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Abstract
We consider a pure exchange economy with a finite number of commodities and households, in which preferences are affected by the welfare of others-measured in terms of wealth-and households are allowed to transfer resources to one another. Our goal is to establish the existence of equilibria under more general and realistic assumptions than those adopted in the literature. Specifically, we substitute an ad hoc limit on individual transfers for a bound that is dual-purpose: it satisfies the mathematical criteria necessary for our proof and adheres to the legal frameworks typically prevalent across various countries. This approach allows us to prove existence under standard assumptions of continuity and quasi-concavity, rather than the stronger assumptions of Lipschitz continuity and concavity imposed in the literature.

Keywords: Exchange economy, interdependent preferences, transfers, competitive equilibrium, generalized game approach.

MSC Classification: 91B24, 91B51, 91A10.

JEL Classification: C62, D50, D51, D62.

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1 Introduction

We consider a pure exchange economy with a finite number of commodities and households, in which preferences are affected by the welfare of others, measured in terms of wealth, and households are allowed to transfer resources to one another. Our goal is to prove the existence of equilibria under more general and realistic assumptions than those adopted in the literature.

The importance of consumption externalities has long been recognized in the literature on other-regarding preferences in a general equilibrium framework, starting with the well-known textbook by Arrow and Hahn (1971). Several contributions have followed; see Laffont and Laroque (1972), Borglin (1973), Shafer and Sonnenschein (1975), Florenzano (2003), Del Mercato (2006), and Balasko (2015). On the other hand, somewhat surprisingly, only a relatively small strand of the literature on other-regarding preferences in general equilibrium acknowledges an obvious implication of such preferences: the possibility of transferring resources to individuals one cares about. A classic example arises when parents care about their children's well-being after they move out. Another important example is philanthropic giving to the nonprofit sector, which has a significant economic impact by enabling organizations to sustain activities in areas such as education and healthcare, where the for-profit sector may fail to provide adequate coverage.
To the best of our knowledge, the only existing papers that consider other-regarding preferences in terms of wealth together with the possibility of transfers are the seminal contribution by Kranich (1988) and the paper by Donato et al. (2025), which identifies and partially resolves some important issues in both the setup of the model and the existence proof presented by Kranich. See also Mercier Ythier (2000) and Donato et al. (2026) for models in which welfare is measured in terms of consumption.

We now briefly describe the above-mentioned issues, how they have been addressed in the literature, and how we deal with them in the present paper. To begin with, we note the following. To obtain our existence result, we apply a standard and powerful theorem based on a fixed-point argument reformulated in terms of a generalized game, following the seminal work of Debreu (1952)[1] : see Theorem 4. To apply this theorem, some key ingredients are required: quasi-concavity of households' utility functions and compactness of the strategy sets of all players, namely, compactness of the price set and of the set of physically feasible consumption and transfers. Compactness conditions are naturally satisfied in the standard exchange economy: quasi-concavity is a reasonable assumption, and compactness of consumption sets follows from resource constraints. However, matters are more complicated in models with other-regarding preferences.

While quasi-concavity in goods and in "the wealth of people we like" is widely accepted and reasonable, this is not the case for bads and for the wealth of people we dislike. Hence, some work is required to address the tension between "needed

[1] Se also Kreps (2013) for a particularly clear exposition of statement and proof of that theorem.

assumptions" and "realistic assumptions." While Kranich does not address this problem, the main idea in Donato et al. (2025) is as follows. If an individual dislikes someone, then she transfers nothing to that person; therefore, there is no loss of generality in ignoring the part of the utility function related to disliked households. Given this observation, it suffices to assume (quasi-)concavity in the wealth of those one likes in order to obtain (quasi-)concavity with respect to each household's choice variables. Thus, no unreasonable concavity assumption is required for the wealth of those one dislikes, since that wealth does not depend on one's own choice variables. In the present paper, we follow this approach.

Another difficulty concerns compactness. In many existence proofs, one temporarily imposes an upper bound on consumption, greater than total resources, and then, using a standard argument, shows that this bound is never binding at equilibrium. Hence, an equilibrium with an upper bound on consumption is also an equilibrium without such a bound. This argument does not extend to transfers, for which no natural physical upper bound exists. Indeed, the presence of transfers may generate a form of "insatiable altruism," leading to a loss of compactness of households' choice sets. As a consequence, robust examples of economies without equilibrium can be constructed (see again Donato et al. (2025)). Following Kranich, Donato et al. (2025) establish existence by imposing an artificial upper bound on transfers.

A further difficulty arises from the potential emptiness of the constraint set in the household maximization problem. For many values of the variables taken as given by a household, the constraint set may be empty. For instance, if a household expects others to transfer more than their own wealth, it may find itself with insufficient resources to guarantee strictly positive consumption to them, rendering the constraint set empty (see Section 2.2.4 in Donato et al. (2025)). Kranich does not address this issue. To address this issue, Donato et al. (2025) introduce a fictitious utility function defined on the entire Euclidean space containing the original domain, thereby allowing for negative wealth and consumption. They then construct a game associated with the original economy endowed with the extended utility function, prove the existence of an equilibrium for this game, and show that it corresponds to an equilibrium of the original economy. However, this approach comes at a cost: it requires a substantial strengthening of the assumptions on households' utility functions. Standard existence proofs only require continuity and quasi-concavity, but it is generally impossible to extend a function while preserving these properties (see De Bernardi and Vesely (2023)). Instead, much stronger assumptions, such as Lipschitz continuity and concavity, are needed to ensure extendability (again, see De Bernardi and Vesely (2023)).

Our strategy addresses both problems in a more realistic way, without imposing the strong assumptions proposed in Donato et al. (2025). We assume that the total amount a household can transfer to others is bounded above by the value of its initial endowment. This assumption (i) restores compactness and (ii) avoids the need to extend utility functions, allowing us to work under standard assumptions of continuity and quasi-concavity. There are several ways to formalize this upper bound, depending

on how transfers are measured—for instance, in units of a numeraire good or as vectors of commodities (see Appendix A.1 for details). However, these formulations do not yield a compact set of transfers when prices lie in the simplex, as required by our proof strategy. Our approach is to model transfers as proportions of the value of a household's endowment: since proportions lie in the simplex, compactness is ensured. This formulation has several advantages. First, it restores compactness, enabling the use of standard existence proofs. Second, since each household expects others to satisfy this requirement, excessively high expectations of transfers cannot arise, and the emptiness problem is avoided. Third, it is not necessary to extend utility functions to the entire Euclidean space, an extension that would require imposing quite strong assumptions. Fourth, the different ways of modeling transfers mentioned above are shown to be equilibrium-equivalent (see Propositions 16 and 17). Fifth, the assumption is economically realistic, as we explain below with reference to legal regulations on donations in several countries. In Italy, according to Article 769 of the Italian Civil Code, a donation is a contract by which a donor enriches another party by disposing of a right belonging to the donor or by assuming an obligation. Article 771 further provides that donations of future goods are null. Taken together, these provisions imply that the object of a valid donation must consist of assets over which the donor has current legal ownership. Although there is no unified European civil code, the principles of European private law (e.g., PECL and DCFR[2]) require that the object of a patrimonial transfer be under the legal control of the transferring party at the time of transfer. In addition, EU rules on creditor protection presuppose that individuals cannot dispose of resources they do not own. In U.S. law, a valid gift requires donative intent, delivery, and acceptance, and the delivery requirement presupposes effective control over the transferred asset.

Our approach yields strictly more general results than those currently available. Both in the present paper and in Donato et al. (2025), existence of equilibrium is established for both nonseparable utility functions and a specific class of separable utility functions.[3] In the nonseparable case, we assume continuity and quasi-concavity instead of Lipschitz continuity and concavity. In the separable case, we assume quasi-concavity instead of concavity for the selfish component, and continuity and quasi-concavity instead of Lipschitz continuity and concavity for the altruistic component; naturally, we maintain quasi-concavity of the overall utility function.[4]

Our existence result provides a useful foundation for the analysis of equilibrium regularity and welfare (see Lombardi et al. (2026)). In particular, the equivalence among different equilibrium concepts allows the model to be studied in terms of normalized prices (and numeraire transfers), which is the standard framework for regularity analysis in general equilibrium theory.

[2] PECL: Principles of European Contract Law; DCFR: Draft Common Frame of Reference.
[3] The specific utility functions used imply that the two specifications are logically independent (see the end of Section 2).
[4] Assumptions not mentioned in the above comparison coincide across the two models.

The remainder of the paper is organized as follows. To set the stage, Section 2, introduces the model and some basic assumptions. Section 3 introduces the equilibrium concept for separable utility functions and proves existence of an equilibrium, while Section 4 extends the result to the nonseparable case. Appendix A establishes an equivalence result among different equilibrium concepts. Furthermore, it also provides some properties of the altruistic part of the utility functions we use. Appendix B collects additional technical results.

2 Set up of the model

We describe an economy in which households, denoted by $h \in \{1, ..., H\} := \mathcal{H}$, exchange goods (or commodities), denoted by $c \in \{1, ..., C\}$, in order to maximize their well-being. Each household h is described by a consumption set $X_h \subseteq \mathbb{R}^C$ with generic element $x_h = (x_h^c)_{c \in \mathcal{C}} \in \mathbb{R}^C$, where $x_h^c \in \mathbb{R}$ is the consumption of good c by household h and by an endowment set included in $\mathbb{R}^C$ with generic element $e_h = (e_h^c)_{c \in \mathcal{C}} \in \mathbb{R}^C$, where $e_h^c \in \mathbb{R}$ is the amount of good c owned by household h. The total resources of the economy are denoted by $r := \sum_{h \in \mathcal{H}} e_h$.

Commodities can be exchanged with other commodities at exchange ratios described by a price vector p belonging to a price set contained in $\mathbb{R}^C$. Without loss of generality, we restrict prices to belong to the set

$$S = \{p \in \mathbb{R}_+^C : pr = 1\}.$$

Each household h can also choose to transfer resources to other households; the transfer vector is denoted by $\tau_h = (\tau_{hh'})_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}_+$, where $\tau_{hh'}$ is the proportion of the endowment of household h transferred to household h'. Assuming $\tau_{hh'}$ is non-negative formalizes the idea that households are not allowed to "steal "goods and must comply with this institutional requirement.

Household h's utility function depends upon her own consumption and anyone else's wealth. Denoting by $\theta_h \in \Theta \subseteq \mathbb{R}$ the household h's wealth, $\theta := (\theta_h)_{h \in \mathcal{H}} \in \Theta^H$ and $\theta_{\setminus h} := (\theta_{h'})_{h' \in \mathcal{H} \setminus \{h\}} \in \Theta^{H-1}$, we define the utility function

$$U_h : X_h \times \Theta^{H-1} \to \mathbb{R}, \ \ (x_h, \theta_{\setminus h}) \mapsto U_h(x_h, \theta_{\setminus h}).$$

Assumptions 1. For any $h \in \mathcal{H}$, $e_h \in \mathbb{R}_{++}^C$ [5].

It is well known that the quasi-concavity of utility functions is key for standard existence theorems, but it fails when the utility function includes "bads"like disliked individuals' consumption. We thus introduce an assumption to address this lack of quasi-concavity. To this aim, we denote by $\mathcal{B}_h$ and $\mathcal{B}_h^{\setminus}$ the sets of individuals that household h likes and dislikes, respectively, and assume the following.

[5] Using a standard "survival assumption", we endow households with a strictly positive vector of goods.

Assumptions 2. For any $h \in \mathcal{H}$, there exists an exogenously given partition[6] of $\mathcal{H}\backslash\{h\}$ in the two sets $\mathcal{B}_h$ and $\mathcal{B}_h^{\setminus}$ such that U_h is strictly increasing in $\theta_{h'}$ for any $h' \in \mathcal{B}_h$ and decreasing in $\theta_{h'}$ for any $h' \in \mathcal{B}_h^{\setminus}$.

We then define a new equilibrium concept in which each household "ignores" those individuals he dislikes, so that households exogenously disliked by h do not receive transfers from h by construction. This new equilibrium is equivalent to the original one (see Donato et al. (2025)). Since we assume that each household's utility is increasing in its own consumption bundle, if household h dislikes household h', then the optimal transfer $\tau_{hh'}$ is necessarily zero. The distinction is thus purely formal: in the original equilibrium $\tau_{hh'} = 0$, whereas in the new equilibrium the variable $\tau_{hh'}$ does not appear in the model (see Donato et al. (2025) for further details). Moreover, we define $\mathcal{B}_{\to h} = \{h' \in \mathcal{H} : h \in \mathcal{B}_{h'}\}$, as the set of households who like h and $\mathcal{B}_{\to h}^{\setminus} = \{h' \in \mathcal{H} : h \in \mathcal{B}_{h'}^{\setminus}\}$, as the set of households who dislike h; $B_h = \#\mathcal{B}_h$, $B = \sum_{h\in\mathcal{H}} B_h$, $B_h^{\setminus} = \#\mathcal{B}_h^{\setminus}$, $B_{\to h} = \#\mathcal{B}_{\to h}$ and $B_{\to h}^{\setminus} = \#\mathcal{B}_{\to h}^{\setminus}$.

As explained in the introduction, a crucial assumption of our model is the legal restriction that households cannot transfer more than the value of the owned endowment, as formalized below.

Assumptions 3. For any $h \in \mathcal{H}$ and $h' \in \mathcal{B}_h$, $\tau_h \in \Delta_h := \{\tau_h \in \mathbb{R}_+^{B_h} : \sum_{h'\in\mathcal{B}_h} \tau_{hh'} \leq 1\}$.

Moreover, we denote by $pe_{h'} + \sum_{h''\in\mathcal{B}_{\to h'}} \tau_{h''h'} \cdot (pe_{h''}) - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''} \cdot (pe_{h'})$, the relative[7] net wealth of household h', i.e., the value of her endowment, plus the value of the transfers she receives, minus the transfers she gives away.

As a consequence of the monotonicity assumption on the utility function, at any maximizing choice, relative net wealth is equal to the value of consumption and therefore it takes values between zero and one. Indeed, if the consumption may be zero, then endpoints of the interval are included; if consumption belongs to $\mathbb{R}_{++}^C$, since $p \in S$, then endpoints are excluded. In our analysis, we present two different specifications of each household h utility function. The first one is

$$\widetilde{U}_h : \mathbb{R}_{++}^C \times (0,1)^{H-1} \to \mathbb{R}, \quad (x_h, \theta_{\setminus h}) \mapsto u_h\,(x_h) + \sum_{h'\in\mathcal{H}\setminus\{h\}} v_{hh'}(\theta_{h'}),$$

and the second one is

$$\widehat{U}_h : \mathbb{R}_{+}^C \times [0,1]^{H-1} \to \mathbb{R}, \quad (x_h, \theta_{\setminus h}) \mapsto \widehat{U}_h(x_h, \theta_{\setminus h}),$$

where definition and assumptions on each utility function presented above are presented and discussed in Section 3 and 4, respectively. The main goal of the paper is to provide an existence results in the two versions of the model with the presented above utility functions.

[6] We allow the possibility that either $\mathcal{B}_h$ or $\mathcal{B}_h^{\setminus}$ may be empty. We also use the standard convention that $\sum_{x\in\varnothing} x = 0$.

[7] "Relative" means with respect to the total wealth of the economy which is $pr = 1$.

3 Model with separable utility functions

In this Section, we analyze the case of separable utility functions: the setting in which the utility function is composed of a selfish component and an other-regarding component. The assumptions on both components are presented below.

Assumptions 4. Selfish preferences of each household $h \in \mathcal{H}$ are described by the utility function $u_h : \mathbb{R}^C_{++} \to \mathbb{R}$ which is: u1. continuous; u2. strictly increasing; u3. quasi-concave; u4. for any $\alpha \in \mathbb{R}$, $\mathrm{Cl}_{\mathbb{R}^C}\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \alpha\} \subseteq \mathbb{R}^C_{++}$[8].

Assumptions 5. For any $h \in \mathcal{H}$ and $h' \in \mathcal{B}_h$, relative wealth-based altruistic preferences are described by the function $v_{hh'} : (0,1) \to \mathbb{R}$ which is: v1. $v_{hh'}$ is continuous; v2. $v_{hh'}$ is strictly increasing; v3. $v_{hh'}$ is bounded.

Observe that a strictly increasing function from an open set of $\mathbb{R}$ to $\mathbb{R}$ is strictly quasi-concave.

For any $h \in \mathcal{H}$, let $\mathcal{U}_h$ denote the set of utility functions u_h satisfying Assumption 4, and $\mathcal{V}_h$ the set of utility functions $(v_{hh'})_{h' \in \mathcal{B}_h}$ satisfying Assumption 5.

Assumptions 6. For any $h \in \mathcal{H}$, and for any $(u_h, (v_{hh'})_{h' \in \mathcal{B}_h}) \in \mathcal{U}_h \times \mathcal{V}_h$, the function

$$u_h + \sum_{h' \in \mathcal{B}_h} v_{hh'} : \mathbb{R}^C_{++} \times (0,1)^{B_h} \to \mathbb{R}, \qquad (x_h, (\theta_{h'})_{h' \in \mathcal{B}_h}) \mapsto u_h(x_h) + \sum_{h' \in \mathcal{B}_h} v_{hh'}(\theta_{h'})$$

is quasi-concave.

For any $h \in \mathcal{H}$, let $\mathcal{W}_h$ denote the set of utilities functions $(u_h, (v_{hh'})_{h' \in \mathcal{B}_h}) \in \mathcal{U}_h \times \mathcal{V}_h$ that satisfy Assumption 6.

