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Regularity in an Exchange Economy with Other-Regarding Preferences and Private Transfers

Nico Lombardi, Paolo Nerini, Vincenzo Platino, and Antonio Villanacci

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Regularity in an Exchange Economy with Other-Regarding Preferences and Private Transfers

Nico Lombardi[*], Paolo Nerini[†], Vincenzo Platino[‡], and Antonio Villanacci[§]

Abstract
We consider a pure exchange economy with a finite number of commodities and households, in which preferences are affected by the welfare of others, measured in terms of wealth, and households are allowed to transfer resources to one another. Economies are described by endowments of commodities and utility functions. We establish generic regularity, that is, for an open and dense subset of economies, the set of equilibria is finite and each equilibrium admits a local differentiable representation as a function of the fundamentals.

Keywords: Interdependent preferences, transfers, competitive equilibrium, regular economies.

JEL classification: C62, D50, D51, D62.

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Chapter 1

Introduction

An important question in general equilibrium theory is whether the foundational properties of competitive equilibria established for classical pure exchange economies persist when agents exhibit other-regarding preferences (ORP) and are permitted to execute voluntary transfers.[1]

While standard general equilibrium models rely on the absence of consumption externalities to ensure market clearing and efficiency, real-world economic interactions - ranging from climate initiatives and philanthropic transfers to tax compliance and political redistribution - are inherently driven by fairness, reciprocity, and social interdependence.

From a general equilibrium perspective, incorporating social preferences fundamentally alters the functioning of exchange. The presence of consumption externalities violates classical assumptions required for equilibrium existence and generic regularity, breaks the First Welfare Theorem, and indeed necessitates a formal redefinition of Pareto optimality.

On the more economic applied side, empirical and macroeconomic evidence underscores the scope of non-selfish allocations - such as individual charitable giving in the United States exceeding $374 billion annually and non-profit sectors generating significant GDP shares across Europe. On the other hand, the rigorous micro-foundations of such economies under voluntary transfers remain analytically challenging.

To systematically analyze exchange economies subject to social preferences and voluntary transfers, our broader research program evaluates three increasingly general theoretical frameworks: the classical pure exchange economy with selfish preferences and no transfers, the exchange economy with other-regarding preferences and no transfers, and the exchange economy with other-regarding preferences and voluntary transfers.

For each specification, three foundational equilibrium properties are crucial for the analysis: existence, generic regularity, and Pareto optimality. In classical environments, all three properties are well-established under standard conditions. Extending these foundations, Bonnisseau et al (2026) demonstrate that under consumption-based ORP without transfers, existence and generic regularity can be preserved, though Pareto optimality generically fails, opening scope for welfare-improving interventions. The goal of the present paper is to analyze those results in economies which allow voluntary inter-agent transfers under both wealth-based and consumption-based preference structures. Specifically, this paper establishes generic regularity in the wealth-based setting - a prerequisite structural stepping stone for subsequent welfare analysis.

Introducing ORP and voluntary transfers invalidates several classical assumptions on individual utility functions. First, monotonicity generally fails; while utility increases in the consumption of agents toward whom an individual feels altruism, it may decrease in the consumption of targets of spite or hostility. Second, quasiconcavity may be violated, as simultaneous altruistic and spiteful motives break the structural convexity of upper contour sets. Third, standard boundary conditions are also disrupted by the presence of other-regarding, spiteful preferences.

When voluntary transfers are introduced, two additional structural complications arise: unrestricted transfers destroy the compactness of aggregate choice correspondences, and particular transfer configurations can render individual budget sets empty.

[1]The first paper to analyze a general equilibrium model with wealth-based other-regarding preferences and transfers is Kranich (1988). Several years later, Mercier Ythier (2000) analyzed a model with transfers in which other-regarding preferences are based on other people's utility from consumption. More recently, several contributions have provided more consistent, accurate, and general existence proofs; see, in particular, Donato, Sperati, and Villanacci (2025a, 2025b) and Donato, Platino, and Villanacci (2026).

Moreover, combining other-regarding preferences (ORP) with voluntary transfers introduces a twofold interdependence. First, utility interdependence implies that household welfare directly depends on the consumption and transfer choices of other households. Second, constraint interdependence implies that available wealth depends directly on transfers chosen across the economy. These non-decomposabilities prevent the equilibrium system from decomposing into independent first-order conditions from individual optimization, forming a crucial barrier to establishing regularity. As a result, the equilibrium function's Jacobian matrix in the standard exchange economy—see, e.g., Villanacci et al. (2002, p. 241)—is considerably simpler than the Jacobian in our framework (see Section 5.3).

The use of a finite local parametrization of utility functions to show both existence and regularity goes back to the seminal works by Smale (1974) and Allen (1981). On the other hand, local parametrizations have been used in the general equilibrium literature (especially in contributions on indeterminacy and Pareto improvement) to study the properties of equilibria after regularity has been obtained. In our paper, we use parametrizations to obtain the regularity result. To the best of our knowledge, the only papers that use our approach are those by Bonnisseau and del Mercato (2010), in a specific case, and Citanna and Villanacci (2002), using a rather heavy machinery approach. Kung (2008) models the functional structure of the economy using a finite local parametrization; this assumption obviates the mathematical difficulties addressed by our proposed framework (see Section 4).

In brief, our main contribution is to establish the regularity of equilibria in a model characterized by strong interdependence among households—an aspect that works against the direct application of standard results derived under the complete absence of such interdependence. We also provide a general proof strategy that, in our view, can be applied to other classes of models beyond the one considered here.

The remainder of the paper is structured as follows. Section 2 formally outlines the general equilibrium framework with other-regarding preferences and voluntary transfers. Section 3 introduces an equivalent definition of equilibrium and states our main theorem on generic regularity. Section 4 establishes the properness of the so-called natural projection (from the equilibrium set to the space of economies). This result is the key tool used to show that the set of regular economies is indeed open (in the appropriately defined topological space). We provide a detailed proof that covers our case and, we hope, serves as a useful pedagogical aid in understanding a proof that is only briefly sketched in the literature. Indeed, we were unable to find a rigorous proof of properness in any article currently available in the literature. Section 5 develops the local finite parametrizations of selfish and altruistic components and proves the density of regular economies. Section 6 synthesizes these results to provide the full regularity result. Finally, Appendix 7 contains auxiliary topological and differential topology lemmas, ensuring a relatively self-contained presentation.

Chapter 2

Set up of the model

We describe an economy in which households exchange goods (or commodities) in order to maximize their well-being. A commodity is denoted by $c \in \{1, ..., C\} := \mathcal{C}$. A household is denoted by $h \in \{1, ..., H\} := \mathcal{H}$ and she is described by the following objects:[1]

a consumption set $X_h \subseteq \mathbb{R}^C$ with generic element $x_h = (x_h^c)_{c \in \mathcal{C}} \in \mathbb{R}^C$, where $x_h^c \in \mathbb{R}$ denotes the consumption of good or commodity c by household h;

an endowment set $E_h \subseteq \mathbb{R}^C$ with generic element $e_h = (e_h^c)_{c \in \mathcal{C}} \in \mathbb{R}^C$, where $e_h^c \in \mathbb{R}$ denotes the amount of good c owned by household h; total resources are denoted and defined as follows: $r := \sum_{h \in \mathcal{H}} e_h$.

Commodities can be exchanged with other commodities at exchange ratios described by a price vector p belonging to the normalized price set $P := \left\{p \in \mathbb{R}^C_{++} : p^1 = 1\right\}$, where commodity 1 is chosen as the numeraire.

Households can also choose to transfer resources to other households: each household $h \in \mathcal{H}$ can choose a transfer vector $\tau_h = (\tau_{hh'})_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}_+^{H-1}$, where $\tau_{hh'}$ denotes the amount of the numeraire good 1 that household h wants to transfer to household $h' \in \mathcal{H} \setminus \{h\}$. Assuming that $\tau_{hh'}$ is non-negative formalizes the idea that households are prohibited from "stealing" goods and that they comply with this institutional requirement.

$\theta_h \in \Theta \subseteq \mathbb{R}$ denotes household h's wealth. We also define $\theta = (\theta_h)_{h \in \mathcal{H}} \in \Theta^H$ and $\theta_{\setminus h} = (\theta_{h'})_{h' \in \mathcal{H} \setminus \{h\}} \in \Theta^{H-1}$.

Household h's utility function depends upon her own consumption and anyone else's relative wealth and it is denoted and defined as follows.

$$U_h : X_h \times \Theta^{H-1} \longrightarrow \mathbb{R}, \quad \left(x_h, \theta_{\setminus h}\right) \mapsto u_h\left(x_h\right) + \sum_{h' \in \mathcal{H} \setminus \{h\}} v_{hh'}\left(\theta_{h'}\right),$$

where

$$u_h : X_h \to \mathbb{R}$$

describes the "selfish" component of the overall utility function of household h, and for any $h' \in \mathcal{H} \setminus \{h\}$,

$$v_{hh'} : \Theta \to \mathbb{R}$$

describes the utility accruing to household h from a given value of the wealth of household h'.

For "physical/biological" reasons, we assume non-negativity of consumption. More precisely, we assume $X_h = \mathbb{R}^C_{++}$. Using a standard "survival assumption", we endow households with a strictly positive vector of goods, as formally assumed below.

Assumption 1. For any $h \in \mathcal{H}$, $e_h \in \mathbb{R}^C_{++}$.

Selfish utility functions satisfy the following assumptions.

Assumption 2. For any $h \in \mathcal{H}$,

u1. $u_h : \mathbb{R}^C_{++} \to \mathbb{R}$ is a three times continuously differentiable function;

[1] Economically meaningful restrictions on the sets defined below will be presented in the remainder of the section.

u2. u_h is differentiably strictly increasing, i.e., for every $x_h \in \mathbb{R}^C_{++}$, $Du_h(x_h) \gg 0$;

u3. u_h is differentiably strictly concave, i.e., for every $x_h \in \mathbb{R}^C_{++}$ and $D^2u_h(x_h)$ is negative definite;

u4. for any $\underline{u} \in \mathbb{R}$, $\left\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \underline{u}\right\}$ is closed in the Euclidean topology of $\mathbb{R}^C$.

The set of all utility functions u_h satisfying the above Assumptions is denoted by $\mathcal{U} \subseteq \mathcal{C}^3(\mathbb{R}^C_{++}, \mathbb{R})$.

Remark 1 *As verified in Appendix 7.1,*

1. $\mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \alpha\right\}\right) \subseteq \mathbb{R}^C_{++} \Leftrightarrow \mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \alpha\right\}\right) \cap \mathcal{F}\left(\mathbb{R}^C_{++}\right) = \varnothing$, *where* $\mathcal{F}\left(\mathbb{R}^C_{++}\right)$ *is the topological boundary of* $\mathbb{R}^C_{++}$.

2. if u_h *satisfies Assumptions u1, u2 and u3, then,*

$$\left\langle \begin{array}{l} \textit{u4': for any } \underline{x} \in \mathbb{R}^C_{++},\ \mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq u(\underline{x})\right\}\right) \subseteq \mathbb{R}^C_{++}, \textit{ and} \\ \\ \textit{u}_h \textit{ is unbounded below.} \end{array} \right\rangle$$

$$1. \Leftrightarrow$$

$$\left\langle \textit{ u4. for any } \alpha \in \mathbb{R},\ \mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \alpha\right\}\right) \subseteq \mathbb{R}^C_{++}. \right\rangle$$

$$2. \Leftrightarrow$$

$$\left\langle \textit{ u4". for any } \alpha \in \mathbb{R},\ \left\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \alpha\right\} \textit{ is } \mathbb{R}^C \textit{ closed.} \right\rangle$$

Remark 2 *Below, we provide some comments on the assumptions introduced in Assumption 2.*

u1. All the results obtained in the present paper require u_h *is* $\mathcal{C}^2$. *We also want to provide results for the case in which* u_h *is* $\mathcal{C}^3$, *as this will be needed for the analysis of the optimality properties of equilibrium allocations that we plan to undertake in future research. The motivation for this stronger assumption can be found, for example, in Footnote 7 in Remark 6 on page 428 of Villanacci et al. (2002), as well as in some of the literature cited there on Pareto-improving analysis.*

u2 and u3. Assumptions u2 and u3 are widely used in general equilibrium theory with smooth preferences. They are strengthening of the monotonicity and convexity assumptions usually imposed and are frequently used in the analysis.

u4. As discussed in the remark above, Assumption u4 is a strengthening of Assumption u4'. It is crucial in our model because the presence of the other-regarding term in the utility function makes it more difficult to apply the standard analysis, which consists in obtaining a lower bound for affordable utility in terms of the agent's own endowment.

Remark 3 *The argument of* $v_{hh'}$ *is the relative wealth of household* h', *i.e.,*

$$\frac{1}{pr}\left(pe_{h'} + \sum_{h'' \in \mathcal{H}\setminus\{h'\}} \tau_{h''h'} - \sum_{h'' \in \mathcal{H}\setminus\{h'\}} \tau_{h'h''}\right).$$

The numerator of the fraction is the value of the household's endowment, plus the value of the transfers it receives, minus the value of the transfers it gives. Therefore, by the monotonicity of the utility function, the household's net wealth equals the value of her consumption at any utility-maximizing choice - see constraint set below in Definition 8. We then have

$$0 < \frac{px_{h'}}{pr} = 1 - \frac{p\sum_{h \in \mathcal{H}\setminus\{h'\}} x_h}{pr} < 1,$$

where the first strict inequality follows from the fact that price, consumption and total resource vectors have strictly positive components, and the first equality follows from market clearing.The above observations imply that the domain of the function $v_{hh'}$ *has to be the open interval* $(0,1)$.

For any $h \in \mathcal{H}$ and $h' \in \mathcal{H}\setminus\{h\}$, the properties which the function

$$v_{hh'} : (0,1) \to \mathbb{R}$$

may satisfy are as follows.

v1. $v_{hh'}$ is three times continuously differentiable;

v2. $v_{hh'}$ is differentiably strictly increasing, i.e., for every $s \in \mathbb{R}_{++}$, $v'_{hh'}(s) > 0$;

v2'. $v_{hh'}$ is decreasing;

v3. $v_{hh'}$ is differentiably strictly concave, i.e., for every $s \in \mathbb{R}_{++}$, $v''_{hh'}(s) < 0$;

v4. $v_{hh'}$ is bounded below.

Remark 4 *Observe that if Assumptions v.1, v.2, v.3., and v.4 are satisfied, then $v_{hh'}$ is bounded above as well - see Proposition 71 in the Appendix 7.1. Observe also that if $v_{hh'}$ is bounded, it could be that $v'_{hh'}$ is unbounded as in the case of $\sqrt[2]{s}$.*

Remark 5 *Below, we provide some comments on the assumptions introduced above on $v_{hh'}$.*

v1. v2 and v3. Comments presented about u1,u2 and u3 apply here as well. Indeed, weaker continuous versions of those assumptions are needed to show the existence of equilibria in Donato, Platino, and Villanacci (2026). Assumption v4 basically says that "you do not care infinitely for the poverty of people you like".

The set of all functions $v_{hh'}$ satisfying the above Assumptions $v1$, $v2$, $v3$, and $v4$ is denoted by $\mathcal{V} \subseteq \mathcal{C}^3((0,1),\mathbb{R})$.

Definition 6 *If $\{X_i : i \in \{1,...,n\}\}$ is a family of pairwise disjoint, possibly empty, subsets of a set X such that $\cup_{i=1}^n X_i = X$, then we say that $\{X_i : i \in \{1,...,n\}\}$ is a generalized partition of X.*

Assumption 3. For any $h \in \mathcal{H}$, there exists a generalized partition $\left\{\mathcal{B}_h, \mathcal{B}_h^{\setminus}\right\}$ of the set $\mathcal{H}\setminus\{h\}$ such that

$\mathcal{B}_h = \{h' \in \mathcal{H}\setminus\{h\} :$ $v_{hh'}$ satisfies properties v1, v2, v3 and v4 $\}$,

$\mathcal{B}_h^{\setminus} = \{h' \in \mathcal{H}\setminus\{h\} : v_{hh'}$ satisfies properties v1 and v2'$\}$.

Remark 7 *We are going to refer to $\mathcal{B}_h$ and $\mathcal{B}_h^{\setminus}$ as the sets of individuals that household h likes and dislikes, respectively.*

We now make a quite important modelling assumption.

Assumption 4. For each household $h \in \mathcal{H}$, the total value of transfers it makes to other households cannot exceed the value of its own endowment, i.e.,

$$\sum_{h' \in \mathcal{H}\setminus\{h\}} \tau_{hh'} \leq pe_h.$$

Given a vector $z = (z^1, \ldots, z^C) \in \mathbb{R}^C$, denote $z^{\setminus} := (z^2, \cdots, z^C) \in \mathbb{R}^{C-1}$. So, in particular,

$$p^{\setminus} = (p^c)_{c \in \mathcal{C}\setminus\{1\}} \in \mathbb{R}^{C-1}_{++},$$

Define also

$$B_h = \#\mathcal{B}_h, \quad B = \textstyle\sum_{h\in\mathcal{H}} B_h,$$

$$B_h^{\setminus} = \#\mathcal{B}_h^{\setminus}, \quad B^{\setminus} = \textstyle\sum_{h\in\mathcal{H}} B_h^{\setminus},$$

$$\Delta_h\left(p^{\setminus}\right) := \left\{ (\tau_{hh'})_{h'\in\mathcal{H}\setminus\{h\}} \in \mathbb{R}^{H-1} : \tau_h \geq 0 \text{ and } \sum_{h'\in\mathcal{H}\setminus\{h\}} \tau_{hh'} \leq pe_h \right\},$$

$$x = (x_h)_{h\in\mathcal{H}} \qquad e = (e_h)_{h\in\mathcal{H}}, \qquad u = (u_h)_{h\in\mathcal{H}}, \qquad v = \left((v_{hh'})_{h'\in\mathcal{B}_h}\right)_{h\in\mathcal{H}}, \qquad v^{\setminus} = \left((v_{hh'})_{h'\in\mathcal{B}_h^{\setminus}}\right)_{h\in\mathcal{H}}.$$

An economy is[2]

$$\mathcal{E}' := \left(e, u, v, v^{\setminus}\right) \in \mathbb{R}^{CH}_{++} \times \mathcal{U}^H \times \mathcal{V}^B \times \left(\mathcal{V}^{\setminus}\right)^{B^{\setminus}} := \mathbb{E}'.$$

Definition 8 *The vector $\left(x^*, p^{*\setminus}, \left((\tau^*_{hh'})_{h'\in\mathcal{B}_h}\right), \left((\tau^*_{hh'})_{h'\in\mathcal{B}_h^{\setminus}}\right)\right) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{C-1}_{++} \times \mathbb{R}^B \times \mathbb{R}^{B^{\setminus}}$ is an equilibrium for the economy $\mathcal{E}' \in \mathbb{E}'$ if*

1. for any $h \in \mathcal{H}$, household h maximizes, i.e.,

for given $\mathcal{E}' \in \mathbb{E}'$, $p^{*\setminus} \in \mathbb{R}^{C-1}_{++}$, $\left((\tau^*_{h'h''})_{h''\in\mathcal{H}\setminus\{h'\}}\right)_{h'\in\mathcal{H}\setminus\{h\}} \in \left(\times_{h'\in\mathcal{H}\setminus\{h\}} \Delta_{h'}\left(p^{*\setminus}\right)\right)$,

[2]We adopt the symbols $\mathcal{E}'$ and $\mathbb{E}'$ rather than the more natural notation $\mathcal{E}$ and $\mathbb{E}$, since $\mathcal{E}'$ and $\mathbb{E}'$ appear only a limited number of times in the remainder of the paper, while $\mathcal{E}$ and $\mathbb{E}$ will be introduced later and used extensively.

$\left(x_h^*, (\tau_{hh'}^*)_{h' \in \mathcal{H}\setminus\{h\}}\right) \in \mathbb{R}^C_{++} \times \mathbb{R}^{H-1}$ *solves*

$$\max_{\left(x_h, (\tau_{hh'})_{h' \in \mathcal{H}\setminus\{h\}}\right) \in \mathbb{R}^C_{++} \times \mathbb{R}^{H-1}}$$

$$u_h(x_h) + \sum_{h' \in \mathcal{H}\setminus\{h\}} v_{hh'} \left(\frac{1}{p^* r} \left(p^* e_{h'} + \tau_{hh'} + \sum_{h'' \in \mathcal{H}\setminus\{h', h\}} \tau_{h''h'}^* - \sum_{h'' \in \mathcal{H}\setminus\{h'\}} \tau_{h'h''}^* \right) \right)$$

subject to

$$p^* e_h + \sum_{h' \in \mathcal{H}\setminus\{h\}} \tau_{h'h}^* - \sum_{h' \in \mathcal{H}\setminus\{h\}} \tau_{hh'} - p^* x_h \geq 0$$

$$p e_h - \sum_{h' \in \mathcal{H}\setminus\{h\}} \tau_{hh'} \geq 0$$

$$\textit{for any } h' \in \mathcal{H}\setminus\{h\}, \quad \tau_{hh'} \geq 0.$$

2. markets clear, i.e.,

$$\sum_{h \in \mathcal{H}} (x_h^* - e_h) = 0.$$

The reason for which the above definition contains no "market clearing conditions for transfers" is a consequence of the simple result below. For any $\tau \in \mathbb{R}^{H(H-1)}$, $p \in S$ and $e \in \mathbb{R}^{CH}_{++}$,

$$\sum_{h \in \mathcal{H}} \sum_{h' \in \mathcal{H}\setminus\{h\}} \tau_{h'h} = \sum_{h \in \mathcal{H}} \sum_{h' \in \mathcal{H}\setminus\{h\}} \tau_{hh'}. \tag{2.1}$$

Theorem 9 *If Assumptions 1, 2 and 3 are satisfied, then for any economy $\mathcal{E}^0 \in \mathbb{E}^0$, an equilibrium $(x^*, (\tau_{hh'}^*)_{h' \in \mathcal{H}\setminus\{h\}}, p^{*\setminus}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{H(H-1)} \times \mathbb{R}^{C-1}_{++}$ exists.*

Proof. The proof is contained in Donato, Platino, and Villanacci (2026) and is based on the equivalence among different types of equilibria and an existence theorem. Observe also that, using Remark 4, we have that the assumptions we make on the functions $v_{hh'}$ imply that they satisfy continuity, strict monotonicity, and boundedness, which are the assumptions made in Donato, Platino, and Villanacci (2026). ■

Chapter 3

The goal of the analysis

A key step in the existence proof presented in Donato, Platino, and Villanacci (2026), as well as in the proof of our main results, is the introduction of a notion of fictitious equilibrium, which is shown to be equivalent to the standard notion of equilibrium while being considerably easier to analyze.

More precisely, this section proceeds as follows. The underlying strategy is to show that, without loss of generality, the equilibrium notion introduced in Definition 8 can be reformulated by effectively disregarding the households belonging to the set $\mathcal{B}_h^{\setminus}$. Accordingly, in Definition 11, we formally introduce this alternative equilibrium notion, referred to as a $\mathcal{B}$ equilibrium. We then characterize $\mathcal{B}$ equilibria by means of Karush–Kuhn–Tucker conditions and, in Proposition 14, establish the equivalence between the standard equilibrium and $\mathcal{B}$ equilibrium notions. Having established this equivalence, we show that the study of regularity properties of the original equilibrium can be reduced to the analysis of the more tractable $\mathcal{B}$ equilibria. We conclude the section by stating the regularity theorem, which constitutes the main result of the paper. This result is an indispensable tool for analyzing the Pareto properties of equilibrium allocations, a topic that will be investigated in a future paper.

Define

$$\mathcal{E} := (e, u, v) \in \mathbb{R}_{++}^{CH} \times \mathcal{U}^H \times \mathcal{V}^B := \mathbb{E},$$

$\mathcal{B}_{\to h} = \{h' \in \mathcal{H} : h \in \mathcal{B}_{h'}\}$, i.e., "the set of households who like h",

$\mathcal{B}_{\to h}^{\setminus} = \{h' \in \mathcal{H} : h \in \mathcal{B}_{h'}^{\setminus}\}$, "the set of households who dislike h",

$\mathcal{B} = (\mathcal{B}_h)_{h\in\mathcal{H}} \in \times_{h\in\mathcal{H}} \mathcal{P}(\mathcal{H}\setminus\{h\})$, where $\mathcal{P}(\mathcal{H}\setminus\{h\})$ is the family of all subsets of $\mathcal{H}\setminus\{h\}$,

$\Delta_h\left(p^{\setminus}\right) := \left\{(\tau_{hh'})_{h'\in\mathcal{B}_h} \in \mathbb{R}^{B_h} : \tau_h \geq 0 \text{ and } \sum_{h'\in\mathcal{H}\setminus\{h\}} \tau_{hh'} \leq pe_h\right\}$

Observe that

$$\mathcal{E}' = \left(\mathcal{E}, v^{\setminus}\right) \in \mathbb{E} \times \mathcal{V}^{\setminus} = \mathbb{E}'.$$

Remark 10 *We endow the set $\mathbb{E}$ of economies with a topology as follows. Consider the topological vector space $\mathbb{R}^{CH} \times \left(\mathcal{C}^3(\mathbb{R}_{++}^C, \mathbb{R})\right)^H \times \left(\mathcal{C}^3((0,1),\mathbb{R})\right)^B$ endowed with the product topology derived from: the standard Euclidean topology on $\mathbb{R}^{CH}$; the $\mathcal{C}^3$ compact-open topology on $\mathcal{C}^3(\mathbb{R}_{++}^C, \mathbb{R})$; and the $\mathcal{C}^3$ compact-open topology on $\mathcal{C}^3((0,1),\mathbb{R})$ - see Section 4 and Appendix 7.3. Then, we endow the subset of economies $\mathbb{E} \subseteq \mathbb{R}^{CH} \times \left(\mathcal{C}^3(\mathbb{R}_{++}^C, \mathbb{R})\right)^H \times \left(\mathcal{C}^3((0,1),\mathbb{R})\right)^B$ with the induced subspace (relative) topology.*

We can then give the following definition of equilibrium - which uses Walras'law in the rewriting of the conditions for market clearing.

Definition 11 *The vector $\left(x, p^{*\setminus}, ((\tau_{hh'}^*)_{h'\in\mathcal{B}_h})_{h\in\mathcal{H}}\right) \in \mathbb{R}_{++}^{CH} \times \mathbb{R}_{++}^{C-1} \times \mathbb{R}^B$ is a $\mathcal{B}$ equilibrium for the economy $\mathcal{E} \in \mathbb{E}$ if*

1. for any $h \in \mathcal{H}$, household h maximizes, i.e.,

for given $\mathcal{E} \in \mathbb{E}$, $p^{*\setminus} \in \mathbb{R}^{C-1}_{++}$, $\left((\tau^*_{h'h''})_{h''\in\mathcal{B}_{h'}}\right)_{h'\in\mathcal{H}\setminus\{h\}} \in \left(\times_{h'\in\mathcal{H}\setminus\{h\}}\Delta_{h'}\left(p^{*\setminus}\right)\right)$,
$(x^*_h, (\tau^*_{hh'})_{h'\in\mathcal{B}_h}) \in \mathbb{R}^C_{++} \times \mathbb{R}^{B_h}$ *solves*

$$\max_{\left(x_h,(\tau_{hh'})_{h'\in\mathcal{B}_h}\right)\in\mathbb{R}^C_{++}\times\mathbb{R}^{B_h}}$$

$$u_h(x_h) + \sum_{h'\in\mathcal{B}_h} v_{hh'}\left(\frac{1}{p^* r}\left(p^* e_{h'} + \tau_{hh'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau^*_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \tau^*_{h'h''}\right)\right)$$

subject to

$$p^* e_h + \sum_{h'\in\mathcal{B}_{\to h}} \tau^*_{h'h} - \sum_{h'\in\mathcal{B}_h} \tau_{hh'} - p^* x_h \geq 0$$

$$p^* e_h - \sum_{h'\in\mathcal{B}_h} \tau_{hh'} \geq 0$$

$$\text{for any } h' \in \mathcal{B}_h, \quad \tau_{hh'} \geq 0$$

2.

$$\sum_{h\in\mathcal{H}} \left(x^{*\setminus}_h - e^{\setminus}_h\right) = 0.$$

Below we present the system of Karush-Kuhn-Tucker conditions and market clearing condition associated with the above definition.

Defined

$$((\tau_{hh'})_{h'\in\mathcal{B}_h})_{h\in\mathcal{H}} = (\tau_h)_{h\in\mathcal{H}} = \tau \qquad (\lambda_h)_{h\in\mathcal{H}} = \lambda \qquad ((\gamma_{hh'})_{h'\in\mathcal{B}_h})_{h\in\mathcal{H}} = \gamma, \qquad (\eta_h)_{h\in\mathcal{H}} = \eta,$$

$$\xi = \left(x, \tau, \lambda, \gamma, \eta, p^{\setminus}\right) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^B \times \mathbb{R}^H_{++} \times \mathbb{R}^B \times \mathbb{R}^H \times \mathbb{R}^{C-1}_{++} = \Xi$$

and $n = \dim \Xi$, we can introduce the so-called extended equilibrium function as follows.[1]

$$\begin{aligned} &\widetilde{F} : \Xi \times \mathbb{E} \to \mathbb{R}^n \\ &\widetilde{F}(\xi, \mathcal{E}) = \text{conditions in (3.2) below,} \end{aligned} \tag{3.1}$$

	For any $h \in \mathcal{H}$,		
C	$Du_h(x_h) - \lambda_h p$		(1)
B_h	$\frac{1}{pr} v'_{hh'}\left(\frac{pe_{h'} + \tau_{hh'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}}{pr}\right) - \lambda_h + \gamma_{hh'} - \eta_h$	for any $h'\in\mathcal{B}_h$	(2)
1	$pe_h + \sum_{h'\in\mathcal{B}_{\to h}} \tau_{h'h} - \sum_{h'\in\mathcal{B}_h} \tau_{hh'} - px_h$		(3)
B_h	$\min\{\tau_{hh'}, \gamma_{hh'}\}$	for any $h'\in\mathcal{B}_h$	(4)
1	$\min\left\{1 - \sum_{h'\in\mathcal{B}_h} \tau_{hh'}, \eta_h\right\}$		(5)
$C-1$	$\sum_{h\in\mathcal{H}} \left(x^{\setminus}_h - e^{\setminus}_h\right)$		(6)

(3.2)

where the first column contains the number of components of the vector shown in the corresponding row.

Definition 12 $\xi \in \Xi$ *is an extended* $\mathcal{B}$ *equilibrium associated with the economy* $\mathcal{E}$ *if* $\widetilde{F}(\xi, \mathcal{E}) = 0$.

Definition 13

$$\begin{aligned} &\breve{F} : \Xi \times \mathbb{R}^{B^{\setminus}} \times \mathbb{E}' \to \mathbb{R}^n \times \mathbb{R}^{B^{\setminus}} \\ &\left(\xi, \left((\tau_{hh'})_{h'\in\mathcal{B}^{\setminus}_h}\right)_{h\in\mathcal{H}}, \mathcal{E}, v^{\setminus}\right) \longmapsto \left(\widetilde{F}(\xi, \mathcal{E}), \left((\tau_{hh'})_{h'\in\mathcal{B}^{\setminus}_h}\right)_{h\in\mathcal{H}}\right). \end{aligned}$$

[1] We use $\widetilde{F}$ instead of the more natural symbol F because the former appears relatively few times in the remainder of the paper, whereas the latter will be introduced below and used extensively throughout.

