbol{x}, z_j, \boldsymbol{y}_{-j}) + \varepsilon_1, \ \forall\, j = 1, .., m \right\}$$

$$\mathcal{N}^\varepsilon : \boldsymbol{x} \in X \rightrightarrows \left\{ \boldsymbol{y} \in Y : d(\boldsymbol{y}, K(\boldsymbol{x})) \le \varepsilon_2 \text{ and } \sum_{j=1}^m F_j(\boldsymbol{x}, \boldsymbol{y}) \le \inf_{\boldsymbol{z} \in K(\boldsymbol{x})} \sum_{j=1}^m F_j(\boldsymbol{x}, z_j, \boldsymbol{y}_{-j}) + \varepsilon_1 \right\}.$$

It is easy to see that, for any $\boldsymbol{x} \in X$, $\mathcal{N}^\varepsilon(\boldsymbol{x}) \subseteq \mathbb{N}^\varepsilon(\boldsymbol{x}) \subseteq \mathcal{N}^{m\varepsilon}(\boldsymbol{x})$, so, even if these maps do not coincide in general, both of them can be used to approach the Nash equilibrium map $\mathcal{N}$ when ε goes to zero [18]. Indeed, for any sequence $(\varepsilon_n)_n = (\varepsilon_n^1, \varepsilon_n^2)_n$, where $(\varepsilon_n^1)_n$ and $(\varepsilon_n^2)_n$ decrease to zero, the following proposition holds.

Proposition 4.1 *If F_j is continuous over $X \times Y$, the map K_j is lower semicontinuous, closed and subcontinuous over X, for any $j = 1, .., m$, then $\mathcal{N}^\varepsilon$ and $\mathbb{N}^\varepsilon$ are closed and subcontinuous over X and, for any sequence $(\boldsymbol{x}_n)_n$ converging in X to $\boldsymbol{x}$, we have*

$$\limsup_n \mathcal{N}^{\varepsilon_n}(\boldsymbol{x}_n) \subseteq \limsup_n \mathbb{N}^{\varepsilon_n}(\boldsymbol{x}_n) \subseteq \mathcal{N}(\boldsymbol{x}).$$

Proof Properties of closedness and subcontinuity are easy to prove, so, let consider $\boldsymbol{x} \in X$ and $(\boldsymbol{x}_n)_n$ which converges to $\boldsymbol{x} \in X$. Let $\boldsymbol{y} \in \limsup_h \mathbb{N}^{\varepsilon_n}(\boldsymbol{x}_n)$, that is $\boldsymbol{y} = \lim_h \boldsymbol{y}_h$, where, for any $h \in \mathbb{N}$, $\boldsymbol{y}_h \in \mathbb{N}^{\varepsilon_{n_h}}(\boldsymbol{x}_{n_h})$, and let us prove that $\boldsymbol{y}$ is a Nash equilibrium of the game $\Omega(\boldsymbol{x})$. If $\boldsymbol{z} \in K(\boldsymbol{x})$, there exists a sequence $(\boldsymbol{z}_h)_h$ converging to $\boldsymbol{z}$ such that $\boldsymbol{z}_h \in K(\boldsymbol{x}_{n_h})$ for h sufficiently large and we have:

$$F_j(\boldsymbol{x}, \boldsymbol{y}) = \lim_h F_j(\boldsymbol{x}_{n_h}, \boldsymbol{y}_h) \le \lim_h \left(F_j(\boldsymbol{x}_{n_h}, z_h^j, \boldsymbol{y}_{-j}^h) + \varepsilon_{n_h}^1 \right) = F_j(\boldsymbol{x}, z_j, \boldsymbol{y}_{-j}) \quad \forall j = 1, .., m$$

and

$$d(\boldsymbol{y}, K(\boldsymbol{x})) \le \liminf_h d(\boldsymbol{y}_h, K(\boldsymbol{x}_{n_h})) = 0$$

due to the above assumptions. $\square$

As shown in [26] and [27], both the maps $\mathbb{N}^\varepsilon$ and $\mathcal{N}^\varepsilon$ may fail to be lower semicontinuous, even if their *strict* versions, $\widetilde{\mathbb{N}}^\varepsilon$ and $\widetilde{\mathcal{N}}^\varepsilon$, do are, in line with was observed in Section 3 for variational inequalities approximations.