Defined, the economy

$$\mathcal{E} = (e_h, u_h, (v_{hh'})_{h' \in \mathcal{B}_h})_{h \in \mathcal{H}} \in \times_{h \in \mathcal{H}} (\mathbb{R}^C_{++} \times \mathcal{W}_h) := \mathbb{E},$$

and, for any $h \in \mathcal{H}$, $\Delta_{\setminus h} = \times_{h' \in \mathcal{H} \setminus \{h\}} \Delta_{h'}$, $\tau_{\setminus h} = (\tau_{h'})_{h' \in \mathcal{H} \setminus \{h\}} \in \Delta_{\setminus h}$ and $\Delta = \times_{h' \in \mathcal{H}} \Delta_{h'}$, $\tau = (\tau_h)_{h \in \mathcal{H}} \in \Delta$, we present the definition of equilibrium.

Definition 1. *The vector $(x^*, \tau^*, p^*) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^B_+ \times S$ is an equilibrium for the economy $\mathcal{E} \in \mathbb{E}$ if*

i. for any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $p^* \in S$, $\tau^*_{\setminus h} \in \Delta_{\setminus h}$, $(x^*_h, \tau^*_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{B_h}$ solves

$$\max_{(x_h, \tau_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{B_h}} u_h(x_h) + \sum_{h' \in \mathcal{B}_h} v_{hh'}(p^* e_{h'} + \tau_{hh'} \cdot (p^* e_h) + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \tau^*_{h'' h'} \cdot (p^* e_{h''}) - \sum_{h'' \in \mathcal{B}_{h'}} \tau^*_{h' h''} \cdot (p^* e_{h'}))$$

$$\text{subject to } (x_h, \tau_h) \in \Gamma_h(p^*, \tau^*_{\setminus h}) :=$$

$$\{ (x'_h, \tau'_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{B_h} : p^* e_h + \sum_{h' \in \mathcal{B}_{\to h}} \tau^*_{h' h} \cdot (p^* e_{h'}) - \sum_{h' \in \mathcal{B}_h} \tau'_{hh'} \cdot (p^* e_h) - p^* x'_h \geq 0,$$
$$1 - \sum_{h' \in \mathcal{B}_h} \tau'_{hh'} \geq 0,\ \tau'_h \geq 0 \}$$

[8] Under Assumptions 4u4, u_h is unbounded below. Suppose there exists k such that $u_h(x_h) \geq k$ for any x_h. Take $(\frac{\mathbf{1}}{n})_{n \in \mathbb{N}}$, with $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^C$. So, $\{\frac{\mathbf{1}}{n} : n \in \mathbb{N}\}$ is included in $\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq k\}$ and 0 belongs to $\mathrm{Cl}_{\mathbb{R}^C}\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq k\} \subseteq \mathbb{R}^C_{++}$. This is a contradiction.

ii. $\sum_{h\in\mathcal{H}}(x_h^* - e_h) = 0$.
The reason for which the above definition contains no "market clearing conditions for transfers" is explained in the simple result below.

Proposition 1. *For any* $\tau \in \mathbb{R}^{HB}$, $p \in S$ *and* $e \in \mathbb{R}^{CH}_{++}$,

$$\sum_{h\in\mathcal{H}}\sum_{h'\in\mathcal{B}_{\to h}} \tau_{h'h}\cdot(pe_{h'}) = \sum_{h\in\mathcal{H}}\sum_{h'\in\mathcal{B}_h} \tau_{hh'}\cdot(pe_h)\,. \tag{1}$$

Proof. The proof is the same as in Proposition 1, in Donato et al. (2025), identifying $t_{hh'}$ with $\tau_{hh'}\cdot(pe_h)$ and $t_{h'h}$ with $\tau_{h'h}\cdot(pe_{h'})$. □

Remark 1. *Usually, the requirement that choice variables belong to the domain of the objective functions is not "repeated" in the constraint set. In some of the literature on the model we are analyzing, that convention lead to ignoring that requirement. We therefore include them explicitly, so the constraint set becomes:*

$$\begin{array}{l}
\{\ (x'_h,\tau'_h)\in\mathbb{R}^C\times\mathbb{R}^{B_h} : p^*e_h + \sum_{h'\in\mathcal{B}_{\to h}}\tau^*_{h'h}\cdot(p^*e_{h'}) - \sum_{h'\in\mathcal{B}_h}\tau'_{hh'}\cdot(p^*e_h) - p^*x'_h \geq 0,\\
x'_h >> 0,\ 1-\sum_{h'\in\mathcal{B}_h}\tau'_{hh'}\geq 0,\ \tau'_h\geq 0,\\
\text{for any } h'\in\mathcal{B}_h,\\
p^*e_{h'} + \tau_{hh'}\cdot(p^*e_h) + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''}) - \sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'}) > 0,\\
p^*e_{h'} + \tau_{hh'}\cdot(p^*e_h) + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''}) - \sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'}) < 1\ \}
\end{array}$$

It is difficult to show that the above set is closed and it is likely not, which hinders the existence strategy we wish to follow. We proceed as follows: (1) we extend the functions $v_{hh'}$ *to functions* $V_{hh'}$ *defined on* $[0,1]$*; (2) we establish the existence of fictitious equilibria defined using this extension; and (3) we show that fictitious equilibria coincide with equilibria.*

Proposition 2. *If* $v_{hh'}:(0,1)\to\mathbb{R}$ *is continuous, strictly increasing and bounded, then there exists an extension* $V_{hh'}$ *on* $[0,1]$ *of* v *which is continuous, strictly increasing and bounded.*

Proof. See Proposition 18 in Appendix A.2. □

We observe that, by Assumption 6 and by the continuity of functions $V_{hh'}$, for any $h'\in\mathcal{B}_h$, the function

$$u_h + \sum_{h'\in\mathcal{B}_h} V_{hh'} : \mathbb{R}^C_{++}\times[0,1]^{B_h}\to\mathbb{R}, \qquad (x_h,(\theta_{h'})_{h'\in\mathcal{B}_h})\mapsto u_h(x_h) + \sum_{h'\in\mathcal{B}_h} V_{hh'}(\theta_{h'})$$

is quasi-concave (see Proposition 19 in Appendix A.2).

Based on the above observations, we introduce the following definition.

Definition 2. *The vector* $(x^*,\tau^*,p^*)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^B_+\times S$ *is an extended utility equilibrium for the economy* $\mathcal{E}\in\mathbb{E}$ *if*

i. for any $h \in \mathcal{H}$, *for given* $\mathcal{E} \in \mathbb{E}$, $p^* \in S$, $\tau^*_{\backslash h} \in \Delta_{\backslash h}$, $(x^*_h, \tau^*_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{B_h}$ *solves*

$$\max_{(x_h,\tau_h)\in\mathbb{R}^C\times\mathbb{R}^{B_h}} u_h(x_h)+\sum_{h'\in\mathcal{B}_h} V_{hh'}(p^*e_{h'}+\tau_{hh'}\cdot(p^*e_h)+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''})-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'}))$$

subject to $(x_h, \tau_h) \in \Gamma^0_h(p^*, \tau^*_{\backslash h}) :=$

$$\{\ (x'_h,\tau'_h)\in\mathbb{R}^C\times\mathbb{R}^{B_h} : p^*e_h+\sum_{h'\in\mathcal{B}_{\to h}}\tau^*_{h'h}\cdot(p^*e_{h'})-\sum_{h'\in\mathcal{B}_h}\tau'_{hh'}\cdot(p^*e_h)-p^*x'_h\geq 0,$$
$$x'_h >> 0,\ 1-\sum_{h'\in\mathcal{B}_h}\tau'_{hh'}\geq 0,\ \tau'_h\geq 0,$$
$$\text{for any } h'\in\mathcal{B}_h,$$
$$p^*e_{h'}+\tau_{hh'}\cdot(p^*e_h)+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''})-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'})\leq 0,$$
$$p^*e_{h'}+\tau_{hh'}\cdot(p^*e_h)+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''})-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'})\leq 1\ \}$$

ii. $\sum_{h\in\mathcal{H}}(x^*_h - e_h) = 0$.

To get compactness of the constraint set, as in the existence proof we presented in Donato et al. (2025), we need to add an "innocuous" constraint, based on the simple observation that $(\frac{e_h}{4H}, 0) \in \Gamma^1_h(p^*, \tau^*_{\backslash h})$.

Definition 3. *The vector* $(x^*, \tau^*, p^*) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^B_+ \times S$ *is an extended utility with additional constraint equilibrium for the economy* $\mathcal{E} \in \mathbb{E}$ *if*
i. for any $h \in \mathcal{H}$, *for given* $\mathcal{E} \in \mathbb{E}$, $p^* \in S$, $\tau^*_{\backslash h} \in \Delta_{\backslash h}$, $(x^*_h, \tau^*_h) \in \mathbb{R}^C \times \mathbb{R}^{B_h}$ *solves*

$$\max_{(x_h,\tau_h)\in\mathbb{R}^C\times\mathbb{R}^{B_h}} u_h(x_h)+\sum_{h'\in\mathcal{B}_h} V_{hh'}(p^*e_{h'}+\tau_{hh'}\cdot(p^*e_h)+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''})-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'}))$$

subject to $(x_h, \tau_h) \in \Gamma^1_h(p^*, \tau^*_{\backslash h}) :=$

$$\Gamma^0_h(p^*,\tau^*_{\backslash h})\cap\{\ (x'_h,\tau'_h)\in\mathbb{R}^C\times\mathbb{R}^{B_h} :$$
$$u_h(x_h)+\sum_{h'\in\mathcal{B}_h} V_{hh'}(p^*e_{h'}+\tau_{hh'}\cdot(p^*e_h)+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''})-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'}))\geq$$
$$u_h(\tfrac{e_h}{4H})+\sum_{h'\in\mathcal{B}_h} V_{hh'}(p^*e_{h'}+0\cdot(p^*e_h)+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''})-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'}))\ \}$$

ii. $\sum_{h\in\mathcal{H}}(x^*_h - e_h) = 0$.

Proposition 3. *The maximization problem presented in Definition 2 and the one in Definition 3 have the same solution sets.*

Proof. The desired results follows from $(\frac{e_h}{4H}, 0) \in \Gamma^1_h(p^*, \tau^*_{\backslash h})$ and Lemma 72 in Section 9.2 of Chapter 8 in Villanacci et al. (2002), page 253-254. □

We finally introduce the following definition of equilibrium, where a standard upper bound on consumption is imposed, namely $k_r := r + \mathbf{1}_C \in \mathbb{R}^C_{++}$.

Definition 4. *The vector* $(x^*, \tau^*, p^*) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^B_+ \times S$ *is a fictitious equilibrium for the economy* $\mathcal{E} \in \mathbb{E}$ *if*

i. for any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $p^* \in S$, $\tau^*_{\setminus h} \in \Delta_{\setminus h}$, $(x^*_h, \tau^*_h) \in \mathbb{R}^C \times \mathbb{R}^{B_h}$ solves

$$\begin{array}{l}\max_{(x_h,\tau_h)\in\mathbb{R}^C\times\mathbb{R}^{B_h}} \\ u_h(x_h)+\sum_{h'\in\mathcal{B}_h} V_{hh'}(p^*e_{h'}+\tau_{hh'}\cdot(p^*e_h)+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''})-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'}))\end{array}$$

$$\text{subject to } (x_h,\tau_h)\in\Gamma^2_h(p^*,\tau^*_{\setminus h}) := \Gamma^1_h(p^*,\tau^*_{\setminus h})\cap\{\,(x'_h,\tau'_h)\in\mathbb{R}^C\times\mathbb{R}^{B_h} : k_r - x_h \geq 0\,\} \tag{2}$$

ii. $\sum_{h\in\mathcal{H}}(x^*_h - e_h) = 0$.

Remark 2. *For any $(x_h,\tau_h)\in\Gamma^2_h(p^*,\tau^*_{\setminus h})$, we have $(x_h,\tau_h)\in\Gamma^1_h(p^*,\tau^*_{\setminus h})$ and thus,*

$$\begin{array}{l} u_h(x_h) \geq \\ u_h(\frac{e_h}{4H})+\sum_{h'\in\mathcal{B}_h} V_{hh'}(p^*e_{h'}+0\cdot(p^*e_h)+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''})-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'}))+ \\ -\sum_{h'\in\mathcal{B}_h} V_{hh'}(p^*e_{h'}+\tau_{hh'}\cdot(p^*e_h)+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''})-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'}))\geq \\ u_h(\frac{e_h}{4H})+\sum_{h'\in\mathcal{B}_h}(V_{hh'}(0)-V_{hh'}(1)) := \underline{u}^e_h \end{array} \tag{3}$$

where the second inequality follows from the fact that $V_{hh'}$ is (strictly) increasing and the domain of $V_{hh'}$ is $[0,1]$.

We now define the generalized game associated with an economy $\mathcal{E}$. To this aim, we recall the definition of generalized game, the related concept of Nash equilibrium and the classical existence result.

Definition 5. *(Kreps (2013), page 339) Given $n\in\mathbb{N}$, an n-player generalized game is a triple $\mathcal{G} = \{A_i, C_i, \pi_i\}_{i=1}^n$, where for any $i\in\{1,...,n\}$,*
1. A_i is a set of strategies or actions with generic element a_i;
2. $C_i : A_{\setminus i} := \times_{j\in\{1,...,n\}\setminus\{i\}} A_j \twoheadrightarrow A_i, \quad a_{\setminus i} = (a_j)_{j\neq i} \mapsto\!\!\!\mapsto C_i\left(a_{\setminus i}\right)$ is a constraint set-valued function;
3. $\pi_i : A := \times_{i\in\{1,..,n\}} A_i \to \mathbb{R}$, $a\mapsto u_i(a)$ is a utility or payoff function.

Definition 6. *A Nash equilibrium for the generalized game $\mathcal{G} = \{A_i, C_i, \pi_i\}_{i=1}^n$ is an n-tuple of actions $a^* := (a^*_i)_{i=1}^n \in A$ such that for any $i\in\{1,...,n\}$, a^*_i solves the following problem. For given $a^*_{\setminus i} := \left(a^*_j\right)_{j\in\{1,...,n\}\setminus\{i\}} \in A_{\setminus i}$,*

$$\max_{a_i\in A_i}\ \pi_i\left(a_i, a^*_{\setminus i}\right) \quad \text{subject to} \quad a_i \in C_i\left(a^*_{\setminus i}\right).$$

Theorem 4. *(Debreu (1952). See also Kreps (2013), page 340) Let a generalized game $\mathcal{G} = \{A_i, C_i, \pi_i\}_{i=1}^n$ be given. If for any $i\in\{1,...,n\}$,*
1. there exists $n_i\in\mathbb{N}$ such that A_i is a nonempty, compact, convex subset of $\mathbb{R}^{n_i}$;
2. C_i is a non-empty valued, convex valued, lower hemicontinuous and upper hemicontinuous set-valued function;
3. π_i is a continuous function and for any $a_{\setminus i}\in A_{\setminus i}$, the function $\pi_i(\cdot, a_{\setminus i}) : A_i \longrightarrow \mathbb{R}$, $a_i\mapsto\pi_i(a_i, a_{\setminus i})$ is quasi-concave,
then $\mathcal{G}$ has a Nash equilibrium.

We define, for any $h\in\mathcal{H}$, $X_h := \{x_h\in\mathbb{R}^C_{++} : u_h(x_h)\geq\underline{u}^e_h \text{ and } x_h\leq k_r\}$.

Remark 3. *From Remark 2, we have $\mathrm{Im}(\Gamma^2_h)\subseteq X_h\times\Delta_h$.*

According Definition 5, the generalized game associated with an economy $\mathcal{E}$ is so defined.