Proposition 14 *1.* $\left(x^*, ((\tau^*_{hh'})_{h'\in\mathcal{B}_h})_{h\in\mathcal{H}}, p^{*\setminus}\right)$ *is a* $\mathcal{B}$ *equilibrium for* $\mathcal{E} \Leftrightarrow$
there exists $(\lambda^*, \gamma^*, \eta^*)$ *such that* $\widetilde{F}\left(x^*, ((\tau^*_{hh'})_{h'\in\mathcal{B}_h})_{h\in\mathcal{H}}, \lambda^*, \gamma^*, \eta^*, p^{*\setminus}, \mathcal{E}\right) = 0$.
2. If $\left(x^*, \left((\tau^*_{hh'})_{h'\in\mathcal{B}_h}, (\tau^*_{hh'})_{h'\in\mathcal{B}^\setminus_h}\right)_{h\in\mathcal{H}}, p^{*\setminus}\right)$ *is an equilibrium for* $\left(\mathcal{E}, v^\setminus\right)$*, then* $\left((\tau^*_{hh'})_{h'\in\mathcal{B}^\setminus_h}\right)_{h\in\mathcal{H}} = 0$.
3. $\left(x^*, \left((\tau^*_{hh'})_{h'\in\mathcal{B}_h}, (\tau^*_{hh'})_{h'\in\mathcal{B}^\setminus_h}\right)_{h\in\mathcal{H}}, p^{*\setminus}\right)$ *is an equilibrium for* $\left(\mathcal{E}, v^\setminus\right) \Leftrightarrow$
$\left(x^*, ((\tau^*_{hh'})_{h'\in\mathcal{B}_h})_{h\in\mathcal{H}}, p^{*\setminus}\right)$ *is a* $\mathcal{B}$ *equilibrium for* $\mathcal{E}$.
4. $\left(x^*, \left((\tau^*_{hh'})_{h'\in\mathcal{B}_h}, (\tau^*_{hh'})_{h'\in\mathcal{B}^\setminus_h}\right)_{h\in\mathcal{H}}, p^{*\setminus}\right)$ *is an equilibrium for* $\left(\mathcal{E}, v^\setminus\right) \Leftrightarrow$
there exists $(\lambda^*, \gamma^*, \eta^*)$ *such that* $\breve{F}\left(x^*, \left((\tau^*_{hh'})_{h'\in\mathcal{B}^\setminus_h}, (\tau^*_{hh'})_{h'\in\mathcal{B}^\setminus_h}\right)_{h\in\mathcal{H}}, \lambda^*, \gamma^*, \eta^*, p^{*\setminus}, \mathcal{E}\right) = 0$.

Proof. 1. It is easy to verify that the set of maximizers of the maximization problem described in the Definition of $\mathcal{B}$ equilibrium is characterized by the associated Karush-Kuhn-Tucker conditions. Indeed, the choice set $\mathbb{R}^C_{++} \times \mathbb{R}^{B_h}$ is open and convex, the objective function is strictly concave, constraint functions are linear and there exists $(x_h, \tau_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{B_h}$ which satisfies each constraint with a strict inequality. Indeed, taken

$$x_h = \frac{e_h}{2H} \gg 0, \qquad \text{and for any } h' \in \mathcal{B}_h, \qquad \tau_{hh'} = \frac{p^* e_h}{2H} > 0,$$

we have

$$p^* x_h + \sum_{h'\in\mathcal{B}_h} \tau_{hh'} \leq p^* e_h \left(\frac{1}{2H} + \frac{B_h}{2H}\right) \leq p^* e_h \left(\frac{H}{2H}\right) < p^* e_h$$

and

$$\sum_{h'\in\mathcal{B}_h} \tau_{hh'} = \sum_{h'\in\mathcal{B}_h} \frac{p^* e_h}{2H} = \frac{\sum_{h'\in\mathcal{B}_h} p^* e_h}{2H} < p^* e_h.$$

Existence of a solution to the maximization problem is not insured: the constraint set is not compact. On the other hand, we do know that an equilibrium exists from the existence analysis presented in Donato, Platino and Villanacci (2026). Uniqueness follows from strict concavity of the objective function with respect to the choice variables and quasi-concavity of the constraints.

2. and 3. are the content of Proposition 15 in Donato, Sperati and Villanacci (2025a).

4. [$\Rightarrow$] It follows from 2. and 1.; [$\Leftarrow$] It follows from 1. and 3. ∎

Proposition 14 allows to say that to study regularity of equilibria, it is enough to analyze $\mathcal{B}$ equilibria instead of equilibria. Indeed to analyze regularity amounts to investigate:

a. the cardinality of the equilibrium set associated with each economy;
b. smooth dependence of the endogenous variables from exogenous variables;
c. zero being a regular value of the extended equilibrium function.

Any of the above described properties about equilibria can be studied using the corresponding characteristic of $\mathcal{B}$ equilibria, as a consequence of Proposition 14. Indeed,

a'. from Proposition 14.3, it is clear that the cardinalities of the two sets of equilibria is the same;

b'. If we show that ξ depends smoothly on $\mathcal{E}$, then we also get that $\xi' := (\xi, \tau')$ depends smoothly on $\mathcal{E}' := \left(\mathcal{E}, v^\setminus\right)$, simply because changes in $v^\setminus \in \mathcal{V}^\setminus$ do not affect ξ and transfers τ' remain zero for any value of the exogenous variables;

c'. Defined $\left((\tau_{hh'})_{h'\in\mathcal{B}^\setminus_h}\right)_{h\in\mathcal{H}} = \tau^\setminus$, observe that the partial Jacobian matrix of $\breve{F}$ is

$$D_{\xi'}\breve{F}\left(\xi, \tau^\setminus, \mathcal{E}, v^\setminus\right) = \begin{array}{c} \\ \widetilde{F} \\ \\ \tau^\setminus \end{array} \begin{array}{c} \xi \\ \left[\begin{array}{c} \\ D_\xi \widetilde{F}\left(\xi, \mathcal{E}\right) \\ \\ 0 \end{array}\right. \end{array} \begin{array}{c} \tau^\setminus \\ \left.\begin{array}{c} \\ 0 \\ \\ I \end{array}\right] \end{array}$$

and then $\det D_{\xi'}\breve{F} = \det D_\xi \widetilde{F} \cdot \det I = \det D_\xi \widetilde{F}$.

The goal of our analysis is to show the following result, which is proved in Section 6. The Theorem below is an application of Glöckner Theorem, which is presented in Section 7.6.

Since Glöckner's theorem relies on the concept of Michal–Bastiani differentiability, we first introduce its definition. For further details, see Appendix 7.3.

Definition 15 *Let E and W be two locally convex and topological real vector spaces.*

A function $f\colon V \to W$ defined on $V \subseteq E$ an open subset of E, is said to be Michal-Bastiani differentiable if:

1. *$f \in \mathcal{C}^0(V, W)$;*
2. *the function*
$$d_M f : V \times E \to W, \qquad (w, s) \longmapsto \lim_{t \to 0} \frac{f(w + ts) - f(w)}{t}$$
is well defined, i.e., the limit always exists and it is component-wise finite;
3. *$d_M f$ is continuous.*

If $f\colon V \to W$ is Michal-Bastiani differentiable, then we set $f \in \mathcal{C}^1_M(V, W)$.

Theorem 16 *There exists an open and dense set $\mathcal{D} \subseteq \mathbb{E}$ such that,*

A. for every $\mathcal{E}^0 \in \mathcal{D}$, there exists $r \in \mathbb{N}$ such that $\left(\widetilde{F}_{\mathcal{E}^0}\right)^{-1}(0) = \left\{\xi_i^0 : i = 1, ..., r_{\mathcal{E}^0}\right\}$.

B. There exist an open neighborhood $V\left(\mathcal{E}^0\right) \subseteq \mathbb{E}$ of $\mathcal{E}^0$, for any $i = 1, ..., r_{\mathcal{E}^0}$ an open neighborhood $\mathcal{O}\left(\xi_i^0\right) \subseteq \Xi$ of ξ_i^0 and a function $\zeta_i : V\left(\mathcal{E}^0\right) \to \mathbb{R}^n$ such that

1. *$\zeta_i \in \mathcal{C}^1_M\left(V\left(\mathcal{E}^0\right), \mathbb{R}^n\right)$,*
2. *$\zeta_i\left(V\left(\mathcal{E}^0\right)\right) \subseteq \mathcal{O}\left(\xi_i^0\right)$,*
3. *$\zeta_i\left(\mathcal{E}^0\right) = \xi_i^0$,*
4. *for any $i, j = 1, ..., r$ such that $i \neq j$, $\mathcal{O}\left(\xi_i^0\right) \cap \mathcal{O}\left(\xi_j^0\right) = \varnothing$,*
5. *for any $i = 1, ..., r_{\mathcal{E}^0}$, $\left\{(\xi, \mathcal{E}) \in \mathcal{O}\left(\xi_i^0\right) \times V\left(\mathcal{E}^0\right) : \widetilde{F}(\xi, \mathcal{E}) = 0\right\} = \left\{(\xi, \mathcal{E}) \in \mathcal{O}\left(\xi_i^0\right) \times V\left(\mathcal{E}^0\right) : \xi = \zeta_i(\mathcal{E})\right\}$,*
6. *$\left\{(\xi, \mathcal{E}) \in \Xi \times V\left(\mathcal{E}^0\right) : \widetilde{F}(\xi, \mathcal{E}) = 0\right\} = \cup_{i=1}^{r} \left\{(\xi, \mathcal{E}) \in \mathcal{O}\left(\xi_i^0\right) \times V\left(\mathcal{E}^0\right) : \xi = \zeta_i(\mathcal{E})\right\}$.*

Remark 17 *Note that from Theorem 16 we obtain a family of functions ζ_i which are continuous on open neighborhoods of $\mathcal{E}$. Openness is meant with respect to the subspace topology described in Remark 10.*

Remark 18 *Roughly speaking, the above Theorem says that for any economy $\mathcal{E}^0$ in the "quite large" set $\mathcal{D}$*

A. the number of associated equilibria is finite;

B. there exists an open neighborhood $V\left(\mathcal{E}^0\right)$ of $\mathcal{E}^0$ for which

(1., 2., 3., 5.) each associated equilibrium depends smoothly from the economies in $V\left(\mathcal{E}^0\right)$;

(4.) each equilibrium is locally isolated;

(6.) the number of equilibria associated with each economy is constant in $V\left(\mathcal{E}^0\right)$.

Chapter 4

Openness: Properness of the natural projection

In this section, we prove that the so-called natural projection is proper.[1] This result plays a fundamental role in the analysis of a broad class of general equilibrium models with smooth utility functions, including the model studied in this paper. To the best of our knowledge, the existing literature either does not provide a proof of this result or provides only a brief argument. We therefore present a complete, detailed, and - to the greatest extent possible - self-contained proof. As will become clear, establishing this result requires a substantial amount of work.

Definition 19 *The natural projection is the function*

$$pr : \widetilde{F}^{-1}(0) \to \mathbb{E}, \quad (\xi, \mathcal{E}) \to \mathcal{E}.$$

Proposition 20 *$\widetilde{F}$ is continuous.*

Proof. Each component function of $\widetilde{F}$ is the algebraic sum or product of linear functions or functions which are proved to be continuous in Lemmas 22, 23, 24 and 25 presented below.

■

Proof. Then the desired result follows from Theorem 18.4, page 110, in Munkres (2000). ■

The following result establishes the continuity of the so-called evaluation map. A proof can be found, for example, in Theorem 46.10, page 286, Munkres (2000).

Theorem 21 *If X is a locally compact Hausdorff space, Y is a metric space and $\mathcal{C}(X, Y)$ is endowed with the compact-open topology, then $g : \mathcal{C}(X, Y) \times X \to Y$, $(f, x) \longmapsto f(x)$ - also called the evaluation map - is continuous.*

Note that $\mathbb{R}^C_{++}$ is a locally compact Hausdorff space[2]. (A topological space X is locally compact if every point has a neighborhood whose closure is compact. $\mathbb{R}$ is a metric space.)

For completeness, we provide also with a proof of the continuity of the evaluation map on $\mathcal{C}^3$. We remark, that in the proof we are going to deal with sequences of functions and vectors. Here and along the paper we use the notation $\left(x^{[k]}\right)_{k\in\mathbb{N}}$ for any sequence in $k \in \mathbb{N}$.

Lemma 22 *If $\mathcal{C}^3 := \mathcal{C}^3\left(\mathbb{R}^C_{++}, \mathbb{R}\right)$ is endowed with the compact-open topology, then*

$$\phi : \mathcal{U} \times \mathbb{R}^C_{++} \to \mathbb{R}, \quad (u, x) \longmapsto u(x)$$

is continuous.

[1] Let (X, d) and (Y, d') be metric spaces. We say that a function $f : X \to Y$ is proper if it is continuous and for every compact set $K \subseteq Y$, the set $f^{-1}(K)$ is compact in X.

[2] In a Hausdorff space: Limits of convergent sequences are unique. Compact sets are closed.

Proof. Since the compact-open topology is metrizable, then sequential continuity implies continuity and it suffices to show that[3]

if $\left(u^{[k]}, x^{[k]}\right)$ is a sequence in $\mathcal{C}^3 \times \mathbb{R}^C_{++}$ converging in

(the compact-open topology) $\times$ (the relative topology of $\mathbb{R}^C$ in $\mathbb{R}^C_{++}$) to $(\overline{u}, \overline{x}) \in \mathcal{C}^3 \times \mathbb{R}^C_{++}$,

then $u^{[k]}\left(x^{[k]}\right)$ is a sequence in $\mathbb{R}$ converging to $u(x)$.

We want to show that $\forall \varepsilon > 0\ \exists\ N_{\varepsilon,\overline{x},\overline{u}} \in \mathbb{N}$ such that $\forall n > N_{\varepsilon,\overline{x},\overline{u}}$, $\left|u^n(x^n) - \overline{u}(\overline{x})\right| < \varepsilon$.

The main idea of the proof is to use the assumptions, i.e., the facts that $u^{[k]} \overset{k}{\to} \overline{u}$ and $x^{[k]} \overset{k}{\to} \overline{x}$, and the following triangle inequality : $\left|u^{[k]}\left(x^{[k]}\right) - \overline{u}(\overline{x})\right| \leq \left|u^{[k]}\left(x^{[k]}\right) - \overline{u}\left(x^{[k]}\right)\right| + \left|\overline{u}\left(x^{[k]}\right) - \overline{u}(\overline{x})\right|$.

Recall that, from Theorem 515 in the "Compact Open Topology" Chapter in the "Notes on Topology",

> Since $\{f_n\} \subseteq \mathcal{C}^3(X,Y)$ converges to $f \in \mathcal{C}^3(X,Y)$ in the compact-open topology,
> then
> $\{f_n\}$ converges uniformly on compact sets to f

Observe that
1. since $x^{[k]} \to \overline{x}$, we have that $x^{[k]} \in Z := \left(\left\{x^{[k']} : k' \in \mathbb{N}\right\} \cup \{\overline{x}\}\right)$, a compact set;
2. Since $u^{[k]}$ converges uniformly on compact sets to $\overline{u}$, we have that $\forall \varepsilon > 0\ \exists N^1_{\varepsilon,Z,\overline{u}} \in \mathbb{N}$ such that $\forall k > N^1_{\varepsilon,Z,\overline{u}}$, $\forall x \in \left(\left\{x^{[k']} : k' \in \mathbb{N}\right\} \cup \{\overline{x}\}\right)$, $\left|\ u^{[k]}(x) - \overline{u}(x)\right| < \frac{\varepsilon}{2}$.

By choosing $x = x^{[k]} \in Z$, we have

$$\forall \varepsilon > 0\ \exists\ N^1_{\varepsilon,Z,\overline{u}} \text{ such that } \forall k > N^1_{\varepsilon,Z,\overline{u}},\ \left|\ u^{[k]}\left(x^{[k]}\right) - \overline{u}\left(x^{[k]}\right)\right| < \tfrac{\varepsilon}{2}$$

From the continuity of $\overline{u}$ and the convergence of $\left(x^{[k]}\right)_{k\in\mathbb{N}}$, we have that

$$\forall \varepsilon > 0\ \exists N^2_{\varepsilon,\overline{x},\overline{u}} \in \mathbb{N} \text{ such that } \forall k > N^2_{\varepsilon,\overline{x},\overline{u}},\ \left|\ \overline{u}\left(x^{[k]}\right) - \overline{u}(\overline{x})\right| < \tfrac{\varepsilon}{2}$$

Observe that

$$\left|u^{[k]}\left(x^{[k]}\right) - \overline{u}(\overline{x})\right| = \left|u^{[k]}\left(x^{[k]}\right) - \overline{u}\left(x^{[k]}\right) + \overline{u}\left(x^{[k]}\right) - \overline{u}(\overline{x})\right| \leq$$

$$\leq \left|u^{[k]}\left(x^{[k]}\right) - \overline{u}\left(x^{[k]}\right)\right| + \left|\overline{u}\left(x^{[k]}\right) - \overline{u}(\overline{x})\right|.$$

Then

$$\forall \varepsilon > 0\ \exists N_{\varepsilon,Z,\overline{u}} = \max\left\{N^1_{\varepsilon,Z,\overline{u}}, N^2_{\varepsilon,\overline{x},\overline{u}}\right\} \in \mathbb{N} \text{ such that } \forall k > N_{\varepsilon,Z\overline{u}},\ \left|\ u^{[k]}\left(x^{[k]}\right) - \overline{u}(\overline{x})\right| < \varepsilon$$

as desired ∎

In a similar manner it is possible to prove the following.

Lemma 23 *Let $r \geq 1$. For every $s \in \{0, \cdots, r\}$, the map*

$$\phi^s \colon \mathcal{C}^r(\mathbb{R}^C_{++}, \mathbb{R}) \times \mathbb{R}^C_{++} \to \mathbb{R}, \quad \phi^s(u,x) = \begin{cases} a\ partial\ s-th\ derivative\ of\ u\ computed\ at\ x, & if\ s \geq 1, \\ u(x), & if\ s = 0. \end{cases}$$

is continuous.

Note that the case $s = 0$ is just the evaluation map.

We consider now the space $\mathcal{U}$ as a subset of $\mathcal{C}^3(\mathbb{R}^C_{++}, \mathbb{R})$ induced by the compact-open topology.

Lemma 24 *Let $s \in \{0,1,2,3\}$. The map*

$$\phi^s \colon \mathcal{U} \times \mathbb{R}^C_{++} \to \mathbb{R}, \quad \phi^s(u,x) = \begin{cases} a\ partial\ s-th\ derivative\ of\ u\ computed\ at\ x, & if\ s = 1,2,3, \\ u(x), & if\ s = 0. \end{cases}$$

is continuous.

[3] To avoid confusion between superscripts and sequence indices, we use the notation $[k]$ for the latter, as in $\xi^{[k]}$.

In a similar manner we consider $\mathcal{V}$ as a subset of $\mathcal{C}^3((0,1),\mathbb{R})$ induced by the compact-open topology.

Lemma 25 *Let $s \in \{0,1,2,3\}$. The map*

$$\phi\colon \mathcal{V}\times(0,1)\to\mathbb{R},\quad \phi^s(v,x)=\begin{cases} v(x), & if\, s=0,\\ a\ s-th\ derivative\ of\ u\ computed\ at\ x, & if\, s=1,2,3.\end{cases}$$

is continuous.

We provide now with some properties of the functions $v\in\mathcal{V}$. Recall that, as mentioned in Remark 4, $v_{hh'}$ and $v'_{hh'}$ are bounded, see Proposition 71 in the Appendix 7.1.

Proposition 26 *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{V}$ which converges in the compact-open topology to $v\in\mathcal{V}$. Then,*

1. there exists $a\in\mathbb{R}$ such that for any $n\in\mathbb{N}$ and for any $x\in(0,1)$, we have $v_n(x)\le a$, i.e., $\{v_n : n\in\mathbb{N}\}$ is uniformly bounded above.

2. it is false that $\{v_n : n\in\mathbb{N}\}$ is uniformly bounded below.

Proof. 1.

For any $x_0\in(0,1)$, from uniform convergence on compact sets (indeed, pointwise convergence is enough) we have that

$$v_n(x_0)\overset{\mathbb{R}}{\to} v(x_0),$$

$$v'_n(x_0)\overset{\mathbb{R}}{\to} v'(x_0).$$

Then, the following sets are compact in $\mathbb{R}$:

$$S_0 := \{v_n(x_0) : n\in\mathbb{N}\}\cup\{v(x_0)\},$$

$$S_1 := \{v'_n(x_0) : n\in\mathbb{N}\}\cup\{v'(x_0)\}$$

Then, it is meaningful to give the following definitions

$$M_0 = \max S_0\in\mathbb{R},$$

$$M_1 = \max S_1\in\mathbb{R}_+.$$

where $M_1\in\mathbb{R}_+$ because the functions v_n are differentiable and strictly increasing and M_0 and M_1 do not depend upon $n\in\mathbb{N}$.

Since v is concave and differentiable, then for any $n\in\mathbb{N}$, any $x\in(x_0,1)$ - we have

$$v_n(x)\le v_n(x_0)+v'_n(x_0)(x-x_0)\le M_0+\overset{\ge 0}{M_1}(x-x_0)\overset{x,x_0\in(0,1)}{\le} M_0+M_1.$$

For any $x\in(0,x_0]$, since v_n is strictly increasing, we have

$$v_n(x)\le v_n(x_0).$$

Then, for any $n\in\mathbb{N}$, v_n is bounded above by M_0+M_1.

2. We define $v_n\colon(0,1)\to\mathbb{R}$ as

$$v_n(x)=\ln(x+1)-ne^{-nx},$$

and $v\colon(0,1)\to\mathbb{R}$ as

$$v(x)=\ln(x+1).$$

We have the following properties (for any $n\in\mathbb{N}$):

- $v_n, v\in\mathcal{C}^3((0,1),\mathbb{R})$;
- $v'_n=\frac{1}{1+x}+n^2e^{-nx}>0$ and $v'=\frac{1}{1+x}>0$, for every $x\in(0,1)$;

- $v_n'' = -\frac{1}{(1+x)^2} - n^3 e^{-nx} < 0$ and $v'' = -\frac{1}{(1+x)^2} < 0$, for every $x \in (0,1)$;
- $v_n''' = \frac{2}{(1+x)^3} + n^4 e^{-nx}$ and $v''' = \frac{2}{(1+x)^3}$, for every $x \in (0,1)$;
- $\lim_{x\to 0^+} v_n'(x) = 1 + n^2 < \infty$ for every $n \in \mathbb{N}$, and $\lim_{x\to 0^+} v'(x) = 1 < \infty$.

Hence $v_n, v \in \mathcal{V}$, for every $n \in \mathbb{N}$.

We study the convergence of the sequence v_n.

- For every $x \in (0,1)$ we have $\lim_{n\to+\infty} v_n(x) = v_(x)$, i.e., there is pointwise convergence of v_n to v on $(0,1)$.
- In order to study the uniform convergence on compacta it is enough to study the convergence on a generic closed interval $[a,b] \subset (0,1)$.

  Indeed, let $K \subseteq (0,1)$ be compact, therefore there exist $0 < a < b < 1$ such that $K \subseteq [a,b]$ and then

  $$\max_{x\in K} |v_n(x) - v(x)| \leq \max_{x\in[a,b]} |v_n(x) - v(x)|.$$

  Hence if the right-hand side converges to zero with respect to n, then we have the convergence to zero also of the left-hand side.
- Let $0 < a < b < 1$, then

  $$\lim_{n\to+\infty} \max_{x\in[a,b]} |v_n(x) - v(x)| = \lim_{n\to+\infty} \max_{x\in[a,b]} ne^{-nx} = \lim_{n\to+\infty} ne^{-na} = 0.$$
- Let $0 < a < b < 1$, then

  $$\lim_{n\to+\infty} \max_{x\in[a,b]} |v_n'(x) - v'(x)| = \lim_{n\to+\infty} \max_{x\in[a,b]} n^2 e^{-nx} = \lim_{n\to+\infty} n^2 e^{-na} = 0.$$
- Let $0 < a < b < 1$, then

  $$\lim_{n\to+\infty} \max_{x\in[a,b]} |v_n''(x) - v''(x)| = \lim_{n\to+\infty} \max_{x\in[a,b]} n^3 e^{-nx} = \lim_{n\to+\infty} n^3 e^{-na} = 0.$$
- Let $0 < a < b < 1$, then

  $$\lim_{n\to+\infty} \max_{x\in[a,b]} |v_n'''(x) - v'''(x)| = \lim_{n\to+\infty} \max_{x\in[a,b]} n^4 e^{-nx} = \lim_{n\to+\infty} n^4 e^{-na} = 0.$$

Hence there is uniform convergence of v_n to v and their derivatives until the third order on every compacts of $(0,1)$. Therefore, v_n converges to v with respect to the compact-open topology in $\mathcal{C}^3((0,1),\mathbb{R})$.
Finally, we show that the sequence v_n is not uniform bounded from below. Indeed, by the increasing property of v_n, we have

$$\inf_{x\in(0,1)} v_n(x) = \lim_{x\to 0^+} v_n(x) = -n,$$

for every $n \in \mathbb{N}$. Therefore the sequence cannot be uniformly bounded: let $M > 0$ be such that for every $x \in (0,1)$, $-M < v_n(x) < M$. We can find $n_0 \in \mathbb{N}$ such that $n_0 > M$. Since

$$\inf_{x\in(0,1)} v_{n_0}(x) = -n_0 < -M,$$

we conclude that v_n is not uniformly bounded. ∎

Proposition 27 *pr is proper.*

Proof. First of all observe that since projections are continuous functions and pr is the restriction of a continuous function to a subset of the domain of the projection, then pr is continuous - see also Theorem 18.2.d, page 107 in Munkres (2000).

Taken a compact set $K \subseteq \mathbb{E}$, we want to show that $pr^{-1}(K)$ is compact. Take a sequence $(\xi^{[k]}, \mathcal{E}^{[k]})$ contained in $pr^{-1}(K)$ and therefore such that $\widetilde{F}(\xi^{[k]}, \mathcal{E}^{[k]}) = 0$ and assume that, up to a subsequence,

$\mathcal{E}^{[k]} \to \overline{\mathcal{E}} \in \mathbb{E}$, i.e., $\left(e^{[k]}\right)_{k\in\mathbb{N}}$ converges in the Euclidean topology to $\overline{e} \in \mathbb{R}^{CH}_{++}$and $\left(u^{[k]}\right)_{k\in\mathbb{N}}$ and $\left(v^{[k]}\right)_{k\in\mathbb{N}}$ in the appropriate subspace of the compact-open topology $\overline{u} \in \mathcal{U}^H$ and $\overline{v} \in \mathcal{V}^B$. It suffices to prove that, up to a subsequence, $(\xi^{[k]})_{k\in\mathbb{N}}$ converges to a well chosen $\overline{\xi} \in \Xi$ such that $\widetilde{F}(\overline{\xi}, \overline{\mathcal{E}}) = 0$.

The proof proceeds by establishing a sequence of claims, the first of which is the most difficult one.

Claim 1.

$$\text{For any } h \in \mathcal{H}, \left(x_h^{[k]}\right)_{k\in\mathbb{N}} \text{ admits a subsequence converging to } \overline{x}_h \in \mathbb{R}^C_{++}, \tag{4.1}$$

which is done below.

Observe that $\left\{x_h^{[k]} : k \in \mathbb{N}\right\}$ is bounded below and above, as verified below.

α. Boundedness below. Since $\left\{x_h^{[k]} : k \in \mathbb{N}\right\} \subseteq \mathbb{R}^C_{++}$ is bounded from below by zero.

β. Boundedness above. From the fact that for any $k \in \mathbb{N}$, $\widetilde{F}\left(\xi^{[k]}, \mathcal{E}^{[k]}\right) = 0$, we have that for any $h' \in \mathcal{H}$, $x_{h'}^{[k]} = \sum_{h\in\mathcal{H}} e_h^{[k]} - \sum_{h\neq h'} x_h^{[k]}$. Therefore, since $\left(e_h^{[k]}\right)_{k\in\mathbb{N}}$, up to a subsequence, converges and $\left\{x^{[k]} : k \in \mathbb{N}\right\}$ is bounded from below , the result follows.

We also provide a different proof of buondedness above. By the fact that $\widetilde{F}(\xi^{[k]}, \mathcal{E}^{[k]}) = 0$ for any $k \in \mathbb{N}$, summing equation (3) in (3.2) with respect to h, using equation (6) in (3.2) and invoking equation (2.1), one easily shows that market clearing condition is satisfied for any commodity. Thus, $0 \ll x_{h'}^{[k]} \leq \sum_{h\in\mathcal{H}} e_h^{[k]} - \sum_{h\neq h'} x_h^{[k]} \ll \sum_{h\in\mathcal{H}} e_h^{[k]}$ holds true for any $k \in \mathbb{N}$. By the convergence of $(e_h^{[k]})_{k\in\mathbb{N}}$ to $\overline{e}_h$, for any commodity $c \in \mathcal{C}$, $(\sum_{h\in\mathcal{H}} e_h^{[k]c})_{k\in\mathbb{N}}$ converges to $\sum_{h\in\mathcal{H}} \overline{e}_h^c$. So, fixing $\varepsilon > 0$ and $c \in \mathcal{C}$, there exists $\mathbb{N} \ni n = n(\varepsilon, c)$ such that, for any $k \geq n$, $\sum_{h\in\mathcal{H}} e_h^{[k]c} < \sum_{h\in\mathcal{H}} \overline{e}_h^c + \varepsilon$. Defining $b := (b^c)_{c\in\mathcal{C}}$, with $b^c := \max\{\sum_{h\in\mathcal{H}} e_h^{[k]c} : k = 1, \ldots, n-1\} \cup \{\sum_{h\in\mathcal{H}} \overline{e}_h^c + \varepsilon\}$, we have $0 \ll x_{h'}^{[k]} \ll \sum_{h\in\mathcal{H}} e_h^{[k]} \leq b$ for any $k \in \mathbb{N}$, which concludes the proof of boundedness above.

Then,

$$x_h^{[k]} \text{ does converge to } \overline{x}_h \in \mathbb{R}^C_+. \tag{4.2}$$

We are then left with showing that $\overline{x}_h \in \mathbb{R}^C_{++}$.

For any $k \in \mathbb{N}$, define $r^{[k]} := \sum_{h\in\mathcal{H}} e_h^{[k]}$ and $\overline{r}$ the limit (up to subsequences).

Claim. For any $k \in \mathbb{N}$, $\left(\underline{x}_h^{[k]}, \underline{\tau}_h^{[k]}\right) := \left(\frac{r^{[k]}}{\pi_h H}, \frac{p^{[k]} r^{[k]}}{\pi_h H} \cdot \mathbf{1}_{B_h}\right)$, where $\pi_h := \max_{c\in\mathcal{C}} \left(\max\left\{\frac{r^{[k]c}}{e_h^{[k]c}} : k \in \mathbb{N}\right\} \cup \left\{\frac{\overline{r}^c}{\overline{e}_h^c}\right\}\right) > 0$, belongs to the constraint set of the maximization problem presented in the Definition of $\mathcal{B}$ equilibrium (see Definition 11) at $\mathcal{E}^{[k]} \in \mathbb{E}$, $p^{[k]\setminus} \in \mathbb{R}^{C-1}_{++}$, $\left((\tau_{h'h''}^{[k]})_{h''\in\mathcal{B}_{h'}}\right)_{h'\in\mathcal{H}\setminus\{h\}} \in \left(\times_{h'\in\mathcal{H}\setminus\{h\}} \Delta_{h'}\left(p^{[k]\setminus}\right)\right)$.

Proof of the Claim.

First of all, let's "construct" the choice of π_h . For any $c \in \mathcal{C}$ and $k \in \mathbb{N}$, let $N_h^{[k]c} \in \mathbb{N}$ be large enough in order to have that $\frac{1}{N_h^{[k]c}} r^{[k]c} \leq e_h^{[k]c}$ or $N_h^{[k]c} \geq \frac{r^{[k]c}}{e_h^{[k]c}}$; then, it is enough to take $N_h^c := \max\left(\left\{\frac{r^{[k]c}}{e_h^{[k]c}} : k \in \mathbb{N}\right\} \cup \left\{\frac{\overline{r}^c}{\overline{e}_h^c}\right\}\right)$, where the max exists because the set $\left\{\frac{r^{[k]c}}{e_h^{[k]c}} : k \in \mathbb{N}\right\} \cup \left\{\frac{\overline{r}^c}{\overline{e}_h^c}\right\}$ is compact, and $\pi_h := \max\{N_h^c : c \in \mathcal{C}\} \in \mathbb{R}_{++}$.