Proposition 4.2 *If F_j is continuous over $X \times Y$, the map K_j is lower semicontinuous, closed and subcontinuous over X, for any $j = 1, .., m$, then the maps $\widetilde{\mathbb{N}}^\varepsilon$ and $\widetilde{\mathcal{N}}^\varepsilon$ are lower semicontinuous over X.*

Proof Let $(\bar{\boldsymbol{x}}_n)_n$ be a sequence in X which converges to $\bar{\boldsymbol{x}} \in X$ and let $\bar{\boldsymbol{y}} \in \mathbb{N}^\varepsilon(\bar{\boldsymbol{x}})$, i.e.

$$d(\bar{\boldsymbol{y}}, K(\bar{\boldsymbol{x}})) \; < \varepsilon_2 \text{ and } F_j(\boldsymbol{x}, \bar{\boldsymbol{y}}) \le \inf_{z_j \in K_j(\bar{\boldsymbol{x}})} F_j(\bar{\boldsymbol{x}}, z_j, \bar{\boldsymbol{y}}_{-j}) + \varepsilon_1 \quad \forall\, j = 1, .., h. \tag{13}$$

Let $\boldsymbol{z} \in K(\bar{\boldsymbol{x}})$ such that $||\bar{\boldsymbol{y}} - \boldsymbol{z}|| < \varepsilon_2$ and let $(\boldsymbol{z}_n)_n$ be a sequence, existing since K is lower semicontinuous over X, which converges to $\boldsymbol{z}$ such that $\boldsymbol{z}_n \in K(\bar{\boldsymbol{x}}_n)$ for n sufficiently large. Then, the sequence $(\bar{\boldsymbol{y}}_n)_n$, defined by $\bar{\boldsymbol{y}}_n = \bar{\boldsymbol{y}}$ for any n, converges to $\bar{\boldsymbol{y}}$ and $\bar{\boldsymbol{y}}_n \in \mathbb{N}^\varepsilon(\bar{\boldsymbol{x}}_n)$ for n large since:

- $\limsup_n d(\bar{\boldsymbol{y}}_n, K(\bar{\boldsymbol{x}}_n)) \le d(\bar{\boldsymbol{y}}, K(\bar{\boldsymbol{x}})) \le ||\bar{\boldsymbol{y}} - \boldsymbol{z}|| < \varepsilon_2$;

- if there exist $j \in \{1,..,m\}$, an increasing sequence of integers $(n_h)_h$ and a sequence $(z_h^j)_h$ in Y_j such that $z_h^j \in K_j(\bar{\boldsymbol{x}}_{n_h})$ and $F_j(\bar{\boldsymbol{x}}_{n_h}, \bar{\boldsymbol{y}}_{n_h}) > F_j(\bar{\boldsymbol{x}}_{n_h}, z_h^j, \bar{\boldsymbol{y}}_{n_h}^{-j}) + \varepsilon_1$ for any $h \in \mathbb{N}$, subcontinuity and closedness assumptions imply that a subsequence of $(z_h^j)_h$ converges to a point $z_j \in K_j(\bar{\boldsymbol{x}})$, so, by continuity of F_j we have $F_j(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \geq F_j(\bar{\boldsymbol{x}}, z_j, \bar{\boldsymbol{y}}_{-j}) + \varepsilon_1$ and we get a contradiction with (13).

The proof for $\widetilde{\mathcal{N}}^{\varepsilon}$ goes similarly and is omitted. $\square$

However, since we are interested in approximating the second stage of MF games, it is important to deal with approximating set-valued maps which are closed-valued in order to get the existence of approximating solutions to the problem (PMF). Thus, neither $\widetilde{\mathcal{N}}^{\varepsilon}$ and $\widetilde{\mathbb{N}}^{\varepsilon}$ can be useful to this purpose. Nevertheless, in [26] and [27], Morgan and Raucci actually proved the lower semicontinuity property of $\mathcal{N}^{\varepsilon}$ and $\mathbb{N}^{\varepsilon}$, but for restricted classes of payoffs and when $\varepsilon_2 = 0$.
More precisely:
– in order to get that $\mathcal{N}^{\varepsilon}$ is lower semicontinuous over X, the function $\sum_{j=1}^{m} F_j(\boldsymbol{x}, \cdot)$ is assumed to be convex on Y and $F_j(\boldsymbol{x}, \cdot, \boldsymbol{y}_{-j})$ is assumed to be concave over Y_j for every $j = 1,..,m$;

– in order to get that $\mathbb{N}^{\varepsilon}$ is lower semicontinuous over X, the function $F_j(\boldsymbol{x}, \cdot)$ is assumed to be convex on Y and the function $\inf_{v_j \in Y_j} F_j(\boldsymbol{x}, v_j, \cdot)$ is assumed to be concave over Y_{-j} for every $j = 1,..,m$.