Definition 7. *Let $n = 1 + H$ be players. For each player $h \in \{0, 1..., H\}$,*
1. Set of actions are
$A_0 = S \subseteq \mathbb{R}^C$
$A_h = X_h \times \Delta_h \subseteq \mathbb{R}^{C+B_h}$ *for any* $h \in \mathcal{H}$*;*
2. Constraint set-valued functions[9] *are:*
$C_0 : (\times_{h\in\mathcal{H}}(X_h \times \Delta_h)) \twoheadrightarrow S \qquad (x, \tau) \mapsto\!\!\mapsto S$
$C_h : S \times (\times_{h'\in\mathcal{H}\setminus\{h\}}(X_{h'} \times \Delta_{h'})) \twoheadrightarrow X_h \times \Delta_h \quad (p, (x_{h'}, \tau_{h'})_{h'\in\mathcal{H}\setminus\{h\}}) \mapsto\!\!\mapsto \Gamma_h^2(p, \tau_{\setminus h})$;
3. Utility functions are:
$\pi_0 : S \times (\times_{h\in\mathcal{H}}(X_h \times \Delta_h)) \to \mathbb{R} \quad (p, (x_h, \tau_h)_{h\in\mathcal{H}}) \mapsto p \cdot \sum_{h\in\mathcal{H}}(x_h - e_h)$
$\pi_h : S \times (\times_{h\in\mathcal{H}}(X_h \times \Delta_h)) \to \mathbb{R} \quad (p, (x_h, \tau_h)_{h\in\mathcal{H}}) \mapsto \overline{U}_h(p, (x_h, \tau_h)_{h\in\mathcal{H}})$
where $\overline{U}_h(p, (x_h, \tau_h)_{h\in\mathcal{H}}) := u_h(x_h) + \sum_{h'\in\mathcal{B}_h} V_{hh'}(pe_{h'} + \tau_{hh'} \cdot (pe_h) + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau_{h''h'} \cdot (pe_{h''}) - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''} \cdot (pe_{h'}))$. *The generalized game is given by*

$$((S, (X_h \times \Delta_h)), (C_0, (C_h)_{h\in\mathcal{H}}), (\pi_0, (\overline{U}_h)_{h\in\mathcal{H}})).$$

Definition 8. *A Nash equilibrium for the generalized game associated with $\mathcal{E} \in \mathbb{E}$, is a vector $(p^*, (x_h^*, \tau_h^*)_{h\in\mathcal{H}}) \in S \times (\times_{h\in\mathcal{H}}(X_h \times \Delta_h))$ such that*
i. for given $(x_h^*, \tau_h^*)_{h\in\mathcal{H}} \in \times_{h\in\mathcal{H}}(X_h \times \Delta_h)$, p^* solves[10] $\max_{p\in S} p \cdot \sum_{h\in\mathcal{H}} (x_h^* - e_h)$,

ii. for any $h \in \mathcal{H}$, for given $p^* \in S$, $(x_{h'}^*, \tau_{h'}^*)_{h'\in\mathcal{H}\setminus\{h\}} \in \times_{h'\in\mathcal{H}\setminus\{h\}}(X_{h'} \times \Delta_{h'})$, (x_h^*, τ_h^*) solves

$$\begin{aligned} &\max_{(x_h,\tau_h)\in X_h\times\Delta_h} \overline{U}_h(p^*, (x_h, \tau_h), (x_{h'}^*, \tau_{h'}^*)_{h\in\mathcal{H}\setminus\{h\}}) \\ &\text{subject to } (x_h, \tau_h) \in C_h(p^*, (x_{h'}^*, \tau_{h'}^*)_{h'\in\mathcal{H}\setminus\{h\}}) = \Gamma_h^2(p^*, \tau_{\setminus h}^*). \end{aligned} \tag{4}$$

Remark 4. *By definition of $\overline{U}_h$, for any $\mathcal{E} \in \mathbb{E}$, $p^* \in S$, $((x_h, \tau_h), (x_{h'}^*, \tau_{h'}^*)_{h\in\mathcal{H}\setminus\{h\}}) \in (X_h \times \Delta_h) \times (\times_{h'\in\mathcal{H}\setminus\{h\}}(X_{h'} \times \Delta_{h'}))$, we have $\overline{U}_h(p^*, ((x_h, \tau_h), (x_{h'}^*, \tau_{h'}^*)_{h\in\mathcal{H}\setminus\{h\}})) = u_h(x_h) + \sum_{h'\in\mathcal{B}_h} V_{hh'}(p^*e_{h'} + \tau_{hh'} \cdot (p^*e_h) + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau_{h''h'}^* \cdot (p^*e_{h''}) - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}^* \cdot (p^*e_{h'}))$, i.e., the objective function in the maximization problem for player $h \in \mathcal{H}$ in the generalized game and the objective function in the maximization problem for player $h \in \mathcal{H}$ in the fictitious equilibrium assume the same values.*
Lemma 1. *For any $(p^*, \tau_{\setminus h}^*) \in S \times \Delta_{\setminus h}$, $\Gamma_h^2(p^*, \tau_{\setminus h}^*) \cap (X_h \times \Delta_h) = \Gamma_h^2(p^*, \tau_{\setminus h}^*)$.*

Proof. Clearly, $\Gamma_h^2(p^*, \tau_{\setminus h}^*) \cap (X_h \times \Delta_h) \subseteq \Gamma_h^2(p^*, \tau_{\setminus h}^*)$. It remains to prove that $\Gamma_h^2(p^*, \tau_{\setminus h}^*) \subseteq \Gamma_h^2(p^*, \tau_{\setminus h}^*) \cap (X_h \times \Delta_h)$. Take $(x_h, \tau_h) \in \Gamma_h^2(p^*, \tau_{\setminus h}^*)$, we have to show that $(x_h, \tau_h) \in X_h \times \Delta_h$. From $(x_h, \tau_h) \in \Gamma_h^2(p^*, \tau_{\setminus h}^*)$, we have that $\tau_h \in \Delta_h$ and, from Remark 2, $u_h(x_h) \geq \underline{u}_h^e$, then $x_h \in X_h$. Hence, $(x_h, \tau_h) \in X_h \times \Delta_h$. □

Proposition 5. *The solution set to the maximization problem for household h in the fictitious equilibrium coincides with the solution set to the maximization problem for household h in the generalized game.*

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[9] Observe that C_h is well-defined because $\Gamma_h^2(p, \tau_{\setminus h}) \subseteq X_h \times \Delta_h$, from Remark 3.
[10] An interpretation of the above condition is that player 0, usually called the auctioneer, wants to choose prices in order to minimize the value of resources which are not consumed, i.e., she wants to solve the problem $\min_{p\in S} pr - p\sum_{h\in\mathcal{H}} x_h^*$.

Proof. It follows from Remark 4 and Lemma 1. □

Below, we show that the generalized game admits a Nash equilibrium. To this aim, we apply Theorem 4. First, we need the following definition and result.

Definition 9. *The set-valued function* $\Gamma_h^{2+} : S \times \Delta_{\setminus h} \twoheadrightarrow \mathbb{R}^C \times \mathbb{R}^{B_h}$ *is defined as* Γ_h^2 *with weak inequalities substituted by strict inequalities.*

Lemma 2. *For any* $h \in \mathcal{H}$*, the set-valued function* $\Gamma_h^2 : S \times \Delta_{\setminus h} \twoheadrightarrow \mathbb{R}^C \times \mathbb{R}^{B_h}$ *is: 1. non-empty valued; 2. convex valued; 3. closed; 4. compact valued; 5. lower hemicontinuous; 6. upper hemicontinuous.*

Proof. To get the desired result, it is enough to check that the Assumptions of Proposition 22 in the Appendix, about the function f used to define Γ_h^2 and Γ_h^{2+} are indeed satisfied. First of all, notice that Γ_h^{2+} is non-empty valued. Indeed, it is enough to consider

$$\widetilde{x}_h := \frac{e_h}{2H} \gg 0 \text{ and for any } h' \in \mathcal{B}_h, \widetilde{\tau}_{hh'} := \frac{1}{2H} > 0$$

Then,

$$p^* e_h + (\textstyle\sum_{h' \in \mathcal{B}_{\to h}} \tau^*_{h'h} \cdot (p^* e_{h'})) - \sum_{h' \in \mathcal{B}_h} \widetilde{\tau}_{hh'} \cdot (p^* e_h) - p^* \widetilde{x}_h \geq p^* e_h (1 - \tfrac{H-1+1}{2H}) > 0;$$

$$\widetilde{x}_h = \tfrac{e_h}{2H} \ll r \ll k_r;$$

$$1 - \textstyle\sum_{h' \in \mathcal{B}_h} \widetilde{\tau}_{hh'} = 1 - \sum_{h' \in \mathcal{B}_h} \frac{1}{2H} \geq 1 - \frac{H-1}{2H} > 0;$$

for any $h' \in \mathcal{B}_h$, $p^* e_{h'} + \widetilde{\tau}_{hh'} \cdot (p^* e_h) + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \tau^*_{h''h'} \cdot (p^* e_{h''}) - \sum_{h'' \in \mathcal{B}_{h'}} \tau^*_{h'h''} \cdot (p^* e_{h'}) \geq \widetilde{\tau}_{hh'} \cdot (p^* e_h) \geq \frac{p^* e_h}{2H} > 0$

for any $h' \in \mathcal{B}_h$, $p^* e_{h'} + \widetilde{\tau}_{hh'} \cdot (p^* e_h) + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \tau^*_{h''h'} \cdot (p^* e_{h''}) - \sum_{h'' \in \mathcal{B}_{h'}} \tau^*_{h'h''} \cdot (p^* e_{h'}) \leq \frac{1}{2H} \cdot (p^* e_h) + \sum_{h'' \in \mathcal{H} \setminus \{h\}} (p^* e_{h''}) < p^* r$

$$\overline{U}_h(p^*, ((\tfrac{e_h}{2H}, \tfrac{1}{2H} \mathbf{1}_{B_h}), (x^*_{h'}, \tau^*_{h'})_{h \in \mathcal{H} \setminus \{h\}})) > \overline{U}_h(p^*, ((\tfrac{e_h}{4H}, \mathbf{0}_{B_h}), (x^*_{h'}, \tau^*_{h'})_{h \in \mathcal{H} \setminus \{h\}}))$$

2. and 3b. are clearly satisfied[11].

4a. Taken an arbitrary $(p^*, \tau^*_{\setminus h})$, we are going to show that $\Gamma_h^2\ (p^*, \tau^*_{\setminus h})$ is sequentially compact. Let $(x_h^n, \tau_h^n)_{n \in \mathbb{N}}$ be a sequence of elements in $\Gamma_h^2\ (p^*, \tau^*_{\setminus h})$. Then, up to a subsequence, we have $(x_h^n, \tau_h^n) \overset{n}{\to} (\overline{x}_h, \overline{\tau}_h) \in [0_C, k_r] \times \Delta_h$. We are left with showing that $\overline{x}_h >> 0$. Since, for any $n \in \mathbb{N}$, $(x_h^n, \tau_h^n) \in \Gamma_h^2(p^*, \tau^*_{\setminus h})$, from Remark 2, then $x_h^n \in \{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \underline{u}_h^e\}$. Since $x_h^n \overset{n}{\to} \overline{x}_h$, then $\overline{x}_h \in \mathrm{Cl}_{\mathbb{R}^C}(\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \underline{u}_h^e\})$ which is contained in $\mathbb{R}^C_{++}$ by Assumption4.u4, as desired. All the remaining constraints describing $\Gamma_h^2\ (p^*, \tau^*_{\setminus h})$ are trivially satisfied, since they are affine in (x_h, τ_h). □

[11] Observe that given an arbitrary differentiable function $f : O \subseteq \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$, if $Df > 0$, or $Df < 0$, then f is locally nonsatiated.

Proposition 6. *For any economy $\mathcal{E} \in \mathbb{E}$, the generalized game*

$$((S, (X_h \times \Delta_h)), (C_0, (C_h)_{h\in\mathcal{H}}), (\pi_0, (\overline{U}_h)_{h\in\mathcal{H}}))$$

has a Nash equilibrium (p^*, x^*, τ^*).

Proof. We show that the Assumptions of Theorem 4 are verified.
1. there exists $n_i \in \mathbb{N}$ such that A_i is a nonempty, compact, convex subset of $\mathbb{R}^{n_i}$.
$A_0 = S = \left\{p \in \mathbb{R}^C_+ : pr = 1\right\}$ satisfies the needed assumptions. To show that A_h has the desired properties, we show that both X_h and Δ_h do so. For any $h \in \mathcal{H}$, Δ_h satisfies the needed assumptions by definition. To show that X_h is nonempty, it is enough to notice that $\frac{e_h}{2H} \in X_h$[12]. The compactness of X_h is verified below. We want to show that X_h is sequentially compact. Since X_h is bounded below by 0_C and above by k_r, up to a subsequence, $x^n_h \overset{n}{\to} \overline{x}_h \in [0, k_r] \subseteq \mathbb{R}^C$. We are left with showing that $\overline{x}_h >> 0$. Since for any $n \in \mathbb{N}$, $x^n_h \in X_h$, then $x^n_h \in \left\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \underline{u}^e_h\right\}$. Since $x^n_h \overset{n}{\to} \overline{x}_h$, then $\overline{x}_h \in \mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \underline{u}^e_h\right\}\right)$ which is contained in $\mathbb{R}^C_{++}$ by Assumption 4.u4, as desired. Convexity of X_h follows from Assumption 4.u3.
2. C_i is a non-empty valued, convex valued, lower hemi-continuous and upper hemi-continuous set-valued function. By definition of S and since $C_0 : (\times_{h\in\mathcal{H}}(X_h \times \Delta_h)) \to S, (x, \tau) \mapsto\mapsto S$, then it continuous by definition of upper and lower hemi-continuity. Verification of the needed properties for C_h goes through two steps: *(i)* If Γ^2_h has the desired properties, then C_h has the desired properties. This is an immediate consequence of Lemma 4 in the Appendix B; *(ii)* Γ^2_h has the desired properties (see Lemma 2).
3. π_i is continuous and for any $a_{\setminus i} \in A_{\setminus i}$, the function $\pi_i(\cdot, a_{\setminus i}) : A_i \to \mathbb{R}$, $a_i \mapsto \pi_i(a_i, a_{\setminus i})$ is quasi-concave. For given $(x_h)_{h\in\mathcal{H}}$, π_0 is linear in p and therefore concave and quasi-concave. $\overline{U}_h$ is continuous because it is sum and composition of continuous functions. We are left with showing that for any $(p, (x_{h'}, \tau_{h'})_{h'\in\mathcal{H}\setminus\{h\}}) \in S \times (\times_{h'\in\mathcal{H}\setminus\{h\}}(X_{h'} \times \Delta_{h'}))$, $\overline{U}_h(x_h, \tau_h, (p, (x_{h'}, \tau_{h'})_{h'\in\mathcal{H}\setminus\{h\}})) := \widehat{U}_h(x_h, \tau_h)$ is quasi-concave. The quasi-concavity of $\widehat{U}_h(x_h, \tau_h)$ is a direct consequence of of the quasi-concavity of $u_h + \sum_{h'\in\mathcal{B}_h} V_{hh'}$. Indeed, for any $h' \in \mathcal{B}_h$, define

$$a_{h'} = pe_{h'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau_{h''h'}(pe_{h''}) - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}(pe_{h'}) \text{ and } w_h = pe_h.$$

Then, $\widehat{U}_h(x_h, \tau_h) = u_h(x_h) + \sum_{h'\in\mathcal{B}_h} V_{hh'}(a_{h'} + \tau_{hh'}w_h)$. Take any $(x'_h, \tau'_h), (x''_h, \tau''_h)$ and any $\lambda \in [0, 1]$, we have

$$\begin{array}{ll}
\widehat{U}_h((1-\lambda)(x'_h, \tau'_h) + \lambda(x''_h, \tau''_h)) & = \\
u_h((1-\lambda)x'_h + \lambda x''_h) + \sum_{h'\in\mathcal{B}_h} V_{hh'}((1-\lambda)(a_{h'} + w_h\tau'_{hh'}) + \lambda(a_{h'} + w_h\tau''_{hh'})) & \geq \\
\min\{u_h(x'_h) + \sum_{h'\in\mathcal{B}_h} V_{hh'}(a_{h'} + w_h\tau'_{hh'}), u_h(x''_h) + \sum_{h'\in\mathcal{B}_h} V_{hh'}(a_{h'} + w_h\tau''_{hh'})\} & = \\
\min\{\widehat{U}_h(x'_h, \tau'_h), \widehat{U}_h(x''_h, \tau''_h)\} &
\end{array}$$

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[12] Indeed, $\frac{e_h}{2H} < k_h = r + \mathbf{1}$. Moreover, $u_h(\frac{e_h}{2H}) \geq u_h(\frac{e_h}{4H}) - (\sum_{h'\in\mathcal{B}_h} \overset{(>0)}{V(1)} - V(0)) := \underline{u}^e_h$, with thewith equality holding if $\mathcal{B}_h = \varnothing$.

where the first inequality follows from the quasi-concavity of $u_h + \sum_{h' \in \mathcal{B}_h} V_{hh'}$. □

Proposition 7. *If (x^*, p^*, τ^*) is a Nash equilibrium for the generalized game, then $p^* \cdot \sum_{h \in \mathcal{H}} (x_h^* - e_h) = 0$.*

Proof. Using the strict monotonicity of the utility function with respect to x_h and the fact that $p^* \in S$, it is easy to show that budget constraints hold as equalities. Then summing up with respect to households and using Proposition 1, we get the desired result. Suppose our claim does not hold, i.e.,

$$p^* x_h^* < p^* e_h + \sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h}^* \cdot (p^* e_{h'}) - \sum_{h' \in \mathcal{B}_h} \tau_{hh'}^* \cdot (p^* e_h) := w_h(p^*, \tau^*) \tag{5}$$

Since $p^* \in S$, then we can define $\mathcal{C}^+ := \{c \in \mathcal{C} : p^{*c} > 0\} \neq \varnothing$. We then distinguish the following two cases: Case a. There exists $\widetilde{c} \in \mathcal{C}^+$ such that $x_h^{*\widetilde{c}} < r^{\widetilde{c}} + 1$; Case b. For any $c \in \mathcal{C}^+$, $x_h^{*c} = r^c + 1$.
Case a. Define $x_h^{**} = (x_h^{**c})_{c \in \mathcal{C}}$ such that

$$x_h^{**c} := \begin{cases} x_h^{*c} & \text{if} \quad c \in \mathcal{C}^+ \backslash \{\widetilde{c}\} \\ x_h^{*\widetilde{c}} + \frac{1}{2} \min\left\{ \frac{w_h(p^*, \tau^*) - p^* x_h^*}{p^{*\widetilde{c}}}, r^{\widetilde{c}} + 1 - x_h^{*\widetilde{c}} \right\} > x_h^{*\widetilde{c}} & \text{if} \quad c = \widetilde{c} \\ x_h^{*c} & \text{otherwise} \end{cases}$$

The strictly inequality follows from the fact that $w_h(p^*, \tau^*) - p^* x_h^* \overset{(5)}{>} 0$ and $r^{\widetilde{c}} + 1 - x_h^{*\widetilde{c}} > 0$. Since $(x_h^{**}, \tau_h^*), (x_h^*, \tau_h^*) \in \Gamma_h^2(p^*, \tau_{\backslash h}^*)$ and $x_h^{**} > x_h^* >> 0$, we have $\overline{U}_h(p^*, ((x_h^{**}, \tau_h^*), (x_{h'}^*, \tau_{h'}^*)_{h \in \mathcal{H} \backslash \{h\}})) > \overline{U}_h(p^*, (x_{h'}^*, \tau_{h'}^*)_{h' \in \mathcal{H}})$. This is a contradiction since (x_h^*, τ_h^*) is a solution to household h's maximization problem in Definition 8.
Case b. This case cannot hold. Assume it does. Then, by definition of $w_h(p^*, \tau^*)$, we get $w_h(p^*, \tau^*) \leq p^* e_h + \sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h}^* \cdot (p^* e_{h'}) \leq p^* r = 1$. Hovewever, by (5), $w_h(p^*, \tau^*) \overset{(5)}{>} p^* x_h^* = \sum_{c \in \mathcal{C}^+} p^{*c}(r^c + 1) = \sum_{c \in \mathcal{C}} p^{*c}(r^c + 1) = p^* r + \sum_{c \in \mathcal{C}} p^{*c} > 1$, which is the desired contradiction. □