Then, we have

$$\text{for any } k \in \mathbb{N}, \frac{1}{\pi_h} r^{[k]} \leq e_h^{[k]}. \tag{4.3}$$

Observe that since for any $h \in \mathcal{H}$, $c \in \mathcal{C}, k \in \mathbb{N}$, $\frac{r^{[k]c}}{e_h^{[k]c}} > 1$ then $\pi_h > 1$.

Let's now check that $\left(\underline{x}_h^{[k]}, \underline{\tau}_h^{[k]}\right)$ belongs to the constraint set described above.

The first constraint is $p^{[k]}\underline{x}_h^{[k]} + \sum_{h'\in\mathcal{B}_h} \underline{\tau}_{hh'}^{[k]} \leq p^{[k]} e_h^{[k]} + \sum_{h'\in\mathcal{B}_{\to h}} \tau_{h'h}^{[k]}$; indeed,

$$p^{[k]}\underline{x}_h^{[k]} + \textstyle\sum_{h'\in\mathcal{B}_h} \underline{\tau}_{hh'}^{[k]} := p^{[k]} \frac{r^{[k]}}{\pi_h H} + \sum_{h'\in\mathcal{B}_h} \frac{p^{[k]} r^{[k]}}{\pi_h H} \overset{B_h \leq H-1}{\leq}$$

$$\leq \frac{p^{[k]} r^{[k]}}{\pi_h H} H = p^{[k]} \frac{1}{\pi_h} r^{[k]} \overset{(4.3)}{\leq} p^{[k]} e_h^{[k]} \leq p^{[k]} e_h^{[k]} + \textstyle\sum_{h'\in\mathcal{B}_{\to h}} \tau_{h'h}^{[k]}$$

where the last inequality follows from the fact $\sum_{h'\in\mathcal{B}_{\to h}} \tau_{h'h}^{[k]} \geq 0$.

The second constraint is $\sum_{h'\in\mathcal{B}_h}\underline{\tau}^{[k]}_{hh'}\leq p^{[k]}e^{[k]}_h$; indeed

$$\sum_{h'\in\mathcal{B}_h}\underline{\tau}^{[k]}_{hh'}:=\sum_{h'\in\mathcal{B}_h}\frac{p^{[k]}r^{[k]}}{\pi_h H}\leq\frac{H-1}{H}\frac{p^{[k]}r^{[k]}}{\pi_h}\overset{(4.3)}{\leq}\frac{H-1}{H}p^{[k]}e^{[k]}_h,$$

and the last term is smaller than $p^{[k]}e^{[k]}_h$, as desired.
The fact $\underline{\tau}^{[k]}_h\geq 0$ is obvious.
End of the proof of the Claim.

Then, since by definition, $\left(x^{[k]}_h,\tau^{[k]}_h\right)$ is solution to the maximization problem presented in the Definition 11 of $\mathcal{B}$ equilibrium at $\mathcal{E}^{[k]}\in\mathbb{E}$, $p^{[k]\setminus}\in\mathbb{R}^{C-1}_{++}$, $\left((\tau^{[k]}_{h'h''})_{h''\in\mathcal{B}_{h'}}\right)_{h'\in\mathcal{H}\setminus\{h\}}\in\left(\times_{h'\in\mathcal{H}\setminus\{h\}}\Delta_{h'}\left(p^{[k]\setminus}\right)\right)$, and using the Claim above, we have that for any $k\in\mathbb{N}$,

$$\begin{array}{l}
u^{[k]}_h\left(x^{[k]}_h\right)+\sum_{h'\in\mathcal{B}_h}v^{[k]}_{hh'}\left(\frac{1}{p^{[k]}r^{[k]}}\left(p^{[k]}e^{[k]}_{h'}-\sum_{h''\in\mathcal{B}_{h'}}\tau^{[k]}_{h'h''}+\left(\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^{[k]}_{h''h'}\right)+\tau^{[k]}_{hh'}\right)\right)\geq\\
u^{[k]}_h\left(\frac{r^{[k]}}{\pi_h H}\right)+\sum_{h'\in\mathcal{B}_h}v^{[k]}_{hh'}\left(\frac{1}{p^{[k]}r^{[k]}}\left(\overset{(\geq 0)}{\left(p^{[k]}e^{[k]}_{h'}-\sum_{h''\in\mathcal{B}_{h'}}\tau^{[k]}_{h'h''}\right)}+\overset{(\geq 0)}{\left(\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^{[k]}_{h''h'}\right)}+\frac{p^{[k]}r^{[k]}}{\pi_h H}\right)\right)\geq\\
\geq u^{[k]}_h\left(\frac{r^{[k]}}{\pi_h H}\right)+\sum_{h'\in\mathcal{B}_h}v^{[k]}_{hh'}\left(\frac{1}{\pi_h H}\right),
\end{array}\tag{4.4}$$

where the last inequality follows from the Assumption v2, i.e., the fact that $v^{[k]}_{hh'}$ is strictly increasing. Observe that, since $\pi_h>1$, then $\frac{1}{\pi_h H}\in(0,1)$.

Moreover, from Proposition 26 (uniform boundedness above), we have that there exists $\widetilde{v}_{hh'}\in\mathbb{R}$ such that

$$\text{for any } k\in\mathbb{N} \text{ and any } \theta^{[k]}_{h'}\in(0,1),\ v^{[k]}_{hh'}\left(\theta^{[k]}_{h'}\right)\leq\widetilde{v}_{hh'}.\tag{4.5}$$

Moreover, from uniform convergence on compact sets, $\left(v^{[k]}_{hh'}\left(\frac{1}{\pi_h H}\right)\right)_{k\in\mathbb{N}}$ converges to $\overline{v}_{hh'}\left(\frac{1}{\pi_h H}\right)\in\mathbb{R}$ and then, the following definition is well given:

$$\widetilde{\widetilde{v}}_{hh'}=\min\left(\left\{v^{[k]}_{hh'}\left(\frac{1}{\pi_h H}\right):k\in\mathbb{N}\right\}\cup\left\{\overline{v}_{hh'}\left(\frac{1}{\pi_h H}\right)\right\}\right).\tag{4.6}$$

Subclaim. For any $h'\in\mathcal{B}_h$, $\widetilde{v}_{hh'}>\widetilde{\widetilde{v}}_{hh'}$.
Proof of the Subclaim.
For any $k\in\mathbb{N}$ and $\theta^+\in\left(\frac{1}{\pi_h H},1\right)$, we have that

$$\widetilde{v}_{hh'}\overset{(4.5)}{\geq}v^{[k]}_{hh'}\left(\theta^+\right)\overset{(1)}{>}v^{[k]}_{hh'}\left(\frac{1}{\pi_h H}\right)\overset{(4.6)}{\geq}\widetilde{\widetilde{v}}_{hh'},$$

where (1) follows from the Assumption v2, i.e., the fact that $v^{[k]}_{hh'}$ is strictly increasing.
End of the proof of the Subclaim.
We then have

$$\begin{array}{l}
u^{[k]}_h\left(x^{[k]}_h\right)\overset{(4.4)}{\geq}\\
u^{[k]}_h\left(\frac{r^{[k]}}{\pi_h H}\right)+\sum_{h'\in\mathcal{B}_h}v^{[k]}_{hh'}\left(\frac{1}{\pi_h H}\right)+\\
-\sum_{h'\in\mathcal{B}_h}v^{[k]}_{hh'}\left(\frac{1}{p^{[k]}r^{[k]}}\left(p^{[k]}e^{[k]}_{h'}-\sum_{h''\in\mathcal{B}_{h'}}\tau^{[k]}_{h'h''}+\left(\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^{[k]}_{h''h'}\right)+\tau^{[k]}_{hh'}\right)\right)\overset{(4.5),(4.6)}{\geq}\\
u^{[k]}_h\left(\frac{r^{[k]}}{\pi_h H}\right)+\sum_{h'\in\mathcal{B}_h}\left(\widetilde{\widetilde{v}}_{hh'}-\widetilde{v}_{hh'}\right).
\end{array}\tag{4.7}$$

Then, defined,

$$l_h := \sum_{h' \in \mathcal{B}_h} \left(\widetilde{v}_{hh'} - \widetilde{\widetilde{v}}_{hh'} \right) \overset{\text{Subclaim}}{>} 0.$$

we have that

$$\text{for any } k \in \mathbb{N}, \qquad u_h^{[k]}\left(x_h^{[k]}\right) \geq u_h^{[k]}\left(\frac{r^{[k]}}{\pi_h H}\right) - l_h.$$

Define $S_h = \left\{ \frac{r^{[k]}}{4\pi_h H} \right\}_{k=1}^{\infty} \cup \left\{ \frac{\overline{r}}{4\pi_h H} \right\}$ which is a compact set. Observe that

$$\text{for any } k \in \mathbb{N}, \qquad u_h^{[k]}(x_h^{[k]}) \geq u_h^{[k]}\left(\frac{r^{[k]}}{\pi_h H}\right) - l_h \geq \min_{r \in S_h} u_h^{[k]}(r) - l_h. \tag{4.8}$$

We need to prove the following key result.

Claim 1'. For any $h \in \mathcal{H}$, there exists a sequence $(\varepsilon_{kh})_{k \in \mathbb{N}}$ with elements in $\mathbb{R}$ such that

$$\begin{aligned} &0.\ \text{for any } k \in \mathbb{N},\ \varepsilon_{kh} \ \geq 0, \\ &\text{a. for any } k \in \mathbb{N},\ \min_{r \in S_h} u_h^{[k]}(r) \geq \min_{r \in S_h} \overline{u}_h(r) - \varepsilon_{kh}, \text{ and} \\ &\text{b. } \varepsilon_{kh} \overset{k}{\to} 0. \end{aligned} \tag{4.9}$$

Proof of Claim 1'.

For any $k \in \mathbb{N}$, define

$$\varepsilon_{kh} = \max_{r \in S_h} \left| u_h^{[k]}(r) - \overline{u}_h(r) \right| \geq 0,$$

and then (4.9.0) is satisfied. For simplicity, let $\varepsilon_{kh} := \varepsilon_k$.

Observe that ε_k does not depend upon r. Then for any $r \in S_h$ and any $k \in \mathbb{N}$, we have $\left| u_h^{[k]}(r) - \overline{u}_h(r) \right| \leq \varepsilon_k$ and then[4] $-\varepsilon_k \leq u_h^{[k]}(r) - \overline{u}_h(r) \leq \varepsilon_k$ and therefore

$$\text{for any } r \in S_h \text{ and any } k \in \mathbb{N}, \text{we have } u_h^{[k]}(r) \geq \overline{u}_h(r) - \varepsilon_k. \tag{4.10}$$

Then, from (4.10),

$$\text{for any } r \in S_h \text{ and any } k \in \mathbb{N}, \text{ we have } u_h^{[k]}(r) \geq \min_{r' \in S_h} \overline{u}_h(r') - \varepsilon_k,$$

and then

$$\text{for any } k \in \mathbb{N}, \text{we have } \min_{r'' \in S_h} u_h^{[k]}(r'') \geq \min_{r' \in S_h} \overline{u}_h(r') - \varepsilon_k,$$

and (4.9.a) is proved.

Moreover, using convergence on compact sets (and the fact that S_h is indeed compact), we do have that

$$\forall \varepsilon > 0 \ \ \exists K(\varepsilon) \in \mathbb{N} \text{ such that } \forall k \geq K(\varepsilon), \forall r \in S_h, \text{ we have } \left| u_h^{[k]}(r) - \overline{u}_h(r) \right| < \varepsilon,$$

$$\forall \varepsilon > 0 \ \ \exists K(\varepsilon) \in \mathbb{N} \text{ such that } \forall k \geq K(\varepsilon), \ \max_{r \in S_h} \left| u_h^{[k]}(r) - \overline{u}_h(r) \right| < \varepsilon,$$

and then

$$\varepsilon_k = \max_{r \in S_h} \left| u_h^{[k]}(r) - \overline{u}_h(r) \right| \longrightarrow^{k} 0,$$

and (4.9.b) is proved.

End of the proof of Claim 1'.

Fact. Let $u_h \in \mathcal{U}$. If there exist $x_h \in \mathbb{R}^C_{++}$ and $\underline{u} \in \mathbb{R}$ such that $u_h(x_h) \geq \underline{u}$, then there exists $\underline{x} \in \mathbb{R}^C_{++}$ such that $u_h(\underline{x}_h)) = \underline{u}$.

[4] For any $a \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_{++}$, we have $|a| \leq \varepsilon$ if and only if $-\varepsilon \leq a \leq +\varepsilon$ if and only if $\varepsilon \geq -a \geq -\varepsilon$. In other word, $|a| \leq \varepsilon$ if and only if $a, -a \in [-\varepsilon, +\varepsilon]$.

Proof of the fact. Since u_h is unbounded below - see Appendix 7.1, there exists $\widetilde{x} \in \mathbb{R}^C_{++}$ such that $u_h(\widetilde{x}) < \underline{u} \leq u_h(x_h)$. Since u_h is continuous and its domain $\mathbb{R}^C_{++}$ is convex and therefore connected, then the desired result follows from the Intermediate Value Theorem for continuous functions.
End of the proof of the fact.

Defined $\widehat{x}_h = \arg\max_{x_h \in S_h} \overline{u}_h(x_h)$, we have that $\overline{u}_h(\widehat{x}_h) \geq \min_{x_h \in S_h} \overline{u}_h(x_h) \overset{l_h>0}{>} \min_{x_h \in S_h} \overline{u}_h(x_h) - l_h$. Then, from the Fact above, we have that

$$\text{there exists } x_h^* \in \mathbb{R}^C_{++} \text{ such that } \overline{u}_h(x_h^*) = \min_{x_h \in S_h} \overline{u}_h(x_h) - l_h. \tag{4.11}$$

Then, using Claim 1', we have that there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with elements in $\mathbb{R}_+$ such that

$$\begin{aligned} &\text{for any } k \in \mathbb{N}, \\ &u_h^{[k]}(x_h^{[k]}) \overset{(4.8)}{\geq} \min_{r \in S_h} u_h^{[k]}(r) - l_h \overset{(4.9.a)}{\geq} \min_{r \in S_h} \overline{u}_h(r) - \varepsilon_k - l_h \overset{(4.11)}{=} \overline{u}_h(x_h^*) - \varepsilon_k \end{aligned} \tag{4.12}$$

Let $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^C$ and $\delta > 0$ small enough such that $x_h^* - 3\delta\mathbf{1} \in \mathbb{R}^C_{++}$. Then by strict monotonicity, we have $\overline{u}_h(x_h^*) > \overline{u}_h(x_h^* - \delta\mathbf{1})$. Then, since $\varepsilon_k \xrightarrow{k} 0$, there exists k_1 such that $k \geq k_1$ implies $\overline{u}_h(x_h^*) - \varepsilon_k \geq \overline{u}_h(x_h^* - \delta\mathbf{1})$ and then, from (4.12),

$$\forall k \geq k_1, \quad u_h^{[k]}(x_h^{[k]}) \geq \overline{u}_h(x_h^* - \delta\mathbf{1}). \tag{4.13}$$

Observe that $x_h^* - 2\delta\mathbf{1} > x_h^* - 3\delta\mathbf{1} \in \mathbb{R}^C_{++}$. Consider any $\widetilde{x}_h \in \mathbb{R}^C_{++}$ such that

$$\overline{u}_h(\widetilde{x}_h) = \overline{u}_h(x_h^* - 2\delta\mathbf{1}). \tag{4.14}$$

Since $\beta := \overline{u}_h(x_h^* - 2\delta\mathbf{1}) < \overline{u}_h(x_h^* - \delta\mathbf{1}) := \alpha$, we have $\varepsilon := \frac{\alpha-\beta}{2} > 0$. Since convergence on compact sets implies pointwise convergence, then we have that

$$u_h^{[k]}(\widetilde{x}_h) \xrightarrow{k} \overline{u}_h(\widetilde{x}_h) \overset{(4.14)}{=} \overline{u}_h(x_h^* - 2\delta\mathbf{1})$$

and then there exists $k_2 \in \mathbb{N}$, $k_2 \geq k_1$ such that for any $k \geq k_2$,

$$u_h^{[k]}(\widetilde{x}_h) < \overline{u}_h(x_h^* - 2\delta\mathbf{1}) + \varepsilon := \beta + \frac{\alpha - \beta}{2} = \frac{\alpha + \beta}{2} \overset{(1)}{<} \alpha := \overline{u}_h(x_h^* - \delta\mathbf{1}) \ , \tag{4.15}$$

where (1) follows from the fact that $\frac{\alpha+\beta}{2} < \alpha \Leftrightarrow \alpha + \beta < 2\alpha \Leftrightarrow \beta - \alpha < 0$, as assumed.
Then,

$$\text{for any } k \geq k_2, \qquad u_h^{[k]}(x_h^{[k]}) \overset{(4.13)}{\geq} \overline{u}_h(x_h^* - \delta\mathbf{1}) \overset{(4.15)}{>} u_h^{[k]}(\widetilde{x}_h)$$

Then, from the concavity of $u_h^{[k]}$, for $k > k_2$,

$$0 \leq u_h^{[k]}(x_h^{[k]}) - u_h^{[k]}(\widetilde{x}_h) \leq D_{x_h} u_h^{[k]}(\widetilde{x}_h) \ \ (x_h^{[k]} - \widetilde{x}_h) \tag{4.16}$$

From the definition of convergence in the compact-open topology, which is equivalent to the uniform convergence on compact sets and implies pointwise convergence, we have that $D_{x_h} u_h^{[k]}(\widetilde{x}_h) \xrightarrow{k} D_{x_h} \overline{u}_h(\widetilde{x}_h)$. Moreover, from (4.2), $x_h^{[k]} \to \overline{x}_h$ and from the continuity of the dot product we then have $D_{x_h} u_h^{[k]}(\widetilde{x}_h) \ \ (x_h^{[k]} - \widetilde{x}_h) \to D_{x_h} \overline{u}_h(\widetilde{x}_h) \ \ (x_h - \widetilde{x}_h)$. Then, from (4.16),

$$D_{x_h} \overline{u}_h(\widetilde{x}_h) \ \ (\overline{x}_h - \widetilde{x}_h) \geq 0.$$

Summarizing, we showed that for any $\widetilde{x}_h \in \mathbb{R}^C_{++}$ such that $\overline{u}_h(\widetilde{x}_h) = \overline{u}_h(x_h^* - 2\delta\mathbf{1})$ - see (4.14) - we do have $D_{x_h} \overline{u}_h(\widetilde{x}_h)(\overline{x}_h - \widetilde{x}_h) \geq 0$, i.e.,

$$\overline{x}_h \in \bigcap_{\widetilde{x}_h \in \left\{z \in \mathbb{R}^C_{++} : \overline{u}_h(z) = u_h(x_h^* - 2\delta\mathbf{1})\right\}} \left\{y \in \mathbb{R}^C : D_{x_h} \overline{u}_h(\widetilde{x}_h)(y - \widetilde{x}_h) \geq 0\right\} := B. \tag{4.17}$$

Define

$$A = \{y \in \mathbb{R}^C_{++} : \overline{u}_h(y) \geq \overline{u}_h(x_h^* - 2\delta\mathbf{1})\}.$$

In Proposition 28 below, we prove that $A = B$. Then, from (4.17) ,

$$\overline{x}_h \in B = A \subseteq \mathrm{Cl}_{\mathbb{R}^C}(A) = \mathrm{Cl}_{\mathbb{R}^C}\left(\{y \in \mathbb{R}^C_{++} : \overline{u}_h(y) \geq \overline{u}_h(x^*_h - 2\delta\mathbf{1})\}\right) \subseteq \mathbb{R}^C_{++},$$

where the last inclusion follows from Assumption u4, and therefore we obtained the desired result (4.1) , i.e., that $\overline{x}_h \in \mathbb{R}^C_{++}$.

Claim 2. $\left(\lambda_h^{[k]}\right)_{[k]\in\mathbb{N}}$ admits a subsequence converging to $\overline{\lambda} \in \mathbb{R}^H_{++}$.

Since $p^1 = 1$, for any $h \in \mathcal{H}$, from equation (1) in (3.2), we have

$$D_{x_h^1} u_h^{[k]}\left(x_h^{[k]}\right) = \lambda_h^{[k]}.$$

Taking $\lim_{k\to+\infty}$ of both sides, we get

$$\lim_{k\to+\infty} \lambda_h^{[k]} = \lim_{k\to+\infty} D_{x_h^1} u_h^{[k]}\left(x_h^{[k]}\right) = D_{x_h^1}\overline{u}_h\left(\overline{x}_h\right) := \overline{\lambda}_h > 0,$$

where last equality follows from the fact that $\overline{u}_h$ is C^3 and the last strict inequality comes from Assumption u2 - we also used the fact that $D_{x_h^1} u_h^{[k]}$ converges uniformly on compact sets.

Claim 3. $\left(p^{[k]\setminus}\right)_{[k]\in\mathbb{N}}$ admits a subsequence converging to $\overline{p}^{\setminus} \in \mathbb{R}^{C-1}_{++}$.

Again from equation (1) in (3.2), we get

$$\lim_{k\to+\infty} p^{[k]\setminus} = \lim_{k\to+\infty} \frac{D_{x_h^\setminus} u_h^{[k]}\left(x_h^{[k]}\right)}{\lambda_h^{[k]}} = \frac{\overset{(>=)}{D_{x_h^\setminus}\overline{u}_h\left(\overline{x}_h\right)}}{\underset{(>0)}{\overline{\lambda}_h}} := \overline{p}^{\setminus} >> 0,$$

where again the last strict inequality comes from Assumption u2.

Claim 4. $\left(\tau_h^{[k]}\right)_{k\in\mathbb{N}}$ admits a subsequence converging to $\overline{\tau}_h \in \mathbb{R}^{B_h}_+$.

Observe that for any $h \in \mathcal{H}$, any $h' \in \mathcal{B}_h$, any $k \in \mathbb{N}$, we have

$$\tau_{hh'}^{[k]} \geq 0 \text{ and } 0 \leq \sum_{h'\in\mathcal{B}_h} \tau_{hh'}^{[k]} \leq p^{[k]} e_h^{[k]}. \tag{4.18}$$

Observe that from Claim 3 and the assumption, up to subsequences, $p^{[k]} \overset{k}{\to} \overline{p}$, $e_h^{[k]} \overset{k}{\to} \overline{e}_h$. Therefore, from the continuity of the dot product, $p^{[k]} e_h^{[k]}$ converges and therefore $\left\{p^{[k]} e_h^{[k]} : k \in \mathbb{N}\right\}$ is a bounded set and therefore it is bounded above. Then, for any $h \in \mathcal{H}$, there exists $\overline{\overline{L}}_h \in (0,1)$ such that for any $k \in \mathbb{N}$, we have $p^{[k]} e_h^{[k]} \leq \overline{\overline{L}}_h$ and from (4.18),

$$0 \leq \sum_{h'\in\mathcal{B}_h} \tau_{hh'}^{[k]} \leq p^{[k]} e_h^{[k]} \leq \overline{\overline{L}}_h.$$

Summarizing, for any $k \in \mathbb{N}$, $\tau_h^{[k]}$ belongs to $\Delta\left(\overline{\overline{L}}_h\right) := \left\{\tau = (\tau_{hh'})_{h'\in\mathcal{B}_h} \in \mathbb{R}^{B_h} :\ \tau_h \geq 0,\ \sum_{h'\in\mathcal{B}_h} \tau_{hh'} \leq \overline{\overline{L}}_h\right\}$. Since $\Delta\left(\overline{\overline{L}}_h\right)$ is a compact set, then, up to a subsequence, $\left(\tau_h^{[k]}\right)_{[k]\in\mathbb{N}}$ does converge to some $\overline{\tau}_h \in \mathbb{R}^{B_h}_+$. Moreover, from (4.18), we have $0 \leq \sum_{h'\in\mathcal{B}_h} \overline{\tau}_{hh'} \leq \overline{p}\overline{e}_h$.

Claim 5. $\left(\gamma_{hh'}^{[k]}\right)_{k\in\mathbb{N}}$ and $\left(\eta_h^{[k]}\right)_{k\in\mathbb{N}}$ admits a subsequence converging to $\overline{\gamma}_{hh'} \in \mathbb{R}_+$ and $\overline{\eta}_h \in \mathbb{R}_+$, respectively.

We distinguish two exhaustive cases.

Case 1. $\sum_{h'\in\mathcal{B}_h}\overline{\tau}_{hh'}<\overline{pe}_h$.

Preliminary remark. If $x^{[k]}\to x$, $y^{[k]}\to y$ and $y>x$, then there exists $N\in\mathbb{N}$ such that for $k>N$, $y^{[k]}>x^{[k]}$.

Then, taking the convergent subsequence to be the sequence itself, from the preliminary remark, we have that there exists $N\in\mathbb{N}$ such that for any $k>N$, $\sum_{h'\in\mathcal{B}_h}\tau^{[k]}_{hh'}<p^{[k]}e^{[k]}_h$ and then, from equation (5) in (3.2), we have

$$\text{for any } k>N,\ \ \eta^{[k]}_h=0 \text{ and therefore } \eta^{[k]}_h\to 0.$$

Then, for any $h'\in\mathcal{B}_h$, we have what follows. From equation (2) in (3.2), we have

$$\begin{array}{l}0\le\gamma^{[k]}_{hh'}=-\frac{1}{p^{[k]}r^{[k]}}\cdot v'^{[k]}_{hh'}\left(\frac{p^{[k]}e^{[k]}_{h'}+\tau^{[k]}_{hh'}+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^{[k]}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\tau^{[k]}_{h'h''}}{p^{[k]}r^{[k]}}\right)+\lambda^{[k]}_h+\eta^{[k]}_h\\ \overset{k}{\to}\\ -\frac{1}{\overline{pr}}\cdot\overline{v}'_{hh'}\left(\frac{\overline{pe}_{h'}+\overline{\tau}_{hh'}+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\overline{\tau}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\overline{\tau}_{h'h''}}{\overline{pr}}\right)+\overline{\lambda}_h+\overset{=0}{\overline{\eta}_h}:=\overline{\gamma}_{hh'}\end{array}$$

Observe that $\frac{p^{[k]}e^{[k]}_{h'}+\tau^{[k]}_{hh'}+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^{[k]}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\tau^{[k]}_{h'h''}}{p^{[k]}r^{[k]}}=\frac{p^{[k]}x^{[k]}_{h'}}{p^{[k]}r^{[k]}}:=\theta^{[k]}_{h'}$ does converge up to subsequence to $\overline{\theta}_{h'}=\frac{\overline{px}_{h'}}{\overline{pr}}\in(0,1)$: indeed

$$0<\frac{\overset{(+)}{\overline{p}}\overset{(+)}{\overline{x}}_{h'}}{\underset{(+)}{\overline{p}}\underset{(+)}{\overline{r}}}=\frac{\overline{p}\left(\overline{r}-\sum_{h''\neq h'}\overset{(+)}{\overline{x}}_{h''}\right)}{\overline{pr}}<1.$$

Then also $v^{[k]}_{hh'}\left(\theta^{[k]}_{h'}\right)$ converges to $\overline{v}_{hh'}\left(\overline{\theta}_{h'}\right)$ - see Lemma 25 - where $\theta_{h'}$ does belong to the domain of $v_{hh'}$.

Case 2. $\sum_{h'\in\mathcal{B}_h}\overline{\tau}_{hh'}=\overline{pe}_h>0$.

Then, there exists $\widehat{h}\in\mathcal{B}_h$ such that $\overline{\tau}_{h\widehat{h}}>0$ and then there exists $N\in\mathbb{N}$ such that for any $k>N$, $\tau^{[k]}_{h\widehat{h}}>0$ and then

$$\text{for any } k>N,\ \gamma^{[k]}_{h\widehat{h}}=0,\quad \text{i.e.}\qquad \gamma^{[k]}_{h\widehat{h}}\longrightarrow 0.$$

From equation (2) in (3.2), we have

$$\begin{array}{l}0\le\eta^{[k]}_h=-\frac{1}{p^{[k]}r^{[k]}}v'^{[k]}_{hh'}\left(\frac{p^{[k]}e^{[k]}_{h'}+\tau^{[k]}_{hh'}+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^{[k]}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\tau^{[k]}_{h'h''}}{p^{[k]}r^{[k]}}\right)-\lambda^{[k]}_h+\gamma^{[k]}_{h\widehat{h}}\\ \longrightarrow\\ -\frac{1}{\overline{pr}}\cdot\overline{v}'_{hh'}\left(\frac{\overline{pe}_{h'}+\overline{\tau}_{hh'}+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\overline{\tau}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\overline{\tau}_{h'h''}}{\overline{pr}}\right)+\overline{\lambda}_h+\overset{=0}{\overline{\gamma}_{hh'}}:=\overline{\eta}_h.\end{array}$$

Then, using the argument presented in Case 1, we can show that for any $h''\in\mathcal{B}_h\setminus\left\{\widehat{h}\right\}$, we have that $\gamma^{[k]}_{hh''}\longrightarrow\overline{\gamma}_{hh''}$.

Claim 6.

Finally since for any $k\in\mathbb{N}$,

$$\widetilde{F}\left(\xi^{[k]},\mathcal{E}^{[k]}\right)=0,$$

taking then limit for $k\longrightarrow+\infty$ and since from Lemma 20, we have that $\widetilde{F}$ is continuous, then we have

$$\widetilde{F}\left(\overline{\xi},\overline{\mathcal{E}}\right)=0.$$

as desired. ∎

Proposition 28 *Let $u_h \in \mathcal{U}$, and x_h^* as in the proof of Proposition 27. Define*[5] $x_h^* - 2\delta\mathbf{1} = x_h^{**}$, $\alpha = u_h(x_h^{**})$,

$$A^I = \left\{ y \in \mathbb{R}^C_{++} : u_h(y) = \alpha \right\},$$

and recall that we defined

$$A = \{ y \in \mathbb{R}^C_{++} : u_h(y) \geq \alpha \},$$

$$B = \bigcap_{\widetilde{x}_h \in A^I} \left\{ y \in \mathbb{R}^C : D_{x_h} u_h(\widetilde{x}_h)(y - \widetilde{x}_h) \geq 0 \right\}.$$

Then $A = B$.