However, the above conditions are not necessary, as shown by the next example.

Example 4.1 Let $l = 1$, $h = m = 2$, $X = [0, +\infty[$, $Y = [0,1] \times [0,1]$, $K(x) = Y$ and $F_1(x, \boldsymbol{y}) = F_2(x, \boldsymbol{y}) = x\, y_1\, y_2$ for any $x \in X$. One can check that $\mathcal{N}$ is not lower semicontinuous at $x = 0$, but both $\mathbb{N}^{\varepsilon}$ and $\mathcal{N}^{\varepsilon}$ do are, even if F_1, F_2 and $F_1 + F_2$ are not convex over Y.

Therefore, from now on, let us consider the class of payoffs that satisfy the following assumptions:

H_1) for any $j = 1,..,m$ and $\boldsymbol{x} \in X$, the payoff $F_j(\boldsymbol{x}, \cdot, \boldsymbol{y}_{-j})$ is differentiable with respect to y_j on Y_j and the map K_j is subcontinuous, lower semicontinuous and closed over X;

H_2) for any $j = 1,..,m$ the set Y_j is convex, the set valued map K_j is convex-valued over X and, for any $(\boldsymbol{x}, \boldsymbol{y}) \in X \times Y$, the function $F_j(\boldsymbol{x}, \cdot, \boldsymbol{y}_{-j})$ is pseudoconvex over Y_j.

Then, also the maps $\mathcal{M}^{\varepsilon}$ and $\mathcal{S}^{\epsilon}$, defined by

$$\mathcal{M}^{\varepsilon} : \boldsymbol{x} \in X \rightrightarrows \mathcal{M}^{\varepsilon}(\boldsymbol{x}) = \left\{ \boldsymbol{y} \in Y : \, d(\boldsymbol{y}, K(\boldsymbol{x})) \leq \varepsilon_2 \;\text{ and }\; \sum_{j=1}^{m} \left\langle \frac{\partial F_j}{\partial v_j}(\boldsymbol{x}, \boldsymbol{v}), y_j - v_j \right\rangle \leq \varepsilon_1 \;\; \forall\, \boldsymbol{v} \in K(\boldsymbol{x}) \right\}$$

$$\mathcal{S}^{\epsilon} : \boldsymbol{x} \in X \rightrightarrows \mathcal{S}^{\epsilon}(\boldsymbol{x}) = \left\{ \boldsymbol{y} \in Y : \, d(\boldsymbol{y}, K(\boldsymbol{x})) \leq \varepsilon_2 \;\text{ and }\; \sum_{j=1}^{m} \left\langle \frac{\partial F_j}{\partial y_j}(\boldsymbol{x}, \boldsymbol{y}), y_j - v_j \right\rangle \leq \varepsilon_1 \;\; \forall\, \boldsymbol{v} \in K(\boldsymbol{x}) \right\},$$

are suitable approximations of the map $\mathcal{N}$, as proved in the below proposition.

Proposition 4.3 *If assumptions H_1) and H_2) hold and* $\dfrac{\partial F_j}{\partial y_j}$ *is continuous over $X \times Y$, for any $j = 1, .., m$, then, for any sequence $(\varepsilon_n)_n = (\varepsilon_n^1, \varepsilon_n^2)_n$, where $(\varepsilon_n^1)_n$ and $(\varepsilon_n^2)_n$ are decreasing to zero, any $\boldsymbol{x} \in X$ and any sequence $(\boldsymbol{x}_n)_n$ converging to $\boldsymbol{x}$ in X, we have*

$$\limsup_n \mathcal{M}^{\varepsilon_n}(\boldsymbol{x}_n) \subseteq \limsup_n \mathcal{S}^{\varepsilon_n}(\boldsymbol{x}_n) \subseteq \mathcal{S}(\boldsymbol{x}) = \mathcal{M}(\boldsymbol{x}) = \mathcal{N}(\boldsymbol{x}).$$

Proof Let $\boldsymbol{x} \in X$ and let $(\boldsymbol{x}_n)_n$ be a sequence converging to $\boldsymbol{x}$ in X. Under the given assumptions, the set of Nash equilibria $\mathcal{N}(\boldsymbol{x})$ of the game $\Omega(\boldsymbol{x})$ coincides with the solution set $\mathcal{S}(\boldsymbol{x})$ to the variational inequality in (6) and to the solution set $\mathcal{M}(\boldsymbol{x})$ to the Minty variational inequality in (9) (see, for example, [6]). Thus, the result follows from Proposition 3.4 applied to the function A defined in (7). $\square$

The next result follows from Proposition 3.1.