Proposition 8. *If (x^*, p^*, τ^*) is a Nash equilibrium for the generalized game, then it is a fictitious equilibrium with $p^* \gg 0$.*

Proof. By definition of Nash equilibrium, each player is maximizing. Therefore, for player $h = 0$, we have that

$$\text{for any } p \in S, \quad p^* \cdot \sum_{h \in \mathcal{H}} (x_h^* - e_h) \geq p \cdot \sum_{h \in \mathcal{H}} (x_h^* - e_h). \tag{6}$$

From Proposition 5, $(x_h^*), \tau_h^*$ solves the household h's maximization problem (2) at p^* and $\tau_{\backslash h}^*$. So, we are left with checking market clearing. Since $(x_h^*, \tau_h^*) \in \Gamma_h^2(p^*, \tau_{\backslash h}^*)$, we get that for any $h \in \mathcal{H}$

$$0 \geq p^* x_h^* - p^* e_h - \sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h}^* (p^* e_{h'}) + \sum_{h' \in \mathcal{B}_h} \tau_{hh'}^* (p^* e_h).$$

Summing up with respect to $h \in \mathcal{H}$, we get

$$0 \geq \sum_{h\in\mathcal{H}} p^*(x_h^* - e_h) - \sum_{h\in\mathcal{H}} \sum_{h'\in\mathcal{B}_{\to h}} \tau^*_{h'h}(p^* e_{h'}) + \sum_{h\in\mathcal{H}} \sum_{h'\in\mathcal{B}_h} \tau^*_{hh'}(p^* e_h) = \sum_{h\in\mathcal{H}} p^*(x_h^* - e_h), \tag{7}$$

where last equality follows from Proposition 1. Then, for any $p \in S$,

$$0 \overset{(7)}{\geq} p^* \cdot \textstyle\sum_{h\in\mathcal{H}}(x_h^* - e_h) \overset{(6)}{\geq} p \cdot \textstyle\sum_{h\in\mathcal{H}}(x_h^* - e_h). \tag{8}$$

For any $c \in \mathcal{C}$, define $p(c) = (p(c)^{c'})_{c'\in\mathcal{C}}$ such that

$$p(c)^{c'} := \begin{cases} \frac{1}{r^{c'}} & \text{if } c' = c \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $p(c) \in S$. Then, from (8), we get $0 \geq \frac{1}{r^{c'}} \sum_{h\in\mathcal{H}}(x_h^{*c'} - e_h^{c'})$, and hence $0 \geq \sum_{h\in\mathcal{H}}(x_h^{*c'} - e_h^{c'})$. Repeating the same argument for any c, we get

$$\sum_{h\in\mathcal{H}}(x_h^* - e_h) \leq 0. \tag{9}$$

Let's now show that $p^* \gg 0$. Suppose our claim is false and without loss of generality, assume that $p^{*1} = 0$. Then, from strict monotonicity of u_h in x_h (and since $x_h^* \in \text{dom } u_h$), we would have for any $h \in \mathcal{H}$, $x_h^{*1} = r^1 + 1$. Then, $\sum_{h\in\mathcal{H}}(x_h^{*1} - e_h^1) = \sum_{h\in\mathcal{H}}(r^1 + 1) - r^1 > 0$, contradicting (9). From Proposition 7, we have $p^* \sum_{h\in\mathcal{H}}(x_h^* - e_h) = 0$. Since $p^* >> 0$, from (9), we also have $\sum_{h\in\mathcal{H}}(x_h^* - e_h) = 0$. □

Proposition 9. *For any economy, a fictitious equilibrium is an extended utility equilibrium.*

Proof. From Proposition 3, any extended utility with additional constraint equilibrium is an extended utility equilibrium. Therefore, it is enough to show that any fictitious equilibrium is an extended utility with additional constraint equilibrium according Definition 3[13]. Let (x^*, τ^*, p^*) be a fictitious equilibrium. We want to show that if

$$\begin{array}{ll} (a)\ (x_h^*, \tau_h^*) \in \Gamma_h^2(p^*, \tau^*_{\setminus h}), \text{ and} & \\ (b)\ \text{for any } (x_h, \tau_h) \in \Gamma_h^2(p^*, \tau^*_{\setminus h}), & \widehat{U}_h(x_h^*, \tau_h^*) \geq \widehat{U}_h(x_h, \tau_h), \end{array} \tag{10}$$

then

$$\begin{array}{ll} (1)\ (x_h^*, \tau_h^*) \in \Gamma_h^1(p^*, \tau^*_{\setminus h}), \text{ and} & \\ (2)\ \text{for any } (x_h, \tau_h) \in \Gamma_h^1(p^*, \tau^*_{\setminus h}), & \widehat{U}_h(x_h^*, \tau_h^*) \geq \widehat{U}_h(x_h, \tau_h). \end{array} \tag{11}$$

Before proceeding to the proof of the desired result, observe that, by market clearing, for any $h \in \mathcal{H}$, $0 << x_h^* \leq \sum_{h'\in\mathcal{H}} e_{h'} << r + \mathbf{1}$. Since $\{(x_h, \tau_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{B_h} : x_h <<$

[13] The proof is quite standard. See for example, Theorem 49 in Donato and Villanacci (2023).

$k_r\}$ is an open set which contains (x_h^*, τ_h^*), then there exists $\delta^* \in \mathbb{R}_{++}$ such that

$$\begin{aligned}&\mathcal{N}_{\delta^*}(x_h^*, \tau_h^*) := \{(x_h, \tau_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{B_h} : d((x_h^*, \tau_h^*), (x_h, \tau_h)) < \delta^*\} \\ &\subseteq \{(x_h, \tau_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{B_h} : x_h << k_r\}.\end{aligned}$$

Then,

$$\mathcal{N}_{\delta^*}(x_h^*, \tau_h^*) \cap (\mathbb{R}^C_{++} \times \Delta_h) \subseteq (0, k_r] \times \Delta_h \tag{12}$$

Let's now proceed to the proof of the desired result. Since $\Gamma_h^2(p^*, \tau_{\backslash h}^*) \subseteq \Gamma_h^1(p^*, \tau_{\backslash h}^*)$, conclusion (11.1) follows from condition (10.a). Now suppose conclusion (11.2) does not hold, i.e.,

$$\exists\ (\widetilde{x}_h, \widetilde{\tau}_h) \in \Gamma_h^1(p^*, \tau_{\backslash h}^*) \backslash \Gamma_h^2(p^*, \tau_{\backslash h}^*) \text{ such that } \widehat{U}_h(\widetilde{x}_h, \widetilde{\tau}_h) > \widehat{U}_h(x_h^*, \tau_h^*). \tag{13}$$

Then

$$(\widetilde{x}_h, \widetilde{\tau}_h) \neq (x_h^*, \tau_h^*). \tag{14}$$

Since $(x_h^*, \tau_h^*), (\widetilde{x}_h, \widetilde{\tau}_h) \in \Gamma_h^1(p^*, \tau_{\backslash h}^*)$ and $\Gamma_h^1(p^*, \tau_{\backslash h}^*)$ is convex, then

$$\forall \lambda \in (0,1),\ \ (x_h^\lambda, \tau_h^\lambda) := (1-\lambda)(x_h^*, \tau_h^*) + \lambda(\widetilde{x}_h, \widetilde{\tau}_h) \in \Gamma_h^1(p^*, \tau_{\backslash h}^*). \tag{15}$$

From (13) and semistrict quasi-concavity of $\widehat{U}_h$, we have[14]

$$\forall\ \lambda \in (0,1), \qquad \widehat{U}_h(x_h^\lambda, \tau_h^\lambda) > \widehat{U}_h(x_h^*, \tau_h^*). \tag{16}$$

Now, $\|(x_h^\lambda, \tau_h^\lambda) - (x_h^*, \tau_h^*)\| \overset{\text{Def. } (x_h^\lambda, \tau_h^\lambda)}{=} \lambda \cdot \|(x_h^*, \tau_h^*) - (\widetilde{x}_h, \widetilde{\tau}_h)\| < \delta^*$ if and only if $\lambda < \frac{\delta^*}{\|(x_h^*, \tau_h^*) - (\widetilde{x}_h, \widetilde{\tau}_h)\|} \in \mathbb{R}_{++}$, where $\|(x_h^*, \tau_h^*) - (\widetilde{x}_h, \widetilde{\tau}_h)\| > 0$ from (14). Then, for any $\lambda \in \big(0, \frac{\delta^*}{\|(x_h^*, \tau_h^*) - (\widetilde{x}_h, \widetilde{\tau}_h)\|}\big)$, and using (15), we have that

$$(x_h^\lambda, \tau_h^\lambda) \in \Gamma_h^1(p^*, \tau_{\backslash h}^*) \cap \mathcal{N}_{\delta^*}(x_h^*, \tau_h^*) \cap (\mathbb{R}^C_{++} \times \Delta_h), \tag{17}$$

where the last intersection follows from the fact that the constraints $\widehat{x}_h >> 0$ and $\widehat{\tau}_h \in \Delta_h$ are part of the definition of $\Gamma_h^1(p^*, \tau_{\backslash h}^*)$. Observe that

$$\Gamma_h^2(p^*, \tau_{\backslash h}^*) = \Gamma_h^1(p^*, \tau_{\backslash h}^*) \cap ((0, k_r] \times \Delta_h). \tag{18}$$

Then,

$$\Gamma_h^1(p^*, \tau_{\backslash h}^*) \cap \mathcal{N}_{\delta^*}(x_h^*, \tau_h^*) \cap (\mathbb{R}^C_{++} \times \Delta_h) \overset{(12)}{\subseteq} \Gamma_h^1(p^*, \tau_{\backslash h}^*) \cap ((0, k_r] \times \Delta_h) \overset{(18)}{=} \Gamma_h^2(p^*, \tau_{\backslash h}^*). \tag{19}$$

[14] $\widehat{U}_h = \overline{U}_h(\cdot; (p, (x_{h'}, \tau_{h'})_{h' \in \mathcal{H} \backslash \{h\}})) : X_h \times \Delta_h \to \mathbb{R}$ is continuous and Locally nonsatiation by Points u1. and u.2 of Assumption 4 and Assumption 2. Furthermore, it is is quasi-concave by of Proposition 19 in Appendix A.2. Therefore, by Proposition 20 in Appendix B, $\widehat{U}_h$ is semistrictly quasi-concave.

From (17) and (19), we have

$$(x_h^\lambda, \tau_h^\lambda) \in \Gamma_h^2(p^*, \tau_{\setminus h}^*). \tag{20}$$

(20) and (16) contradict assumption (10.b). □

Remark 5. *From Propositions 8 and 9, in any extended utility function equilibrium, we have* $p^* \in \mathbb{R}^C_{++}$, $x_h^* \in \mathbb{R}^C_{++}$, *and* $x_h^* << r$. *Therefore, for any* $h \in \mathcal{H}$, *we do have* $0 < \frac{p^* x_h^*}{p^* r^*} < 1$ *and an extended utility equilibrium is indeed an equilibrium.*

We can then get the desired result of the section.

Theorem 10. *For any economy* $\mathcal{E}$, *an equilibrium* $(x^*, p^*, \tau^*) \in \mathbb{R}^{CH}_{++} \times S \times \Delta$ *exists and* $p^* >> 0$.

Proof. From Proposition 6, a generalized Nash equilibria exists; from Proposition 8, a fictitious equilibrium exists; from Proposition 9, and Remark 5 an equilibrium exists. □

4 Model with nonseparable utility functions

In this section we deal with the version of the model with nonseparable utility functions defined as follows. For any $h \in \mathcal{H}$,

$$\widehat{U}_h : \mathbb{R}^C_+ \times [0,1]^{B_h} \times [0,1]^{B_h^\setminus} \to \mathbb{R}, (x_h, \theta_{\mathcal{B}_h}, \theta_{\mathcal{B}_h^\setminus}) \mapsto \widehat{U}_h(x_h, \theta_{\mathcal{B}_h}, \theta_{\mathcal{B}_h^\setminus}). \tag{21}$$

Assumptions 7. For any $h \in \mathcal{H}$, the utility function $\widehat{U}_h$ is: U1. continuous; U2. strictly increasing in x_h; U3. strictly increasing in $\theta_{h'}$ for any $h' \in \mathcal{B}_h$ and decreasing in $\theta_{h'}$ for any $h' \in \mathcal{B}_h^\setminus$; U4. quasi-concave in $(x_h, \theta_{\mathcal{B}_h})$.

We denote by $\widehat{\mathcal{U}}_h$ the set of functions satisfying Assumption 7.

As defined in Donato et al. (2025) (see Section 3, page 11), we describe the vector of "wealth" of individuals household h does not like.

Definition 10. *For any* $h \in \mathcal{H}$ *and any* $h' \in \mathcal{B}_h^\setminus$, *the wealth function of individuals that household* h *dislikes is* $q_h : S \times \Delta_{\setminus h} \to \mathbb{R}^{B_h^\setminus}$,

$$q_h(p, \tau_{\setminus h}) := \Big(pe_{h'} + 0 + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \tau_{h'' h'}(pe_{h''}) - \sum_{h'' \in \mathcal{B}_{h'}} \tau_{h' h''}(pe_{h'}) \Big)_{h' \in \mathcal{B}_h^\setminus}.$$

Defined, the economy $\widehat{\mathcal{E}} = (e_h, \widehat{U}_h)_{h \in \mathcal{H}} \in \times_{h \in \mathcal{H}} (\mathbb{R}^C_{++} \times \widehat{\mathcal{U}}_h) := \widehat{\mathbb{E}}$, we present the definition of equilibrium.

Definition 11. *$(x^*, \tau^*, p^*) \in \mathbb{R}^{CH}_+ \times \Delta \times S$ is an equilibrium for the economy $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$ if a. for any $h \in \mathcal{H}$, for given $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$, $p^* \in S$ and $\tau^*_{\setminus h} \in \Delta_{\setminus h}$, $(x^*_h, \tau^*_h) \in \mathbb{R}^C \times \Delta_h$ solves*

$$\max_{(x_h,\tau_h)\in\mathbb{R}^C_+\times\mathbb{R}^{B_h}} \widehat{U}_h(x_h,\ (p^*e_{h'}+\tau_{hh'}(p^*e_h)+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}(p^*e_{h''})-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}(p^*e_{h'}))_{h'\in\mathcal{B}_h},\ q_h(p^*,\tau^*_{\setminus h}))$$

$$\text{subject to}(x_h,\tau_h) \in \Gamma^3_h(p^*,\tau^*_{\setminus h}) :=$$

$$\{\ (x'_h,\tau'_h) \in \mathbb{R}^C \times \mathbb{R}^{B_h} : p^*e_h + \sum_{h'\in\mathcal{B}_{\to h}} \tau^*_{h'h}(p^*e_{h'}) - \sum_{h'\in\mathcal{B}_h}\tau'_{hh'}(p^*e_h) - p^*x'_h \geq 0,$$
$$x'_h \geq 0,\ \ 1-\sum_{h'\in\mathcal{B}_h}\tau'_{hh'} \geq 0,\ \ \tau'_h \geq 0\ \}$$

b. $\sum_{h\in\mathcal{H}}(x^*_h - e_h) = 0$.

To show existence of the above equilibrium, also in this case, we need to introduce the upper bound k_x on the consumption.

Definition 12. *$(x^*, \tau^*, p^*) \in \mathbb{R}^{CH}_+ \times \Delta \times S$ is an equilibrium with upper bound on consumption for the economy $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$ if*
a. for any $h \in \mathcal{H}$, for given $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$, $p^* \in S$ and $\tau^*_{\setminus h} \in \Delta_{\setminus h}$, $(x^*_h, \tau^*_h) \in \mathbb{R}^C_+ \times \Delta_h$ solves

$$\max_{(x_h,\tau_h)\in\mathbb{R}^C_+\times\mathbb{R}^{B_h}} \widehat{U}_h(x_h,(p^*e_{h'}+\tau_{hh'}\cdot(p^*e_h)+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''})-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'}))_{h'\in\mathcal{B}_h},\ q_h(p^*,\tau^*_{\setminus h}))$$

$$\text{subject to}(x_h,\tau_h) \in \Gamma^4_h(p^*,\tau^*_{\setminus h}) :=$$

$$\Gamma^3_h(p^*,\tau^*_{\setminus h}) \cap \{\ (x'_h,\tau'_h) \in \mathbb{R}^C \times \mathbb{R}^{B_h} : k_x - x'_h \geq 0,$$

$$p^*e_{h'} + \tau_{hh'}(p^*e_h) + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}(p^*e_{h''}) - \sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}(p^*e_{h'}) \in [0,1]\ \};$$

b. $\sum_{h\in\mathcal{H}}(x^*_h - e_h) = 0$.

In this model, taking into account Definition 5, the generalized game associated with an economy $\widehat{\mathcal{E}}$ is defined as follows.

Definition 13. *Given $n = 1+H$ players, indexed by $h \in \{0,1,...,H\}$, the generalized game is the triple*

$$((S,(\widehat{X}_h\times\Delta_h)_{h\in\mathcal{H}}),(C_0,(\widehat{C}_h)_{h\in\mathcal{H}}),(\pi_0,(\widetilde{U}_h)_{h\in\mathcal{H}})),$$

where 1. the sets of actions are $A_0 = S$ and, for any $h \in \mathcal{H}$, $A_h = \widehat{X}_h \times \Delta_h$ with $\widehat{X}_h = \{x_h \in \mathbb{R}^C : 0 \leq x_h \leq k_r\}$;
2. the constraint set-valued functions are $C_0(x,\tau) = S$ and $\widehat{C}_h(p,(x_{h'},\tau_{h'})_{h'\in\mathcal{H}\setminus\{h\}}) = \Gamma^4_h(p,\tau_{\setminus h})$;
3. the utility functions are $\pi_0(p,(x_h,\tau_h)_{h\in\mathcal{H}}) = p\cdot\sum_{h\in\mathcal{H}}(x_h - e_h)$ and $\pi_h(p,(x_h,\tau_h)_{h\in\mathcal{H}}) = \widetilde{U}_h(p,(x_h,\tau_h)_{h\in\mathcal{H}}) =$
$\widehat{U}_h(x_h,(pe_{h'} + \tau_{hh'}(pe_h) + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau_{h''h'}(pe_{h''}) - \sum_{h''\in\mathcal{B}_{h'}}\tau_{h'h''}(pe_{h'}))_{h'\in\mathcal{B}_h}, q_h(p,\tau_{\setminus h}))$.