Proof. For lighter notation, write u instead of u_h and x instead of x_h.
$[A \subseteq B]$
Take $y \in A$; then by definition of A, we have $u(y) \geq \alpha$, or

$$u(y) - \alpha \geq 0. \tag{4.19}$$

We want to show that for any $\overline{x} \in A^I$, i.e., $u(\overline{x}) = \alpha$, we have that $D_x u(\overline{x})(y - \overline{x}) \geq 0$. Take an arbitrary

$$\overline{x} \in A^I. \tag{4.20}$$

Since u is concave, then $u(y) \leq u(\overline{x}) + Du(\overline{x})(y - \overline{x})$ or

$$Du(\overline{x})(y - \overline{x}) \geq u(y) - u(\overline{x}). \tag{4.21}$$

Then,

$$Du(\overline{x})(y - \overline{x}) \overset{(4.21)}{\geq} u(y) - u(\overline{x}) \overset{(4.20)}{\geq} u(y) - \alpha \overset{(4.19)}{\geq} 0,$$

as desired.
$[B \subseteq A]$
We want to show what follows. Let $x \in \mathbb{R}^C$ be such that for any $x^0 \in A^I$, we have $Du(x_0) \cdot (x - x_0) \geq 0$. Then $\boxed{1.}$ $x \in \mathbb{R}^C_{++}$ and $\boxed{2.}$ $u(x) \geq u(x_0) = \alpha = u(x^{**})$.
$\boxed{\text{Proof of 1.}}$
Suppose otherwise, i.e., $x \notin \mathbb{R}^C_{++}$., i.e., $x \in \mathbb{R}^C \backslash \mathbb{R}^C_{++}$. By assumption, $x^{**} \in \mathbb{R}^C_{++}$. Then,

$$\exists \beta \in \mathbb{R}_{++} \text{ such that } x + \beta \cdot \mathbf{1} >> x^{**}, \tag{4.22}$$

as verified below.
First Proof.
Take $\beta > \max\{x^{**c} - x^c : c \in \mathcal{C}\}$; then,
a. since $x^{**} \in \mathbb{R}^C_{++}$ and $x \notin \mathbb{R}^C_{++}$., i.e., $\exists c' \in \mathcal{C}$ such that $x^{c'} \leq 0$, we have that

$$\text{there exists } c' \in \mathcal{C} \text{ such that } \overset{(>0)}{x^{**c'}} + \left(\overset{\geq 0}{-x^{c'}} \right) > 0, \tag{4.23}$$

and therefore $\beta > 0$;
b. for any $c \in \mathcal{C}$, $x^c + \beta > x^c + x^{**c} - x^c = x^{**c}$.
End of the proofs of (4.22).
Then, by monotonicity of u, we have

$$u(x + \beta \cdot \mathbf{1}) > u(x^{**}). \tag{4.24}$$

Moreover, there exists

$$\underline{\alpha} \in \mathbb{R}_+ \tag{4.25}$$

such that

$$x + \underline{\alpha} \cdot \mathbf{1} \in \mathcal{F}\left(\mathbb{R}^C_{++}\right) = \left\{ x \in \mathbb{R}^C : x \geq 0 \text{ and } \exists c \in \mathcal{C} \text{ such that } x^c = 0 \right\} \tag{4.26}$$

[5] Below, A^I is an Indifference surface.

as[6] verified below. Define

$$\underline{\alpha} = \max\{-x^c : c \in \mathcal{C}\} := -x^{\widehat{c}} \geq 0,$$

where the inequality follows from the fact that by assumption

$$x \in \mathbb{R}^C \backslash \mathbb{R}^C_{++} = \left\{x \in \mathbb{R}^C : \exists \widetilde{c} \in \mathcal{C} \text{ such that } x^{\widetilde{c}} \leq 0\right\}.$$

Then $x + \underline{\alpha} \cdot \mathbf{1} = (x^c + \underline{\alpha})_{c \in \mathcal{C}}$ is such that

$$x^c + \underline{\alpha} = \begin{cases} 0 & \text{if} \quad c = \widehat{c} \\ \\ x^c + \max\left\{-x^{c'} : c' \in \mathcal{C}\right\} \geq x^c + (-x^c) = 0 & \text{if} \quad c \neq \widehat{c}. \end{cases}$$

Observe that

$$\beta > \underline{\alpha}, \tag{4.27}$$

as verified below. Observe that since $x^{**} >> 0$, then for any $c \in \mathcal{C}$, $x^{**c} - x^c > -x^c$, and $\beta := \max\{x^{**c} - x^c : c \in \mathcal{C}\} > \max\{-x^c : c \in \mathcal{C}\} := \underline{\alpha}$.[7]

Then, from (4.25) and (4.27), we have that

$$x \leq x + \underline{\alpha} \cdot \mathbf{1} << x + \beta \cdot \mathbf{1}.$$

Claim. There exists $\gamma^* \in (0, \beta)$ such that $x + (\underline{\alpha} + \gamma^*) \cdot \mathbf{1} >> x + \underline{\alpha} \cdot \mathbf{1}$ and therefore $x + (\alpha + \gamma^*) \cdot \mathbf{1} \in \mathbb{R}^C_{++}$ and

$$\text{such that } u\left(x + (\underline{\alpha} + \gamma^*) \cdot \mathbf{1}\right) < u\left(x^{**}\right). \tag{4.28}$$

Proof of the Claim.

First of all observe that, for any $\gamma \in \mathbb{R}_{++}$, from (4.26), $x + (\underline{\alpha} + \gamma) \cdot \mathbf{1} \in \mathbb{R}^C_{++}$. Now suppose that (4.28) does not hold

Suppose otherwise; then for any $n \in \mathbb{N}$ such that $\frac{1}{n} < \beta$, we have

$$u\left(x + \left(\underline{\alpha} + \frac{1}{n}\right) \cdot \mathbf{1}\right) \geq u\left(x^{**}\right).$$

Then, $x + \underline{\alpha} \cdot \mathbf{1} \in \mathrm{Cl}_{\mathbb{R}^C}\left(\left\{y \in \mathbb{R}^C_{++} : u(y) \geq u\left(x^{**}\right)\right\}\right)$, as verified below. For any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $x + \left(\underline{\alpha} + \frac{1}{n_\varepsilon}\right) \cdot \mathbf{1} \in B\left(x + \underline{\alpha} \cdot \mathbf{1}, \varepsilon\right)$ and $u\left(x + \left(\underline{\alpha} + \frac{1}{n_\varepsilon}\right) \cdot \mathbf{1}\right) \geq u\left(x^{**}\right)$. But then, from Assumption u4, $x + \underline{\alpha} \cdot \mathbf{1} \subseteq \mathbb{R}^C_{++}$, contradicting (4.26).

End of the proof of the Claim.

Summarizing

$$u\left(x + (\underline{\alpha} + \gamma^*) \cdot \mathbf{1}\right) \overset{(4.28)}{<} u\left(x^{**}\right) \overset{(4.24)}{<} u\left(x + \beta \cdot \mathbf{1}\right) \overset{(4.25)}{\leq} u\left(x + (\underline{\alpha} + \beta) \cdot \mathbf{1}\right).$$

Then, by continuity of the function $U : \mathbb{R}_+ \to \mathbb{R}$, $\alpha \longmapsto u\left(x + \alpha \cdot \mathbf{1}\right)$, we have that

$$\exists \gamma_0 \in \begin{pmatrix} (>0) \\ \gamma^*, \beta \end{pmatrix} \text{ such that}$$

$$x + (\alpha + \gamma_0) \cdot \mathbf{1} >> x + \alpha \cdot \mathbf{1} \text{ and therefore } x + (\alpha + \gamma_0) \cdot \mathbf{1} \in \mathbb{R}^C_{++} \text{ and}$$

$$u\left(x^{**}\right) = u\left(x + (\underline{\alpha} + \gamma_0) \cdot \mathbf{1}\right),$$

where

$$\underline{\alpha} + \gamma_0 > 0 \tag{4.29}$$

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[6] $\mathcal{F}\left(\mathbb{R}^C_{++}\right) \overset{\text{Math 2}}{=} \left(\mathrm{Cl}\left(\mathbb{R}^C_{++}\right)\right) \backslash \mathrm{Int}\left(\mathbb{R}^C_{++}\right) = \mathbb{R}^C_+ \backslash \mathbb{R}^C_{++} = \left\{x \in \mathbb{R}^C : (\forall i \in \{1, ..., n\}, x_i \geq 0) \wedge \neg (\forall i \in \{1, ..., n\}, x_i > 0)\right\} =$
$\left\{x \in \mathbb{R}^C : (\forall i \in \{1, ..., n\}, x_i \geq 0) \wedge (\exists j \in \{1, ..., n\} \text{ such that } x_j \leq 0)\right\} = \left\{x \in \mathbb{R}^C : \forall i \in \{1, ..., n\}, x_i \geq 0 \wedge (\exists j \in \{1, ..., n\} \text{ such that } x_j = 0)\right\}$.

[7] Fact. If for any $c \in \mathcal{C}$, $x^c > y^c$, then $M_x := \max\{x^c : c \in \mathcal{C}\}$ and $M_y := \max\{y^c : c \in \mathcal{C}\}$ exist (from the extreme value theorem) and $M_x > M_y$.

Proof. There exist $k \in \mathcal{C}$ such that $M_y = y^k$. Then, $M_x \overset{\text{Def. } M_x}{\geq} x^k \overset{\text{Assu.}}{>} y^k = M_y$.

because $\underline{\alpha} \geq 0$ and $\gamma_0 > \gamma^* \overset{\text{from Claim}}{>} 0$.

Then, defined $x_0 := x + (\underline{\alpha} + \gamma_0) \cdot \mathbf{1} \overset{(4.29)}{\in} \mathbb{R}^C_{++}$, we have

$$u(x_0) = u(x^{**}) \text{ and}$$

$$0 \overset{\text{Assu. u4}}{\leq} Du(x_0) \cdot (x - x_0) = \overset{(>>0)}{Du}(x_0) \cdot \left(\overset{(>>0)}{-(\underline{\alpha} + \gamma_0)} \cdot \mathbf{1} \right) < 0,$$

which is the desired contradiction.

Proof of 2.

Now suppose that $x \in \mathbb{R}^C_{++}$ and our Claim is false, i.e., $u(x) < u(x^{**})$.

Therefore,

$$\text{there exists } c' \in \mathcal{C} \text{ such that } x^{**c'} > x^{c'}: \tag{4.30}$$

suppose otherwise, i.e., then $x^{**} \leq x$ and, by monotonicity, $u(x^{**}) \leq u(x)$, a contradiction. Then there exists

$$\gamma \in \mathbb{R}_{++} \text{ such that } x + \gamma \cdot \mathbf{1} >> x^{**}.$$

Indeed, is enough to take $\gamma \in \mathbb{R}$ such that $\gamma \cdot \mathbf{1} >> x^{**} - x$, i.e., $\gamma > \max\{x^{**c} - x^c : c \in \mathcal{C}\} \overset{(4.30)}{>} 0$.

Then, $u(x) < u(x^{**}) < u(x + \gamma \cdot \mathbf{1})$. By continuity there exists $\gamma^* \in (0, \gamma)$ such that $u(x^{**}) = u(x + \gamma^* \cdot \mathbf{1})$. Then, defined $x_0 = x + \gamma^* \cdot \mathbf{1} \in \mathbb{R}^C_{++}$, we have

$$u(x_0) = u(x^{**}) \text{ and}$$

$$Du(x_0) \cdot (x - x_0) = \overset{(+)}{Du}(x_0) \cdot (-\gamma^* \cdot \mathbf{1}) < 0.$$

contradicting our assumption that for any $x^0 \in A^I$, $Du(x_0) \cdot (x - x_0) \geq 0$. ■

Chapter 5

Density: The $\alpha - \beta$ model

In general equilibrium theory, to show density of a given property in the economy space amounts to use the following result - see Theorem 26, page 151, in Villanacci and others (2002).

Theorem 29 *Let M, Ω and N be C^r manifolds of dimensions m, p and n, respectively. Let $f : M \times \Omega \to N$, $(x, \omega) \longmapsto f(x, \omega)$ be a C^r function. Assume $r > \max\{m - n, 0\}$. For any $\omega \in \Omega$, we set $f_\omega \colon M \to N$ by $f_\omega(x) := f(x, \omega)$. If y is a regular value for f, then $\Omega_f^* := \{\omega \in \Omega : \ y \ \text{is a regular value for } f_\omega\}$ has full measure in Ω and, therefore, is dense in Ω.*

In the case in which M, Ω are open subsets of Euclidean spaces and N is an Euclidean space, to show that say 0 is a regular value for f amounts to compute the rank of the Jacobian $Df(x, \omega) \in \mathbb{M}(m + p, n)$ and show that rank is equal to n.

On the other hand, in many (and perhaps all) applications of general equilibrium theory, the domain of the equilibrium function is the Cartesian product of open subsets of Euclidean spaces and (subsets of) function spaces. At first sight, the presence of function spaces prevents a direct application of Theorem 29. However, by introducing a suitable finite parameterization of the functions involved, it is possible to:

1. define a new equilibrium function whose domain is an open subset of a Euclidean space;
2. simplify (or, in some cases, make possible) the proof of regularity, since finite parameterizations effectively increase the number of columns of the relevant Jacobian matrix, thereby facilitating the establishment of the desired rank condition.

A rigorous treatment of these ideas is provided in Section 5.1. Section 5.2 introduces the $\alpha-\beta$ economy, which provides the finite parameterization of our framework. Section 5.3 shows that borderline cases are absent on an open and dense subset of the economy space. Section 5.4 introduces the so-called partition functions $\widetilde{F}_P$, which are smooth and allow us to apply Glöckner's theorem. Finally, by exploiting the properties of these functions, we show that the set of regular economies is an open and dense subset of the space of all economies.

5.1 Recipe to get density using finite parametrizations

In this Section we want to present a Recipe to get density using finite parametrizations of the functions defining economies.

Let the following objects be given:

an open subset Π of $\mathbb{R}^k$; to fix ideas take $\Pi = \mathbb{R}^{CH}_{++}$ with generic element e;

a subset $\breve{\mathcal{U}}$ of $\left(\mathcal{C}^3\left(\mathbb{R}^C_{++}, \mathbb{R}\right), \mathcal{T}^1_{co}\right)$, where $\mathcal{T}^1_{co}$ is the compact-open topology on $\mathcal{C}^3\left(\mathbb{R}^C_{++}, \mathbb{R}\right)$;

a subset $\breve{\mathcal{V}}$ of $\left(\mathcal{C}^3\left((0,1), \mathbb{R}\right), \mathcal{T}^2_{co}\right)$, where $\mathcal{T}^2_{co}$ is the compact-open topology on $\mathcal{C}^3\left((0,1), \mathbb{R}\right)$;

$\mathbb{E} := \mathbb{R}^{CH}_{++} \times \breve{\mathcal{U}}^H \times \breve{\mathcal{V}}^B$ is the set of economies with generic element $\left((e_h)_{h\in\mathcal{H}}, (u_h)_{h\in\mathcal{H}}, \left((v_{hh'})_{h'\in\mathcal{B}_h}\right)_{h\in\mathcal{H}}\right)$;

an open set Ξ of $\mathbb{R}^n$.

Since the same construction is employed to prove density in both Lemma 48 and Lemma 62, corresponding to the equilibrium functions $\widetilde{F}$ and $\widetilde{F}_P$, respectively, throughout this section we use the generic notation

$\widetilde{G} : \Xi \times \mathbb{E} \to \mathbb{R}^n$, $(\xi, \mathcal{E}) \longmapsto \widetilde{G}(\xi, \mathcal{E})$ or

$$\widetilde{G} : \Xi \times \mathbb{R}_{++}^{CH} \times \breve{\mathcal{U}}^H \times \breve{\mathcal{V}}^B \to \mathbb{R}^n, \quad (\xi, e, u, v) \longmapsto \widetilde{G}(\xi, e, u, v).$$

to cover both settings.

We can now define the following so-called "finite parametrization" of the functions u_h and $v_{hh'}$. For the selfish part of the utility function, we follow closely Allen (1981), Lemma 4.1, page 292 (and Lemma 38, page 35 in Carosi, Gori, Villanacci (2007)).

Definition 30 *For any $\alpha_h \in \mathbb{R}_{++}^C$, define*

$$l_{\alpha_h} : \mathbb{R}_{++}^C \to \mathbb{R},\ x_h \longmapsto \sum_{c=1}^{C} \alpha_h^c \cdot \log(x_h^c).$$

and

$$u + l_\alpha := (u_h + l_{\alpha_h})_{h \in \mathcal{H}}.$$

Also set $u_\alpha = (u_{a_h})_{h \in \mathcal{H}}$.
For any $h \in \mathcal{H}$ and $h' \in \mathcal{B}_h$, take $\beta_{hh'} \in \mathbb{R}_{++}$ and define $\beta_h \odot v_h = (\beta_{hh'} \cdot v_{hh'})_{h' \in \mathcal{B}_h}$, and

$$\beta \odot v := (\beta_h \odot v_h)_{h \in \mathcal{H}} = \left((\beta_{hh'} v_{hh'})_{h' \in \mathcal{B}_h}\right)_{h \in \mathcal{H}}.$$

Observe that since $\alpha_h \in \mathbb{R}_{++}^C$, then $u_{0_{\mathbb{R}^C}}$ is not defined.

The goal of the analysis is as follows. Defined a condition $p(\xi, e, u, v)$, we want to show that there exists a dense subset $\mathcal{D}^*$ of $\mathbb{E}$ such that for any $(e, u, v) \in \mathcal{D}^*$, we have that for any $\xi \in \left(\widetilde{G}_{(e,u,v)}\right)^{-1}(0)$ condition $p(\xi, e, u, v)$ holds.

Condition 1. For any $\alpha_h \in \mathbb{R}_{++}^C$, $u_h + l_{\alpha_h} \in \breve{\mathcal{U}}$. For any $\beta_{hh'} \in \mathbb{R}_{++}$, $\beta_{hh'} v_{hh'} \in \breve{\mathcal{V}}$.

Remark 31 *In our framework, Condition 1 is verified in Lemma 38 and Lemma 39.*

Definition 32 $\Omega = \mathbb{R}_{++}^{CH} \times \mathbb{R}_{++}^{CH} \times \mathbb{R}_{++}^{B}$ *with generic element* $\omega = \left(e, (\alpha_h)_{h \in \mathcal{H}}, \left((\beta_{hh'})_{h' \in \mathcal{B}_h}\right)_{h \in \mathcal{H}}\right) := (e, \alpha, \beta)$.

Condition 1 allows to present the following definition.

Definition 33 *For any $(u, v) \in \breve{\mathcal{U}}^H \times \breve{\mathcal{V}}^B$, define the corresponding finite parametrization version of the equilibrium system function as follows.*

$$\widehat{G}_{(u,v)} : \Xi \times \mathbb{R}_{++}^{CH} \times \mathbb{R}_{++}^{CH} \times \mathbb{R}^B \to \mathbb{R}^n,$$

$$\widehat{G}_{(u,v)}(\xi, e, \alpha, \beta) = \widetilde{G}\left(\xi, e, (u_h + l_{\alpha_h})_{h \in \mathcal{H}}, \left((\beta_{hh'} v_{hh'})_{h' \in \mathcal{B}_h}\right)_{h \in \mathcal{H}}\right) = \widetilde{G}(\xi, e, u + l_\alpha, \beta \odot v)$$

Definition 34 *For any $(u, v) \in \breve{\mathcal{U}}^H \times \breve{\mathcal{V}}^B$,*

$$\Omega^*_{(u,v)} = \left\{\omega \in \Omega :\ \textit{for any } \xi \in \left(\widehat{G}_{(u,v)|\omega}\right)^{-1}(0),\ \textit{condition } p(\xi, e, u + l_a, \beta \odot v)\ \textit{holds}\right\}.$$

Condition 2. For any $(u, v) \in \breve{\mathcal{U}}^H \times \breve{\mathcal{V}}^B$, $\Omega^*_{(u,v)}$ is full measure in Ω.

Remark 35 *In our framework, Condition 2 is verified in Lemmas 45 and 61, for different definitions of $p(\xi, e, u, v)$.*

Conditions 3. a. For any $u_h \in \breve{\mathcal{U}}$ and for any (compact-open topology) open neighborhood N_{u_h} of u_h, there exists $\varepsilon_h \in \mathbb{R}_{++}$ such that for any $\alpha_h \in \mathbb{R}_{++}^C \cap B_{\mathbb{R}^C}(0, \varepsilon_h)$, we have that $u_h + l_{\alpha_h} \in N_{u_h}$;

b. For any $v_{hh'} \in \breve{\mathcal{V}}$ and for any (compact-open topology) open neighborhood $N_{v_{hh'}}$ of $v_{hh'}$, there exists $\delta_{hh'} \in \mathbb{R}_{++}$ such that for any $\beta_{hh'} \in (1 - \delta_{hh'}, 1 + \delta_{hh'})$, we have that $\beta_{hh'} v_{hh'} \in N_{v_{hh'}}$.

Remark 36 *In our framework, Condition 3.a. is verified in Proposition 122, and Condition 3.b. is verified in Proposition 111 and in Lemma 114, Propositions 119 and 120 contained in the Section 7.3.3, which is taken from the Chapter on "Compact-open topology" in Villanacci (2026) which is cut and pasted here.*

We are now ready to prove the desired result.

Proposition 37 *If Conditions 1., 2., 3a. and 3b. hold true, then*

$$\mathcal{D}^* = \left\{ (e,u,v) \in \mathbb{E} : \text{ for any } \xi \in \left(\widetilde{G}_{(e,u,v)}\right)^{-1}(0), \text{ condition } p(\xi,e,u,v) \text{ holds} \right\}$$

is dense in $\mathbb{E}$.

Proof. Let an arbitrary$(e,u,v) \in \mathbb{E}$ be given; let an arbitrary open neighborhood

$$N^* = B_{\mathbb{R}^{CH}}(e,\varepsilon) \times \left(\times_{h\in\mathcal{H}} N^*_{u_h}\right) \times \left(\times_{h\in\mathcal{H}} \left(\times_{h'\in\mathcal{B}_h} N^*_{v_{hh'}}\right)\right)$$

of (e,u,v) and $\varepsilon > 0$ be given, where "open" means that for any $h \in \mathcal{H}$ and any $h' \in \mathcal{B}_h$, $N^*_{u_h}$ is open in the relative topology of $\breve{\mathcal{U}}$ with respect the compact-open topology on $\mathcal{C}^3\left(\mathbb{R}^C_{++},\mathbb{R}\right)$ and $N^*_{v_{hh'}}$ is open in the relative topology of $\breve{\mathcal{V}}$ with respect the compact-open topology on $\mathcal{C}^3((0,1),\mathbb{R})$.

We want to show that

$$\text{there exists } (e',u',v') \in N^* \text{ such that for any } \xi \in \left(\widetilde{G}_{(e',u',v')}\right)^{-1}(0), \text{ condition } p(\xi,e',u',v') \text{ holds.} \tag{5.1}$$

Since $N^*_{u_h}$ and $N^*_{v_{hh'}}$ are open in the relative topologies described above, then there exists a compact-open topology open neighborhood N_{u_h} and $N_{v_{hh'}}$ such that

$$N^*_{u_h} = N_{u_h} \cap \breve{\mathcal{U}} \text{ and } N = N_{v_{hh'}} \cap \breve{\mathcal{V}}. \tag{5.2}$$

Observe that since, from Condition 2, for any $(u,v) \in \breve{\mathcal{U}}^H \times \breve{\mathcal{V}}^B$, we have that $\Omega^*_{(u,v)}$ has full measure in Ω, then $\Omega^*_{(u,v)}$ is dense in Ω, as well. Therefore, for any Ω open subset S of Ω, we do have $S \cap \Omega^*_{(u,v)} \neq \varnothing$. For any $h \in \mathcal{H}$ and $h' \in \mathcal{B}_h$, define the open sets

$$S_h = \mathbb{R}^C_{++} \cap B_{\mathbb{R}^C}(0,\varepsilon_h),$$

$$T_{hh'} = (1-\delta_{hh'}, 1+\delta_{hh'}),$$

where $\varepsilon_h \in \mathbb{R}_{++}$ and $\delta_{hh'} \in \mathbb{R}_{++}$ are defined in Conditions 3a. and 3b.

Then $(B_{\mathbb{R}^{CH}}(e,\varepsilon) \times (\times_{h\in\mathcal{H}} S_h) \times (\times_{h\in\mathcal{H}} (\times_{h'\in\mathcal{B}_h} T_{hh'})))$ is Ω open and since $\Omega^*_{(u,v)}$ is dense in Ω, we have that

$$(B_{\mathbb{R}^{CH}}(e,\varepsilon) \times (\times_{h\in\mathcal{H}} S_h) \times (\times_{h\in\mathcal{H}} (\times_{h'\in\mathcal{B}_h} T_{hh'}))) \quad \cap \quad \Omega^*_{(u,v)} \neq \varnothing.$$

Therefore, by definition of $\Omega^*_{(u,v)}$, there exists $(e^*,\alpha^*,\beta^*) \in (B_{\mathbb{R}^{CH}}(e,\varepsilon) \times (\times_{h\in\mathcal{H}} S_h) \times (\times_{h\in\mathcal{H}} (\times_{h'\in\mathcal{B}_h} T_{hh'})))$ such that

$$\text{for any } \xi \in \left(\widehat{G}_{(u,v)|(e^*,\alpha^*,\beta^*)}\right)^{-1}(0), \text{ i.e., for any } \xi \in \Xi \text{ such that } \widetilde{G}\left(\xi, e^*, \left(u_h + l_{\alpha^*_h}\right)_{h\in\mathcal{H}}, (\beta^*_h \odot v_h)\right) = 0,$$

$$\text{condition } p\left(\xi, e^*, \left(u_h + l_{\alpha^*_h}\right)_{h\in\mathcal{H}}, (\beta^*_h \odot v_h)\right) \text{ holds.}$$

Observe that, from Condition 1, for any $h \in \mathcal{H}$, $u_h + l_{\alpha^*_h} \in \breve{\mathcal{U}}$ and for any $h \in \mathcal{H}$ and $h' \in \mathcal{B}_h$, $\beta_{hh'} v_{hh'} \in \breve{\mathcal{V}}$.

Since from Condition 3.a and 3.b, $u_h + l_{\alpha_h} \in N_{u_h}$ and $\beta_{hh'} v_{hh'} \in N_{v_{hh'}}$, we did obtain the desired result.

Indeed, summarizing, we found (e^*,α^*,β^*) such that

$$e' := e^* \in B_{\mathbb{R}^{CH}}(e,\varepsilon),$$

$$u' := \left(u_h + l_{\alpha^*_h}\right)_{h\in\mathcal{H}} \in \times_{h\in\mathcal{H}} \left(N_{u_h} \cap \breve{\mathcal{U}}\right) \overset{(5.2)}{=} \times_{h\in\mathcal{H}} N^*_{u_h},$$

$$v' := (\beta^*_h \odot v_h)_{h\in\mathcal{H}} \in \times_{h\in\mathcal{H}} \left(\times_{h'\in\mathcal{B}_h} \left(N_{v_{hh'}} \cap \breve{\mathcal{V}}\right)\right) \overset{(5.2)}{=} \times_{h\in\mathcal{H}} \left(\times_{h'\in\mathcal{B}_h} N^*_{v_{hh'}}\right),$$

i.e.,

$$(e', u', v') \in N^*. \tag{5.3}$$

Moreover,

$$\text{for any } \xi \in \Xi \text{ such that } \widetilde{G}(\xi, e', u', v') = 0, \text{ condition } p(\xi, e', u', v') \text{ holds.} \tag{5.4}$$

(5.3) and (5.4) are exactly the desired result (5.1). ■

Summarizing, the Recipe is as follows.
0. Specialize the Notation presented above;
1. Define condition $p(\xi, e, u, v)$;
2. Check Conditions 1., 2., 3.a. and 3.b.

5.2 The $\alpha - \beta$ model

Following the ideas provided in Section 5.1, we now introduce the finite parameterization that will be used throughout the paper. For each $h \in \mathcal{H}$, we consider finite parameterizations of both the selfish and the altruistic components of household h's utility function, see Definition 30.

Lemma 38 *For any $h \in \mathcal{H}$, $h' \in \mathcal{B}_h$, $\beta_{hh'} \in \mathbb{R}_{++}$ and $v_{hh'} \in \mathcal{V}$, we have that $\beta_{hh'} v_{hh'} \in \mathcal{V}$.*

Proof. We have to check that $\beta_{hh'} v_{hh'}$ satisfies Assumptions v1-v4. Indeed,
v1. $\beta_{hh'} v_{hh'}$ is three times continuously differentiable; v2. $\beta_{hh'} v'_{hh'} > 0$ and $\beta_{hh'} v_{hh'}$ is differentiably strictly increasing; v3. $\beta_{hh'} v''_{hh'} < 0$ and $\beta_{hh'} v_{hh'}$ is differentiably strictly concave; v4. the right derivative of $\beta_{hh'} v_{hh'}$ is $\beta_{hh'} v'^{+}_{hh'}(0)$, which is bounded above. ■

Lemma 39 *For any $u_h \in \mathcal{U}$ and $\alpha_h \in \mathbb{R}^C_{++}$, we have $u_{\alpha_h} \in \mathcal{U}$.*

Proof. u1. u_{α_h} is obviously $C^3(\mathbb{R}^C_{++}, \mathbb{R})$.
u2. Simply observe that $D_{x_h^c} u_{\alpha_h}(x_h) = D_{x_h^c} u(x_h) + \frac{\alpha_h^c}{x_h^c} > 0$.
u3. $D^2 u_{\alpha_h} = D^2 u_h + \operatorname{diag}\left(\left(-\frac{\alpha_h^c}{(x_h^c)^2}\right)_{c \in \mathcal{C}}\right)$ is clearly negative definite.
u4. Define $l_{\alpha_h}(x_h) = \sum_{c \in \mathcal{C}} \alpha_h^c \log(x_h^c)$. From Lemma 72, we want to show that, for any $\underline{u} \in \operatorname{Im} u_{\alpha_h}$, $\operatorname{Cl}_{\mathbb{R}^C}\left((u_{\alpha_h})^{-1}([\underline{u}, +\infty))\right) \cap \partial \mathbb{R}^C_{++} = \varnothing$, where $\partial \mathbb{R}^C_{++}$ denotes the boundary of $\mathbb{R}^C_{++}$. If $(u_{\alpha_h})^{-1}([\underline{u}, +\infty)) = \varnothing$, then we are done. Otherwise, we have what follows.

First of all observe that

$$(u_{\alpha_h})^{-1}([\underline{u}, +\infty)) \subseteq u_h^{-1}\left(\left[\frac{\underline{u}}{2}, +\infty\right)\right) \cup l_{\alpha_h}^{-1}\left(\left[\frac{\underline{u}}{2}, +\infty\right)\right), \tag{5.5}$$

as verified below. Take $x_h \in (u_{\alpha_h})^{-1}([\underline{u}, +\infty))$, i.e., $u_h(x_h) + l_{\alpha_h}(x_h) \geq \underline{u}$, and suppose our desired claim does not hold, i.e., $u_h(x_h) < \frac{\underline{u}}{2}$ and $l_{\alpha_h}(x_h) < \frac{\underline{u}}{2}$; then, $u_h(x_h) + l_{\alpha_h}(x_h) < \frac{\underline{u}}{2} + \frac{\underline{u}}{2} = \underline{u}$, which is the desired contradiction. Then,

$$\operatorname{Cl}_{\mathbb{R}^C}\left((u_{\alpha_h})^{-1}([\underline{u}, +\infty))\right) \overset{(1)}{=} (u_{\alpha_h})^{-1}([\underline{u}, +\infty)) \overset{(5.5)}{\subseteq} u_h^{-1}\left(\left[\tfrac{\underline{u}}{2}, +\infty\right)\right) \cup l_{\alpha_h}^{-1}\left(\left[\tfrac{\underline{u}}{2}, +\infty\right)\right) \overset{(2)}{=}$$

$$\operatorname{Cl}_{\mathbb{R}^C}\left(u_h^{-1}\left(\left[\tfrac{\underline{u}}{2}, +\infty\right)\right)\right) \cup \operatorname{Cl}_{\mathbb{R}^C}\left(l_{\alpha_h}^{-1}\left(\left[\tfrac{\underline{u}}{2}, +\infty\right)\right)\right)$$

where (1) and (2) follow from the facts that u_h and l_{α_h} are continuous functions and that the closure of a closed set is the set itself.