Proposition 4.4 *If assumptions H_1) holds and* $\dfrac{\partial F_j}{\partial y_j}$ *is continuous over $X \times Y$, for any $j = 1, .., m$, then, the map $\widetilde{\mathcal{M}}^\varepsilon$ is lower semicontinuous over X.*
If also assumption H_2) holds, then the map $\mathcal{M}^\varepsilon$ is lower semicontinuous over X and both the maps $\widetilde{\mathcal{M}}^\varepsilon$ and $\mathcal{M}^\varepsilon$ are convex-valued.

Now, we are able to clarify why the above proposition is crucial for investigating MF games. As observed in the Introduction, a *really good* approximation map of exact equilibria of the second stage of an MF game should be closed-valued, lower semicontinuous and, simultaneously, it should asymptotically reach the exact equilibria, so, neither $\mathcal{N}^\varepsilon$ and $\mathbb{N}^\varepsilon$, as well $\widetilde{\mathcal{N}}^\varepsilon$ and $\widetilde{\mathbb{N}}^\varepsilon$, can be useful. Under differentiability assumption, both the maps $\mathcal{S}^\epsilon$ and $\mathcal{M}^\varepsilon$, defined before, allow to approach exact Nash equilibria of the second stage, but, as seen in Section 3, $\mathcal{S}^\epsilon$ may fail to be lower semicontinuous under non-restrictive assumptions. In the case of Example 4.1, all approximations $\mathbb{N}^\varepsilon$, $\mathcal{N}^\varepsilon$, $\mathcal{S}^\epsilon$ and $\mathcal{M}^\varepsilon$ are lower semicontinuous, but one has to explicitly check $\mathbb{N}^\varepsilon$, $\mathcal{N}^\varepsilon$, $\mathcal{S}^\epsilon$ in order to get their lower semicontinuity, whereas the lower semicontinuity of $\mathcal{M}^\varepsilon$ follows from Proposition 4.4. Therefore, regularizing the equilibria map at the second stage by considering the map $\mathcal{M}^\varepsilon$ seems to be the most suitable approach.

5 Approximate and Viscosity Solutions in MF Games via VI's

In this section we consider the Pessimistic MF problem defined in (3) assuming that H_1) and the following hold:

H_3) MF is a weighted potential game, in the sense of Definition 2.1, with Π as a weighted potential and $\beta_i \in \mathbb{R}_{++}$, so that for the leader payoffs L_i we have

$$L_i(\boldsymbol{x}, \boldsymbol{y}) = \beta_i \Pi(\boldsymbol{x}, \boldsymbol{y}) + \Psi_i(\boldsymbol{x}_{-i}), \ \forall\, (\boldsymbol{x}, \boldsymbol{y}) \in X \times Y,\ i = 1, .., k.$$

Then, in line with previous papers [20], [21], [22], we could introduce several concepts of viscosity solutions in the class of weighted potential MF games, each of which linked to one of the approximations of the follower's Nash equilibrium map considered in Section 4. However, as already said above, we do not use here $\mathcal{N}^{\varepsilon}$, $\widetilde{\mathcal{N}}^{\varepsilon}$, $\mathbb{N}^{\varepsilon}$ and $\widetilde{\mathbb{N}}^{\varepsilon}$, addressing to [26] for results linked to such approximations in the case of only one leader.

Definition 5.1 *Let the MF game be a weighted potential game with* Π *as a weighted potential. A point* $\bar{\boldsymbol{x}} \in X$ *is a (pessimistic)* $\mathcal{M}$*-viscosity solution to the problem* (PMF) *if for every sequence of couples of positive numbers decreasing to zero* $(\varepsilon_n^1, \varepsilon_n^2)_n = (\varepsilon_n)_n$ *there exists a sequence* $(\bar{\boldsymbol{x}}_n)_n$, $\bar{\boldsymbol{x}}_n \in X$ *for any* $n \in \mathbb{N}$, *such that:*

$V_1)$ *a subsequence* $(\bar{\boldsymbol{x}}_{n_k})_k$ *converges towards* $\bar{\boldsymbol{x}}$*;*

$V_2)$ *for any* $n \in \mathbb{N}$, $\bar{\boldsymbol{x}}_n$ *is a minimum point for the function*

$$\mathbb{P}^{\varepsilon_n} : \boldsymbol{x} \in X \to \mathbb{P}^{\varepsilon_n}(\boldsymbol{x}) \;=\; \sup_{\boldsymbol{y} \in \mathcal{M}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y})$$

i.e.
$$\mathbb{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \leq \mathbb{P}^{\varepsilon_n}(\boldsymbol{x}) \quad \forall\, \boldsymbol{x} \in X;$$