Definition 14. *A Nash equilibrium for the generalized game associated with* $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$ *is a vector* $(p^*, (x_h^*, \tau_h^*)_{h\in\mathcal{H}}) \in S \times (\times_{h\in\mathcal{H}}(\widehat{X}_h \times \Delta_h))$ *such that,*
i. for given $(x_h^*, \tau_h^*)_{h'\in\mathcal{H}\setminus\{h\}}$, p^* solves $\max_{p\in S} p \cdot \sum_{h\in\mathcal{H}}(x_h^* - e_h)$,
ii. and for any $h \in \mathcal{H}$, *for given* p^*, $(x_{h'}^*, \tau_{h'}^*)_{h'\in\mathcal{H}\setminus\{h\}}$, (x_h^*, τ_h^*) *solves*

$$\max_{(x_h,\tau_h)\in\widehat{X}_h\times\Delta_h} \widehat{U}_h(x_h, (p^*e_{h'} + \tau_{hh'}(p^*e_h) + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau^*_{h''h'}(p^*e_{h''}) - \sum_{h''\in\mathcal{B}_{h'}} \tau^*_{h'h''}(p^*e_{h'}))_{h'\in\mathcal{B}_h}, q_h(p^*, \tau^*_{\setminus h}))$$

$$\text{subject to } (x_h, \tau_h) \in \widehat{C}_h(p^*, (x_{h'}^*, \tau_{h'}^*)_{h'\in\mathcal{H}\setminus\{h\}}) = \Gamma_h^4(p^*, \tau^*_{\setminus h}).$$

Lemma 3. *For any* $h \in \mathcal{H}$, *the set-valued function* Γ_h^4 *is non-empty valued; convex valued; closed; compact valued; lower hemi-continuous; and upper hemi-continuous.*

Proof. Defined the set-valued map $\Gamma_h^{4+} : S \times \Delta_{\setminus h} \twoheadrightarrow \mathbb{R}^C \times \mathbb{R}^{B_h}$ as Γ_h^4 with weak inequalities substituted by strict inequalities, the desired result follows from Proposition 22 in Appendix B. Γ_h^{4+} is nonempty valued, since $(\widetilde{x}_h, \widetilde{\tau}_h) \in \Gamma_h^{4+}$, with $\widetilde{x}_h = \frac{e_h}{2H} \gg 0$ and for any $h' \in \mathcal{B}_h$, $\widetilde{\tau}_{hh'} = \frac{1}{2H} \gg 0$. Trivially, all constraint functions satisfy the desired assumption because they are either affine and not constant or concave and also locally nonsatiated. Moreover, by definition Γ_h^4 is closed valued and $Im(\Gamma_h^4) \subseteq \widehat{X}_h \times \Delta_h$, where $\widehat{X}_h \times \Delta_h$ is a compact set, and closed subsets of compact sets are compact. Hence all assumptions of Proposition 22 hold true, then Γ_h^4 satisfies the desired properties. □

Proposition 11. *For any economy* $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$ *, the generalized game*

$$((S, (X_h \times \Delta_h)), (C_0, (\widehat{C}_h)_{h\in\mathcal{H}}), (\pi_0, (\widetilde{U}_h)_{h\in\mathcal{H}}))$$

has a Nash equilibrium (p^*, x^*, τ^*).

Proof. The desired result is obtained by applying Theorem 4 and the proof closely follows that of Theorem 6.
1. $A_0 = S$ and for any $h \in \mathcal{H}$, $A_h = \widehat{X}_h \times \Delta_h$ are nonempty, compact and convex sets.
2. $C_0 = S$ and, for any $h \in \mathcal{H}$, $\widehat{C}_h = \Gamma_h^4$ are nonempty and convex valued, lower and upper hemicontinuous set-valued function.
3. For given $(x_h)_{h\in\mathcal{H}}$, $\pi_0 = p\sum_{h\in\mathcal{H}}(x_h - e_h)$ is quasi-concave and continuous and, for any $h \in \mathcal{H}$, $\pi_h = \widetilde{U}_h$ is continuous. We prove that for any $(p, (x_{h'}, \tau_{h'})_{h'\in\mathcal{H}\setminus\{h\}}) \in S \times (\times_{h'\in\mathcal{H}\setminus\{h\}}(\widehat{X}_{h'} \times \Delta_{h'}))$, $\widetilde{U}_h(x_h, \tau_h, (p, (x_{h'}, \tau_{h'})_{h'\in\mathcal{H}\setminus\{h\}}))$ is quasi-concave in (x_h, τ_h), i.e., for any $(x_h', \tau_h'), (x_h'', \tau_h'')$ and any $\lambda \in [0, 1]$, we have

$$\widetilde{U}_h((1-\lambda)x_h' + \lambda x_h'', (1-\lambda)\tau_h' + \lambda\tau_h'', (p, (x_{h'}, \tau_{h'})_{h'\in\mathcal{H}\setminus\{h\}})) \geq$$

$$\min\{\widetilde{U}_h(x_h', \tau_h', (p, (x_{h'}, \tau_{h'})_{h'\in\mathcal{H}\setminus\{h\}})), \widetilde{U}_h(x_h'', \tau_h'', (p, (x_{h'}, \tau_{h'})_{h'\in\mathcal{H}\setminus\{h\}}))\}.$$

For any $h' \in \mathcal{H}\setminus\{h\}$, define

$$a_{h'} = pe_{h'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau_{h''h'}(pe_{h''}) - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}(pe_{h'}).$$

Take any $(x'_h, \tau'_h), (x''_h, \tau''_h)$ and any $\lambda \in [0,1]$, we have

$$\widetilde{U}_h((1-\lambda)x'_h + \lambda x''_h, (1-\lambda)\tau'_h + \lambda\tau''_h, (p, (x_{h'}, \tau_{h'})_{h' \in \mathcal{H}\setminus\{h\}})) \overset{\text{Def. } \widetilde{U}_h}{=}$$
$$\widehat{U}_h((1-\lambda)x'_h + \lambda x''_h, ((1-\lambda)(a_{h'} + (pe_h)\tau'_{hh'}) + \lambda(a_{h'} + (pe_h)\tau''_{hh'}))_{h' \in \mathcal{B}_{h'}}, q_h(p, \tau_{\setminus h})) \overset{\widehat{U}_h \text{ quasi-concave}}{\geq}$$
$$\min\{\widehat{U}_h(x'_h, (a_{h'} + (pe_h)\tau'_{hh'})_{h' \in \mathcal{B}_{h'}}, q_h(p, \tau_{\setminus h})), \widehat{U}_h(x''_h, (a_{h'} + (pe_h)\tau''_{hh'})_{h' \in \mathcal{B}_{h'}}, q_h(p, \tau_{\setminus h})\} =$$
$$\min\{\widetilde{U}_h(x'_h, \tau'_h, (p, (x_{h'}, \tau_{h'})_{h' \in \mathcal{H}\setminus\{h\}})), \widetilde{U}_h(x''_h, \tau''_h, (p, (x_{h'}, \tau_{h'})_{h' \in \mathcal{H}\setminus\{h\}}))\}.$$

□

Proposition 12. *If (x^*, p^*, τ^*) is a Nash equilibrium for the generalized game, then $p^* \cdot \sum_{h \in \mathcal{H}}(x^*_h - e_h) = 0$.*

Proof. The proof follows the same argument as the proof of Proposition 7, with only minor adjustments to the details. □

Proposition 13. *If (x^*, p^*, τ^*) is a Nash equilibrium for the generalized game, then it is an equilibrium with an upper bound on consumption and $p^* \gg 0$.*

Proof. The proof proceeds along the same lines as the proof of Proposition 8. □

Proposition 14. *For any economy, an equilibrium with upper bound on consumption, is such that $(x^*, p^*, \tau^*) \in \widehat{\Gamma}^3_h(p^*, \tau^*_{\setminus h})$ and for any $(x_h, \tau_h) \in \widehat{\Gamma}^3_h(p^*, \tau^*_{\setminus h})$,*

$$\widehat{U}_h(x^*_h, (a_{h'} + (p^*e_h)\tau^*_{hh'})_{h' \in \mathcal{B}_h}, q_h(p^*, \tau^*_{\setminus h})) > \widehat{U}_h(x_h, (a_{h'} + (p^*e_h)\tau_{hh'})_{h' \in \mathcal{B}_h}, q_h(p^*, \tau^*_{\setminus h}))$$

where $\widehat{\Gamma}^3_h(p^*, \tau^*_{\setminus h}) := \Gamma^3_h(p^*, \tau^*_{\setminus h}) \cap \{(x'_h, \tau'_h) \in \mathbb{R}^C \times \mathbb{R}^{B_h} :$

$$p^*e_{h'} + \tau_{hh'}(p^*e_h) + \sum_{h'' \in \mathcal{B}_{\to h'}\setminus\{h\}} \tau^*_{h''h'}(p^*e_{h''}) - \sum_{h'' \in \mathcal{B}_{h'}} \tau^*_{h'h''}(p^*e_{h'}) \in [0,1]\}$$

Proof. The proof is standard and follows the same steps as the proof of Proposition 9, with the appropriate adjustments concerning the utility functions and the constraint sets. □

We can then get the desired result of the section.

Theorem 15. *For any economy $\mathcal{E}$, an equilibrium $(x^*, p^*, \tau^*) \in \mathbb{R}^{CH}_+ \times S \times \Delta$ exists and $p^* >> 0$.*

Proof. From Proposition 11, a generalized Nash equilibria exists; from Proposition 13, an equilibria with upper bound on consumption exists; from Proposition 14 and taking into account that $p^*e_{h'} + \tau^*_{hh'}(p^*e_h) + \sum_{h'' \in \mathcal{B}_{\to h'}\setminus\{h\}} \tau^*_{h''h'}(p^*e_{h''}) - \sum_{h'' \in \mathcal{B}_{h'}} \tau^*_{h'h''}(p^*e_{h'}) \in [0,1]$, then an equilibrium exists. □

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Appendix A

A.1 Equivalent definitions of equilibria

Preliminary definitions.

$$\Delta^{num}_{\setminus h}(p) = \{t_{\setminus h}:=((t_{h'h''})_{h''\in\mathcal{B}_{h'}})_{h'\in\mathcal{H}\setminus\{h\}}\in\mathbb{R}^{\sum_{h'\in\mathcal{H}\setminus\{h\}} B_{h'}}: t_{\setminus h}\geq 0 \text{ and } \forall h'\in\mathcal{H}\setminus\{h\},\ p^1\sum_{h''\in\mathcal{B}_{h'}} t_{h'h''}\leq pe_{h'}\}$$

$$\Delta^{vec}_{\setminus h}(p) = \{\mathbf{t}_{\setminus h}:=((\mathbf{t}_{h'h''})_{h''\in\mathcal{B}_{h'}})_{h'\in\mathcal{H}\setminus\{h\}}\in\mathbb{R}^{C\sum_{h'\in\mathcal{H}\setminus\{h\}} B_{h'}}: \mathbf{t}_{\setminus h}\geq 0 \text{ and } \forall h'\in\mathcal{H}\setminus\{h\},\ p\sum_{h''\in\mathcal{B}_{h'}} \mathbf{t}_{h'h''}\leq pe_{h'}\}.$$

Remark 6. *In equilibrium $p^1 > 0$. That is a simple consequence of strict monotonicity of the utility function.*

A.1.1 The case of separable utility functions

Definition 15. *The vector $(x^*, t^*, p^*) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^B_+ \times S$ is a numeraire equilibrium for the economy $\mathcal{E} \in \mathbb{E}$ if*
i. for any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $p^* \in S, t^*_{\setminus h} \in \Delta^{num}_{\setminus h}(p^*)$

$$\max_{(x_h,t_h)\in\mathbb{R}^C_{++}\times\mathbb{R}^{B_h}} u_h(x_h) + \sum_{h'\in\mathcal{B}_h} v_{hh'}(p^*e_{h'} + p^{*1}t_{hh'} + p^{*1}\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} t^*_{h''h'} - p^{*1}\sum_{h''\in\mathcal{B}_{h'}} t^*_{h'h''})$$

subject to $(x_h, t_h) \in \Gamma^{num}_h(p^*, t^*_{\setminus h}) :=$

$$\{(x_h,t_h)\in\mathbb{R}^C_{++}\times\mathbb{R}^{B_h} : p^*e_h + p^{*1}\sum_{h'\in\mathcal{B}_{\to h}} t^*_{h'h} - p^{*1}\sum_{h'\in\mathcal{B}_h} t_{hh'} - p^*x_h \geq 0 \text{ for any } h'\in\mathcal{B}_h,\ t_{hh'}\geq 0,\ p^*e_h - \sum_{h'\in\mathcal{B}_h} p^{*1}t_{hh'}\geq 0\}$$

ii. $\sum_{h\in\mathcal{H}}(x^*_h - e_h) = 0$.
Definition 16. *The vector $(x^*, \mathbf{t}^*, p^*) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CB}_+ \times S$ is a vector equilibrium for the economy $\mathcal{E} \in \mathbb{E}$ if*
i. for any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $p^* \in S, \mathbf{t}^*_{\setminus h} \in \Delta^{vec}_{\setminus h}(p^*)$

$$\max_{(x_h,\mathbf{t}_h)\in\mathbb{R}^C_{++}\times\mathbb{R}^{CB_h}} u_h(x_h) + \sum_{h'\in\mathcal{B}_h} v_{hh'}(p^*(e_{h'} + \mathbf{t}_{hh'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \mathbf{t}^*_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \mathbf{t}^*_{h'h''}))$$

subject to $(x_h, t_h) \in \Gamma^{vec}_h(p^*, \mathbf{t}^*_{\setminus h}) :=$

$$\{(x_h,\mathbf{t}_h)\in\mathbb{R}^C_{++}\times\mathbb{R}^{CB_h} : p^*e_h + p^*\sum_{h'\in\mathcal{B}_{\to h}} \mathbf{t}^*_{h'h} - p^*\sum_{h'\in\mathcal{B}_h} \mathbf{t}_{hh'} - p^*x_h \geq 0 \text{ for any } h'\in\mathcal{B}_h,\ \mathbf{t}_{hh'}\geq 0,\ p^*e_h - \sum_{h'\in\mathcal{B}_h} p^*\mathbf{t}_{hh'}\geq 0\}$$

ii. $\sum_{h\in\mathcal{H}}(x^*_h - e_h) = 0$.
Proposition 16. *Definition 1, 15 and 16 are equivalent, i.e., the following statements hold true.*

1a. If $(x^*,\tau^*,p^*)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{B}_{+}\times S$ *is an equilibrium for the economy* $\mathcal{E}\in\mathbb{E}$*, then* $(x^*,t^*,p^*)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{B}_{+}\times S$ *is a numeraire equilibrium for the economy* $\mathcal{E}\in\mathbb{E}$ *, where for any* $h\in\mathcal{H}$ *and* $h'\in\mathcal{B}_h$*, we have* $t^*_{hh'}=\frac{\tau^*_{hh'}\cdot(p^*e_h)}{p^{*1}}$*;*
1b. If $(x^*,t^*,p^*)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{B}_{+}\times S$ *is a numeraire equilibrium for the economy* $\mathcal{E}\in\mathbb{E}$*, then* $(x^*,\tau^*,p^*)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{B}_{+}\times S$ *is an equilibrium for the economy* $\mathcal{E}\in\mathbb{E}$ *, where for any* $h\in\mathcal{H}$ *and* $h'\in\mathcal{B}_h$*, we have* $\tau^*_{hh'}=\frac{p^{*1}t^*_{hh'}}{p^*e_h}$*.*
2a. If $(x^*,\tau^*,p^*)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{B}_{+}\times S$ *is an equilibrium for the economy* $\mathcal{E}\in\mathbb{E}$*, then* $(x^*,\mathbf{t}^*,p^*)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{BC}_{+}\times S$ *is a vector equilibrium for the economy* $\mathcal{E}\in\mathbb{E}$ *, where for any* $h\in\mathcal{H}$ *and* $h'\in\mathcal{B}_h$*, we have* $\mathbf{t}^*_{hh'}=(\frac{\tau^*_{hh'}\cdot(p^*e_h)}{p^{*c}\cdot C})_{c\in\mathcal{C}}$*;*
2b. If $(x^*,\mathbf{t}^*,p^*)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{BC}_{+}\times S$ *is a vector equilibrium for the economy* $\mathcal{E}\in\mathbb{E}$*, then* $(x^*,\tau^*,p^*)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{B}_{+}\times S$ *is an equilibrium for the economy* $\mathcal{E}\in\mathbb{E}$ *, where for any* $h\in\mathcal{H}$ *and* $h'\in\mathcal{B}_h$*, we have* $\tau^*_{hh'}=\frac{p^*\mathbf{t}^*_{hh'}}{p^*e_h}$*.*