Then,

$$\operatorname{Cl}_{\mathbb{R}^C}\left((u_{\alpha_h})^{-1}([\underline{u}, +\infty))\right) \cap \partial \mathbb{R}^C_{++} \subseteq$$

$$\left(\operatorname{Cl}_{\mathbb{R}^C}\left(u_h^{-1}\left(\left[\tfrac{\underline{u}}{2}, +\infty\right)\right)\right) \cup \operatorname{Cl}_{\mathbb{R}^C}\left(l_{\alpha_h}^{-1}\left(\left[\tfrac{\underline{u}}{2}, +\infty\right)\right)\right)\right) \cap \partial \mathbb{R}^C_{++} =$$

$$\left(\operatorname{Cl}_{\mathbb{R}^C}\left(u_h^{-1}\left(\left[\tfrac{\underline{u}}{2}, +\infty\right)\right)\right) \cap \partial \mathbb{R}^C_{++}\right) \cup \left(\operatorname{Cl}_{\mathbb{R}^C}\left(l_{\alpha_h}^{-1}\left(\left[\tfrac{\underline{u}}{2}, +\infty\right)\right)\right) \cap \partial \mathbb{R}^C_{++}\right)$$

By assumption, it is $\mathrm{Cl}_{\mathbb{R}^C}\left(u_h^{-1}\left(\left[\frac{u}{\overline{2}},+\infty\right)\right)\right)\cap\partial\mathbb{R}^C_{++}=\varnothing$. Let us prove that $\mathrm{Cl}_{\mathbb{R}^C}\left(l_{\alpha_h}^{-1}\left(\left[\frac{u}{\overline{2}},+\infty\right)\right)\right)\cap\partial\mathbb{R}^C_{++}=\varnothing$.

By contradiction, suppose there exists $x_h\in\mathrm{Cl}_{\mathbb{R}^C}\left(l_{\alpha_h}^{-1}\left(\left[\frac{u}{\overline{2}},+\infty\right)\right)\right)\cap\partial\mathbb{R}^C_{++}$. Since $x_h\in\partial\mathbb{R}^C_{++}$, then there exists $c\in\mathcal{C}$ such that $x_h^c=0$. Since $x_h\in\mathrm{Cl}_{\mathbb{R}^C}\left(l_{\alpha_h}^{-1}\left(\left[\frac{u}{\overline{2}},+\infty\right)\right)\right)$, then there exists a sequence $(x_h^{[k]})_{k\in\mathbb{N}}$ converging to x_h, such that $l_{\alpha_h}(x_h^{[k]})=\alpha_h^c\log(x_h^{[k]c})+\sum_{c'\neq c}\alpha_h^{c'}\log(x_h^{[k]c'})\geq\frac{u}{\overline{2}}$ for any $k\in\mathbb{N}$. Since $(x_h^{[k]c})_{k\in\mathbb{N}}$ converges to 0^+, then, since $\alpha_h^c>o$, it is easy to check that $\alpha_h^c\log(x_h^{[k]c})\to-\infty$ and therefore $\lim_{k\to+\infty}l_{\alpha_h}\left(x_h^{[k]}\right)=-\infty$. Hence, there exists $N\in\mathbb{N}$ such that $l_{\alpha_h}(x_h^{[k]})<\frac{u}{\overline{2}}$, for any $k\geq N$, contradicting the fact that $\left\{x_h^{[k]}:k\in\mathbb{N}\right\}\subseteq l_{\alpha_h}^{-1}([\frac{u}{\overline{2}},+\infty))$. This concludes the proof of Lemma 39. ∎

Definition 40 *An $\alpha-\beta$ economy is*

$$\mathcal{E}^1=\ (e,u,v,\alpha,\beta)\in\mathbb{R}^{CH}_{++}\times\mathcal{U}^H\times\mathcal{V}^B\ \times\mathbb{R}^{CH}_{++}\times\mathbb{R}^B_{++}=\mathbb{E}^1.$$

Definition 41 *The vector $\left(x^*,p^{*\setminus},((\tau^*_{hh'})_{h'\in\mathcal{B}_h})_{h\in\mathcal{H}}\right)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{C-1}_{++}\times\mathbb{R}^B$ is an $\alpha-\beta$ equilibrium for the economy $\mathcal{E}^1\in\mathbb{E}^1$ if*

1. for any $h\in\mathcal{H}$, household h maximizes, i.e.,

for given $\mathcal{E}^1\in\mathbb{E}^1$, $p^{*\setminus}\in\mathbb{R}^{C-1}_{++}$, $\left((\tau^*_{h'h''})_{h''\in\mathcal{B}_{h'}}\right)_{h'\in\mathcal{H}\setminus\{h\}}\in\left(\times_{h'\in\mathcal{H}\setminus\{h\}}\Delta_{B_{h'}}\left(p^{*\setminus}\right)\right)$, we have that $(x_h^*,(\tau^*_{hh'})_{h'\in\mathcal{B}_h})\in\mathbb{R}^C_{++}\times\mathbb{R}^{B_h}$ solves

$$\max_{\left(x_h,(\tau_{hh'})_{h'\in\mathcal{B}_h}\right)\in\mathbb{R}^C_{++}\times\mathbb{R}^{B_h}}$$

$$u_{\alpha_h}(x_h)+\sum_{h'\in\mathcal{B}_h}\beta_{hh'}v_{hh'}\left(\tfrac{1}{p^*r}\left(p^*e_{h'}+\tau_{hh'}+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau^*_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\tau^*_{h'h''}\right)\right)$$

subject to

$$p^*e_h+\sum_{h'\in\mathcal{B}_{\to h}}\tau^*_{h'h}-\sum_{h'\in\mathcal{B}_h}\tau_{hh'}-p^*x_h\geq0\qquad\lambda_h$$

$$pe_h-\sum_{h'\in\mathcal{B}_h}\tau_{hh'}\geq0\qquad\eta_h$$

$$\text{for any } h'\in\mathcal{B}_h,\quad\tau_{hh'}\geq0\qquad\gamma_{hh'}$$

2.

$$\sum_{h\in\mathcal{H}}\left(x_h^{*\setminus}-e_h^{\setminus}\right)=0.$$

We can introduced the so-called extended equilibrium function as follows.

$$\widehat{F}:\Xi\times\mathbb{R}^{CH}_{++}\times\mathcal{U}^H\times\mathcal{V}^B\ \times\mathbb{R}^{CH}_{++}\times\mathbb{R}^B_{++}\to\mathbb{R}^n,$$

$$(\xi,e,u,v,\alpha,\beta)\longmapsto\widetilde{F}(\xi,e,u_\alpha,\beta\odot v)=\text{ last column in (5.6) below}$$

C		$Du_{\alpha_h}(x_h)-\lambda_hp$	(1)
B_h	$h'\in\mathcal{B}_h$	$\frac{1}{pr}\beta_{hh'}v'_{hh'}\left(\frac{1}{pr}\left(pe_{h'}+\tau_{hh'}+\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\tau_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\tau_{h'h''}\right)\right)-\lambda_h+\gamma_{hh'}-\eta_h$	(2)
1		$pe_h+\sum_{h'\in\mathcal{B}_{\to h}}\tau_{h'h}-\sum_{h'\in\mathcal{B}_h}\tau_{hh'}-px_h$	(3)
B_h	$h'\in\mathcal{B}_h$	$\min\{\tau_{hh'},\gamma_{hh'}\}$	(4)
1		$\min\left\{pe_h-\sum_{h'\in\mathcal{B}_h}\tau_{hh'},\eta_h\right\}$	(5)
$C-1$		$\sum_{h\in\mathcal{H}}\left(x_h^{\setminus}-e_h^{\setminus}\right)$	(6)

(5.6)

For any $(u, v) \in \mathcal{U}^H \times \mathcal{V}^B$, we define the finite-dimensional parameter space

$$\Omega := \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CH}_{++} \times \mathbb{R}^{B}_{++}$$

with generic element $\omega = (e, \alpha, \beta) \in \Omega$. We then define the finite-dimensional extended equilibrium function F as the restriction of $\widehat{F}$ to a fixed (u, v):

$$F := \widehat{F}_{(u,v)} : \Xi \times \Omega \longrightarrow \mathbb{R}^n \qquad (\xi, \omega) \longmapsto \widehat{F}(\xi, u, v, \omega) = \widetilde{F}(\xi, e, u_\alpha, \beta \odot v). \tag{5.7}$$

Roughly speaking, the goal is to show that 0 is a regular value for F.

5.3 The analysis of borderline cases

Remark 42 *An obvious immediate problem in showing that for any $\xi \in \Xi$ such that $F(\xi, \omega) = 0$, $\det D_\xi F(\xi, \omega) \neq 0$ is that the* min *function used in the definition of F is not even differentiable when both the constraint and the multiplier are equal. Very simply, consider the following two examples.*

Example 1.

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \min\{-(x-1)+1, (x-1)+1\} = \min\{-x+2, x\}$ is not differentiable if $x = 1$.

Example 2.

$g : \mathbb{R} \to \mathbb{R}$, $g(x) = \min\left\{-x^2+1, \left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)\right\}$, or

$h : \mathbb{R} \to \mathbb{R}$, $h(x) = \min\left\{-x^2+1-\frac{3}{8}, \left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)-\frac{3}{8}\right\}$ are not differentiable in $-\frac{1}{4}\sqrt{10}, \frac{1}{4}\sqrt{10}$.

Based on Remark 42, if we take $\xi \in \Xi$ such that $F(\xi, \omega) = 0$, then, since the minimum of the value of a constraint and the associated multiplier have to be equal *zero*, then it may occur that *both of them* are *zero*, in which case differentiability is lost. This situation is commonly referred to as a borderline case. The strategy we adopt to address this issue follows the approach developed by Cass, Siconolfi, and Villanacci (2001). Roughly speaking, we show that borderline cases do not arise on an open and dense subset of the economy space. Once this result is established, we can return to a standard approach. We now formalize these ideas and prove the desired density result

First of all, we then present a general definition of regular value as follows.

Definition 43 *Given a function $g : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $x \longmapsto g(x)$ which is $\mathcal{C}^0$ on A and $\mathcal{C}^1$ in an open neighborhood of $g^{-1}(0)$, then 0 is a regular value for g if for any $x \in g^{-1}(0)$, we have dg_x is onto.*

The key step in handling the borderline cases is provided by a result whose proof, given in Appendix 7.4, adapts the proof strategy developed by Cass, Siconolfi and Villanacci (2001). We begin by introducing the necessary notation, then state the main result of those authors in Proposition 44, and finally apply it to establish the desired conclusion in Lemma 45.

Observe that, in the definitions of $\widetilde{F}$, $\widehat{F}$, and F, for every $h \in \mathcal{H}$, there are $B_h + 1$ component functions written in the form of min. For each agent h, we refer to these as the min-constraints associated with agent h (see, for instance, (4) and (5) in (5.6) under $\widehat{F}$). Notice also that, under $\widehat{F}$, component function (3) of (5.6) takes the form $\min\{pe_h + \sum_{h' \in \mathcal{B} \to h} \tau_{h'h} - \sum_{h' \in \mathcal{B}h} \tau_{hh'} - px_h, \lambda_h\}$. However, for any element belonging to $(\widehat{F})^{-1}(0)$, component function (1) of (5.6) implies that $\lambda_h > 0$, by Point (u2) of Assumption 2 together with the fact that $p \gg 0$. Therefore, $\min\{pe_h + \sum_{h' \in \mathcal{B} \to h} \tau_{h'h} - \sum_{h' \in \mathcal{B}h} \tau_{hh'} - px_h, \lambda_h\} = pe_h + \sum_{h' \in \mathcal{B} \to h} \tau_{h'h} - \sum_{h' \in \mathcal{B}h} \tau_{hh'} - px_h$. A similar argument applies to $\widetilde{F}$ and F.

It is is useful to rewrite the functions $\widehat{F}$ and F separating the $\mathcal{C}^2$ component functions from the min functions, as formally done below.

Define

$$\widetilde{\varphi} : \Xi \times \mathbb{R}^{CH}_{++} \times \mathcal{U}^H \times \mathcal{V}^B \to \mathbb{R}^{n-m},$$

$$(\xi,e,u,v)\longmapsto\left(\begin{array}{l}\left(\begin{array}{l}Du_h\left(x_h\right)-\lambda_h p\\ \\ \frac{1}{pr}v'_{hh'}\left(\frac{1}{pr}\left(pe_{h'}+\sum_{h''\in\mathcal{B}_{\to h'}}\tau_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\tau_{h'h''}\right)\right)-\lambda_h+\gamma_{hh'}-\eta_h\\ \\ pe_h+\sum_{h'\in\mathcal{B}_{\to h}}\tau_{h'h}-\sum_{h'\in\mathcal{B}_h}\tau_{hh'}-px_h\end{array}\right)_{h\in\mathcal{H}}\\ \sum_{h\in\mathcal{H}}\left(x_h^{\backslash}-e_h^{\backslash}\right);\end{array}\right)$$

$$m=\sum_{h\in\mathcal{H}}\left(B_h+1\right)=B+H;\qquad M=\{1,...,m\};$$

for any $h\in\mathcal{H}$,

$$g_h:\mathbb{R}_{++}^{CH}\times\mathbb{R}^B\times\mathbb{R}_{++}^{C-1}\times\mathbb{R}_{++}^{CH}\to\mathbb{R}^{B_h+1},\qquad\left(x,\tau,p^{\backslash},e\right)\longmapsto\left(\begin{array}{l}\left(\tau_{hh'}\right)_{h'\in\mathcal{B}_h}\\ \\ pe_h-\sum_{h'\in\mathcal{B}_h}\tau_{hh'}\end{array}\right);\tag{5.8}$$

$$\mu_h=\left(\begin{array}{l}\gamma_{hh'}\\ \\ \eta_h\end{array}\right)\in\mathbb{R}^{B_h+1},\tag{5.9}$$

and

$$g:\mathbb{R}_{++}^{CH}\times\mathbb{R}^B\times\mathbb{R}_{++}^{C-1}\times\mathbb{R}_{++}^{CH}\to\mathbb{R}^m,\quad\left(x,\tau,p^{\backslash},e\right)\longmapsto\left(g_h\left(x,\tau,p^{\backslash},e\right)\right)_{h\in\mathcal{H}}=\left(\begin{array}{l}\left(\tau_{hh'}\right)_{h'\in\mathcal{B}_h}\\ \\ pe_h-\sum_{h'\in\mathcal{B}_h}\tau_{hh'}\end{array}\right)_{h\in\mathcal{H}};$$

$$\mu=\left(\mu_h\right)_{h\in\mathcal{H}}\in\mathbb{R}^m.$$

Then, without loss of generality, up to reordering of the component functions, we have

$$\widetilde{F}:\Xi\times\mathbb{R}_{++}^{CH}\times\mathcal{U}^H\times\mathcal{V}^B\to\mathbb{R}^n,\qquad(\xi,e,u,v)\longmapsto\left(\begin{array}{l}\widetilde{\varphi}\left(\xi,e,u,v\right)\\ \\ \left(\min\left\{g_j\left(x,\tau,p^{\backslash},e\right),\mu_j\right\}\right)_{j\in M}\end{array}\right)$$

Similarly, we can define

$$\widehat{\varphi}:\Xi\times\mathbb{R}_{++}^{CH}\times\mathcal{U}^H\times\mathcal{V}^B\times\mathbb{R}_{++}^{CH}\times\mathbb{R}_{++}^{B}\to\mathbb{R}^{n-m},$$

$$(\xi,e,u,v,\alpha,\beta)\longmapsto$$

$$\left(\begin{array}{l}\left(\begin{array}{l}Du_{\alpha_h}\left(x_h\right)-\lambda_h p\\ \\ \frac{1}{pr}\beta_{hh'}v'_{hh'}\left(\frac{1}{pr}\left(pe_{h'}+\sum_{h''\in\mathcal{B}_{\to h'}}\tau_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\tau_{h'h''}\right)\right)-\lambda_h+\gamma_{hh'}-\eta_h\\ \\ pe_h+\sum_{h'\in\mathcal{B}_{\to h}}\tau_{h'h}-\sum_{h'\in\mathcal{B}_h}\tau_{hh'}-px_h\end{array}\right)_{h\in\mathcal{H}}\\ \sum_{h\in\mathcal{H}}\left(x_h^{\backslash}-e_h^{\backslash}\right).\end{array}\right)$$

Then, without loss of generality, up to reordering of the component functions, we have

$$\widehat{F}:\Xi\times\mathbb{R}_{++}^{CH}\times\mathcal{U}^H\times\mathcal{V}^B\times\mathbb{R}_{++}^{CH}\times\mathbb{R}_{++}^{B}\to\mathbb{R}^n,$$

$$(\xi,e,u,v,\alpha,\beta)\longmapsto\left(\begin{array}{l}\widehat{\varphi}\left(\xi,e,u,v,\alpha,\beta\right)\\ \\ \left(\min\left\{g_j\left(x,\tau,p^{\backslash},e\right),\mu_j\right\}\right)_{j\in M}\end{array}\right)$$

Define

$$P(M) =$$

$$\{ \left(M^{+0}, M^{0+}, \{j\}\right) \in \left(\mathcal{P}(M)\right)^3 : j \in M \text{ and}$$

$$\left\{M^{+0}, M^{0+}\right\} \text{ is a generalized partition of } M \backslash \{j\} \}$$

The $+0$ subscript means that multiplier is strictly positive and associated constraint is zero and similarly for the subscript $0+$. Observe that defined $m^{+0} = \#M^{+0}$ and $m^{0+} = \#M^{0+}$, we have $m^{0+}, m^{+0} \in \{0, 1...., m-1\}$ and $m^{0+} + m^{+0} = m - 1$. Then, for any $\left(M^{+0}, M^{0+}, \{j\}\right) \in P(M)$, define

$$\widehat{F}_{(M^{+0}, M^{0+}, \{j\})} : \Xi \times \mathbb{R}^{CH}_{++} \times \mathcal{U}^H \times \mathcal{V}^B \times \mathbb{R}^{CH}_{++} \times \mathbb{R}^B_{++} \to \mathbb{R}^{n-m} \times \mathbb{R}^{m^{0+}+m^{+0}+2=m+1} = \mathbb{R}^{n+1},$$

$$(\xi, e, u, v, \alpha, \beta) \mapsto \begin{pmatrix} \widehat{\varphi}(\xi, e, u, v, \alpha, \beta) \\ \left(g_k\left(x, \tau, p^{\backslash}, e\right)\right)_{k \in M^{+0}} \\ (\mu_l)_{l \in M^{0+}} \\ g_j\left(x, \tau, p^{\backslash}, e\right) \\ \mu_j \end{pmatrix}$$

For any $(u, v) \in \mathcal{U}^H \times \mathcal{V}^B$, define also

$$\widehat{F}_{(M^{+0}, M^{0+}, \{j\}), (u,v)} := F_{(M^{+0}, M^{0+}, \{j\})}. \tag{5.10}$$

We can now state the needed result, recalling the definition in (5.7):

$$F := \widehat{F}_{(u,v)} : \Xi \times \Omega \longrightarrow \mathbb{R}^n \qquad (\xi, \omega) \longmapsto \widehat{F}(\xi, u, v, \omega) = \widetilde{F}(\xi, e, u_\alpha, \beta \odot v).$$

Proposition 44 *If*
1. (existence) For any $\omega \in \Omega$, $(F_\omega)^{-1}(0) \neq \varnothing$.
2. For any $\left(M^{+0}, M^{0+}, \{j\}\right) \in P(M)$, 0 is a regular value for $F_{(M^{+0}, M^{0+}, \{j\})}$,
then

$$\Omega^* := \left\{\omega \in \Omega : \forall (x, \mu) \in (F_\omega)^{-1}(0), \forall j \in M, \text{ either } \mu_j > 0 \text{ or } g_j(\xi, \omega) > 0\right\}$$

has full measure in Ω.

Then we can apply Proposition 44, to get the following result, which says that " typically in Ω, borderline ω economies do not exist".

Lemma 45 *For any $(u, v) \in \mathcal{U}^H \times \mathcal{V}^B$,*

$$\Omega^*_{(u,v)} = \left\{ \begin{array}{l} \omega \in \Omega : \text{for any } \xi \in (F_\omega)^{-1}(0) := \left(\widehat{F}_{(u,v)|\omega}\right)^{-1}(0), \\ \\ \text{for any } h \in \mathcal{H},\ h' \in \mathcal{B}_h \\ \\ \text{either } (\ \gamma_{hh'} > 0 \text{ and } \tau_{hh'} = 0\ ) \text{ or } (\ \gamma_{hh'} = 0 \text{ and } \tau_{hh'} > 0\ ) \\ \\ \text{and} \\ \\ \text{either } (\ pe_h - \sum_{h' \in \mathcal{B}_h} \tau_{hh'} > 0 \text{ and } \eta_h = 0\ ) \text{ or } (\ pe_h - \sum_{h' \in \mathcal{B}_h} \tau_{hh'} = 0 \text{ and } \eta_h > 0\ ) \end{array} \right\}$$

has full measure in Ω.

Proof. To get the desired result, we have to check that the two assumptions in Proposition 44 are verified in the case we are analyzing:
1. Existence: see Theorem 9;
2. It is showed in Proposition 46 below. ∎

Proposition 46 *For any $\left(M^{+0}, M^{0+}, \{j\}\right) \in P(M)$, 0 is a regular value for $F_{(M^{+0}, M^{0+}, \{j\})}$.*

Proof. Consistently with Proposition 127 and Remark 128 in the Appendix 7.4, the strategy of the proof is as follows. All households but one are outside borderline cases. Define $\left\{\mathcal{B}_h^1, \mathcal{B}_h^2\right\}$ a generalized partition of $\mathcal{B}_h$. Conditions about the constraints and the multipliers for households who are not a borderline case are as follows:

$$\tau_{hh'} = 0 \qquad \text{for } h' \in \mathcal{B}_h^1,$$

$$\gamma_{hh'} = 0 \qquad \text{for } h' \in \mathcal{B}_h^2,$$

and

$$\begin{array}{l} \text{either } pe_h - \sum_{h' \in \mathcal{B}_h} \tau_{hh'} = 0 \\ \text{or } \eta_h = 0 \end{array}$$

Conditions about the maximization problems for the households described above are the first equations of the extended systems presented in the following pages. Moreover, the household h^* who is at a borderline case can be in one of the following three situations.

i. there exists $h' \in \mathcal{B}_{h^*}$ such that $\tau_{h^*h'} = \gamma_{h^*h'} = 0$ and $pe_h - \sum_{h' \in \mathcal{B}_{h^*}} \tau_{h^*h'} = 0$; or

ii. there exists $h' \in \mathcal{B}_{h^*}$ such that $\tau_{h^*h'} = \gamma_{h^*h'} = 0$ and $\eta_{h^*} = 0$; or

iii. $pe_h - \sum_{h' \in \mathcal{B}_{h^*}} \tau_{h^*h'} = 0$ and $\eta_{h^*} = 0$.

No other qualitatively different situations are possible. Below, we analyze those three cases. In other words, there exists three qualitatively different types of households, depending on how they can be at a borderline case. Without loss of generality, those households are denoted by $h = 1$, $h = 2$ and $h = 3$ in the matrices describing situations i., ii. and iii. in the following pages. We have to show that the associated Jacobian matrix has full row rank, a goal which is accomplished performing elementary super-rows operations in the order specified in the last column of the matrices we present below and using the matrix in a box in the corresponding column. First of all, we compute the Jacobian of an important version of the equilibrium function and then we accomplish what said above. We also use the Lemma 89 in the appendix.

We proceed as follows: 1. we rewrite the extended system; 2. we compute the Jacobian matrices associated with the two typical no-borderline case households; 3. we analyze case i., ii. and iii. described above.

Observe that an important sub-block of the Jacobian we want to compute is as follows.

		C	B_h	1	B_h	1		$B_{h'}$	$B_{h''}$		1	$C-1$	B_h
		x_h	τ_h	λ_h	γ_h	η_h	...	$\tau_{h'}$	$\tau_{h''}$	...	e_h^1	$e_h^{\setminus}$	β_h
C	$Du_h(x_h) - \lambda_h p$	D^2u_h		$-p$									
B_h	$\frac{1}{pr}\cdot\beta_{hh'}$ $v'_{hh'}$ $\frac{1}{pr}$ $(pe_{h'}+\tau_{hh'}+$ $\sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \tau_{h''h'}$ $-\sum_{h'' \in \mathcal{B}_{h'}} \tau_{h'h''})$ $-\lambda_h+\gamma_{hh'}-\eta_h$		dg $\frac{1}{(pr)^2}$ $\beta_{hh'}$ $v''_{hh'}$	$-\mathbf{1}$	I	$-\mathbf{1}$		$*$	$*$		$*$	$*$	dg $\frac{1}{pr}$ $\beta_{hh'}$ $v'_{hh'}$
1	$pe_h-\sum_{h' \in \mathcal{B}_h} \tau_{hh'}+$ $\sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h}-px_h$	$-p$	$-\mathbf{1}$					$*$	$*$		1	$p^{\setminus}$	
B_h^1	$\tau_{hh'}$		$I_{B_h^1} \mid 0$										
B_h^2	$\gamma_{hh'}$				$0 \mid I_{B_h^2}$								
1	η_h					1							
or	$pe_h-\sum_{h' \in \mathcal{B}_h} \tau_{hh'}$		$-\mathbf{1}$								1	$p^{\setminus}$	

Let's consider the more difficult case in which $\eta_h > 0$ and perform the elementary **row operation** "subtract the last row from the row of the budget constraint".

		C	B_h	1	B_h	1		$\tau_{h'}$	$\tau_{h''}$		1	$C-1$	B_h
		x_h	τ_h	λ_h	γ_h	η_h	...			...	e_h^1	$e_h^{\setminus}$	β_h
C	$Du_h(x_h) - \lambda_h p$	$D^2 u_{\alpha_h}$		$-p$									
B_h	$\frac{1}{pr} \cdot \beta_{hh'}$ $v'_{hh'}$ $\frac{1}{pr}$ $(pe_{h'} + \tau_{hh'} + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \tau_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \tau_{h'h''})$ $-\lambda_h + \gamma_{hh'} - \eta_h$		dg $\frac{1}{(pr)^2}$ $\beta_{hh'}$ $v''_{hh'}$	$\mathbf{-1}$	I	$\mathbf{-1}$		$*$	$*$		$*$	$*$	dg $\frac{1}{pr}$ $\beta_{hh'}$ $v'_{hh'}$
1	$pe_h - \sum_{h' \in \mathcal{B}_h} \tau_{hh'} + \sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h} - px_h$	$-p$						$*$	$*$				
B_h^1	$\tau_{hh'}$		$I_{B_h^1} \mid 0$										
B_h^2	$\gamma_{hh'}$				$0 \mid I_{B_h^2}$								
1	η_h					1							
or	$pe_h - \sum_{h' \in \mathcal{B}_h} \tau_{hh'}$		$\mathbf{-1}$								1	$p^{\setminus}$	

Observe that there are only zeros on the left of the two involved super-rows. Another, equivalent, possibility would be to consider the equivalent system in which substituting in the upper bound on transfer constraint in the budget equation constraint. In other word, you observe that the following two systems are equivalent - with obvious notation.

$$\left\{ \begin{array}{lcl} \varphi(x,\pi) & = & 0 \\ ax + bx & = & 0 \\ ax & = & 0 \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{lcl} \varphi(x,\pi) & = & 0 \\ bx & = & 0 \\ ax & = & 0 \end{array} \right.$$

Case i.

	x_1	τ_1	λ_1	$\gamma_{1h'}$	η_1	x_2	τ_2	λ_2	$\gamma_{2h'}$	η_2	x_3	τ_3	λ_3	$\gamma_{3h'}$	η_3	β_1	β_2	β_3	$e_1^{\setminus}$	e_1^1	e_2^1	e_3^1	α_1	α_2	α_3	
Du_{α_1} $-\lambda_1 p$	D^2		$-p$																				A_1			0
$\frac{1}{pr}\beta_{1h'}$ $v_{1h'}$ $-\lambda_1+$ $\gamma_{1h'}$ $-\eta_1$		$*$	$\mathbf{-1}$	I	$\mathbf{-1}$		$*$					$*$				dg				$*$	$*$	$*$				0
pe_1+ $\Sigma\tau_{h'1}$ $-\Sigma\tau_{1h'}-$ px_1	$-p$						$*$					$*$														3
$\tau_{1h'}$		$I\ 0$																								4
$\gamma_{1h''}$				$0\ I$																						4
pe_1- $\Sigma\tau_{1h'}$		$\mathbf{-1}\vert\mathbf{-1}$																	$p^{\setminus}$	1						1
Du_{α_2} $-\lambda_2 p$						D^2		$-p$																A_2		0
$\frac{1}{pr}\beta_{2h'}$ $\cdot v_{2h'}$ $-\lambda_2+$ $\gamma_{2h'}$ $-\eta_2$		$*$					$*$	$\mathbf{-1}$	I	$\mathbf{-1}$		$*$					dg		$*$	$*$	$*$	$*$				0
pe_2+ $\Sigma\tau_{h'2}$ $-\Sigma\tau_{2h'}$ $-px_2$		$*$				$-p$	$\mathbf{-1}$					$*$									1					1
$\tau_{2h'}$							$I\ 0$																			4
$\gamma_{2h''}$									$0\ I$																	4
η_2										1																4
Du_{α_3} $-\lambda_3 p$											D^2		$-p$												A_3	0
$\frac{pe_3}{pr}$ $\beta_{3h'}$ $v_{3h'}$ $-\gamma_{3h'}$ $-\eta_3$		$*$					$*$					$*$	$\mathbf{-1}$	I	$\mathbf{-1}$			dg	$*$	$*$	$*$	$*$				0
pe_3+ $\Sigma\tau_{h'3}$ $-\Sigma\tau_{3h'}$ $-px_3$		$*$					$*$				$-p$															3
$\tau_{3h'}$												$I\ 0$														4
$\gamma_{3h''}$														$0\ I$												4
pe_3 $-\Sigma\tau_{3h'}$												$\mathbf{-1}$										1				1
γ_{31}														10												4
$\Sigma x_h^{\setminus}-$ $e_h^{\setminus}$	$0I$					$0I$					$0I$								$0\ I$							2

In detail you have what follows. Do 0:

	x_1	τ_1	λ_1	$\gamma_{1h'}$	η_1	x_2	τ_2	λ_2	$\gamma_{2h'}$	η_2	x_3	τ_3	λ_3	$\gamma_{3h'}$	η_3	$e_1^\setminus$	e_1^1	e_2^1	e_3^1	
$pe_1+\Sigma\tau_{h'1}$ $-\Sigma\tau'_{1h'}-px_1$	$\boxed{-p}$						$*$					$*$								
$\tau_{1h'}$		$\boxed{I\ 0}$																		
$\gamma_{1h''}$				$\boxed{0\ I}$																
$pe_1-\Sigma\tau_{1h'}$		$-\mathbf{1}\vert-\mathbf{1}$														$p^\setminus$	$\boxed{1}$			1
$pe_2+\Sigma\tau_{h'2}$ $-\Sigma\tau'_{2h'}-px_2$		$*$				$-p$	$-\mathbf{1}$					$*$						$\boxed{1}$		1
$\tau_{2h'}$							$\boxed{I\ 0}$													
$\gamma_{2h''}$									$\boxed{0\ I}$											
η_2										$\boxed{1}$										
$pe_3+\Sigma\tau_{h'3}$ $-\Sigma\tau'_{3h'}-px_3$		$*$					$*$				$\boxed{-p}$									
$\tau_{3h'}$												$\boxed{I\ 0}$								
$\gamma_{3h''}$														$\boxed{0\ I}$						
$pe_3-\Sigma\tau_{3h'}$												$-\mathbf{1}$							$\boxed{1}$	1
γ_{31}														$\boxed{10}$						
$\Sigma x_h^\setminus - e_h^\setminus$	$0I$					$0I$					$0I$					$\boxed{0\ I}$				

Do 1:

	x_1	τ_1	λ_1	$\gamma_{1h'}$	η_1	x_2	τ_2	λ_2	$\gamma_{2h'}$	η_2	x_3	τ_3	λ_3	$\gamma_{3h'}$	η_3	$e_1^{\setminus}$	
$pe_1+\Sigma\tau_{h'1}$ $-\Sigma\tau'_{1h'}-px_1$	$-p$						$*$					$*$					
$\tau_{1h'}$		$I\ 0$															
$\gamma_{1h''}$				$0\ I$													
$\tau_{2h'}$							$I\ 0$										
$\gamma_{2h''}$									$0\ I$								
η_2										1							
$pe_3+\Sigma\tau_{h'3}$ $-\Sigma\tau'_{3h'}-px_3$		$*$					$*$				$-p$						
$\tau_{3h'}$												$I\ 0$					
$\gamma_{3h''}$														$0\ I$			
γ_{31}														10			
$\Sigma x_h^{\setminus}-e_h^{\setminus}$	$0I$					$0I$					$0I$					$0\ I$	2

Do 2 and erase columns of zero:

	x_1	τ_1	$\gamma_{1h'}$	τ_2	$\gamma_{2h'}$	η_2	x_3	τ_3	$\gamma_{3h'}$	
$pe_1+\Sigma\tau_{h'1}$ $-\Sigma\tau'_{1h'}-px_1$	$-p$			$*$				$*$		3
$\tau_{1h'}$		$I\ 0$								
$\gamma_{1h''}$			$0\ I$							
$\tau_{2h'}$				$I\ 0$						
$\gamma_{2h''}$					$0\ I$					
η_2						1				
$pe_3+\Sigma\tau_{h'3}$ $-\Sigma\tau'_{3h'}-px_3$		$*$		$*$			$-p$			3
$\tau_{3h'}$								$I\ 0$		
$\gamma_{3h''}$									$0\ I$	
γ_{31}									10	

Do 3

	τ_1	$\gamma_{1h'}$	τ_2	$\gamma_{2h'}$	η_2	τ_3	$\gamma_{3h'}$	
$\tau_{1h'}$	$I\ 0$							
$\gamma_{1h''}$		$0\ I$						
$\tau_{2h'}$			$I\ 0$					
$\gamma_{2h''}$				$0\ I$				
η_2					1			
$\tau_{3h'}$						$I\ 0$		
$\gamma_{3h''}$							$0\ I$	
γ_{31}							10	

and then we are done.