$V_3)$
$$\lim_n \mathbb{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \;=\; \inf_{\boldsymbol{x} \in X} \sup_{\boldsymbol{y} \in \mathcal{N}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y}) = \omega.$$

Definition 5.2 *Let the MF game be a weighted potential game with* Π *as a weighted potential. Then, the function*

$$\mathbb{P}^{\varepsilon} : \boldsymbol{x} \in X \to \mathbb{P}^{\varepsilon}(\boldsymbol{x}) = \sup_{\boldsymbol{y} \in \mathcal{M}^{\varepsilon}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y}) \tag{14}$$

is called an ε*-*$\mathcal{M}$*(pessimistic) weighted potential and any minimum point, if existing, for* $\mathbb{P}^{\varepsilon}$ *is said an* ε*-*$\mathcal{M}$*(pessimistic) solution to the problem (PMF).*

An $\mathcal{M}$-viscosity solution $\bar{\boldsymbol{x}}$ can be seen as the limit of a sequence of approximate solutions to a two-stage game with one leader, having the weighted potential function Π as payoff, and m followers playing the game $\Omega(\bar{\boldsymbol{x}})$. Then, condition $V_3)$ is significant since it requires that the values of those approximating sequences should converge to the security value of a bilevel problem with Nash equilibrium constraint $\Omega(\bar{\boldsymbol{x}})$, so that $\bar{\boldsymbol{x}}$ can be seen as a security strategy for a One-Leader-Multi-Follower game and we address to [4] for a deep review about approximation in One-Leader games.

Proposition 5.1 *If the set* X *is compact,* $F_j \in C^1(X \times Y)$ *for any* $j = 1, .., m$ *and the weighted potential* Π *is lower semicontinuous over* $X \times Y$, *then for any couple* $\varepsilon = (\varepsilon_1, \varepsilon_2)$ *there exists an* ε*-*$\mathcal{M}$ *solution to the problem (PMF).*

Proof The assumptions on the followers payoff F_j imply that the function A in (7) is continuous and by Proposition 3.1 the map $\mathcal{M}^{\varepsilon}$ is lower semicontinuous over X. Then the function $\mathbb{P}^{\varepsilon}$ is lower semicontinuous over X [1] and has a minimum point which is an ε-$\mathcal{M}$ solution to the problem (PMF). $\square$

For notations simplicity, from now on, we denote by **H**) the set of assumptions $H_1) + H_2) + H_3)$.

Proposition 5.2 *Under the assumptions of Proposition 5.1, if, moreover,* **H**) *and the following hold:*
i) for every $\boldsymbol{x} \in X$ *there exists a sequence* $(\boldsymbol{x}_n)_n$ *converging to* $\boldsymbol{x}$ *in* X *such that for every* $\boldsymbol{y} \in Y$ *and every sequence* $(\boldsymbol{y}_n)_n$ *converging to* $\boldsymbol{y}$ *in* Y *one has*

$$\limsup_n \Pi(\boldsymbol{x}_n, \boldsymbol{y}_n) \ \leq \ \Pi(\boldsymbol{x}, \boldsymbol{y});$$

then, there exists an $\mathcal{M}$*-viscosity solution to the problem (PMF).*

Proof Let $(\varepsilon_n^1, \varepsilon_n^2)_n = (\varepsilon_n)_n$ be a sequence of couples of positive numbers decreasing to zero and let $(\bar{\boldsymbol{x}}_n)_n$, $\bar{\boldsymbol{x}}_n \in X$ for any $n \in \mathbb{N}$, be a sequence such that $\mathbb{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \ = \ \min_{\boldsymbol{x}\in X} \sup_{\boldsymbol{y}\in\mathcal{M}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y})$.
Since the set X is compact, a subsequence $(\bar{\boldsymbol{x}}_{n_h})_h$ of $(\bar{\boldsymbol{x}}_n)_n$ converges to $\bar{\boldsymbol{x}} \in X$. Let us show that condition V_3) is satisfied, so we could conclude that $\bar{\boldsymbol{x}}$ is an $\mathcal{M}$-viscosity solution to the problem (PMF).
In fact, since, for any $n \in \mathbb{N}$, $\mathcal{N}(\bar{\boldsymbol{x}}) = \mathcal{S}(\bar{\boldsymbol{x}}) = \mathcal{M}(\bar{\boldsymbol{x}}) \subseteq \mathcal{M}^{\varepsilon_n}(\bar{\boldsymbol{x}})$, the following inequalities hold:

$$\inf_{\boldsymbol{x}\in X} \sup_{\boldsymbol{y}\in\mathcal{N}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y}) \ \leq \ \liminf_n \inf_{\boldsymbol{x}\in X} \sup_{\boldsymbol{y}\in\mathcal{M}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y}) \ = \ \liminf_n \mathbb{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n).$$