Proof. For all statements the market clearing conditions are clearly satisfied.
1a. For any $h\in\mathcal{H}$, for given $\mathcal{E}\in\mathbb{E}$, $p^*\in S$ and $\tau^*_{\setminus h}=(\tau^*_{h'})_{h'\in\mathcal{H}\setminus\{h\}}\in\Delta_{\setminus h}$, (x^*_h,τ^*_h) solves the maximization problem of Definition 1. We want to prove that, for any $h\in\mathcal{H}$, for given $\mathcal{E}\in\mathbb{E}$, $p^*\in S$ and $t^*_{\setminus h}=((\frac{\tau^*_{h'h''}(p^*e_{h'})}{p*1})_{h''\in\mathcal{B}_{h'}})_{h'\in\mathcal{H}\setminus\{h\}}\in\Delta^{num}_{\setminus h}(p^*)$, $(x^*_h,t^*_h=(\frac{\tau^*_{hh'}(p^*e_h)}{p*1})_{h'\in\mathcal{B}_h})$ solves the maximization problem of Definition 15. First of all, we have to verify that $(x^*_h,t^*_h=(\frac{\tau^*_{hh'}(p^*e_h)}{p*1})_{h'\in\mathcal{B}_h})$ satisfies the constraint set in Problem of Definition 15 at $t^*_{h'h''}=\frac{\tau_{h'h''}pe_{h'}}{p^{*1}}$ for any $h'\in\mathcal{H}\setminus\{h\}$ and any $h''\in\mathcal{B}_{h'}$. Indeed, $(x^*_h,t^*_h=(\frac{\tau^*_{hh'}(p^*e_h)}{p*1})_{h'\in\mathcal{B}_h})\in\Gamma^{num}_h(p^*,t^*_{\setminus h})$, because

$$\begin{array}{l}
p^*e_h+p^{*1}\sum_{h'\in\mathcal{B}_{\to h}}\frac{\tau^*_{h'h}p^*e_{h'}}{p^{*1}}-p^{*1}\sum_{h'\in\mathcal{B}_h}\frac{\tau^*_{hh'}pe_h}{p^{*1}}-p^*x_h\overset{\text{Def. }1}{\geq}0\\
\text{for any } h'\in\mathcal{B}_h,\ t_{hh'}=\frac{\tau^*_{hh'}(p^*e_h)}{p*1}\geq 0,\\
p^*e_h-p^{*1}\sum_{h'\in\mathcal{B}_h}\frac{\tau^*_{hh'}(p^*e_h)}{p*1}=p^*e_h(1-\sum_{h'\in\mathcal{B}_h}\tau^*_{hh'})\overset{\text{Def. }1}{\geq}0
\end{array}$$

We assume, by contradiction, that there exists (x^{**}_h,t^{**}_h) such that $(x^{**}_h,t^{**}_h)\in\Gamma^{num}_h(p^*,t^*_{\setminus h})$ and it gives a higher utility than (x^*_h,t^*_h). Then we get
i. $\left(x^{**}_h,\tau^{**}_h=(\frac{p^{*1}t^{**}_{hh'}}{p^*e_h})_{h'\in\mathcal{B}_h}\right)$ belongs to $\Gamma_h(p^*,\tau^*_{\setminus h})$;
ii. it gives a higher utility in the maximization problem of Definition 1 than (x^*_h,τ^*_h)
where i. and ii. contradict the definition of (x^*_h,τ^*_h).
i. From $(x^{**}_h,t^{**}_h)\in\Gamma^{num}_h(p^*,t^*_{\setminus h})$, we get

$$0\leq p^*e_h+p^{*1}\sum_{h'\in\mathcal{B}_{\to h}}t^*_{h'h}-p^{*1}\sum_{h'\in\mathcal{B}_h}t^{**}_{hh'}-p^*x^{**}_h=p^*e_h+\sum_{h'\in\mathcal{B}_{\to h}}\tau^*_{h'h}\cdot(p^*e_{h'})-\sum_{h'\in\mathcal{B}_h}\tau^{**}_{hh'}\cdot(p^*e_h)-p^*x^{**}_h;$$

for any $h'\in\mathcal{B}_h$, $0\leq t^{**}_{hh'}=\frac{\tau^{**}_{hh'}(p^*e_h)}{p^{*1}}$ or $\tau^{**}_{hh'}\geq 0$;

$$0\leq p^*e_h-\sum_{h'\in\mathcal{B}_h}p^{*1}t^{**}_{hh'}=(p^*e_h)(1-\sum_{h'\in\mathcal{B}_h}\tau^{**}_{hh'})\ \text{ or }\ (1-\sum_{h'\in\mathcal{B}_h}\tau^{**}_{hh'})\geq 0.$$

That is $(x_h^{**}, \tau_h^{**}) \in \Gamma_h(p^*, \tau_{\setminus h}^*)$.
ii. From the fact that (x_h^{**}, t_h^{**}) gives a higher utility than (x_h^*, t_h^*), we get

$$u_h(x_h^{**}) + \sum_{h' \in \mathcal{B}_h} v_{hh'} \Big(p^* e_{h'} + p^{*1} t_{hh'}^{**} + p^{*1} \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} t_{h''h'}^* - p^{*1} \sum_{h'' \in \mathcal{B}_{h'}} t_{h'h''}^* \Big) >$$

$$u_h(x_h^*) + \sum_{h' \in \mathcal{B}_h} v_{hh'} \Big(p^* e_{h'} + p^{*1} t_{hh'}^* + p^{*1} \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} t_{h''h'}^* - p^{*1} \sum_{h'' \in \mathcal{B}_{h'}} t_{h'h''}^* \Big)$$

that becomes, from definition of $t_{hh'}^{**}$

$$u_h(x_h^{**}) + \sum_{h' \in \mathcal{B}_h} v_{hh'} \Big(p^* e_{h'} + \tau_{hh'}^{**} \cdot (p^* e_h) + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \tau_{h''h'}^* \cdot (p^* e_{h''}) - \sum_{h'' \in \mathcal{B}_{h'}} \tau_{h'h''}^* \cdot (p^* e_{h'}) \Big) >$$

$$u_h(x_h^*) + \sum_{h' \in \mathcal{B}_h} v_{hh'} \Big(p^* e_{h'} + \tau_{hh'}^* \cdot (p^* e_h) + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \tau_{h''h'}^* \cdot (p^* e_{h''}) - \sum_{h'' \in \mathcal{B}_{h'}} \tau_{h'h''}^* \cdot (p^* e_{h'}) \Big)$$

contradicting the definition of (x_h^*, τ_h^*).
1b. For any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $p^* \in S$ and $t_{\setminus h}^* \in \Delta_{\setminus h}^{num}(p^*)$, $(x_h^*, t_h^*) \in \Gamma_h^{num}(p^*, t_{\setminus h}^*)$ solves the maximization problem of Definition 15, we want to prove that, for any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $p^* \in S$ and $\tau_{\setminus h}^* = ((\frac{p^{*1} t_{h'h''}^*}{p^* e_{h'}})_{h'' \in \mathcal{B}'_h})_{h \in \mathcal{H} \setminus \{h\}} \in \Delta_{\setminus h}$, $\big(x_h^*, \tau_h^* = ((\frac{p^{*1} t_{hh'}^*}{p^* e_h})_{h' \in \mathcal{B}_h})$ solves the maximization problem of Definition 1. First of all, notice that $\big(x_h^*, \tau_h^* = ((\frac{p^{*1} t_{hh'}^*}{p^* e_h})_{h' \in \mathcal{B}_h})$ belongs to the constraint set of problem of Definition 1 at $\tau_{h'h''}^* = \frac{p^{*1} t_{h'h''}^*}{p^* e_h}$ for any $h' \in \mathcal{H} \setminus h$ and any $h'' \in \mathcal{B}_{h'}$. Indeed, we have $\big(x_h^*, \tau_h^* = ((\frac{p^{*1} t_{hh'}^*}{p^* e_h})_{h' \in \mathcal{B}_h}) \in \Gamma_h(p^*, \tau_{\setminus h}^*)$, because

$$\begin{array}{l} p^* e_h + p^{*1} \sum_{h' \in \mathcal{B}_{\to h}} \frac{t_{h'h}^*}{p^* e_{h'}} (p^* e_{h'}) - p^{*1} \sum_{h' \in \mathcal{B}_h} \frac{t_{hh'}^*}{p^* e_h} (p e_h) - p^* x_h \overset{\text{Def. } 15}{\geq} 0 \\ \text{for any } h' \in \mathcal{B}_h,\ \tau_{hh'} = \frac{p^* 1 t_{hh'}^*}{p^* e_h} \geq 0, \\ 1 - \sum_{h' \in \mathcal{B}_h} \frac{p^{*1} t_{hh'}^*}{p^* e_h} = p^* e_h - p^{*1} \sum_{h' \in \mathcal{B}_h} t_{hh'}^* \overset{\text{Def. } 15}{\geq} 0 \end{array}$$

We assume, by contradiction, that there exists (x_h^{**}, τ_h^{**}) such that $(x_h^{**}, \tau_h^{**}) \in \Gamma_h(p^*, \tau_{\setminus h}^*)$ and it gives a higher utility than (x_h^*, τ_h^*). Then we get
i. $\big(x_h^{**}, t_h^{**} = (\frac{\tau_{hh'}^{**} (p^* e_h)}{p^{*1}})_{h' \in \mathcal{B}_h}\big)$ belongs to $\Gamma_h^{num}(p^*, \tau_{\setminus h}^*)$;
ii. it gives a higher utility in the maximization problem of Definition 15 than (x_h^*, t_h^*),
where i. and ii. contradict the definition of (x_h^*, t_h^*).
i. From $(x_h^{**}, \tau_h^{**}) \in \Gamma_h(p^*, \tau_{\setminus h}^*)$, we get

$$0 \leq p^* e_h + \sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h}^* \cdot (p^* e_{h'}) - \sum_{h' \in \mathcal{B}_h} \tau_{hh'}^{**} \cdot (p^* e_h) - p^* x_h^{**} = p^* e_h + p^{*1} \sum_{h' \in \mathcal{B}_{\to h}} t_{h'h}^* - p^{*1} \sum_{h' \in \mathcal{B}_h} t_{hh'}^{**} - p^* x_h^{**};$$

for any $h' \in \mathcal{B}_h$, $0 \leq \tau^{**}_{hh'} = \frac{t^{**}_{hh'} p^{*1}}{p^* e_h}$ or $t^{**}_{hh'} \geq 0$;

$$0 \leq 1 - \sum_{h' \in \mathcal{B}_h} \tau^{**}_{hh'} = 1 - \sum_{h' \in \mathcal{B}_h} \frac{t^{**}_{hh'} p^{*1}}{p^* e_h} \quad or \quad p^* e_h - \sum_{h' \in \mathcal{B}_h} p^{*1} t^{**}_{hh'} \geq 0.$$

That is $(x^{**}_h, t^{**}_h) \in \Gamma^{num}_h(p^*, t^*_{\setminus h})$.
ii. From the fact that (x^{**}_h, τ^{**}_h) and it gives a higher utility than (x^*_h, τ^*_h), we get

$$u_h(x^{**}_h) + \sum_{h' \in \mathcal{B}_h} v_{hh'} \Big(p^* e_{h'} + \tau^{**}_{hh'} \cdot (p^* e_h) + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \tau^*_{h''h'} \cdot (p^* e_{h''}) - \sum_{h'' \in \mathcal{B}_{h'}} \tau^*_{h'h''} \cdot (p^* e_{h'}) \Big) >$$

$$u_h(x^*_h) + \sum_{h' \in \mathcal{B}_h} v_{hh'} \Big(p^* e_{h'} + \tau^*_{hh'} \cdot (p^* e_h) + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \tau^*_{h''h'} \cdot (p^* e_{h''}) - \sum_{h'' \in \mathcal{B}_{h'}} \tau^*_{h'h''} \cdot (p^* e_{h'}) \Big),$$

that becomes, from definition of $\tau^{**}_{hh'}$

$$u_h(x^{**}_h) + \sum_{h' \in \mathcal{B}_h} v_{hh'} \Big(p^* e_{h'} + p^{*1} t^{**}_{hh'} + p^{*1} \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} t^*_{h''h'} - p^{*1} \sum_{h'' \in \mathcal{B}_{h'}} t^*_{h'h''} \Big) >$$

$$u_h(x^*_h) + \sum_{h' \in \mathcal{B}_h} v_{hh'} \Big(p^* e_{h'} + p^{*1} t^*_{hh'} + p^{*1} \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} t^*_{h''h'} - p^{*1} \sum_{h'' \in \mathcal{B}_{h'}} t^*_{h'h''} \Big),$$

contradicting the definition of (x^*_h, t^*_h).
2a. For any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $p^* \in S$ and $\tau^*_{\setminus h} = (\tau^*_{h'})_{h' \in \mathcal{H} \setminus \{h\}} \in \Delta_{\setminus h}$, (x^*_h, τ^*_h) solves the maximization problem of Definition 1. We want to prove that, for any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $p^* \in S$ and $\mathbf{t}^*_{\setminus h} = \Big(\big((\frac{\tau^*_{h'h''}(p^* e_{h'})}{C p^{*c}})_{c \in \mathcal{C}} \big)_{h'' \in \mathcal{B}_{h'}} \Big)_{h' \in \mathcal{H} \setminus \{h\}} \in \Delta^{vec}_{\setminus h}(p^*)$, $\big(x^*_h, \mathbf{t}^*_h = ((\frac{\tau^*_{hh'}(p^* e_h)}{C p^{*c}})_{c \in \mathcal{C}})_{h' \in \mathcal{B}_h}\big)$ solves the maximization problem of Definition 16. First of all, we show that $\big(x^*_h, \mathbf{t}^*_h = ((\frac{\tau^*_{hh'}(p^* e_h)}{C p^{*c}})_{c \in \mathcal{C}})_{h' \in \mathcal{B}_h}\big)$ belongs to the constraint set of problem of Definition 16 at $\mathbf{t}^*_{\setminus h} = \Big(\big((\frac{\tau^*_{h'h''}(p^* e_{h'})}{C p^{*c}})_{c \in \mathcal{C}} \big)_{h'' \in \mathcal{B}_{h'}} \Big)_{h' \in \mathcal{H} \setminus \{h\}} \in \Delta^{vec}_{\setminus h}(p^*)$. Indeed,

$$p^* e_h + \textstyle\sum_{c \in \mathcal{C}} p^{*c} \sum_{h' \in \mathcal{B}_{\to h}} \frac{\tau^*_{h'h}(p^* e_{h'})}{C p^{*c}} - \sum_{c \in \mathcal{C}} p^{*c} \sum_{h' \in \mathcal{B}_h} \frac{\tau^*_{hh'}(p^* e_h)}{C p^{*c}} - p^* x^*_h =$$

$$p^* e_h + \textstyle\sum_{h' \in \mathcal{B}_{\to h}} \tau^*_{h'h}(p^* e_{h'}) - \sum_{h' \in \mathcal{B}_h} \tau^*_{hh'}(p^* e_h) - p^* x^*_h \overset{\text{Def. } 1}{\geq} 0$$

$$\text{for any } h' \in \mathcal{B}_h,\ \mathbf{t}^*_{hh'} = \big(\tfrac{\tau^*_{hh'}(p * e_h)}{C p^{*c}}\big)_{c=1}^{C} \geq 0,$$

$$p^* e_h - \textstyle\sum_{c \in \mathcal{C}} p^{*c} \sum_{h \in \mathcal{B}_h} \frac{\tau^*_{hh'}(p^* e_h)}{C p^{*c}} = p^* e_h (1 - \sum_{h \in \mathcal{B}_h} \tau^*_{hh'}) \overset{\text{Def. } 1}{\geq} 0$$

We assume, by contradiction, that there exists $(x^{**}_h, \mathbf{t}^{**}_h)$ such that $(x^{**}_h, \mathbf{t}^{**}_h) \in \Gamma^{vec}_h(p^*, \mathbf{t}^*_{\setminus h})$ and it gives a higher utility than $(x^*_h, \mathbf{t}^*_h)$. Then we get

i. $\left(x_h^{**}, \tau_h^{**} = (\frac{Cp^{*c}t_{hh'}^{**c}}{p^*e_h})_{h'\in\mathcal{B}_h}\right)$ belongs to $\Gamma_h(p^*, \tau_{\setminus h}^*)$;
ii. it gives a higher utility in the maximization problem of Definition 1 than (x_h^*, τ_h^*),
where i. and ii. contradict the definition of (x_h^*, τ_h^*).
i. From $(x_h^{**}, \mathbf{t}_h^{**}) \in \Gamma_h^{vec}(p^*, t_{\setminus h}^*)$, we get

$$0 \leq p^*e_h + p^* \sum_{h'\in\mathcal{B}_{\to h}} \mathbf{t}_{h'h}^* - p^* \sum_{h'\in\mathcal{B}_h} \mathbf{t}_{hh'}^{**} - p^*x_h = p^*e_h + \sum_{h'\in\mathcal{B}_{\to h}} \tau_{h'h}^* \cdot (p^*e_{h'}) - \sum_{h'\in\mathcal{B}_h} \tau_{hh'}^{**} \cdot (p^*e_h) - p^*x_h^{**};$$

for any $h' \in \mathcal{B}_h$ and $c \in \mathcal{C}$, $0 \leq \mathbf{t}_{hh'}^{**c} = \frac{\tau_{hh'}^{**}(p^*e_h)}{Cp^{*c}}$ or $\tau_{hh'}^{**} \geq 0$;

$$0 \leq p^*e_h - \sum_{h'\in\mathcal{B}_h} p^*\mathbf{t}_{hh'}^{**} = (p^*e_h)(1 - \sum_{h'\in\mathcal{B}_h} \tau_{hh'}^{**}) \quad or \quad 1 - \sum_{h'\in\mathcal{B}_h} \tau_{hh'}^{**} \geq 0.$$