Case iii.

	x_1	τ_1	λ_1	$\gamma_{1h'}$	η_1	x_2	τ_2	λ_2	$\gamma_{2h'}$	η_2	x_3	τ_3	λ_3	$\gamma_{3h'}$	η_3	β_1	β_2	β_3	$e_1^{\setminus}$	e_1^1	e_2^1	e_3^1	α_1	α_2	α_3	
Du_1 $-\lambda_1 p$	D^2		$-p$																				$\boxed{A_1}$			0
$\frac{1}{pr}\beta_{1h'}$ $v_{1h'}$ $-\lambda_1+$ $\gamma_{1h'}-$ η_1		$*$	$\mathbf{-1}$	I	$\mathbf{-1}$		$*$					$*$				$\boxed{\text{dg}}$				$*$	$*$	$*$				0
pe_1+ $\Sigma\tau_{h'1}$ $-\Sigma\tau'_{1h'}-$ px_1	$\boxed{-p}$						$*$					$*$														3
$\tau_{1h'}$		$\boxed{I\ 0}$																								4
$\gamma_{1h''}$				$\boxed{0\ I}$																						4
pe_1- $\Sigma\tau_{1h'}$		$\mathbf{-1}\vert\mathbf{-1}$																	$p^{\setminus}$	$\boxed{1}$						1
Du_{α_2} $-\lambda_2 p$						D^2		$-p$																$\boxed{A_2}$		0
$\frac{1}{pr}\beta_{2h'}$ $v_{2h'}$ $-\lambda_2+$ $\gamma_{2h'}$ $-\eta_2$		$*$					$*$	$\mathbf{-1}$	I	$\mathbf{-1}$		$*$					$\boxed{\text{dg}}$		$*$	$*$	$*$	$*$				0
pe_2+ $\Sigma\tau_{h'2}$ $-\Sigma\tau_{2h'}-$ px_2		$*$				$-p$	$\mathbf{-1}$					$*$									$\boxed{1}$					1
$\tau_{2h'}$							$\boxed{I\ 0}$																			4
$\gamma_{2h''}$									$\boxed{0\ I}$																	4
η_2										$\boxed{1}$																4
Du_{α_3} $-\lambda_3 p$											D^2		$-p$												$\boxed{\alpha_3}$	0
$\frac{pe_3}{pr}$ $\beta_{3h'}$ $v_{3h'}$ $-\gamma_{3h'}-$ η_3		$*$					$*$					$*$	$\mathbf{-1}$	I	$\mathbf{-1}$			$\boxed{\text{dg}}$	$*$	$*$	$*$	$*$				0
pe_3+ $\Sigma\tau_{h'3}$ $-\Sigma\tau_{3h'}$ $-px_3$		$*$					$*$				$\boxed{-p}$															3
$\tau_{3h'}$												$\boxed{I\ 0}$															4
$\gamma_{3h''}$														$\boxed{0\ I}$													4
pe_3 $-\Sigma\tau_{3h'}$												$\mathbf{-1}$										$\boxed{1}$				1
η_3															$\boxed{1}$												4
$\Sigma x_h^{\setminus}$ $-e_h^{\setminus}$	$0I$					$0I$					$0I$								$\boxed{0\ I}$								2

Case ii.

	x_1	τ_1	λ_1	$\gamma_{1h'}$	η_1	x_2	τ_2	λ_2	$\gamma_{2h'}$	η_2	x_3	τ_3	λ_3	$\gamma_{3h'}$	η_3	β_1	β_2	β_3	$e_1^{\setminus}$	e_1^1	e_2^1	e_3^1	α_1	α_2	α_3	
Du_{α_1} $-\lambda_1 p$	D^2		$-p$																				A_1			0
$\frac{1}{pr}\beta_{1h'}$ $v_{1h'}$ $-\lambda_1+$ $\gamma_{1h'}-$ η_1		$*$	$\mathbf{-1}$	I	$\mathbf{-1}$		$*$					$*$				dg				$*$	$*$	$*$				0
pe_1+ $\Sigma\tau_{h'1}$ $-\Sigma\tau_{1h'}-$ px_1	$-p$						$*$					$*$														3
$\tau_{1h'}$		$I\ 0$																								4
$\gamma_{1h''}$				$0\ I$																						4
pe_1- $\Sigma\tau_{1h'}$		$\mathbf{-1}\vert\mathbf{-1}$																	$p^{\setminus}$	1						1
Du_{α_2} $-\lambda_2 p$						D^2		$-p$																A_2		0
$\frac{1}{pr}\beta$ $\cdot v_{2h'}$ $-\lambda_2+$ $\gamma_{2h'}$ $-\eta_2$		$*$					$*$	$\mathbf{-1}$	I	$\mathbf{-1}$		$*$					dg		$*$	$*$	$*$	$*$				0
pe_2 $+\Sigma\tau_{h'2}$ $-\Sigma\tau_{2h'}-$ px_2		$*$				$-p$	$\mathbf{-1}$					$*$									1					1
$\tau_{2h'}$							$I\ 0$																			4
$\gamma_{2h''}$									$0\ I$																	4
η_2										1																4
Du_{α_3} $-\lambda_3 p$											D^2		$-p$												A_3	0
$\frac{pe_3}{pr}$ $\beta_{3h'}$ $v_{3h'}$ $-\gamma_{3h'}$ $-\eta_3$		$*$					$*$					$*$	$\mathbf{-1}$	I	$\mathbf{-1}$			dg	$*$	$*$	$*$	$*$				0
pe_3+ $\Sigma\tau_{h'3}$ $-\Sigma\tau'_{hh'}$ $-px'_h$		$*$					$*$				$-p$															3
$\tau_{3h'}$												$I\ 0$														4
$\gamma_{3h''}$														$0\ I$												4
η_3												$\mathbf{-1}$														4
γ_{31}															1											4
$\Sigma x_h^{\setminus}$ $-e_h^{\setminus}$	$0I$					$0I$					$0I$								$0\ I$							2

■

Definition 47 *For any $(\xi, \mathcal{E}) \in \Xi \times \mathbb{E}$ such that $\widetilde{F}(\xi, \mathcal{E}) = 0$, we say that $(\xi, \mathcal{E})$ is not at a borderline case if "the conditions in Definition of $\Omega^*_{(u,v)}$ are satisfied", i.e.,*

for any $h \in \mathcal{H}$, $h' \in \mathcal{B}_h$, either ($\gamma_{hh'} > 0$ and $\tau_{hh'} = 0$) or ($\gamma_{hh'} = 0$ and $\tau_{hh'} > 0$) and either ($pe_h - \sum_{h' \in \mathcal{B}_h} \tau_{hh'} > 0$ and $\eta_h = 0$) or ($pe_h - \sum_{h' \in \mathcal{B}_h} \tau_{hh'} = 0$ and $\eta_h > 0$).

Lemma 48 *The set*

$$\mathcal{D}_1 = \left\{ \mathcal{E} \in \mathbb{E} : \text{for any } \xi \in \left(\widetilde{F}_{\mathcal{E}}\right)^{-1}(0),\ (\xi, \mathcal{E}) \text{ is not at a borderline case} \right\}$$

is dense in $\mathbb{E}$.

Proof. We want to apply the Recipe presented in Section 5.1.
Condition $p(\xi, e, u, v)$ is "$(\xi, \mathcal{E})$ is not at a borderline case".
Condition 1 is verified in Lemma 38 and Lemma 39.
Condition 2 is verified in Lemma 45.
Condition 3.a. is verified in Proposition 122.
Condition 3.b. is verified in Proposition 111 and in Lemma 114, Propositions 119 and 120 contained in the Section 7.3.3 which is taken from the Chapter on "Compact-open topology" in Villanacci (2026). ■

5.4 Auxiliary P Economies

In this section, exploiting the fact that equilibria associated with economies in $\mathcal{D}_1$ are not borderline cases, we introduce the so-called partition functions $\widetilde{F}_P$ (see Definition 51). These functions are obtained by replacing the components of $\widetilde{F}$ involving minimum operator with one of the two functions appearing in the corresponding minimum. By construction, the partition functions are smooth, allowing us to apply Glöckner's theorem (see Lemma 68). Moreover, for an appropriate choice of P, the functions $\widetilde{F}$ and $\widetilde{F}_P$ coincide locally (see Lemma 52), and the zero set of $\widetilde{F}$ is a strict subset of the union of the zero sets of all $\widetilde{F}_P$ (see Lemma 53). Exploiting these properties, we show that the set of so-called regular economies is open and dense in the space of all economies (see Corollary 66). The above results are then used to establish regularity in the full sense; see Theorem 16 and its proof in the next section.

To continue our analysis, we need to introduce some definitions.

Definition 49 *For any $h \in \mathcal{H}$, $\mathcal{J}_h = \{1, ..., B_h + 1\}$, where $B_h + 1$ is the cardinality of the set of min-constraints of household h 's maximization problem.*

Definition 50 *For any $(\xi, \mathcal{E}) \in \Xi \times \mathbb{E}$ such that $F(\xi, \mathcal{E}) = 0$, define the following generalized partition of $\mathcal{J}_h$:*

$$\begin{aligned}
\mathcal{J}_h^1(\xi, \mathcal{E}) &= \{j \in \mathcal{J}_h : g_{hj}\left(x, \tau, p^{\setminus}, e\right) > 0, \mu_{hj} = 0\}, \\
\mathcal{J}_h^2(\xi, \mathcal{E}) &= \{j \in \mathcal{J}_h : g_{hj}\left(x, \tau, p^{\setminus}, e\right) = 0, \mu_{hj} > 0\}, \\
\mathcal{J}_h^3(\xi, \mathcal{E}) &= \{j \in \mathcal{J}_h : g_{hj}\left(x, \tau, p^{\setminus}, e\right) = 0, \mu_{hj} = 0\}.
\end{aligned} \tag{5.11}$$

Instead of restating each constraint explicitly, we prefer to provide a more general description of the constraints. See, for instance, the definitions of g_h and μ_h given in (5.8) and (5.9).

Following the approach of Cass, Siconolfi, and Villanacci (2001), and motivated by our focus on the economies in the set $\mathcal{D}_1$, we exclude by definition the exceptional case in which both a multiplier and its associated constraint are equal to zero. Accordingly, we exogenously fix a generalized bipartition corresponding to the remaining, generic cases.

Definition 51 *For any $h \in \mathcal{H}$, let $\mathcal{P}_h$ be the family of generalized bipartitions $P_h = \left\{P_h^1, P_h^2\right\}$ of $\mathcal{J}_h$. Defined $\mathcal{P} = \times_{h\in\mathcal{H}}\mathcal{P}_h$ with generic element $P := (P_h)_{h\in\mathcal{H}}$. For any $P \in \mathcal{P}$, define*

$$\widetilde{F}_P : \Xi \times \mathbb{E} \to \mathbb{R}^n,$$

$$(\xi, \mathcal{E}) \longmapsto \left(\begin{array}{l} \widetilde{\varphi}(\xi, \mathcal{E}) \\ \left(\begin{array}{l} (\mu_{hj})_{j\in P_h^1} \\ \left(g_{hj}\left(x, \tau, p^{\setminus}, e\right)\right)_{j\in P_h^2} \end{array} \right)_{h\in\mathcal{H}} \end{array} \right)$$

We now present two results linking the zeros of $\widetilde{F}$ and $\widetilde{F}_P$.

Lemma 52 *Assume that $(\xi, \mathcal{E}) \in \Xi \times \mathbb{E}$ is such that $\widetilde{F}(\xi, \mathcal{E}) = 0$. If there exist $P', P'' \in \mathcal{P}$ such that $P' \neq P''$ and $\widetilde{F}_{P'}(\xi, \mathcal{E}) = \widetilde{F}_{P''}(\xi, \mathcal{E}) = 0$, then $\mathcal{E} \notin \mathcal{D}_1$.*

Proof. If $P' \neq P''$, then there exists $h \in \mathcal{H}$ such that $P_h' := \left(P_h'^1, P_h'^2\right) \neq \left(P_h''^1, P_h''^2\right) := P_h''$. Therefore, being P_h' and P_h'' generalized partitions of $M_h := \{1, ..., B_h + 1\}$, we have that $P_h'^1 \neq P_h''^1$ and therefore, without loss of generality[1], there exists $j \in P_h'^1$ such that $j \notin P_h''^1$ and therefore $j \in P_h''^2$. Since $\widetilde{F}_{P'}(\xi, \mathcal{E}) = 0$ and $j \in P_h'^1$, then $\mu_{hj} = 0$; since $\widetilde{F}_{P''}(\xi, \mathcal{E}) = 0$ and $j \in P_h''^2$, then $g_{hj}\left(x, \tau, p^{\setminus}, e\right) = 0$.

Summarizing,

there exist $h \in \mathcal{H}$ and $j \in M_h$ such that $\mu_{hj} = 0$ and $g_{hj}\left(x, \tau, p^{\setminus}, e\right) = 0$
and
$\widetilde{F}(\xi, \mathcal{E}) = 0$.

Therefore, from Definition 47, we do have that $\mathcal{E} \notin \mathcal{D}_1$. ∎

Lemma 53 $\widetilde{F}^{-1}(0) \subseteq \cup_{P\in\mathcal{P}}\left(\widetilde{F}_P\right)^{-1}(0)$.

Proof. We want to show that if $\widetilde{F}(\xi, \mathcal{E}) = 0$, then there exists $P \in \mathcal{P}$ such that $\widetilde{F}_P(\xi, \mathcal{E}) = 0$. Indeed,

$$0 = \widetilde{F}(\xi, \mathcal{E}) \overset{\text{Def. } \widetilde{F}}{=} \left(\begin{array}{l} \widetilde{\varphi}(\xi, \mathcal{E}) \\ \\ \left(\min\left\{g_j\left(x, \tau, p^{\setminus}, e\right), \mu_j\right\}\right)_{j\in M} \end{array} \right).$$

Then, by definition of min, for any $h \in \mathcal{H}$, there exists a generalized partition[2] $\left\{M_h^{+0}, M_h^{0+}, M_h^{00}\right\}$ of M_h such that

$$\widetilde{F}(\xi, \mathcal{E}) = \left(\begin{array}{l} \widetilde{\varphi}(\xi, \mathcal{E}) \\ \\ \left(\begin{array}{l} \left(g_{j_{h_1}}\left(x, \tau, p^{\setminus}, e\right)\right)_{j_{h_1}\in M_h^{+0}} \\ \\ \left(\mu_{j_{h_2}}\left(x, \tau, p^{\setminus}, e\right)\right)_{j_{h_2}\in M_h^{0+}} \\ \\ \left(\begin{array}{l} \left(g_{j_{h_3}}\left(x, \tau, p^{\setminus}, e\right)\right) \\ \left(\mu_{j_{h_3}}\left(x, \tau, p^{\setminus}, e\right)\right) \end{array} \right)_{j_{h_3}\in M_h^{00}} \end{array} \right)_{h\in\mathcal{H}} \end{array} \right).$$

Let $\mathcal{P}\left(M_h^{00}\right)$ be the family of generalized bipartition $\left\{P_h^{001}, P_h^{002}\right\}$ of M_h^{00}. Then, for any $h \in \mathcal{H}$, define the family of sets

$$P_h^1 = \left\{M_h^{0+} \cup P_h^{001}\right\}$$

$$P_h^2 = \left\{M_h^{+0} \cup P_h^{002}\right\}$$

Then, for any $P = \left(P_h^1, P_h^2\right)_{h\in\mathcal{H}}$ constructed as described above, we do have $\widetilde{F}(\xi, \mathcal{E}) = \widetilde{F}_P(\xi, \mathcal{E}) = 0$. ∎

[1]Similar argument holds if there exists $j \in P_h''^1$ such that $j \notin P_h'^1$; indeed, then $j \in P_h'^2$.
[2]This partition depends upon $(\xi, \mathcal{E})$.

Remark 54 *The inclusion established in the previous result is generally strict, since the zero set of F_P may contain points at which either the constraint or the corresponding multiplier is negative, a possibility that is excluded by the definition of F.*

Remark 55 *A somehow interesting, difficult problem is what follows. Given $P \in \mathcal{P}$, and consider the associated projection $pr_P : (F_P)^{-1}(0) \to \Omega$, $(\xi, \omega) \to \omega$. It is hard to show that pr_P is proper, and we strongly conjecture that pr_P is not proper. Then, the question is as follows. What are the properties on equilibria, regular economies, smooth dependence which survive to that loss? We loose openness. Therefore the set of regular economies is full measure but not open. That is the case of $\mathbb{R}\backslash\mathbb{Q}$.*

Definition 56

$$\widehat{F}_P : \Xi \times \mathbb{R}^{CH}_{++} \times \mathcal{U}^H \times \mathcal{V}^B \times \mathbb{R}^{CH}_{++} \times \mathbb{R}^B_{++} \to \mathbb{R}^n, \qquad \widehat{F}_P(\xi, e, u, v, \alpha, \beta) = \widetilde{F}_P(\xi, e, u_\alpha, \beta \odot v)$$

For any $(u, v) \in \mathcal{U}^H \times \mathcal{V}^B$, $F_P = \widehat{F}_{P,(u,v)}$;

$$F_P : \Xi \times \Omega \to \mathbb{R}^n, \qquad (\xi, e, \alpha, \beta) \longmapsto F_P(\xi, e, \alpha, \beta)$$

Remark 57 *For any $P \in \mathcal{P}$, F_p is $\mathcal{C}^1$.*

Remark 58 *As a consequence of Lemmas 38 and 39, the results stated in Lemmas 52 and 53 also apply to F_P.*

Recall that from (5.7), $F := \widehat{F}_{(u,v)} : \Xi \times \Omega \to \mathbb{R}^n \qquad (\xi, \omega) \longmapsto \widehat{F}(\xi, u, v, \omega) = \widetilde{F}(\xi, e, u_\alpha, \beta \odot v)$.

Definition 59 *For any $(u, v) \in \mathcal{U}^H \times \mathcal{V}^B$,*

$$\Omega^{**}_{(u,v)} =$$

$$\left\{ \ \omega \in \Omega : \text{for any } \xi \in (F_\omega)^{-1}(0), \text{ for any } P \in \mathcal{P}, \ F_P(\xi, \omega) = 0 \Rightarrow \ \det D_\xi F_P(\xi, \omega) \neq 0 \ \right\}$$

Remark 60

$$\Omega^{**}_{(u,v)} =$$

$$\left\{ \ \omega \in \Omega : \text{for any } P \in \mathcal{P}, \text{ for any } \xi \in (F_\omega)^{-1}(0), \ \text{ for any } \xi \in (F_{P\omega})^{-1}(0), \det \ D_\xi F_P(\xi, \omega) \neq 0 \ \right\} =$$

$$\cap_{P \in \mathcal{P}} \left\{ \omega \in \Omega : \ \text{for any } \xi \in (F_\omega)^{-1}(0) \cap (F_{P\omega})^{-1}(0), \det D_\xi F_P(\xi, \omega) \neq 0 \right\}.$$

Lemma 61 *For any $(u, v) \in \mathcal{U}^H \times \mathcal{V}^B$, the set $\Omega^{**}_{(u,v)}$ is a full measure subset of Ω.*

Proof. First of all, we show that for any $P \in \mathcal{P}$, 0 is a regular value for F_P. Indeed, the proof of the desired result is an easier version of the proofs presented in Proposition 46. Using the same technique as the one used there, with the only exception that here we have two types of households, it is sufficient to consider the following table, where without loss of generality, we consider only two households denoted by $h = 1$ and $h = 2$. Observe that, by definition of F_P, no borderline case exists.

	x_1	τ_1	λ_1	$\gamma_{1h'}$	η_1	x_2	τ_2	λ_2	$\gamma_{2h'}$	η_2	τ_h	β_1	β_2	$e_1^{\setminus}$	e_1^1	e_2^1	α_1	α_2	
$Du_{\alpha_1}(x_1)$ $-\lambda_1 p$	D^2		$-p$														$\boxed{A_1}$		0
$\frac{1}{p^r}\beta_{1h'}\cdot v_{1h'}$ $-\lambda_1+\gamma_{1h'}-\eta_1$		$*$	$\mathbf{-1}$	I	$\mathbf{-1}$		$*$				$*$	$\boxed{\text{dg}}$			$*$	$*$			0
$pe_1+\Sigma\tau_{h'1}$ $-\Sigma\tau_{1h'}-px_1$	$\boxed{-p}$						$*$				$*$								3
$\tau_{1h'}$		$\boxed{I\ 0}$																	4
$\gamma_{1h''}$				$\boxed{0\ I}$															4
$pe_1-\Sigma\tau_{1h'}$		$\mathbf{-1}\vert\mathbf{-1}$												$p^{\setminus}$	$\boxed{1}$				1
$Du_{\alpha_2}(x_2)$ $-\lambda_2 p$						D^2		$-p$										$\boxed{A_2}$	0
$\frac{1}{p^r}\beta_{2h'}\cdot v_{2h'}$ $-\lambda_2+\gamma_{2h'}-\eta_2$		$*$					$*$	$_{\mathbf{-1}}$	I	$\mathbf{-1}$	$*$		$\boxed{\text{dg}}$	$*$	$*$	$*$			0
$pe_2+\Sigma\tau_{h'2}$ $-\Sigma\tau'_{2h'}-px'_2$		$*$				$-p$	$\mathbf{-1}$				$*$					$\boxed{1}$			1
$\tau_{2h'}$							$\boxed{I\ 0}$												4
$\gamma_{2h''}$									$\boxed{0\ I}$										4
η_2										$\boxed{1}$									4
$\Sigma x_h^{\setminus}-e_h^{\setminus}$	$0I$					$0I$								$\boxed{0\ I}$					2

Then, we have verified that for any $P \in \mathcal{P}$, 0 is a regular value for F_P, and from Theorem 123, we have that there exists a full measure subset Ω_P^* of Ω such that for any $\omega \in \Omega_P^*$, 0 is a regular value for $F_{P\omega}$.

Define

$$\widetilde{\Omega} = \cap_{P\in\mathcal{P}}\Omega_P^* = \cap_{P\in\mathcal{P}}\left\{\omega \in \Omega : \text{ for any } \xi \in (F_{P\omega})^{-1}(0), \quad \det D_\xi F_P(\xi,\omega) \neq 0\right\}.$$

Since

$$\Omega_{(u,v)}^{**} := \cap_{P\in\mathcal{P}}\left\{\omega \in \Omega : \text{ for any } \xi \in (F_\omega)^{-1}(0) \cap (F_{P\omega})^{-1}(0), \quad \det D_\xi F_P(\xi,\omega) \neq 0\right\},$$

then $\widetilde{\Omega} \subseteq \Omega_{(u,v)}^{**}$[3] and, using the fact that the intersection of a finite number of full measure sets is itself full measure, we are done. ■

Lemma 62 *The set*

$$\mathcal{D}_2 =$$

$$\{\, \mathcal{E} \in \mathbb{E} : \textit{for any } \xi \in \left(\widetilde{F}_{\mathcal{E}}\right)^{-1}(0),$$

$$\textit{for any } P \in \mathcal{P},\ \widetilde{F}_P(\xi,\ \mathcal{E}) = 0 \Rightarrow\ \det D_\xi \widetilde{F}_P(\xi,\mathcal{E}) \neq 0 \,\}$$

is a dense subset of $\mathbb{E}$.

Proof. Again we apply the Recipe to get density presented in Section 5.1.

Condition $p(\xi,\mathcal{E})$ is as follows: for any $P \in \mathcal{P}$, $\widetilde{F}_P(\xi,\mathcal{E}) = 0 \Rightarrow\ \det D_\xi \widetilde{F}_P(\xi,\mathcal{E}) \neq 0$.

Observe that Condition 1, 3a and 3b are the same as in the proof of Lemma 48. For completeness, we repeat what said in that Lemma about those conditions.

Condition 1 is verified in Lemma 38 and Lemma 39.

Condition 2 is verified in Lemma 61.

Condition 3.a. is verified in Proposition 122.

Condition 3.b. is verified in Proposition 111 and in Lemma 114, Propositions 119 and 120 contained in the Section 7.3.3 which is taken from the Chapter on "Compact-open topology" in Villanacci (2026). ■

[3]To better understand the inclusion $\widetilde{\Omega} \subseteq \Omega_{(u,v)}^{**}$, simplifying notation, let two sets $A(\omega)$ and $B(\omega)$ depending on $\omega \in \Omega$ and a generic condition $p(\xi,\omega)$, with $\xi \in \Xi$, be given. Define

$$S^{**} = \{\omega \in \Omega : \text{for any } \xi \in A(\omega) \cap B(\omega), \quad p(\xi,\omega)\}$$

and

$$\widetilde{S} = \{\omega \in \Omega : \text{for any } \xi \in B(\omega), \quad p(\xi,\omega)\}.$$

Then, we have $\widetilde{S} \subseteq S^{**}$.

Chapter 6

Generic Regularity

The present Section is devoted to prove the main result of the paper which is Theorem 16.

Definition 63

$$\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2.$$

Proposition 64 *$\mathcal{D}$ is $\mathbb{E}$ open and dense.*

Proof. Openness. Observe that

$$\mathcal{D}_1 = \left\{ \mathcal{E} \in \mathbb{E} : \text{for any } \xi \in \left(\widetilde{F}_{\mathcal{E}}\right)^{-1}(0),\ (\xi, \mathcal{E}) \text{ is not at a borderline case} \right\} =$$

$$= \left\{ \begin{array}{l} \mathcal{E} \in \mathbb{E} : \text{for any } \xi \in \left(\widetilde{F}_{\mathcal{E}}\right)^{-1}(0), \\ \text{for any } h \in \mathcal{H}, h' \in \mathcal{B}_h \\ \text{either } (\ \gamma_{hh'} > 0 \text{ and } \tau_{hh'} = 0\ ) \text{ or } (\ \gamma_{hh'} = 0 \text{ and } \tau_{hh'} > 0\ ) \text{ and} \\ \text{either } (\ pe_h - \sum_{h' \in \mathcal{B}_h} \tau_{hh'} > 0 \text{ and } \ \eta_h = 0\ ) \text{ or } (pe_h - \sum_{h' \in \mathcal{B}_h} \tau_{hh'} = 0 \text{ and } \ \eta_h > 0\ ) \end{array} \right\}$$

Then $\mathcal{D}_1$ is open from properness of the natural projection, i.e., Proposition 27, and Proposition 133. Observe also that

$$\mathcal{D}_2 =$$

$$\{\ \mathcal{E} \in \mathbb{E} :\ \text{for any } (\xi, \mathcal{E}) \in \widetilde{F}^{-1}(0) \cap \left(\cap_{P \in \mathcal{P}} \left(\widetilde{F}_P\right)^{-1}(0) \right),$$

$$\det D_\xi \widetilde{F}_P(\xi, \mathcal{E}) \neq 0\ \}$$

Then the desired result follows again from the properness of the projection, Proposition 132[1], together with the continuity of the determinant and the continuity of $D_\xi \widetilde{F}_P$ on $\widetilde{F}^{-1}(0) \cap (\cap_{P \in \mathcal{P}} (\widetilde{F}_P)^{-1}(0))$.

density.

First Proof.

From Lemma 45 and Lemma 61, for any $(u, v) \in \mathcal{U}^H \times \mathcal{V}^B$, $\Omega^*_{(u,v)} \cap \Omega^{**}_{(u,v)}$ has full measure in Ω. Then, following the same strategy used in Proposition 37, $\mathcal{D}$ is dense in $\mathbb{E}$.

Second Proof.

From the above argument, both $\mathcal{D}_1$ and $\mathcal{D}_2$ are open. From Lemmas 48 and 62, $\mathcal{D}_1$ and $\mathcal{D}_2$ are dense in $\mathbb{E}$. Then, the desired result follows from the fact that the intersection of a finite number of open and dense sets is (open and) dense - see Proposition 83. ■

[1]Note that in Proposition 132 is showed that the set $\mathcal{O}$ is $(\Xi \times \mathbb{E})$-open, where $\mathcal{O}$ is defined only via a (generic) constraint of the form $g((\xi, \mathcal{E})) > 0$. In our specific case of the $\mathcal{D}_2$, the condition is given by $\det D_\xi \widetilde{F}_P(\xi, \mathcal{E}) \neq 0$. We remark that Proposition 132 implies anyway the openness of $\mathcal{D}_2$. Indeed, we can write

$$\mathcal{D}_2 = \mathcal{D}_2^+ \cup \mathcal{D}_2^-,$$

where $\mathcal{D}_2^+$ (resp. $\mathcal{D}_2^-$) is given by the condition $\det D_\xi \widetilde{F}_P(\xi, \mathcal{E}) > 0$ (respc. $-\det D_\xi \widetilde{F}_P(\xi, \mathcal{E}) > 0$).