If there exists a real number c such that $\inf_{\boldsymbol{x}\in X} \sup_{\boldsymbol{y}\in\mathcal{N}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y}) < c < \limsup_n \mathbb{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n)$, we can determine a point $\boldsymbol{x}' \in X$ and a subsequence $(\bar{\boldsymbol{x}}_{n_k})_k$ of $(\bar{\boldsymbol{x}}_n)_n$ such that

$$\sup_{\boldsymbol{y}\in\mathcal{N}(\boldsymbol{x}')} \Pi(\boldsymbol{x}', \boldsymbol{y}) < c < \mathbb{P}^{\varepsilon_{n_k}}(\bar{\boldsymbol{x}}_{n_k}) \tag{15}$$

for any $k \in N$. Moreover, condition i) implies that, corresponding to $\boldsymbol{x}'$, there exists a sequence $(\boldsymbol{x}'_n)_n$ converging to $\boldsymbol{x}'$ in X such that $\limsup_n \Pi(\boldsymbol{x}'_n, \boldsymbol{y}_n) \leq \Pi(\boldsymbol{x}', \boldsymbol{y})$ for any $\boldsymbol{y} \in Y$ and any sequence $(\boldsymbol{y}_n)_n$ converging to $\boldsymbol{y}$ in Y.
Since $\mathbb{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \leq \mathbb{P}^{\varepsilon_n}(\boldsymbol{x}'_n)$ for any $n \in \mathbb{N}$, right inequality in (15) implies that, for any $k \in \mathbb{N}$,

$$c < \mathbb{P}^{\varepsilon_{n_k}}(\boldsymbol{x}'_{n_k}) \ = \ \sup_{\boldsymbol{y}\in\mathcal{M}^{\varepsilon_{n_k}}(\boldsymbol{x}'_{n_k})} \Pi(\boldsymbol{x}'_{n_k}, \boldsymbol{y}),$$

so there exists a sequence $(\boldsymbol{y}'_k)_k$, $\boldsymbol{y}'_k \in \mathcal{M}^{\varepsilon_{n_k}}(\boldsymbol{x}'_{n_k})$, such that $c < \Pi(\boldsymbol{x}'_{n_k}, \boldsymbol{y}'_k)$ for any $k \in \mathbb{N}$. A subsequence of $(\boldsymbol{y}'_k)_k$, still denoted by $(\boldsymbol{y}'_k)_k$, converges to a point $\boldsymbol{y}' \in Y$ and $\boldsymbol{y}' \in \mathcal{M}(\boldsymbol{x}')$ due to Proposition 3.4. Therefore, since our assumptions imply that $\mathcal{M}(\boldsymbol{x}') = \mathcal{S}(\boldsymbol{x}') = \mathcal{N}(\boldsymbol{x}')$, $\boldsymbol{y}' \in \mathcal{N}(\boldsymbol{x}')$ and we get by i) that $c \leq \Pi(\boldsymbol{x}', \boldsymbol{y}') \leq \sup_{\boldsymbol{y}\in\mathcal{N}(\boldsymbol{x}')} \Pi(\boldsymbol{x}', \boldsymbol{y})$, contradicting (15). □

The above results have been obtained under non-restrictive assumptions on the data: for instance, condition i) holds if, for any $\boldsymbol{x} \in X$, the weighted potential $\Pi(\boldsymbol{x}, \cdot)$ is upper semicontinuous over Y.
Now, for sake of completeness, we present also viscosity solutions obtained by using $\mathcal{S}^\epsilon(\boldsymbol{x})$, instead of $\mathcal{M}^\varepsilon(\boldsymbol{x})$.