That is $(x_h^{**}, \tau_h^{**}) \in \Gamma_h(p^*, \tau_{\setminus h}^*)$.
ii. From the fact that $(x_h^{**}, \mathbf{t}_h^{**})$ gives a higher utility than $(x_h^*, \mathbf{t}_h^*)$, we get

$$u_h(x_h^{**}) + \sum_{h'\in\mathcal{B}_h} v_{hh'}\Big(p^*(e_{h'} + \mathbf{t}_{hh'}^{**} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \mathbf{t}_{h''h'}^* - \sum_{h''\in\mathcal{B}_{h'}} \mathbf{t}_{h'h''}^*)\Big) >$$

$$u_h(x_h^{**}) + \sum_{h'\in\mathcal{B}_h} v_{hh'}\Big(p^*(e_{h'} + \mathbf{t}_{hh'}^{*} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \mathbf{t}_{h''h'}^* - \sum_{h''\in\mathcal{B}_{h'}} \mathbf{t}_{h'h''}^*)\Big),$$

that becomes, from definition of $\mathbf{t}_{hh'}^{**}$

$$u_h(x_h^{**}) + \sum_{h'\in\mathcal{B}_h} v_{hh'}\Big(p^*e_{h'} + \tau_{hh'}^{**} \cdot (p^*e_h) + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau_{h''h'}^* \cdot (p^*e_{h''}) - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}^* \cdot (p^*e_{h'})\Big) >$$

$$u_h(x_h^{*}) + \sum_{h'\in\mathcal{B}_h} v_{hh'}\Big(p^*e_{h'} + \tau_{hh'}^{*} \cdot (p^*e_h) + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau_{h''h'}^* \cdot (p^*e_{h''}) - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}^* \cdot (p^*e_{h'})\Big)$$

contradicting the definition of (x_h^*, τ_h^*).
2b. We assume that for any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $p^* \in S$ and $\mathbf{t}_{\setminus h}^* = (\mathbf{t}_{h'}^*)_{h'\in\mathcal{H}\setminus\{h\}} \in \Delta_{\setminus h}^{vec}(p^*)$, $(x_h^*, \mathbf{t}_h^*)$ solves the maximization problem of Definition 16. We want to prove that, for any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $p^* \in S$ and

$$\tau_{\setminus h}^* = \Big((\frac{p^*\mathbf{t}_{h'h''}^*}{p^*e_{h'}})_{h''\in\mathcal{B}_{h'}}\Big)_{h'\in\mathcal{H}\setminus\{h\}} \in \Delta_{\setminus h}, \tag{A1}$$

$\left(x_h^*, \tau_h^* = (\frac{p^*\mathbf{t}_{hh'}^*}{p^*e_h})_{h'\in\mathcal{B}_h}\right)$ solves the maximization problem of Definition 1. First of all, we show that (x_h^*, τ_h^*) belongs to the constraint set of problem of Definition 1 at

$\tau^*_{\setminus h} = \left(\left(\frac{p^* \mathbf{t}^*_{h'h''}}{p^* e_{h'}} \right)_{h'' \in \mathcal{B}_{h'}} \right)_{h' \in \mathcal{H} \setminus \{h\}} \in \Delta_{\setminus h}$. Indeed,

$$p^* e_h + \sum_{h' in \mathcal{B}_{\to h}} \frac{p^* \mathbf{t}^*_{h'h}}{p^* e_{h'}} (p^* e_{h'}) - \sum_{h' \in \mathcal{B}_h} \frac{p^* \mathbf{t}^*_{hh'}}{p^* e_h} (p^* e_h) - p^* x^*_h =$$

$$p^* e_h + p^* \sum_{h' in \mathcal{B}_{\to h}} \mathbf{t}^*_{h'h} - p^* \sum_{h' \in \mathcal{B}_h} \mathbf{t}^*_{hh'} - p^* x^*_h \overset{\text{Def. } 16}{\geq} 0$$

for any $h' \in \mathcal{B}_h$, $\tau^*_{hh'} = (\frac{p^* \mathbf{t}^*_{hh'}}{p^* e_h}) \geq 0$

$$1 - \sum_{h' \in \mathcal{B}_h} \frac{p^* \mathbf{t}^*_{hh'}}{p^* e_h} = p^* e_h (1 - p^* \sum_{h' \in \mathcal{B}_h} \mathbf{t}_{hh'}) \overset{\text{Def. } 16}{\geq} 0$$

We assume, by contradiction, that there exists (x^{**}_h, τ^{**}_h) such that $(x^{**}_h, \tau^{**}_h) \in \Gamma_h(p^*, \tau^*_{\setminus h})$ and which gives a higher utility than (x^*_h, τ^*_h). Define

$$\mathbf{t}^{**}_h := \left(\left(\frac{\tau^{**}_{hh'}(p^* e_h)}{C p^{*c}} \right)_{c \in \mathcal{C}} \right)_{h' \in \mathcal{B}_h}, \tag{A2}$$

where the equilibrium price p^{*c} of commodity c is strictly positive by the monotonicity assumption. Notice that, from (A2)

$$p^* \mathbf{t}^{**}_{hh'} = \sum_{c \in \mathcal{C}} p^{c*} \frac{\tau^{**}_{hh'}(p^* e_h)}{C p^{*c}} = \tau^{**}_{hh'}(p^* e_h), \tag{A3}$$

and from (A1)

$$p^* \mathbf{t}^*_{h'h} = \tau^*_{h'h}(p^* e_{h'}). \tag{A4}$$

We want to show that i. $(x^{**}_h, \mathbf{t}^{**}_h) \in \Gamma^{vec}_h(p^*, \mathbf{t}^*_{\setminus h})$ and ii. it gives a higher utility in the maximization problem of Definition 16, than (x^*_h, t^*_h), as verified below.
i. From $(x^{**}_h, \tau^{**}_h) \in \Gamma_h(p^*, t^*_{\setminus h})$ and taking into account (A3) and (A4), we get

$$0 \leq p^* e_h + \sum_{h' \in \mathcal{B}_{\to h}} \tau^*_{h'h} \cdot (p^* e_{h'}) - \sum_{h' \in \mathcal{B}_h} \tau^{**}_{hh'} \cdot (p^* e_h) - p^* x^{**}_h = p^* e_h + p^* \sum_{h' \in \mathcal{B}_{\to h}} \mathbf{t}^*_{h'h} - p^* \sum_{h' \in \mathcal{B}_h} \mathbf{t}^{**}_{hh'} - p^* x^{**}_h;$$

for any $h' \in \mathcal{B}_h$ and $c \in \mathcal{C}$, $\mathbf{t}^{**c}_{hh'} = \frac{\tau^{**}_{hh'}(p^* e_h)}{C p^{*c}} \geq 0$ or $\mathbf{t}^{**}_{hh'} \geq 0$;

$$0 \leq 1 - \sum_{h' \in \mathcal{B}_h} \tau^{**}_{hh'} = 1 - \sum_{h' \in \mathcal{B}_h} \frac{p^* \mathbf{t}^{**}_{hh'}}{p^* e_h} \text{ that is, } 0 \leq p^* e_h - \sum_{h' \in \mathcal{B}_h} p^* \mathbf{t}^{**}_{hh'},$$

that is $(x^{**}_h, \mathbf{t}^{**}_h) \in \Gamma^{vec}_h(p^*, \mathbf{t}^*_{\setminus h})$.
ii. From the fact that (x^{**}_h, τ^{**}_h) gives a higher utility than (x^*_h, τ^*_h), we get

$$u_h(x^{**}_h) + \sum_{h' \in \mathcal{B}_h} v_{hh'} \left(p^* e_{h'} + \tau^{**}_{hh'} \cdot (p^* e_h) + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \tau^*_{h''h'} \cdot (p^* e_{h''}) - \sum_{h'' \in \mathcal{B}_{h'}} \tau^*_{h'h''} \cdot (p^* e_{h'}) \right) >$$

$$u_h(x_h^*)+\sum_{h'\in\mathcal{B}_h} v_{hh'}\Big(p^*e_{h'}+\tau^*_{hh'}\cdot(p^*e_h)+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}\cdot(p^*e_{h''})-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\cdot(p^*e_{h'})\Big)$$

that becomes, from $\tau^{**}_{hh'}=\frac{p^*t_{hh'}}{p^*e_h}$, for any $h\in\mathcal{H}, h'\in\mathcal{B}_h$ and $\tau^*_{h'h''}=\frac{p^*t^*_{h'h''}}{p^*e_{h'}}$, for any $h'\in\mathcal{H}\setminus\{h\}, h''\in\mathcal{B}_{h'}$,

$$u_h(x_h^{**})+\sum_{h'\in\mathcal{B}_h} v_{hh'}\Big(p^*\big(e_{h'}+\mathbf{t}^{**}_{hh'}+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\mathbf{t}^*_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\mathbf{t}^*_{h'h''}\big)\Big)>$$

$$u_h(x_h^{**})+\sum_{h'\in\mathcal{B}_h} v_{hh'}\Big(p^*\big(e_{h'}+\mathbf{t}^{*}_{hh'}+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\mathbf{t}^*_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\mathbf{t}^*_{h'h''}\big)\Big).$$

contradicting the definition of $(x_h^*,\mathbf{t}_h^*)$. □

A.1.2 The case of nonseparable utility functions

For any $h\in\mathcal{H}$ and any $h'\in\mathcal{B}_h^{\setminus}$, we define

$$q_h^{num}(p,t_{\setminus h}):=\Big(pe_{h'}+0+p^1\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}t_{h''h'}-p^1\sum_{h''\in\mathcal{B}_{h'}}t_{h'h''}\Big)_{h'\in\mathcal{B}_h^{\setminus}},\ \ \forall(p,t_{\setminus h})\in S\times\Delta^{num}_{\setminus h}(p)$$

and

$$q_h^{vec}(p,\mathbf{t}_{\setminus h}):=\Big(pe_{h'}+0+p\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\mathbf{t}_{h''h'}-p\sum_{h''\in\mathcal{B}_{h'}}\mathbf{t}_{h'h''}\Big)_{h'\in\mathcal{B}_h^{\setminus}},\forall(p,\mathbf{t}_{\setminus h})\in S\times\Delta^{vec}_{\setminus h}(p).$$

Definition 17. *The vector $(x^*,t^*,p^*)\in\mathbb{R}^{CH}_+\times\mathbb{R}^B_+\times S$ is a numeraire equilibrium for the economy $\widehat{\mathcal{E}}\in\widehat{\mathbb{E}}$ if*
i. for any $h\in\mathcal{H}$, for given $\widehat{\mathcal{E}}\in\widehat{\mathbb{E}}$, $p^*\in S$, $t^*_{\setminus h}\in\Delta^{num}_{\setminus h}(p^*)$

$$\max_{(x_h,t_h)\in\mathbb{R}^C_+\times\mathbb{R}^{B_h}}\widehat{U}_h(x_h,(p^*e_{h'}+p^{*1}t_{hh'}+p^{*1}\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}t^*_{h''h'}-p^{*1}\sum_{h''\in\mathcal{B}_{h'}}t^*_{h'h''}),q_h^{num}(p^*,t^*_{\setminus h}))$$

$$\text{subject to }\ (x_h,t_h)\in\Gamma_h^{num}(p^*,t^*_{\setminus h})$$

ii. $\sum_{h\in\mathcal{H}}(x_h^*-e_h)=0$.

Definition 18. *The vector $(x^*,\mathbf{t}^*,p^*)\in\mathbb{R}^{CH}_+\times\mathbb{R}^{CB}_+\times S$ is a vector equilibrium for the economy $\widehat{\mathcal{E}}\in\widehat{\mathbb{E}}$ if*
i. for any $h\in\mathcal{H}$, for given $\widehat{\mathcal{E}}\in\widehat{\mathbb{E}}$, $p^*\in S$, $\mathbf{t}^*_{\setminus h}\in\Delta^{vec}_{\setminus h}(p^*)$

$$\max_{(x_h,\mathbf{t}_h)\in\mathbb{R}^C_+\times\mathbb{R}^{CB_h}}\widehat{U}_h(x_h,(p^*(e_{h'}+\mathbf{t}_{hh'}+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\mathbf{t}^*_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\mathbf{t}^*_{h'h''})),q_h^{vec}(p^*,\mathbf{t}^*_{\setminus h}))$$

$$\text{subject to }\ (x_h,\mathbf{t}_h)\in\Gamma_h^{vec}(p^*,\mathbf{t}^*_{\setminus h})$$

ii. $\sum_{h\in\mathcal{H}}(x_h^* - e_h) = 0$.

Proposition 17. *Definition 11, 17 and 18 are equivalent, i.e., the following statements hold true.*

1a. If $(x^*, \tau^*, p^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^B \times S$ *is an equilibrium for the economy* $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$*, then* $(x^*, t^*, p^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^B \times S$ *is a numeraire equilibrium for the economy* $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$ *, where for any* $h \in \mathcal{H}$ *and* $h' \in \mathcal{B}_h$*, we have* $t_{hh'}^* = \frac{\tau_{hh'}^* \cdot (p^* e_h)}{p^{*1}}$*;*

1b. If $(x^*, t^*, p^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^B \times S$ *is a numeraire equilibrium for the economy* $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$*, then* $(x^*, \tau^*, p^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^B \times S$ *is an equilibrium for the economy* $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$*, where for any* $h \in \mathcal{H}$ *and* $h' \in \mathcal{B}_h$*, we have* $\tau_{hh'}^* = \frac{p^{*1} t_{hh'}^*}{p^* e_h}$*.*

2a. If $(x^*, \tau^*, p^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^B \times S$ *is an equilibrium for the economy* $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$*, then* $(x^*, \mathbf{t}^*, p^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^{BC} \times S$ *is a vector equilibrium for the economy* $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$*, where for any* $h \in \mathcal{H}$ *and* $h' \in \mathcal{B}_h$*, we have* $\mathbf{t}_{hh'}^* = (\frac{\tau_{hh'}^* \cdot (p^* e_h)}{p^{*c} \cdot C})_{c\in\mathcal{C}}$*;*

2b. If $(x^*, \mathbf{t}^*, p^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^{BC} \times S$ *is a vector equilibrium for the economy* $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$*, then* $(x^*, \tau^*, p^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^B \times S$ *is an equilibrium for the economy* $\widehat{\mathcal{E}} \in \widehat{\mathbb{E}}$ *, where for any* $h \in \mathcal{H}$ *and* $h' \in \mathcal{B}_h$*, we have* $\tau_{hh'}^* = \frac{p^* \mathbf{t}_{hh'}^*}{p^* e_h}$*.*

Proof. The proof is a simple adaptation of the proof of Proposition 16. □

A.2 The extension of the functions $v_{hh'}$

Proposition 18. *If a function* $v : (0,1) \to \mathbb{R}$ *is continuous, strictly increasing and bounded, then there exists an extension* V *of* v *on* $[0,1]$*, so defined,* $V : [0,1] \to \mathbb{R}$*,*

$$x \mapsto \begin{cases} \lim_{x\to 0^+} v(x) & \text{if } x = 0, \\ v(x) & \text{if } x \in (0,1), \\ \lim_{x\to 1^-} v(x) & \text{if } x = 1. \end{cases}$$

which is continuous, strictly increasing and bounded.

Proof. This proposition represents a simpler case of the one already described in Proposition 23 of Donato et al. (2025). Moreover, the boundedness of function V follows directly from the boundedness of function v. □

Proposition 19. *Under the maintained assumptions, for any* $h \in \mathcal{H}$*, the function* $u_h + \sum_{h'\in\mathcal{B}_h} V_{hh'} : \mathbb{R}_{++}^C \times [0,1]^{B_h} \to \mathbb{R}$ *is continuous and quasi-concave.*

Proof. Continuity follows immediately. The proof of quasi-concavity involves some notational complexity and it is presented below. We have to show that, for any $h \in \mathcal{H}$, $h' \in \mathcal{B}_h$, for any $(x'_h, w'_{h'}), (x''_h, w''_{h'}) \in \mathbb{R}_{++}^C \times [0,1]^{B_h}$ and for any $\lambda \in (0,1)$,

$$u_h((1-\lambda)x'_h + \lambda x''_h) + \textstyle\sum_{h'\in\mathcal{B}_h} V_{hh'}((1-\lambda)w'_{h'} + \lambda w''_{h'}) \geq$$

$$\min\{u_h(x'_h) + \textstyle\sum_{h'\in\mathcal{B}_h} V_{hh'}(w'_{h'}), \quad u_h(x''_h) + \textstyle\sum_{h'\in\mathcal{B}_h} V_{hh'}(w''_{h'})\}.$$

For any $w = (w_{h'})_{h' \in \mathcal{B}_h} \in [0,1]^{B_h}$, define $\mathcal{B}_h^0(w) = \{h' \in \mathcal{B}_h : w_{h'} = 0\}$, $\mathcal{B}_h^{w_{h'}}(w) = \{h' \in \mathcal{B}_h : w_{h'} \in (0,1)\}$, $\mathcal{B}_h^1(w) = \{h' \in \mathcal{B}_h : w_{h'} = 1\}$, we have

$$\mathcal{B}_h = \cup_{i \in \{0, w_{h'}, 1\}} \mathcal{B}_h^i(w). \tag{A5}$$

For any $w' = (w'_{h'})_{h' \in \mathcal{B}_h}, w'' = (w''_{h'})_{h' \in \mathcal{B}_h} \in [0,1]^{B_h}$, $i \in \{0, w'_{h'}, 1\}$, $j \in \{0, w''_{h'}, 1\}$, defined $\mathcal{B}_h^{i,j}(w', w'') = \mathcal{B}_h^i(w') \cap \mathcal{B}_h^j(w'')$, we can prove that

$$\mathcal{B}_h^i(w') = \cup_{j \in \{0, w''_{h'}, 1\}} \mathcal{B}_h^{i,j}(w', w''). \tag{A6}$$