Lemma 65 *For any $(\xi,\mathcal{E})\in\Xi\times\mathcal{D}$ such that $\widetilde{F}(\xi,\mathcal{E})=0$, there exists a generalized partition $P(\xi,\mathcal{E})\in\mathcal{P}$ and an open neighborhood $N(\xi,\mathcal{E})$ of ξ such that for any $\xi'\in N(\xi,\mathcal{E})$, $\widetilde{F}(\xi',\mathcal{E})=\widetilde{F}_{P(\xi,\mathcal{E})}(\xi',\mathcal{E})$, and $\widetilde{F}_{\mathcal{E}}$ is $\mathcal{C}^1$ on $N(\xi,\mathcal{E})$, $D_\xi\widetilde{F}(\xi,\mathcal{E})=D_\xi\widetilde{F}_{P(\xi,\mathcal{E})}(\xi,\mathcal{E})$ and $\det\ D_\xi\widetilde{F}(\xi,\mathcal{E})=\det D_\xi\widetilde{F}_{P(\xi,\mathcal{E})}(\xi,\mathcal{E})\neq 0$.*

Proof. Since, by assumption, $\widetilde{F}(\xi,\mathcal{E})=0$ and $\mathcal{E}:=(e,u,v)\in\mathcal{D}$, then, from the Definition of $\mathcal{D}_1$, for any $h\in\mathcal{H}$, there is a partition $\left\{\mathcal{J}_h^1(\xi,\mathcal{E}),\mathcal{J}_h^2(\xi,\mathcal{E})\right\}$ of $\mathcal{J}_h$ such that

$$\widetilde{F}(\xi,\mathcal{E})=$$

$$\left(\begin{array}{l}\widetilde{\varphi}(\xi,\mathcal{E})\\ \left(\begin{array}{l}\left(\min\left\{g_{hj}\left(x,\tau,p^{\backslash},e\right),\mu_{hj}\right\}\right)_{j\in\mathcal{J}_h^1(\xi,\mathcal{E})}\\ \left(\min\left\{g_{hj}\left(x,\tau,p^{\backslash},e\right)\right\}\right)_{j\in\mathcal{J}_h^2(\xi,\mathcal{E})}\end{array}\right)_{h\in\mathcal{H}}\end{array}\right)=$$

$$=\left(\begin{array}{l}\widetilde{\varphi}(\xi,e,u,v)\\ \left(\begin{array}{l}\left(\mu_{hj}\right)_{j\in\mathcal{J}_h^1(\xi,\mathcal{E})}\\ \left(g_{hj}\left(x,\tau,p^{\backslash},e\right)\right)_{j\in\mathcal{J}_h^2(\xi,\mathcal{E})}\end{array}\right)_{h\in\mathcal{H}}\end{array}\right),$$

and, still by definition of $\mathcal{D}_1$, we have that for any $h\in\mathcal{H}$,

$$\begin{array}{ll}\left(\mu_{hj}\right)_{j\in\mathcal{J}_h^2(\xi,\mathcal{E})} & >>0\\ \left(g_{hj}\left(x,\tau,p^{\backslash},e\right)\right)_{j\in\mathcal{J}_h^1(\xi,\mathcal{E})} & >>0.\end{array}\tag{6.1}$$

Therefore, choosing $P(\xi,\mathcal{E})=\left(\mathcal{J}_h^1(\xi,e),\mathcal{J}_h^2(\xi,e)\right)_{h\in\mathcal{H}}$, we have that

$$\widetilde{F}(\xi,\mathcal{E})=\widetilde{F}_{P(\xi,\mathcal{E})}(\xi,\mathcal{E}).$$

Then, (since the identity function is continuous) and since g_{hj} are continuous functions, from Proposition 91,

$$\begin{array}{l}\text{there exists an open neighborhood } N(\xi) \text{ of } \xi \text{ such that}\\ \text{for any } \xi'\in N(\xi), \text{ the strict inequalities in (6.1) are satisfied.}\end{array}\tag{6.2}$$

Therefore for any $\xi'\in N(\xi)$,

$$\widetilde{F}(\xi',\mathcal{E}):=\left(\begin{array}{l}\widetilde{\varphi}(\xi',\mathcal{E})\\ \\ \left(\left(\min\left\{g_{hj}\left(x,\tau,p^{\backslash},e\right),\mu_{hj}\right\}\right)_{j\in M_h}\right)_{h\in\mathcal{H}}\end{array}\right)\overset{(6.2)}{=}$$

$$=\left(\begin{array}{l}\widetilde{\varphi}(\xi,\mathcal{E})\\ \left(\begin{array}{l}\left(\mu'_{hj}\right)_{j\in\mathcal{J}_h^1(\xi,\mathcal{E})}\\ \left(g_{hj}\left(x',\tau',p'^{\backslash},e\right)\right)_{j\in\mathcal{J}_h^2(\xi,\mathcal{E})}\end{array}\right)_{h\in\mathcal{H}}\end{array}\right):=\widetilde{F}_{P(\xi,\mathcal{E})}(\xi',\mathcal{E})$$

and then all the other desired results follow. ∎

Corollary 66 *For any $\mathcal{E}\in\mathcal{D}$, where $\mathcal{D}$ is an open and dense subset of $\mathbb{E}$, 0 is a regular value for $\widetilde{F}_{\mathcal{E}}$ (and therefore, for any $\xi\in\left(\widetilde{F}_{\mathcal{E}}\right)^{-1}(0)$, we have $\det D_\xi\widetilde{F}(\xi,\mathcal{E})\neq 0$).*

Proof. From Proposition 64, $\mathcal{D}$ is open and dense in $\mathbb{E}$. Then, "just taking a subset of the statement" of Lemma 65, we have what follows.

For any $(\xi,\mathcal{E})\in\Xi\times\mathcal{D}$ such that $\widetilde{F}(\xi,\mathcal{E})=0$, there exists an open neighborhood $N(\xi,\mathcal{E})$ of ξ such that $\widetilde{F}_{\mathcal{E}}$ is $\mathcal{C}^1$ on $N(\xi,\mathcal{E})$ and $\det\ D_\xi\widetilde{F}(\xi,\mathcal{E})\neq 0$.

Therefore, $\widetilde{F}$ is $\mathcal{C}^0$, and it is $\mathcal{C}^1$ in a neighborhood N of $\left(\widetilde{F}_{\mathcal{E}}\right)^{-1}(0)$ (indeed $N=\cup_{\xi\in\left(\widetilde{F}_{\mathcal{E}^0}\right)^{-1}(0)}N_P(\xi)$, which is an open set) and for any $\xi\in\left(\widetilde{F}_{\mathcal{E}}\right)^{-1}(0)$, we do have $\det D_\xi\widetilde{F}(\xi,\mathcal{E})\neq 0$. Using Definition 43, we can conclude that 0 is a regular value for $\widetilde{F}_{\mathcal{E}}$. ∎

Definition 67 $T := \mathbb{R}^{CH}_{++} \times \left(\times_{h \in \mathcal{H}} \left(\mathcal{C}^3 \left(\mathbb{R}^C_{++}, \mathbb{R} \right) \times \left(\mathcal{C}^3 \left((0,1), \mathbb{R} \right) \right)^{B_h} \right) \right)$ *and for any* $P \in \mathcal{P}$,

$$\mathcal{F}_P : \Xi \times T \to \mathbb{R}^n,$$

$$(\xi, e, u, v) \longmapsto \textit{Component functions of } \widetilde{F}_P$$

Lemma 68 *For any* $P \in \mathcal{P}$, $\mathcal{F}_P \in \mathcal{C}^1_M$.

Proof. Following what was said in Definition 134 in Appendix 7.6, we need to show that
1. $\mathcal{F}_P \in \mathcal{C}^0 \left(\Xi \times T, \mathbb{R}^n \right)$.
2. The function

$$d_M \mathcal{F}_P \colon (\Xi \times T) \times \left(\mathbb{R}^n \times \left(\mathbb{R}^{CH} \times \left(\times_{h \in \mathcal{H}} \left(\mathcal{C}^3 \left(\mathbb{R}^C_{++}, \mathbb{R} \right) \times \left(\mathcal{C}^3 \left((0,1), \mathbb{R} \right) \right)^{B_h} \right) \right) \right) \right) \to \mathbb{R}^n,$$

$$\left((\xi, e, u, v), (\xi^*, e^*, u^*, v^*) \right) \longmapsto \lim_{t \to 0} \frac{\mathcal{F}_P((\xi,e,u,v) + t(\xi^*, e^*, u^*, v^*)) - \mathcal{F}_P((\xi,e,u,v))}{t}$$

is well defined, i.e., the limit always exists and is component-wise finite.

3. $d_M \mathcal{F}_P$ is continuous.

Below, we present the proofs of the above mentioned results.

1.

It follows from the same argument used to show properness.

2.

Take $t \neq 0$, $(\xi, e, u, v) \in (\Xi \times T)$, and $(\xi^*, e^*, u^*, v^*) \in \mathbb{R}^n \times \left(\mathbb{R}^{CH} \times \left(\times_{h \in \mathcal{H}} \left(\mathcal{C}^3 \left(\mathbb{R}^C_{++}, \mathbb{R} \right) \times \left(\mathcal{C}^3 \left((0,1), \mathbb{R} \right) \right)^{B_h} \right) \right) \right)$, we have that $\mathcal{F}_P \left((\xi, e, u, v) + t \left(\xi^*, e^*, u^*, v^* \right) \right) - \mathcal{F} \left((\xi, e, u, v) \right)$ is equal to the list of component functions below.

$$D \left(u_h + t u^*_h \right) \left(x_h + t x^*_h \right) - \left(\lambda_h + t \lambda^*_h \right) \left(p + t p^* \right) - D u_h \left(x_h \right) + \lambda_h p \tag{1}$$

$$\begin{array}{l} \frac{1}{(p+tp^*)(r+tr^*)} \left(v'_{hh'} + t \left(v^*_{hh'} \right)' \right) \\ \left(\frac{\left((p+tp^*) \left(e_{h'} + t e^*_{h'} \right) \right) + \sum_{h'' \in \mathcal{B}_{\to h'}} \left(\tau_{h''h'} + t \tau^*_{h''h'} \right) - \sum_{h'' \in \mathcal{B}_{h'}} \left(\tau_{h'h''} + t \tau^*_{h'h''} \right)}{(p+tp^*)(r+tr^*)} \right) + \\ - \left(\lambda_h + t \lambda^*_h \right) + \gamma_{hh'} + t \gamma^*_{hh'} - \eta_h - t \eta^*_h + \\ - \left(\frac{1}{pr} v'_{hh'} \left(\frac{p e_{h'} + \sum_{h'' \in \mathcal{B}_{\to h'}} \tau_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \tau_{h'h''}}{pr} \right) - \lambda_h + \gamma_{hh'} - \eta_h \right) \end{array} \tag{2}$$

$$\begin{array}{l} - \left(p + t p^* \right) \left(\left(x_h + t x^*_h \right) - \left(e_h + t e^*_h \right) \right) - p \left(x_h - e_h \right) + \\ + \sum_{h' \in \mathcal{B}_{\to h}} \left(\tau_{h'h} + t \tau^*_{h'h} \right) - \sum_{h' \in \mathcal{B}_h} \left(\tau_{hh'} + t \tau^*_{hh'} \right) + \\ - \sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h} + \sum_{h' \in \mathcal{B}_h} \tau'_{hh'} \end{array} \tag{3}$$

$$\tau_{hh'} + t \tau^*_{hh'} - \tau_{hh'} \quad \text{or} \quad \gamma_{hh'} + t \gamma^*_{hh'} - \gamma_{hh'} \tag{4}$$

$$\begin{array}{l} \left(p + t p^* \right) \left(e_h + t e^*_h \right) - \sum_{h' \in \mathcal{B}_h} \left(\tau_{hh'} + t \tau^*_{hh'} \right) - p e_h + \sum_{h' \in \mathcal{B}_h} \tau_{hh'} \\ \text{or} \quad \eta_h + t \gamma^*_h - \eta_h \end{array} \tag{5}$$

$$- \sum_{h \in \mathcal{H}} \left(x^{\setminus}_h + t x^{*\setminus}_h \right) + \sum_{h \in \mathcal{H}} \left(e^{\setminus}_h + t e^{*\setminus}_h \right) - \sum_{h \in \mathcal{H}} x^{\setminus}_h + \sum_{h \in \mathcal{H}} e^{\setminus}_h \tag{6}$$

Below, we proceed as follows. For each of the six component functions defined above, we first simplify the corresponding quotient $\frac{\mathcal{F}_P((\xi,e,u,v) + t(\xi^*, e^*, u^*, v^*)) - \mathcal{F}((\xi,e,u,v))}{t}$ and then compute the limit of the resulting expression as $t \to 0$.

(1)

$$\frac{D \left(u_h + t u^*_h \right) \left(x_h + t x^*_h \right) - \left(\lambda_h + t \lambda^*_h \right) \left(p + t p^* \right) - D u_h \left(x_h \right) + \lambda_h p}{t}$$

$$\frac{D u_h \left(x_h + t x^*_h \right) - D u_h \left(x_h \right) + t D u^*_h \left(x_h + t x^*_h \right) - \lambda_h p - \lambda_h t p^* - t \lambda^*_h p - t^2 \lambda^*_h p^* + \lambda_h p}{t}$$

$$\frac{D u_h \left(x_h + t x^*_h \right) - D u_h \left(x_h \right)}{t} + D u^*_h \left(x_h + t x^*_h \right) - \lambda_h p^* - \lambda^*_h p - t \lambda^*_h p^*.$$

Taking limits of the above expression and using the fact that u_h is $\mathcal{C}^3$, we get

$$\lim_{t\to 0}\frac{Du_h\left(x_h+tx_h^*\right)-Du_h\left(x_h\right)}{t}+Du_h^*\left(x_h\right)+\lambda_h p^*-\lambda_h^* p=D^2u_h\left(x_h\right)\cdot x_h^*+Du_h^*\left(x_h\right)+\lambda_h p^*-\lambda_h^* p.$$

In more detail, observe that the last equality above follows from what was said below. Dropping h, we want to verify that $\lim_{t\to 0}\frac{Du(x+tx^*)-Du(x)}{t}=D^2u\left(x\right)\cdot x^*$. Observe that, given an arbitrary $\mathcal{C}^3$ function $g:\mathbb{R}^C_{++}\to\mathbb{R}$, then, from the definition and properties of the directional derivative, we have $\lim_{t\to 0}\frac{g(x+tx^*)-g(x)}{t}=Dg\left(x\right)\cdot x^*$. Then we are left with showing that $D\left(Du\left(x\right)\right)=D^2u\left(x\right)$. Indeed,

		x^1	...	x^C
$D_{x^1}u\left(x\right)$		$D_{x^1x^1}u\left(x\right)$		$D_{x^1x^C}u\left(x\right)$
...				
$D_{x^C}u\left(x\right)$		$D_{x^Cx^1}u\left(x\right)$		$D_{x^Cx^C}u\left(x\right)$

(2)

Here we use the fact that elementary operations on continuous functions give rise to continuous functions as well and the computations below. To simplify notation, we use the following definition. For any $h\in\mathcal{H}$ and $h'\in\mathcal{B}_h$,

$$\theta_{h'}:\Xi\times\mathbb{R}^{CH}_{++}\to\mathbb{R}$$

$$(\xi,e)\longmapsto\frac{pe_{h'}+\sum_{h''\in\mathcal{B}_{\to h'}}\tau_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\tau_{h'h''}}{pr}.$$

Then,

$$\theta\left(\left(\xi,e\right)+t\left(\xi^*,e^*\right)\right):=\frac{\left(\left(p+tp^*\right)\left(e_{h'}+te^*_{h'}\right)\right)+\sum_{h''\in\mathcal{B}_{\to h'}}\left(\tau_{h''h'}+t\tau^*_{h''h'}\right)\cdot\left(\left(\left(p+tp^*\right)\left(e_{h''}+te^*_{h''}\right)\right)\right)-\sum_{h''\in\mathcal{B}_{h'}}\left(\tau_{h'h''}+t\tau^*_{h'h''}\right)\cdot\left(\left(p+tp^*\right)\left(e_{h'}+te^*_{h'}\right)\right)}{\left(p+tp^*\right)\left(r+tr^*\right)},$$

observe that θ is a $\mathcal{C}^2$ function because it is the ratio of an affine function with a denominator strictly positive.

Below we consider different terms in (2) one by one.

Observe that

$$\begin{aligned}&\frac{1}{\left(p+tp^*\right)\left(r+tr^*\right)}=\frac{1}{pr+tpr^*+tp^*r+t^2p^*r^*}=\frac{1}{pr}+\left(\frac{1}{pr+tpr^*+tp^*r+t^2p^*r^*}-\frac{1}{pr}\right)=\\&\frac{1}{pr}\left(1-\frac{tpr^*+tp^*r+t^2p^*r^*}{pr+tpr^*+tp^*r+t^2p^*r^*}\right)\end{aligned}\tag{6.3}$$

Then,

$$\frac{1}{\left(p+tp^*\right)\left(r+tr^*\right)}\left(v'_{hh'}+t\left(v^*_{hh'}\right)'\right)\left(\theta\left(\left(\xi,e\right)+t\left(\xi^*,e^*\right)\right)\right)-\frac{1}{pr}v'_{hh'}\left(\theta\left(\xi,e\right)\right)\overset{(6.3)}{=}$$

$$\frac{1}{pr}\left(1-\frac{tpr^*+tp^*r+t^2p^*r^*}{pr+tpr^*+tp^*r+t^2p^*r^*}\right)\left(v'_{hh'}+t\left(v^*_{hh'}\right)'\right)\left(\theta\left(\left(\xi,e\right)+t\left(\xi^*,e^*\right)\right)\right)-\frac{1}{pr}v'_{hh'}\left(\theta\left(\xi,e\right)\right)=$$

$$\frac{1}{pr}\left(v'_{hh'}\left(\theta\left(\left(\xi,e\right)+t\left(\xi^*,e^*\right)\right)\right)-v'_{hh'}\left(\theta\left(\xi,e\right)\right)\right)-\frac{1}{pr}\frac{tpr^*+tp^*r+t^2p^*r^*}{pr+tpr^*+tp^*r+t^2p^*r^*}t\cdot\left(v^*_{hh'}\right)'\left(\theta\left(\xi,e\right)\right)$$

Then dividing by t and taking the limit for $t\to 0$, we get

$$\frac{1}{pr}\lim_{t\to 0}\frac{v'_{hh'}\left(\theta\left(\left(\xi,e\right)+t\left(\xi^*,e^*\right)\right)\right)-v'_{hh'}\left(\theta\left(\xi,e\right)\right)}{t}-\frac{1}{pr}\lim_{t\to 0}\frac{tpr^*+tp^*r+t^2p^*r^*}{pr+tpr^*+tp^*r+t^2p^*r^*}\cdot\left(v^*_{hh'}\right)'\left(\theta\left(\xi,e\right)\right)\overset{(1)}{=}$$

$$=\frac{1}{pr}D\left(v'_{hh'}\circ\theta\right)\left(\xi,e\right)\cdot\left(\xi^*,e^*\right)-\frac{1}{pr}\frac{0}{pr}\cdot\left(v^*_{hh'}\right)'\left(\theta\left(\xi,e\right)\right)=\frac{1}{pr}D\left(v'_{hh'}\circ\theta\right)\left(\xi,e\right)\cdot\left(\xi^*,e^*\right)$$

where (1) follows from the fact that

$$\lim_{t\to 0}\frac{v'_{hh'}\left(\theta\left(\left(\xi,e\right)+t\left(\xi^*,e^*\right)\right)\right)-v'_{hh'}\left(\theta\left(\left(\xi,e\right)\right)\right)}{t}=$$

$$\lim_{t\to 0}\frac{\left(v'_{hh'}\circ\ \theta\right)\left(\left(\xi,e\right)+t\left(\xi^*,e^*\right)\right)-\left(v'_{hh'}\circ\ \theta\right)\left(\xi,e\right)}{t}=D\left(v'_{hh'}\circ\theta\right)\left(\xi,e\right)\cdot\left(\xi^*,e^*\right),$$

and the last equality follows from the fact that compositions of $\mathcal{C}^1$ functions are $\mathcal{C}^1$ and from a well-known property of the directional derivative.

Let us now study the other terms.

$$-\lambda_h - t\lambda_h^* + \gamma_{hh'} + t\gamma_{hh'}^* - \eta_h - t\eta_h^* + \lambda_h - \gamma_{hh'} + \eta_h =$$

$$-t\lambda_h^* + t\gamma_{hh'}^* - t\eta_h^*$$

Summarizing, the limit of the term in (2) as $t \to 0$ is

$$\frac{1}{pr} D\left(v'_{hh'} \circ \theta\right)(\xi, e) \cdot (\xi^*, e^*) - \lambda_h^* + \gamma_{hh'}^* - \eta_h^*.$$

(3)

$$\begin{array}{l} -\left(p + tp^*\right)\left(\left(x_h + tx_h^*\right) - \left(e_h + te_h^*\right)\right) - p\left(x_h - e_h\right) + \\ \sum_{h' \in \mathcal{B}_{\to h}} \left(\tau_{h'h} + t\tau_{h'h}^*\right) - \sum_{h' \in \mathcal{B}_h} \left(\tau_{hh'} + t\tau_{hh'}^*\right) - \sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h} + \sum_{h' \in \mathcal{B}_h} \tau_{hh'} = \end{array}$$

$$\begin{array}{l} -\left(px_h + tp^*x_h + tpx_h^* + t^2p^*x_h^* - pe_h - tp^*e_h - tpe_h^* - t^2p^*e_h^* - px_h + pe_h\right) + \\ \sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h} + t\sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h}^* - \sum_{h' \in \mathcal{B}_h} \tau_{hh'} + t\sum_{h' \in \mathcal{B}_h} \tau_{hh'}^* - \sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h} + \sum_{h' \in \mathcal{B}_h} \tau_{hh'} = \end{array}$$

$$-\left(tp^*x_h + tpx_h^* + t^2p^*x_h^* - tp^*e_h - tpe_h^* - t^2p^*e_h^*\right) + t\sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h}^* + t\sum_{h' \in \mathcal{B}_h} \tau_{hh'}^*$$

Then,

$$\lim_{t \to 0} \frac{1}{t}\left(-\left(tp^*x_h + tpx_h^* + t^2p^*x_h^* - tp^*e_h - tpe_h - t^2p^*e_h\right) + t\sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h}^* - t\sum_{h' \in \mathcal{B}_h} \tau_{hh'}^*\right) =$$

$$\lim_{t \to 0}\left(-\left(p^*x_h + px_h^* + p^*tx_h^* - p^*e_h - pe_h - tp^*e_h\right) + \sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h}^* - \sum_{h' \in \mathcal{B}_h} \tau_{hh'}^*\right) =$$

$$-\left(p^*x_h + px_h^* - p^*e_h - pe_h\right) + \sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h}^* - \sum_{h' \in \mathcal{B}_h} \tau_{hh'}^*$$

(4)

$$\lim_{t \to 0} \frac{\tau_{hh'} + t\tau_{hh'}^* - \tau_{hh'}}{t} = \tau_{hh'}^*.$$

(5)

$$\left(p + tp^*\right)\left(e_h + te_h^*\right) - \sum_{h' \in \mathcal{B}_h}\left(\tau_{hh'} + t\tau_{hh'}^*\right) - pe_h + \sum_{h' \in \mathcal{B}_h} \tau_{hh'} =$$

$$pe_h + tp^*e_h + tpe_h^* + t^2p^*e_h^* - \sum_{h' \in \mathcal{B}_h} \tau_{hh'} - t\sum_{h' \in \mathcal{B}_h} \tau_{hh'}^* - pe_h + \sum_{h' \in \mathcal{B}_h} \tau_{hh'} =$$

$$tp^*e_h + tpe_h^* + t^2p^*e_h^* - t\sum_{h' \in \mathcal{B}_h} \tau_{hh'}^*$$

and

$$\lim_{t \to 0} \frac{1}{t}\left(tp^*e_h + tpe_h^* + t^2p^*e_h^* - t\sum_{h' \in \mathcal{B}_h} \tau_{hh'}^*\right) =$$

$$\lim_{t \to 0}\left(p^*e_h + pe_h^* + tp^*e_h^* - \sum_{h' \in \mathcal{B}_h} \tau_{hh'}^*\right) =$$

$$p^*e_h + pe_h^* - \sum_{h' \in \mathcal{B}_h} \tau_{hh'}^*$$

(6)

$$\sum_{h \in \mathcal{H}}\left(x_h^{\setminus} + tx_h^{*\setminus}\right) - \sum_{h \in \mathcal{H}}\left(e_h^{\setminus} + te_h^{*\setminus}\right) - \sum_{h \in \mathcal{H}} x_h^{\setminus} + \sum_{h \in \mathcal{H}} e_h^{\setminus} =$$

$$\sum_{h \in \mathcal{H}} x_h^{\setminus} + t\sum_{h \in \mathcal{H}} x_h^{*\setminus} - \sum_{h \in \mathcal{H}} e_h^{\setminus} + t\sum_{h \in \mathcal{H}} e_h^{*\setminus} - \sum_{h \in \mathcal{H}} x_h^{\setminus} + \sum_{h \in \mathcal{H}} e_h^{\setminus} =$$

$$t\sum_{h \in \mathcal{H}} x_h^{*\setminus} + t\sum_{h \in \mathcal{H}} e_h^{*\setminus} = t\sum_{h \in \mathcal{H}}\left(x_h^{*\setminus} - e_h^{*\setminus}\right)$$

Then,

$$\lim_{t \to 0} \frac{t\sum_{h \in \mathcal{H}}\left(x_h^{*\setminus} - e_h^{*\setminus}\right)}{t} = \sum_{h \in \mathcal{H}}\left(x_h^{*\setminus} - e_h^{*\setminus}\right).$$

3.

Summarizing what was said above, we have that

$$\lim_{t\to 0}\frac{\mathcal{F}_P((\xi,e,u,v)+t(\xi^*,e^*,u^*,v^*))-\mathcal{F}_P((\xi,e,u,v))}{t}=$$

$$\begin{pmatrix} D^2u_h\left(x_h\right)\cdot x_h^*+Du_h^*\left(x_h\right)+\lambda_h p^*-\lambda_h^* p \\ \frac{1}{pr}D\left(v'_{hh'}\circ\theta\right)(\xi,e)\cdot(\xi^*,e^*)-\lambda_h^*+\gamma_{hh'}^*-\eta_h^* \\ -\left(p^*x_h+px_h^*-p^*e_h-pe_h\right)+\sum_{h'\in\mathcal{B}_{\to h}}\tau_{h'h}^*-\sum_{h'\in\mathcal{B}_h}\tau_{hh'}^* \\ \tau_{hh'}^* \text{ or } \gamma_{hh'}^* \\ p^*e_h+pe_h^*-\sum_{h'\in\mathcal{B}_h}\tau_{hh'}^* \text{ or } \eta_h^* \\ \sum_{h\in\mathcal{H}}\left(x_h^{*\setminus}-e_h^{*\setminus}\right) \end{pmatrix}$$

The proof of continuity of the function $d\mathcal{F}_P$ then follows from the same argument used to show that $\widetilde{F}$ is continuous. In more detail, we have the following:

We remark that the continuity with respect to the Euclidean norm is clearly satisfied. We observe that $\mathcal{C}^3\left(\mathbb{R}^C_{++},\mathbb{R}\right),\mathcal{C}^3\left((0,1),\mathbb{R}\right)$ are equipped with the compact-open topology, which implies, in the case of $\mathcal{C}^3$ functions, the uniform convergence on compacta of the functions themselves, their gradients and Hessians.

Let, for instance, (w_n,s_n) be a converging sequence in $V\times T$. Then there is a sequence of functions $(u_n)_{n\in\mathbb{N}}$ with elements in $\mathcal{U}^H$ converging to some $u\in\mathcal{U}^H$ and a sequence $(x_n)_{n\in\mathbb{N}}$ with elements in $\mathbb{R}^{CH}_{++}$ converging to some $x\in\mathbb{R}^{CH}_{++}$. Then we have that $Du_{h,n}$ converges uniformly on compacta to Du_h, and $D^2u_{h,n}$ converges uniformly on compacta to D^2u_h.

From the previous point,

$$D^2u_{h,n}(x_{h,n})\cdot x_{h,n}^*+Du_{h,n}^*(x_{h,n})-\lambda_{h,n}p_n^*-\lambda_{h,n}^*p_n,$$

By the property of uniform convergence on compacta we have

$$\lim_{n\to+\infty}D^2u_{h,n}(x_{h,n})=D^2u_h(x_h),$$

■

Lemma 69 *Let us consider any $P\in\mathcal{P}$, $\mathcal{E}\in\mathbb{E}$ and $\xi\in\Xi$ such that $\widetilde{F}_P(\xi,\mathcal{E})=0$ and*

$$\det D_\xi\widetilde{F}_P(\xi,\mathcal{E})\neq 0.$$

Then there exists an $\mathbb{E}$ open neighborhood $V^*(\mathcal{E})$ of $\mathcal{E}$, a Ξ open neighborhood $O(\xi)$ of ξ and a function $\zeta^*:V^*(\mathcal{E})\to\mathbb{R}^n$ such that

1. *$\zeta^*\in\mathcal{C}^1_M(V^*(\mathcal{E});\mathbb{R}^n)$,*
2. *$\zeta^*(V^*(\mathcal{E}))\subseteq O(\xi)$,*
3. *$\zeta^*(\mathcal{E})=\xi$,*
4. *$\left\{(\xi,\mathcal{E})\in O(\xi)\times V^*(\mathcal{E}):\widetilde{F}_P(\xi,\mathcal{E})=0\right\}=\{(\xi,\mathcal{E})\in O(\xi)\times V^*(\mathcal{E}):\xi=\zeta^*(\mathcal{E})\}$.*

Proof. The result is an application of Glöckner's Theorem 139 to the function

$$\mathcal{F}_P:\Xi\times T\to\mathbb{R}^n$$

presented in Definition 67.

Indeed, all the assumptions of Glöckner's Theorem, i.e., Theorem 139, are satisfied, as verified below.

$\mathcal{O}$ corresponds to Ξ which is an open subset of an Euclidean space;

V corresponds to $\mathbb{R}^{CH}_{++} \times \left(\times_{h \in \mathcal{H}} \left(\mathcal{C}^3 \left(\mathbb{R}^C_{++}, \mathbb{R} \right) \times \left(\mathcal{C}^3 \left((0,1), \mathbb{R} \right) \right)^{B_h} \right) \right)$ which is an open subset of a topological vector space [2];

$\mathcal{F}_P$ is $\mathcal{C}^1_M$ from Lemma 68;

the other two assumptions are explicitly assumed here.

Therefore, we can conclude that there exists a $\mathbb{R}^{CH}_{++} \times \left(\times_{h \in \mathcal{H}} \left(\mathcal{C}^3 \left(\mathbb{R}^C_{++}, \mathbb{R} \right) \times \left(\mathcal{C}^3 \left((0,1), \mathbb{R} \right) \right)^{B_h} \right) \right)$ open neighborhood $V(\mathcal{E})$ of $\mathcal{E}$, a Ξ open neighborhood $O(\xi)$ of ξ and a function $\zeta : V(\mathcal{E}) \to \mathbb{R}^n$ such that

1'. $\zeta \in \mathcal{C}^1_M(V(\mathcal{E}); \mathbb{R}^n)$,

2'. $\zeta(V(\mathcal{E})) \subseteq O(\xi)$,

3'. $\zeta(\mathcal{E}) = \xi$, and

4'. $\{(\xi, \mathcal{E}) \in O(\xi) \times V(\mathcal{E}) : \mathcal{F}_P(\xi, \mathcal{E}) = 0\} = \{(\xi, \mathcal{E}) \in O(\xi) \times V(\mathcal{E}) : \xi = \zeta(\mathcal{E})\}$.

Consider now the set

$$V^*(\mathcal{E}) = V(\mathcal{E}) \cap \mathbb{E}.$$

Clearly $V^*(\mathcal{E})$ is an $\mathbb{E}$ open subset of $\mathbb{E}$ and $\mathcal{E} \in V^*(\mathcal{E})$. Define

$$\zeta^* = \zeta_{|V^*(\mathcal{E})}.$$

Then

1. $\zeta^* \in \mathcal{C}^1_M \left(V^*(\mathcal{E}), \mathbb{R}^n \right)$, by definition of Michal-Bastiani differentiability for not necessarily open sets (which is based on the fact that the restrictions of continuous functions are continuous);

2. $\zeta^*(V^*(\mathcal{E})) \subseteq O(\xi)$, because $V^*(\mathcal{E}) \subseteq V(\mathcal{E})$ and from 2';

3. $\zeta^*(\mathcal{E}) \overset{\mathcal{E} \in V^*(\mathcal{E})}{=} \zeta(\mathcal{E}) = \xi$.

We are left with showing[3] that

4. $A := \left\{ (\xi, \mathcal{E}) \in O(\xi) \times V^*(\mathcal{E}) : \widetilde{F}_P(\xi, \mathcal{E}) = 0 \right\} = \{(\xi, \mathcal{E}) \in O(\xi) \times V^*(\mathcal{E}) : \xi = \zeta^*(\mathcal{E})\} := B$.