Definition 5.3 *Let the MF game be a weighted potential game with* Π *as a weighted potential. A point* $\bar{\boldsymbol{x}} \in X$ *is a (pessimistic) viscosity solution to the problem (PMF) if for every sequence of couples of positive numbers decreasing to zero* $(\varepsilon_n^1, \varepsilon_n^2)_n = (\varepsilon_n)_n$ *there exists a sequence* $(\bar{\boldsymbol{x}}_n)_n$, $\bar{\boldsymbol{x}}_n \in X$ *for any* $n \in \mathbb{N}$, *such that:*

V_1) *a subsequence* $(\bar{\boldsymbol{x}}_{n_k})_k$ *converges towards* $\bar{\boldsymbol{x}}$;

V_2) *for any* $n \in \mathbb{N}$, $\bar{\boldsymbol{x}}_n$ *is a minimum point for the function*

$$\mathfrak{P}^{\varepsilon_n} : \boldsymbol{x} \in X \to \mathfrak{P}^{\varepsilon_n}(\boldsymbol{x}) \;=\; \sup_{\boldsymbol{y} \in \mathcal{S}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y})$$

i.e.

$$\mathfrak{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \leq \mathfrak{P}^{\varepsilon_n}(\boldsymbol{x}) \quad \forall\, \boldsymbol{x} \in X;$$

V_3)

$$\lim_n \mathfrak{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \;=\; \inf_{\boldsymbol{x} \in X} \sup_{\boldsymbol{y} \in \mathcal{N}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y}) = \omega.$$

Definition 5.4 *Let the MF game be a weighted potential game with* Π *as a weighted potential. Then, the function*

$$\mathfrak{P}^{\varepsilon} : \boldsymbol{x} \in X \to \mathfrak{P}^{\varepsilon}(\boldsymbol{x}) = \sup_{\boldsymbol{y} \in \mathcal{S}^{\epsilon}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y}) \tag{16}$$

is called ε*-(pessimistic) weighted potential and any minimum point, if existing, for* $\mathfrak{P}^{\varepsilon}$ *is said an* ε*-(pessimistic) solution to the problem (PMF).*

Proposition 5.3 *Assume that the set* X *is compact and* $F_j \in C^1(X \times Y)$ *is a quadratic function, for any* $j = 1, .., m$. *If, moreover, for any* $\boldsymbol{x} \in X$:
ii) *the operator* $A(\boldsymbol{x}, \cdot)$ *defined in (7) is monotone over* Y;
iii) *the weighted potential* Π *is lower semicontinuous over* $X \times Y$;
then, for any couple $\varepsilon = (\varepsilon_1, \varepsilon_2)$ *there exists an* ε*-solution to the problem (PMF).*

Proof Since $F_j \in C^1(X \times Y)$ and $F_j(\boldsymbol{x}, \cdot)$ is quadratic, the function A in (7) is continuous over $X \times Y$ and $A(\boldsymbol{x}, \cdot)$ is affine over Y. Then, Proposition 3.3 can be applied, so that the map $\mathcal{S}^{\epsilon}$ is lower semicontinuous over X and, from *iii*) the function $\mathfrak{P}^{\varepsilon}$ is lower semicontinuous over X and has a minimum point which turns out to be an ε-solution to the problem (PMF). □

Therefore, by arguing as in Proposition 5.2, one proves the existence of viscosity solutions.

Proposition 5.4 *Assume that the set* X *is compact and* $F_j \in C^1(X \times Y)$ *is a quadratic function, for any* $j = 1, .., m$. *If, moreover,* **H**), *i*), *ii*) *and iii*) *hold, then, there exists a viscosity solution to the problem (PMF).*

Corollary 5.1 *Assume that the set* X *is compact and* $F_j \in C^1(X \times Y)$ *is a quadratic function, for any* $j = 1, .., m$. *If* **H**), *i*) *and iii*) *are satisfied and, for any* $\boldsymbol{x} \in X$, *the followers game* $\Omega(\boldsymbol{x})$ *is potential and has a convex potential, then, for any couple* $\varepsilon = (\varepsilon_1, \varepsilon_2)$, *there exist* ε*-solutions and viscosity solutions to the problem (PMF).*

Proof Since the game $\Omega(\boldsymbol{x})$ has a convex potential, then the function $A(\boldsymbol{x},\cdot)$ is monotone over Y [18], so, condition $ii)$ in Proposition 5.3 is satisfied. $\square$

To complete this section, we show in the following example how simple is determining $\mathcal{M}$-viscosity solutions.