Them from (A5) and (A6), we get that for any $w', w'' \in [0,1]^{B_h}$,

$$\mathcal{B}_h = \cup_{i \in \{0, w_{h'}, 1\}} \cup_{j \in \{0, w''_{h'}, 1\}} \mathcal{B}_h^{i,j}(w', w''). \tag{A7}$$

and, moreover, $\sum_{h' \in \mathcal{B}_h} V_{hh'}(w') =$

$$\begin{array}{l} \sum_{h' \in \mathcal{B}_h^0(w)} V_{hh'}(0) + \sum_{h' \in \mathcal{B}_h^{w_{h'}}(w)} V_{hh'}(w_{h'}) + \sum_{h' \in \mathcal{B}_h^1(w)} V_{hh'}(1) = \\ \sum_{h' \in \cup_{j \in \{0, w''_{h'}, 1\}} \mathcal{B}_h^{0,j}(w', w'')} V_{hh'}(0) + \sum_{h' \in \cup_{j \in \{0, w''_{h'}, 1\}} \mathcal{B}_h^{w'_{h'}, j}(w', w'')} V_{hh'}(w'_{h'}) + \\ \sum_{h' \in \cup_{j \in \{0, w''_{h'}, 1\}} \mathcal{B}_h^{1,j}(w', w'')} V_{hh'}(1). \end{array} \tag{A8}$$

For a lighter notation, we drop the term (w', w'') in the symbols $\mathcal{B}_h^{i,j}(w', w'')$. From (A7), for any $w', w'' \in [0,1]^{B_h}$,

$$\begin{array}{l} \sum_{h' \in \mathcal{B}_h} V_{hh'}((1-\lambda) w'_{h'} + \lambda w''_{h'}) = \\ \sum_{h' \in \mathcal{B}_h^{0,0}} V_{hh'}((1-\lambda)0 + \lambda 0) + \sum_{h' \in \mathcal{B}_h^{0, w''_{h'}}} V_{hh'}((1-\lambda)0 + \lambda w''_{h'}) + \sum_{h' \in \mathcal{B}_h^{0,1}} V_{hh'}((1-\lambda)0 + \lambda 1) + \\ \sum_{h' \in \mathcal{B}^{w'_{h'}, 0}} V_{hh'}((1-\lambda) w'_{h'} + \lambda 0) + \sum_{h' \in \mathcal{B}_h^{w'_{h'}, w''_{h'}}} V_{hh'}((1-\lambda) w'_{h'} + \lambda w''_{h'}) + \sum_{h' \in \mathcal{B}_h^{w'_{h'}, 1}} V_{hh'}((1-\lambda) w'_{h'} + \lambda 1) + \\ \sum_{h' \in \mathcal{B}_h^{1,0}} V_{hh'}((1-\lambda)1 + \lambda 0) + \sum_{h' \in \mathcal{B}_h^{1, w''_{h'}}} V_{hh'}((1-\lambda)1 + \lambda w''_{h'}) + \sum_{h' \in \mathcal{B}_h^{1,1}} V_{hh'}((1-\lambda)1 + \lambda 1). \end{array} \tag{A9}$$

Taking into account of quasi-concavity of $u_h + \sum_{h' \in \mathcal{B}_h} v_{hh'}$ and that the argument of each addendum belongs to $(0,1)$ and $V_{hh'} = v_{hh'}$, we get, for any $(x'_h, w'_{h'}), (x''_h, w''_{h'}) \in \mathbb{R}^C_{++} \times [0,1]^{B_h}$, for any $n \in \mathbb{N}$ and for any $\lambda \in [0,1]$,

$$\begin{array}{l} u_h((1-\lambda) x'_h + \lambda x''_h) + \\ \sum_{h' \in \mathcal{B}_h^{0,0}} V_{hh'}((1-\lambda)\frac{1}{n} + \lambda \frac{1}{n}) + \sum_{h' \in \mathcal{B}_h^{0, w''_{h'}}} V_{hh'}((1-\lambda)\frac{1}{n} + \lambda w''_{h'}) + \sum_{h' \in \mathcal{B}_h^{0,1}} V_{hh'}((1-\lambda)\frac{1}{n} + \lambda(1-\frac{1}{n})) + \\ \sum_{h' \in \mathcal{B}_h^{w'_{h'}, 0}} V_{hh'}((1-\lambda) w'_{h'} + \lambda \frac{1}{n}) + \sum_{h' \in \mathcal{B}_h^{w'_{h'}, w''_{h'}}} V_{hh'}((1-\lambda) w'_{h'} + \lambda w''_{h'}) + \sum_{h' \in \mathcal{B}_h^{w'_{h'}, 1}} V_{hh'}((1-\lambda) w'_{h'} + \lambda(1-\frac{1}{n})) + \\ \sum_{h' \in \mathcal{B}_h^{1,0}} V_{hh'}((1-\lambda)(1-\frac{1}{n}) + \lambda \frac{1}{n}) + \sum_{h' \in \mathcal{B}_h^{1, w''_{h'}}} V_{hh'}((1-\lambda)(1-\frac{1}{n}) + \lambda w''_{h'}) + \sum_{h' \in \mathcal{B}_h^{1,1}} V_{hh'}((1-\lambda)(1-\frac{1}{n}) + \lambda(1-\frac{1}{n})) = \end{array}$$

$$
\begin{aligned}
&u_h((1-\lambda)x'_h+\lambda x''_h)+\\
&\textstyle\sum_{h'\in\mathcal{B}_h^{0,0}} v_{hh'}((1-\lambda)\frac{1}{n}+\lambda\frac{1}{n})+\sum_{h'\in\mathcal{B}_h^{0,w''_{h'}}} v_{hh'}((1-\lambda)\frac{1}{n}+\lambda w''_{h'})+\sum_{h'\in\mathcal{B}_h^{0,1}} v_{hh'}((1-\lambda)\frac{1}{n}+\lambda(1-\frac{1}{n}))+\\
&\textstyle\sum_{h'\in\mathcal{B}_h^{w'_{h'},0}} v_{hh'}((1-\lambda)w'_{h'}+\lambda\frac{1}{n})+\sum_{h'\in\mathcal{B}_h^{w'_{h'},w''_{h'}}} v_{hh'}((1-\lambda)w'_{h'}+\lambda w''_{h'})+\sum_{h'\in\mathcal{B}_h^{w'_{h'},1}} v_{hh'}((1-\lambda)w'_{h'}+\lambda(1-\frac{1}{n}))+\\
&\textstyle\sum_{h'\in\mathcal{B}_h^{1,0}} v_{hh'}((1-\lambda)(1-\frac{1}{n})+\lambda\frac{1}{n})+\sum_{h'\in\mathcal{B}_h^{1,w''_{h'}}} v_{hh'}((1-\lambda)(1-\frac{1}{n})+\lambda w''_{h'})+\sum_{h'\in\mathcal{B}_h^{1,1}} v_{hh'}((1-\lambda)(1-\frac{1}{n})+\lambda(1-\frac{1}{n}))\geq\\
&\quad\min\{u_h(x'_h)+\textstyle\sum_{h'\in\mathcal{B}_h^{0,0}} v_{hh'}(\frac{1}{n})+\sum_{h'\in\mathcal{B}_h^{0,w''_{h'}}} v_{hh'}(\frac{1}{n})+\sum_{h'\in\mathcal{B}_h^{0,1}} v_{hh'}(\frac{1}{n})+\\
&\quad\textstyle\sum_{h'\in\mathcal{B}_h^{w'_{h'},0}} v_{hh'}(w'_{h'})+\sum_{h'\in\mathcal{B}_h^{w'_{h'},w''_{h'}}} v_{hh'}(w'_{h'})+\sum_{h'\in\mathcal{B}_h^{w'_{h'},1}} v_{hh'}(w'_{h'})+\\
&\quad\textstyle\sum_{h'\in\mathcal{B}_h^{1,0}} v_{hh'}(1-\frac{1}{n})+\sum_{h'\in\mathcal{B}_h^{1,w''_{h'}}} v_{hh'}(1-\frac{1}{n})+\sum_{h'\in\mathcal{B}_h^{1,1}} v_{hh'}(1-\frac{1}{n}),\\
&\quad u_h(x''_h)+\textstyle\sum_{h'\in\mathcal{B}_h^{0,0}} v_{hh'}(\frac{1}{n})+\sum_{h'\in\mathcal{B}_h^{0,w''_{h'}}} v_{hh'}(w''_{h'})+\sum_{h'\in\mathcal{B}_h^{0,1}} v_{hh'}(1-\frac{1}{n})+\\
&\quad\textstyle\sum_{h'\in\mathcal{B}_h^{w'_{h'},0}} v_{hh'}(\frac{1}{n})+\sum_{h'\in\mathcal{B}_h^{w'_{h'},w''_{h'}}} v_{hh'}(w''_{h'})+\sum_{h'\in\mathcal{B}_h^{w'_{h'},1}} v_{hh'}(1-\frac{1}{n})+\\
&\quad\textstyle\sum_{h'\in\mathcal{B}_h^{1,0}} v_{hh'}(1-\frac{1}{n})+\sum_{h'\in\mathcal{B}_h^{1,w''_{h'}}} v_{hh'}(w''_{h'})+\sum_{h'\in\mathcal{B}_h^{1,1}} v_{hh'}(1-\frac{1}{n})\}.
\end{aligned}
\tag{A10}
$$

Taking limits for $n\to 0$ of both sides of (A10) and using the continuity of the involved functions and the min function, the facts that the inequality in (A10) holds in the limit, and that $V_{hh'|(0,1)}=v_{hh'}$ and $V_{hh'}(0)$ and $V_{hh'}(1)$, we get the desired result. Indeed,

$$
\begin{aligned}
&u_h((1-\lambda)x'_h+\lambda x''_h)+\textstyle\sum_{h'\in\mathcal{B}_h} V_{hh'}((1-\lambda)w'_{h'}+\lambda w''_{h'})=u_h((1-\lambda)x'_h+\lambda x''_h)+\\
&\textstyle\sum_{h'\in\mathcal{B}_h^{0,0}} V_{hh'}(0)+\sum_{h'\in\mathcal{B}_h^{0,w''_{h'}}} V_{hh'}(\lambda w''_{h'})+\sum_{h'\in\mathcal{B}_h^{0,1}} V_{hh'}(\lambda)+\\
&\textstyle\sum_{h'\in\mathcal{B}_h^{w'_{h'},0}} V_{hh'}((1-\lambda)w'_{h'})+\sum_{h'\in\mathcal{B}_h^{w'_{h'},w''_{h'}}} V_{hh'}((1-\lambda)w'_{h'}+\lambda w''_{h'})+\sum_{h'\in\mathcal{B}_h^{w'_{h'},1}} V_{hh'}((1-\lambda)w'_{h'}+\lambda)+\\
&\textstyle\sum_{h'\in\mathcal{B}_h^{(1,0)}} V_{hh'}((1-\lambda))+\sum_{h'\in\mathcal{B}_h^{(1,w''_{h'})}} V_{hh'}((1-\lambda)+\lambda w''_{h'})+\sum_{h'\in\mathcal{B}_h^{(1,1)}} V_{hh'}(1)\geq\\
&\min\{u_h(x'_h)+\textstyle\sum_{h'\in\mathcal{B}_h^{0,0}\cup\mathcal{B}_h^{0,w''_{h'}}\cup\mathcal{B}_h^{0,1}} V_{hh'}(0)+\sum_{h'\in\mathcal{B}_h^{w'_{h'},0}\cup\mathcal{B}_h^{w'_{h'},w''_{h'}}\cup\mathcal{B}_h^{w'_{h'},1}} V_{hh'}(w'_{h'})+\sum_{h'\in\mathcal{B}_h^{1,0}\cup\mathcal{B}_h^{1,w''_{h'}}\cup\mathcal{B}_h^{1,1}} V_{hh'}(1),\\
&u_h(x''_h)+\textstyle\sum_{h'\in\mathcal{B}_h^{0,0}\cup\mathcal{B}_h^{w'_{h'},0}\cup\mathcal{B}_h^{1,0}} V_{hh'}(0)+\sum_{h'\in\mathcal{B}_h^{0,w''_{h'}}\cup\mathcal{B}_h^{w'_{h'},w''_{h'}}\cup\mathcal{B}_h^{1,w''_{h'}}} V_{hh'}(w''_{h'})+\sum_{h'\in\mathcal{B}_h^{0,1}\cup\mathcal{B}_h^{w'_{h'},1}\cup\mathcal{B}_h^{1,1}} V_{hh'}(1)\}=\\
&\min\{u_h(x'_h)+\textstyle\sum_{h'\in\mathcal{B}_h} V_{hh'}(w'_{h'}),\quad u_h(x''_h)+\sum_{h'\in\mathcal{B}_h} V_{hh'}(w''_{h'})\}.
\end{aligned}
$$

□

Appendix B Properties of set-valued functions

Consider $D:\Pi\subseteq\mathbb{R}^p\to\to X\subseteq\mathbb{R}^n$, $\pi\mapsto\mapsto\{x\in X: f(\pi,x)\geq 0\}$, where $f:\Pi\times X\to\mathbb{R}^m$. We provide a list of conditions to be satisfied by f which imply that D has nice properties.

Definition 19. *A function $F:X\subseteq\mathbb{R}^n\to\mathbb{R}$, with X nonempty and convex, is said to be*

(i) quasiconcave *iff for any $x,y\in X$ and $\lambda\in[0,1]$ one has*

$$F(\lambda x+(1-\lambda)y)\geq\min\{F(x),F(y)\};$$

(ii) semistrictly quasiconcave *iff for any* $x, y \in X$ *such that* $F(x) \neq F(y)$*, one has*

$$F(\lambda x + (1-\lambda)y) > \min\{F(x), F(y)\}, \quad \forall \lambda \in (0,1)\,;$$

iii) Locally NonSatiated if $\forall x \in X$ *and* $\forall \varepsilon > 0$, $\exists x' \in B_X(x, \varepsilon)$ *such that* $F(x') > F(x)$*;*
iv) (Globally) NonSatiated if $\forall x \in X\ \exists x' \in X$ *such that* $F(x') > F(x)$*.*

Proposition 20. *(Villanacci (2022), Corollary 42, page 15) If* X *is a convex metric space and* $F : X \to \mathbb{R}$ *is continuous, then* f *is semistrictly quasi-concave and NonSatiated* $\Leftrightarrow$ f *is quasi-concave and Locally NonSatiated.*

The following two results provide a slight generalization of Propositions 17 and 18 in Donato et al. (2025) extending the domain of the set-valued maps defining the constraints from Euclidean spaces to nonempty, convex subsets of Euclidean spaces.

Proposition 21. *Let* X *be a nonempty convex subset of* $\mathbb{R}^n$ *and the function* $f := (f_i)_{i=1}^m : X \to \mathbb{R}^m$*,* $x \mapsto f(x)$ *be given. Moreover,*

$$\begin{array}{ll} \text{for any } i \in \{1,...,m\}, & C_i := \{x \in X : f_i(x) \geq 0\}, \\ & C := \{x \in X : f(x) \geq 0\} = \cap_{i=1}^m C_i; \\ \text{for any } i \in \{1,...,m\}, & \widetilde{C}_i := \{x \in X : f_i(x) > 0\}, \\ & \widetilde{C} := \{x \in X : f(x) >> 0\} = \cap_{i=1}^m \widetilde{C}_i; \end{array}$$

If for any $i \in \{1,...,m\}$*,* f_i *is either i. continuous, ii. semistrictly quasi-concave, iii. such that there exists* $z \in X$ *such that* $f_i(z) > 0$*, or i. continuous, ii'. quasi-concave and iii'. such that there exists* $z \in X$ *such that* $f_i(z) > 0$ *and Locally nonsatiated, then 1)* $\widetilde{C} = \mathrm{Int}(C)$ *and 2)* $C = Cl(\widetilde{C})$*.*

Proof. The proof follows the same steps as the proof of the Proposition 17 in Donato et al. (2025). □

Proposition 22. *Let a subset* Π *of* $\mathbb{R}^p$ *a nonempty convex subset* X *of* $\mathbb{R}^C$ *and a function* $f : \Pi \times X \to \mathbb{R}^m$*,* $x \mapsto (f_j(\pi, x))_{j=1}^m$ *be given (with* $f_j : \Pi \times X \to \mathbb{R}$*). Let also the following set valued function be given.*

$$\begin{array}{l} D : \Pi \to\to X, \quad \pi \mapsto\mapsto \{x \in X : f(\pi, x) \geq 0\}; \\ \widetilde{D} : \Pi \to\to X, \quad \pi \mapsto\mapsto \{x \in X : f(\pi, x) >> 0\}. \end{array}$$

I. If 1. $\widetilde{D}$ *is non-empty valued, 2.* f *is i. continuous and for any* $j = 1, ..., m$ *and for any* $\pi \in \Pi$*, either 3a.* $f_j(\pi, \cdot)$ *is ii. semistrictly quasi-concave or 3b.* $f_j(\pi, \cdot)$ *is ii'. quasi-concave and iii'. Locally nonsatiated, then* D *is non-empty valued convex valued, closed graph and lower hemicontinuous. II. If in addition either 4a.* D *is compact valued or 4b.* $\mathrm{Im}(D)$ *is contained in a compact set, then* D *is upper hemicontinuous.*

Proof. The proof follows the same steps as the proof of the Proposition 18 in Donato et al. (2025). □

Lemma 4. *(see Donato et al. (2025))Let* X_1, X_2 *and* Y *be metric spaces and* $\varphi : X_1 \to\to Y$*,* $x_1 \mapsto\mapsto \varphi(x_1)$ *and* $\psi : X_1 \times X_2 \to\to Y$*,* $(x_1, x_2) \mapsto\mapsto \varphi(x_1)$ *be set-valued functions. Then if* φ *satisfies any of the properties listed below, then* ψ *does*

as well: 1. non-empty valued; 2. convex valued; 3. closed; 4. compact valued; 5. lower hemi-continuous; 6. upper hemi-continuous.

Declarations

- Conflict of interest. The authors declare that they have no conflicts of interest relevant to the content of this article.

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