Example 5.1 Let $l = h = 2, k = h = 2, X_1 = X_2 = [0,1], Y_1 = Y_2 = [0,1]$, and consider the MF game defined by

$$K_1(\boldsymbol{x}) = K_2(\boldsymbol{x}) = [0,1], \quad F_1(\boldsymbol{x},\boldsymbol{y}) = x_1\,y_1 + x_2\,y_2, \quad F_2(\boldsymbol{x},\boldsymbol{y}) = y_1\,(x_1 + x_2),$$

$$L_1(\boldsymbol{x},\boldsymbol{y}) = x_1 + x_2 + y_1, \qquad L_2(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}(x_1 + y_1) - x_1.$$

Then, the MF game is a weighted potential game and a weighted potential is $\Pi(\boldsymbol{x},\boldsymbol{y}) = x_1 + y_1$ with $\beta_1 = 1$ and $\beta_2 = \dfrac{1}{2}$. One can check that

$$\mathcal{N}(\boldsymbol{x}) = [0,1]\times[0,1] \quad \text{if } x_1 = 0,$$
$$\mathcal{N}(\boldsymbol{x}) = \{0\}\times[0,1] \quad \text{if } x_1 > 0,$$

so that

$$\mathcal{P}_1(\boldsymbol{x}) = 1 + x_2, \qquad \mathcal{P}_2(\boldsymbol{x}) = \frac{1}{2} \qquad \text{when } x_1 = 0,$$

$$\mathcal{P}_1(\boldsymbol{x}) = x_1 + x_2, \qquad \mathcal{P}_2(\boldsymbol{x}) = -\frac{x_1}{2} \qquad \text{when } x_1 > 0.$$

Thus, the game $(X_1, X_2, \mathcal{P}_1, \mathcal{P}_2)$ does not have Nash equilibria and problem (PMF) does not have pessimistic solutions.
Since, for every $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{v}$ in $[0,1]\times[0,1]$, we have

$$\langle A(\boldsymbol{x},\boldsymbol{v}), \boldsymbol{y}-\boldsymbol{v}\rangle = \frac{\partial F_1}{\partial v_1}(\boldsymbol{x},\boldsymbol{v})(y_1 - v_1) + \frac{\partial F_2}{\partial v_2}(\boldsymbol{x},\boldsymbol{v})(y_2 - v_2) = x_1(y_1 - v_1),$$

it is easy to check that

$$\mathcal{M}^{\varepsilon}(x) = [0,1]\times[0,1] \qquad \text{if } \boldsymbol{x} \in [0,\varepsilon]\times[0,1],$$
$$\mathcal{M}^{\varepsilon}(x) = \left[0, \frac{\varepsilon}{x_1}\right]\times[0,1] \qquad \text{if } \boldsymbol{x} \in\,]\varepsilon,1]\times[0,1].$$

Therefore, for any sequence $(\varepsilon_n)_n$ decreasing to zero we have

$$\mathbb{P}^{\varepsilon_n}(\boldsymbol{x}) = \sup_{\boldsymbol{y}\in\mathcal{M}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x},\boldsymbol{y}) = x_1 + 1 \quad \text{if } x_1 \in [0,\varepsilon_n]$$

$$\mathbb{P}^{\varepsilon_n}(\boldsymbol{x}) = \sup_{\boldsymbol{y}\in\mathcal{M}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x},\boldsymbol{y}) = x_1 + \frac{\varepsilon_n}{x_1} \quad \text{if } x_1 \in\,]\varepsilon_n, 1]$$

and the set of $\mathcal{M}$-viscosity solutions, in this case, is $\{0\}\times[0,1]$, since conditions $V_1) - V_3)$ are satisfied by these points. Indeed, for any $x_2 \in [0,1]$:

$$\lim_n(\sqrt{\varepsilon_n}, x_2) \ = \ (0, x_2)$$

$$\mathbb{P}^{\varepsilon_n}(\sqrt{\varepsilon_n}) \ = \ 2\sqrt{\varepsilon_n} \ = \ \inf_{\boldsymbol{x}\in X} \sup_{\boldsymbol{y}\in\mathcal{M}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y})$$

$$\lim_n \mathbb{P}^{\varepsilon_n}(\sqrt{\varepsilon_n}) \ = \ 0 \ = \ \inf_{\boldsymbol{x}\in X} \sup_{\boldsymbol{y}\in\mathcal{N}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y}).$$

6 Conclusions

Regularizing the follower's Nash equilibria map by using approximate solutions to suitable variational inequalities is here proven to be fruitful in investigating pessimistic problems (PMF) associated to weighted potential Multi-Leader-Multi-Follower games. In fact, this approach allows to define two types of approximate and viscosity solutions for such problems, by using classical or Minty variational inequalities, whose existence is guaranteed under not too restrictive assumptions. More precisely, existence of approximate and viscosity solutions defined via Minty variational inequalities is assured for a larger class of followers payoffs and, moreover, this type of approximation seems to be more handy and simple to determine.

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