Indeed, that is a consequence of Claim 0 below, but we also provide a simple direct proof.

a. $(\xi, \mathcal{E}) \in A \Rightarrow \widetilde{F}_P(\xi, \mathcal{E}) = 0 \overset{\text{Def. } \mathcal{F}_P}{\Rightarrow} \mathcal{F}_P(\xi, \mathcal{E}) = 0 \overset{4'}{\Rightarrow} \xi = \zeta(\mathcal{E}) \overset{\text{def. } A\,,\ \mathcal{E} \in V^*(\mathcal{E})}{\Rightarrow} \xi = \zeta^*(\mathcal{E})$.

b. $(\xi, \mathcal{E}) \in B \Rightarrow \xi = \zeta^*(\mathcal{E}) \overset{\zeta \text{ extension of } \zeta^*}{\Rightarrow} \xi = \zeta(\mathcal{E}) \overset{4'}{\Rightarrow} \mathcal{F}_P(\xi, \mathcal{E}) = 0 \overset{\mathcal{E} \in \mathbb{E}}{\Rightarrow} \widetilde{F}_P(\xi, \mathcal{E}) = 0$. ■

Proof. of Theorem 16.

Observe that from Proposition 64, $\mathcal{D}$ is open and dense in $\mathbb{E}$.

We can now show that for any $\mathcal{E}^0 \in \mathcal{D}$, all conditions in the statement of Theorem 16 are satisfied.

A.

Take $\mathcal{E}^0 \in \mathcal{D}$ (and therefore to $\mathcal{E}^0 \in \mathcal{D}_2$). Since, from the existence result, pr is onto, then $\# \left(\widetilde{F}_{\mathcal{E}^0} \right)^{-1}(0) \neq 0$. Moreover,

$$\left(\widetilde{F}_{\mathcal{E}^0} \right)^{-1}(0) := \left\{ \xi \in \Xi : \widetilde{F}\left(\xi, \mathcal{E}^0 \right) = 0 \right\}$$

and

$$pr^{-1}\left(\mathcal{E}^0 \right) := \left\{ \left(\xi, \mathcal{E}^0 \right) \in \Xi \times \mathbb{E} : \widetilde{F}_{\mathcal{E}^0}(\xi) = 0 \right\} = \left(\widetilde{F}_{\mathcal{E}^0} \right)^{-1}(0) \times \left\{ \mathcal{E}^0 \right\}.$$

Then, $\# \left(pr \right)^{-1}\left(\mathcal{E}^0 \right) = \# \left(\widetilde{F}_{\mathcal{E}^0} \right)^{-1}(0)$. From Corollary 66, we have that 0 is a regular value for $\widetilde{F}_{\mathcal{E}^0}$ and since the dimensions of the domain and the dimension of the codomain of the function $\widetilde{F}_{\mathcal{E}^0}$ are the same, then, from Proposition 34, page 42, in Villanacci and others (2002)[4], $\left(pr \right)^{-1}\left(\mathcal{E}^0 \right)$ and $\left(\widetilde{F}_{\mathcal{E}^0} \right)^{-1}(0)$ are a set of isolated points. Since, pr is proper, then $\left(pr \right)^{-1}\left(\mathcal{E}^0 \right)$ is a compact set of isolated points and from Corollary 16, page 86 in Villanacci and others (2002)[5], we get the desired result.

B.

Take $\mathcal{E}^0 \in \mathcal{D}$. Observe that by definition of $\mathcal{D}$, from A., and Lemma 65, for any $i \in \{1, ..., r_{\mathcal{E}^0}\}$, there exists a partition $P^0_i = \left\{ \left(\mathcal{J}^1_h\left(\xi^0_i, \mathcal{E}^0 \right), \mathcal{J}^2_h\left(\xi^0_i, \mathcal{E}^0 \right) \right)_{h \in \mathcal{H}} \right\} \in \mathcal{P}$ and a Ξ open neighborhood $N\left(\xi^0_i \right)$ of ξ^0_i such that for any $\xi \in N\left(\xi^0_i \right)$, $\widetilde{F}\left(\xi, \mathcal{E}^0 \right) = \widetilde{F}_{P^0_i}\left(\xi, \mathcal{E}^0 \right)$, and $\widetilde{F}_{\mathcal{E}^0}$ is $\mathcal{C}^1$ on $N\left(\xi^0_i \right)$, $D_\xi \widetilde{F}\left(\xi^0_i, \mathcal{E}^0 \right) = D_\xi \widetilde{F}_{P^0_i}\left(\xi^0_i, \mathcal{E}^0 \right)$

[2] It is obvious that T is a subset of $\mathbb{R}^{CH} \times \left(\times_{h \in \mathcal{H}} \left(\mathcal{C}^3(\mathbb{R}^C, \mathbb{R}) \times (\mathcal{C}^3((0,1), \mathbb{R}))^{B_h} \right) \right)$.

[3] For a different proof of the desired result, see Claim 0 in the next section.

[4] A set M is a zero-dimensional manifold if and only if it is a set of isolated points.

[5] Let M and N be $\mathcal{C}^s$ manifolds of the same dimension, respectively, let $f : M \to N$ be a $\mathcal{C}^s$ function, and let y be a regular value for f such that $f^{-1}(y)$ is compact. Then $f^{-1}(y)$ is a finite subset of M.

and $\det\ D_{\xi}\widetilde{F}\left(\xi_i^0,\mathcal{E}^0\right)=\det D_{\xi}\widetilde{F}_{P_i^0}\left(\xi_i^0,\mathcal{E}^0\right)\neq 0$. Then the assumptions of Lemma 69 are satisfied and we can conclude that there exists an $\mathbb{E}$-open neighborhood $V_i^*(\mathcal{E}^0)$ of $\mathcal{E}^0$, a Ξ-open neighborhood $O(\xi_i^0)$ of ξ_i^0 and a function $\zeta_i^*:V_i^*(\mathcal{E}^0)\to\mathbb{R}^n$ such that

1^0. $\zeta_i^*\in\mathcal{C}_M^1(V_i^*(\mathcal{E}^0);\mathbb{R}^n)$,
2^0. $\zeta_i^*(V_i^*(\mathcal{E}^0))\subseteq O(\xi_i^0)$,
3^0. $\zeta_i^*(\mathcal{E}^0)=\xi_i^0$, and
4^0. $\left\{(\xi,\mathcal{E})\in O(\xi_i^0)\times V_i^*(\mathcal{E}^0):\widetilde{F}_{P_i^0}(\xi,\mathcal{E})=0\right\}=\left\{(\xi,\mathcal{E})\in O(\xi_i^0)\times V_i^*(\mathcal{E}^0):\xi=\zeta_i^*(\mathcal{E})\right\}$.

Observe that, still by definition of $\mathcal{D}\subseteq\mathcal{D}_1$, we have that for any $h\in\mathcal{H}$,

$$\begin{aligned}&\text{for any } j\in\mathcal{J}_h^2\left(\xi_i^0,\mathcal{E}^0\right), &&\mu_{hj}^0>g_{hj}\left(x^0,\tau^0,p^{0\backslash},e^0\right)=0\\&\text{for any } j\in\mathcal{J}_h^1\left(\xi_i^0,\mathcal{E}^0\right), &&g_{hj}\left(x^0,\tau^0,p^{0\backslash},e^0\right)>\mu_{hj}^0=0\end{aligned}\tag{6.4}$$

Then, defined $\xi_i^0=\left(x^0,\tau^0,p^{0\backslash},\mu^0\right)$ and $\mathcal{E}^0=\left(e^0,u^0,v^0\right)\in\mathcal{D}$, we have

$$\begin{aligned}&\widetilde{F}_{P_i^0}\left(\xi_i^0,\mathcal{E}^0\right)\overset{\text{Def. } P_i^0}{=}\\&\left(\begin{array}{c}\widetilde{\varphi}\left(\xi_i^0,\mathcal{E}^0\right)\\\left(\begin{array}{c}\left(\mu_{hj}^0\right)_{j\in\mathcal{J}_h^1\left(\xi_i^0,\mathcal{E}^0\right)}\\\left(g_{hj}\left(x^0,\tau^0,p^{0\backslash},e^0\right)\right)_{j\in\mathcal{J}_h^2\left(\xi_i^0,\mathcal{E}^0\right)}\end{array}\right)_{h\in\mathcal{H}}\end{array}\right)\overset{(6.4)}{=}\\&=\left(\begin{array}{c}\widetilde{\varphi}\left(\xi^0,\mathcal{E}^0\right)\\\left(\begin{array}{c}\left(\min\left\{g_{hj}\left(x^0,\tau^0,p^{0\backslash},e^0\right),\mu_{hj}^0\right\}\right)_{j\in\mathcal{J}_h^1\left(\xi_i^0,\mathcal{E}^0\right)}\\\left(\min\left\{g_{hj}\left(x^0,\tau^0,p^{0\backslash},e^0\right),\mu_{hj}^0\right\}\right)_{j\in\mathcal{J}_h^2\left(\xi_i^0,\mathcal{E}^0\right)}\end{array}\right)_{h\in\mathcal{H}}\end{array}\right)\overset{\text{Def. } \widetilde{F}}{=}\\&=\widetilde{F}\left(\xi_i^0,\mathcal{E}^0\right)=0.\end{aligned}$$

Defined $\zeta_i^*\left(\mathcal{E}\right)=\left(x^*\left(\mathcal{E}\right),\tau^*\left(\mathcal{E}\right),\mu^*\left(\mathcal{E}\right),p^{*\backslash}\left(\mathcal{E}\right)\right)$ and since ζ_i^* and the functions g_{hj} are continuous, then there exists an open neighborhood $N\left(\mathcal{E}^0\right)$ of $\mathcal{E}^0$ such that for any $\mathcal{E}\in N\left(\mathcal{E}^0\right)$, the strict inequalities in (6.4) are still satisfied, i.e.,

$$\begin{aligned}&\text{for any } \mathcal{E}\in N\left(\mathcal{E}^0\right)\cap V_i^*(\mathcal{E}^0),\\&\text{for any } j\in\mathcal{J}_h^2\left(\xi_i^0,\mathcal{E}^0\right), &&\mu_{hj}^*\left(\mathcal{E}\right)>g_{hj}\left(x^*\left(\mathcal{E}\right),\tau^*\left(\mathcal{E}\right),p^{*\backslash}\left(\mathcal{E}\right),e\right)\\&\text{for any } j\in\mathcal{J}_h^1\left(\xi_i^0,\mathcal{E}^0\right), &&g_{hj}\left(x^*\left(\mathcal{E}\right),\tau^*\left(\mathcal{E}\right),p^{*\backslash}\left(\mathcal{E}\right),e\right)>\mu_{hj}^*\left(\mathcal{E}\right).\end{aligned}\tag{6.5}$$

Then, for any $\mathcal{E}\in N\left(\mathcal{E}^0\right)\cap V_i^*(\mathcal{E}^0):=\widetilde{V}_i\left(\mathcal{E}^0\right)$,

$$\begin{aligned}&0\overset{4^0,\,2^0}{=}\widetilde{F}_{P_i^0}\left(\zeta_i^*\left(\mathcal{E}\right),\mathcal{E}\right)\overset{\text{Def. } \widetilde{F}_{P_i^0}}{=}\left(\begin{array}{c}\widetilde{\varphi}\left(\zeta_i^*\left(\mathcal{E}\right),\mathcal{E}\right)\\\left(\begin{array}{c}\left(\mu_{hj}^*\left(\mathcal{E}\right)\right)_{j\in\mathcal{J}_h^1\left(\xi_i^0,\mathcal{E}^0\right)}\\\left(g_{hj}\left(x^*\left(\mathcal{E}\right),\tau^*\left(\mathcal{E}\right),p^{*\backslash}\left(\mathcal{E}\right),e\right)\right)_{j\in\mathcal{J}_h^2\left(\xi_i^0,\mathcal{E}^0\right)}\end{array}\right)_{h\in\mathcal{H}}\end{array}\right)\overset{(6.5)}{=}\\&=\left(\begin{array}{c}\widetilde{\varphi}\left(\zeta_i^*\left(\mathcal{E}\right),\mathcal{E}\right)\\\left(\begin{array}{c}\left(\min\left\{\mu_{hj}^*\left(\mathcal{E}\right),g_{hj}\left(x^*\left(\mathcal{E}\right),\tau^*\left(\mathcal{E}\right),p^{*\backslash}\left(\mathcal{E}\right),e\right)\right\}\right)_{j\in\mathcal{J}_h^1\left(\xi_i^0,\mathcal{E}^0\right)}\\\left(\min\left\{g_{hj}\left(x^*\left(\mathcal{E}\right),\tau^*\left(\mathcal{E}\right),p^{*\backslash}\left(\mathcal{E}\right),e\right),\mu_{hj}^*\left(\mathcal{E}\right)\right\}\right)_{j\in\mathcal{J}_h^2\left(\xi_i^0,\mathcal{E}^0\right)}\end{array}\right)_{h\in\mathcal{H}}\end{array}\right)\overset{\text{Def. } \widetilde{F}}{=}\widetilde{F}\left(\zeta_i^*\left(\mathcal{E}\right),\mathcal{E}\right).\end{aligned}$$

Summarizing,

$$\text{for any } i\in\{1,...,r_{\mathcal{E}^0}\},\ \text{for any } \mathcal{E}\in\widetilde{V}_i\left(\mathcal{E}^0\right),\qquad\widetilde{F}\left(\zeta_i^*\left(\mathcal{E}\right),\mathcal{E}\right)=\widetilde{F}_{P_i^0}\left(\zeta_i^*\left(\mathcal{E}\right),\mathcal{E}\right)=0.\tag{6.6}$$

Claim. Defined $\widehat{\zeta}_i = \zeta^*_{i|\widetilde{V}_i(\mathcal{E}^0)}$, we have

$$\begin{aligned} A &:= \left\{(\xi,\mathcal{E}) \in O\left(\xi_i^0\right) \times \widetilde{V}_i\left(\mathcal{E}^0\right) : \xi = \widehat{\zeta}_i\left(\mathcal{E}\right)\right\} = \\ B &:= \left\{(\xi,\mathcal{E}) \in O\left(\xi_i^0\right) \times \widetilde{V}_i\left(\mathcal{E}^0\right) : \widetilde{F}_{P_i^0}(\xi,\mathcal{E}) = 0\right\} = \\ C &:= \left\{(\xi,\mathcal{E}) \in O\left(\xi_i^0\right) \times \widetilde{V}_i\left(\mathcal{E}^0\right) : \widetilde{F}(\xi,\mathcal{E}) = 0\right\}, \end{aligned} \tag{6.7}$$

Proof of the Claim.
The first equality in (6.7) follows from 4^0 and the fact that $\widetilde{V}_i\left(\mathcal{E}^0\right) \subseteq V_i^*(\mathcal{E}^0)$.
The second equality follows from a. and b. below.
a. If $(\xi,\mathcal{E}) \in A = B$, then $(\xi,\mathcal{E}) \in C$.
From the fact that $A = B$, we have that for any $(\xi,\mathcal{E}) \in O\left(\xi_i^0\right) \times \widetilde{V}_i\left(\mathcal{E}^0\right)$, $\xi = \widehat{\zeta}_i\left(\mathcal{E}\right) \Leftrightarrow \widetilde{F}_{P_i^0}\left(\xi,\mathcal{E}\right) = 0$.
From (6.6), we have that for any $i \in \{1,\ldots,r_{\mathcal{E}^0}\}$, for any $\mathcal{E} \in \widetilde{V}_i\left(\mathcal{E}^0\right) \subseteq V_i^*\left(\mathcal{E}^0\right)$,

$$\xi = \widehat{\zeta}_i\left(\mathcal{E}\right) \text{ implies } \widetilde{F}\left(\xi,\mathcal{E}\right) = \widetilde{F}_{P_i^0}\left(\xi,\mathcal{E}\right) = 0 \text{ and } (\xi,\mathcal{E}) \in O\left(\xi_i^0\right) \times \widetilde{V}_i\left(\mathcal{E}^0\right),$$

as desired.
b. If $(\xi,\mathcal{E}) \in C$, then $(\xi,\mathcal{E}) \in B$.
Indeed, we are going to show that there exists an open neighborhood $O^{**}\left(\xi_i^0\right) \times \left(V_i^{**}\left(\mathcal{E}^0\right) \cap \mathcal{D}\right)$[6] of $\left(\xi_i^0,\mathcal{E}^0\right)$ such that

$$\text{for any } (\xi,\mathcal{E}) \in O^{**}\left(\xi_i^0\right) \times \left(V_i^{**}\left(\mathcal{E}^0\right) \cap \mathcal{D}\right), \quad \widetilde{F}\left(\xi,\mathcal{E}\right) = 0 \Rightarrow \widetilde{F}_{P_i^0}\left(\xi,\mathcal{E}\right) = 0. \tag{6.8}$$

Observe that for any $P' \in \mathcal{P} \backslash \left\{P_i^0\right\}$, from the contrapositive of Lemma 52, we have

$$\widetilde{F}_{P'}\left(\xi_i^0,\mathcal{E}^0\right) \neq \widetilde{F}_{P_i^0}\left(\xi_i^0,\mathcal{E}^0\right) = 0.$$

Then, by continuity of $\widetilde{F}_{P'}$ for any $P' \in \mathcal{P}$, there exists a neighborhood $O^{**}\left(\xi_i^0\right) \times \left(V_i^{**}\left(\mathcal{E}^0\right) \cap \mathcal{D}\right) \subseteq O\left(\xi_i^0\right) \times \widetilde{V}_i\left(\mathcal{E}^0\right)$ of $\left(\xi_i^0,\mathcal{E}^0\right)$ such that

$$\begin{aligned} &\text{for any } (\xi,\mathcal{E}) \in O^{**}\left(\xi_i^0\right) \times \left(V_i^{**}\left(\mathcal{E}^0\right) \cap \mathcal{D}\right) \text{ and for any } P' \in \mathcal{P} \backslash \left\{P_i^0\right\}, \\ &\widetilde{F}_{P'}\left(\xi,\mathcal{E}\right) \text{ has the same sign of } \widetilde{F}_{P'}\left(\xi_i^0,\mathcal{E}^0\right), \text{ hence we have } \widetilde{F}_{P'}\left(\xi,\mathcal{E}\right) \neq 0. \end{aligned} \tag{6.9}$$

By the existence result and by definition of $\mathcal{D}$, we have that

$$\begin{aligned} &\text{for any } \mathcal{E} \in \left(V_i^{**}\left(\mathcal{E}^0\right) \cap \mathcal{D}\right) \text{ there exists } \xi \in \Xi \text{ such that } \widetilde{F}\left(\xi,\mathcal{E}\right) = 0, \\ &\text{and there exists } P\left(\xi,\mathcal{E}\right) \in \mathcal{P} \text{ such that } \widetilde{F}_{P(\xi,\mathcal{E})}\left(\xi,\mathcal{E}\right) = 0 \text{ (see the proof of Lemma 65 to construct } P(\xi,\mathcal{E})). \end{aligned} \tag{6.10}$$

We want to show now that $P\left(\xi,\mathcal{E}\right) = P_i^0$, which provides us with (6.8).
Then, from (6.9) and (6.10), we have that

$$\text{for any } \mathcal{E} \in \left(V_i^{**}\left(\mathcal{E}^0\right) \cap \mathcal{D}\right), \text{ for any } \xi \in O^{**}\left(\xi_i^0\right) \subseteq \Xi \text{ such that } \widetilde{F}\left(\xi,\mathcal{E}\right) = 0, \text{ we have } \widetilde{F}_{P_i^0}\left(\xi,\mathcal{E}\right) = 0,$$

as desired.
End of the proof of the Claim.
For lighter notation, below we denote $O^{**}\left(\xi_i^0\right)$ and $\left(V_i^{**}\left(\mathcal{E}^0\right) \cap \mathcal{D}\right)$ by $\widehat{O}\left(\xi_i^0\right)$ and $\widehat{V}_i\left(\mathcal{E}^0\right)$, respectively.
Then, using 1^0 - 4^0 and (6.7), we can conclude that there exists an $\mathbb{E}$-open neighborhood $\widehat{V}_i\left(\mathcal{E}^0\right)$ of $\mathcal{E}^0$, a Ξ-open neighborhood $\widehat{O}\left(\xi_i^0\right)$ of ξ_i^0 and a function $\widehat{\zeta}_i : \widehat{V}_i\left(\mathcal{E}^0\right) \to \mathbb{R}^n$ such that
1'. $\widehat{\zeta}_i \in \mathcal{C}^1_M(\widehat{V}_i\left(\mathcal{E}^0\right), \mathbb{R}^n)$,
2'. $\widehat{\zeta}_i(\widehat{V}_i\left(\mathcal{E}^0\right)) \subseteq \widehat{O}\left(\xi_i^0\right)$,
3'. $\widehat{\zeta}_i(\mathcal{E}^0) = \xi_i^0$, and
4'.1. $\left\{(\xi,\mathcal{E}) \in \widehat{O}\left(\xi_i^0\right) \times \widehat{V}_i\left(\mathcal{E}^0\right) : \widetilde{F}(\xi,\mathcal{E}) = 0\right\} = \left\{(\xi,\mathcal{E}) \in \widehat{O}\left(\xi_i^0\right) \times \widehat{V}_i\left(\mathcal{E}^0\right) : \xi = \widehat{\zeta}_i(\mathcal{E})\right\}$.

[6] Recall that $\mathcal{D}$ is an open set and, by assumption, $\mathcal{E}^0 \in \mathcal{D}$.

Moreover, since $\mathcal{E}^0 \in \mathcal{D} \subseteq \mathcal{D}_2$, from Corollary 66, we have that

$$\det\ D_\xi \widetilde{F}\left(\xi, \mathcal{E}^0\right) \neq 0.$$

Since Ξ is a Hausdorff space, and for any $i \in \{1, ..., r_{\mathcal{E}^0}\}$, ξ_i^0 are isolated points, then for any $i \in \{1, ..., r_{\mathcal{E}^0}\}$, there exists an open neighborhood $O\left(\xi_i^0\right)$ such that

4'.2 $O\left(\xi_i^0\right) \cap O\left(\xi_j^0\right) = \varnothing$ for any $i, j \in \{1, ..., r_{\mathcal{E}^0}\}$ and $i \neq j$.

With a slight abuse of notation, we denote $O\left(\xi_i^0\right) := O\left(\xi_i^0\right) \cap \widehat{O}\left(\xi_i^0\right)$.

Then for any $i \in \{1, ..., r_{\mathcal{E}^0}\}$, $\left(\widehat{\zeta}_i\right)^{-1}\left(O\left(\xi_i^0\right)\right)$ is an open neighborhood of $\mathcal{E}^0$ contained[7] in $\widehat{V}_i\left(\mathcal{E}^0\right)$, without loss of generality $\widehat{V}_i\left(\mathcal{E}^0\right)$ itself. Then, we can rewrite (6.7) as follows.

5'. $\left\{(\xi, \mathcal{E}) \in O(\xi_i^0) \times \widehat{V}_i(\mathcal{E}^0) : \widetilde{F}(\xi, \mathcal{E}) = 0\right\} = \left\{(\xi, \mathcal{E}) \in O(\xi_i^0) \times \widehat{V}_i(\mathcal{E}^0) : \xi = \widehat{\zeta}_i(\mathcal{E})\right\}$.

Define now[8]

$$V\left(\mathcal{E}^0\right) = \bigcap_{i=1}^{r_{\mathcal{E}^0}} \left(\widehat{\zeta}_i\right)^{-1}\left(O\left(\xi_i^0\right)\right).$$

We remark that $V\left(\mathcal{E}^0\right)$ is not empty. In particular $\mathcal{E}^0 \in \left(\widehat{\zeta}_i\right)^{-1}\left(O\left(\xi_i^0\right)\right)$ and $V\left(\mathcal{E}^0\right)$ is an open neighborhood of $\mathcal{E}^0$ in $\mathcal{D}$.

Since $\widehat{V}_i\left(\mathcal{E}^0\right)$ is $\mathbb{E}$-open and $\widehat{\zeta}_i$ is continuous, then $V\left(\mathcal{E}^0\right)$ is $\mathbb{E}$-open. Moreover, since for any $i \in \{1, ..., r_{\mathcal{E}^0}\}$, it is the case that $\mathcal{E}^0 \in \widehat{V}_i\left(\mathcal{E}^0\right)$, $\widehat{\zeta}_i\left(\mathcal{E}^0\right) = \xi_i^0$ and $\mathcal{E}^0 \in \mathbb{E}$, then $\mathcal{E}^0 \in V\left(\mathcal{E}^0\right)$ - shortly $V\left(\mathcal{E}^0\right)$ is an open neighborhood of $\mathcal{E}^0$ contained in $\widehat{V}_i\left(\mathcal{E}^0\right)$. For any $i \in \{1, ..., r_{\mathcal{E}^0}\}$, define

$$\zeta_i = \widehat{\zeta}_{i|V(\mathcal{E}^0)}$$

which is $\mathcal{C}_M^1$, because it is the restriction of a $\mathcal{C}_M^1$ function. Therefore, we can rewrite 5'. above "substituting the smaller neighborhood $V\left(\mathcal{E}^0\right)$ " as follows.

5. for any $i \in \{1, ..., r_{\mathcal{E}^0}\}$, $\left\{(\xi, \mathcal{E}) \in O\left(\xi_i^0\right) \times V\left(\mathcal{E}^0\right) : \widetilde{F}\left(\xi, \mathcal{E}\right) = 0\right\} = \left\{(\xi, \mathcal{E}) \in O\left(\xi_i^0\right) \times V\left(\mathcal{E}^0\right) : \xi = \zeta_i\left(\mathcal{E}\right)\right\}$.

The possibility of "substituting the smaller neighborhood", which will be also used again below, can be verified as follows, using a lighter notation.

Claim 0. If

$$\{(x, y) \in A \times B : F(x, y) = 0\} = \{(x, y) \in A \times B : y = \zeta(x)\} \tag{6.11}$$

and $B' \subseteq B$, then $S := \{(x, y) \in A \times B' : F(x, y) = 0\} = \{(x, y) \in A \times B' : y = \zeta(x)\} := T$.

Proof of Claim 0.

$$\begin{array}{l} (x, y) \in S \Rightarrow \\ (x, y) \in A \times B' \subseteq A \times B \text{ and } F(x, y) = 0 \overset{(6.11)}{\Rightarrow} \\ (x, y) \in A \times B \text{ and } y = \zeta(x) \overset{y \in B'}{\Rightarrow} \\ (x, y) \in T; \end{array}$$

$$\begin{array}{l} (x, y) \in T \Rightarrow \\ (x, y) \in A \times B' \subseteq A \times B \text{ and } y = \zeta(x) \overset{(6.11)}{\Rightarrow} \\ (x, y) \in A \times B \text{ and } F(x, y) = 0 \overset{y \in B'}{\Rightarrow} \\ (x, y) \in S. \end{array}$$

End of the proof of Claim 0.

We can also rewrite 1'., 2'. and 3'., 4'.1 and 4'.2, as follows.

$$\begin{array}{l} 1.\ \zeta_i \in \mathcal{C}_M^1(V\left(\mathcal{E}^0\right); \mathbb{R}^n), \\ 2.\ \zeta_i(V\left(\mathcal{E}^0\right)) \subseteq O\left(\xi_i^0\right), \\ 3.\ \zeta_i(\mathcal{E}^0) = \xi_i^0, \\ 4.\ O\left(\xi_i^0\right) \cap O\left(\xi_j^0\right) = \varnothing \text{ for any } i, j \in \{1, ..., r_{\mathcal{E}^0}\} \text{ and } i \neq j. \end{array} \tag{6.12}$$

[7] $\widehat{\zeta}_i$ is continuous, $\xi_i^0 \in O\left(\xi_i^0\right)$ and $\widehat{\zeta}_i\left(\mathcal{E}^0\right) = \xi_i^0$.

[8] Observe that even choosing $V^0\left(\mathcal{E}^0\right) = \cap_{i=1}^{r_{\mathcal{E}^0}} \widehat{V}_i\left(\mathcal{E}^0\right)$ it could be the case that Conclusion 6. does not hold.

We are left with showing 6, i.e.,

$$\left\{(\xi,\mathcal{E})\in\Xi\times V\left(\mathcal{E}^0\right):\widetilde{F}\left(\xi,\mathcal{E}\right)=0\right\}=\cup_{i=1}^{r_{\mathcal{E}^0}}\left\{(\xi,\mathcal{E})\in O\left(\xi_i^0\right)\times V\left(\mathcal{E}^0\right):\xi=\zeta_i\left(\mathcal{E}\right)\right\}. \tag{6.13}$$

Below, we present two different proofs: the first one is close to the approach provided in Villanacci and others (2002), which combines the proof of Proposition 34 and Remark 35, pages 94 and 95 and Corollary 37, pages 98 and 99 there; the other proof tries to rewrite the proof presented in Gori (2007).

Basic idea of the proofs.

Step 1. Show that there exists a well chosen open neighborhood $V\left(\mathcal{E}^0\right)$ such that

$$\left\{(\xi,\mathcal{E})\in\Xi\times V\left(\mathcal{E}^0\right):\widetilde{F}\left(\xi,\mathcal{E}\right)=0\right\}=\cup_{i=1}^{r_{\mathcal{E}^0}}\left\{(\xi,\mathcal{E})\in O\left(\xi_i^0\right)\times V\left(\mathcal{E}^0\right):\widetilde{F}\left(\xi,\mathcal{E}\right)=0\right\}.$$

Step 2. From 5.,

$$\text{for any } i\in\{1,...,r_{\mathcal{E}^0}\},\ \left\{(\xi,\mathcal{E})\in O\left(\xi_i^0\right)\times V\left(\mathcal{E}^0\right):\widetilde{F}\left(\xi,\mathcal{E}\right)=0\right\}=\left\{(\xi,\mathcal{E})\in O\left(\xi_i^0\right)\times V\left(\mathcal{E}^0\right):\xi=\zeta_i\left(\mathcal{E}\right)\right\}$$

Of course, the difference between the two proofs lies in the accomplishment of Step 1. In the first proof, the key ingredients are those developed by Villanacci et al. (2002), mentioned above, which rely on the properness of the projection. In the second proof, the argument begins by assuming, by contradiction, that the desired result does not hold. This assumption makes it possible to construct a sequence which, by properness, converges to an equilibrium associated with $\mathcal{E}^0$, in addition to the $r_{\mathcal{E}^0}$ equilibria that had already been shown to be the only ones existing.

First proof. Observe that from part A., we do have

$$\begin{aligned}&pr^{-1}\left(\mathcal{E}^0\right):=\left\{\left(\xi,\mathcal{E}^0\right)\in\Xi\times\mathbb{E}:\widetilde{F}_{\mathcal{E}^0}\left(\xi\right)=0\right\}\\&=\left(\widetilde{F}_{\mathcal{E}^0}\right)^{-1}\left(0\right)\times\left\{\mathcal{E}^0\right\}=\left\{\xi_i^0:i\in\{1,...,r_{\mathcal{E}^0}\}\right\}\times\left\{\mathcal{E}^0\right\}.\end{aligned} \tag{6.14}$$

Moreover,

$$\begin{aligned}&\cup_{i=1}^{r_{\mathcal{E}^0}}\left(\left(O\left(\xi_i^0\right)\times V\left(\mathcal{E}^0\right)\right)\cap\widetilde{F}^{-1}\left(0\right)\right)=\\&=\cup_{i=1}^{r_{\mathcal{E}^0}}\left\{(\xi,\mathcal{E})\in O\left(\xi_i^0\right)\times V\left(\mathcal{E}^0\right):\widetilde{F}\left(\xi,\mathcal{E}\right)=0\right\}\\&\text{is an }\widetilde{F}^{-1}\left(0\right)\text{ open neighborhood of }pr^{-1}\left(\mathcal{E}^0\right),\end{aligned}$$

as verified below. Observe that the set above is clearly $\widetilde{F}^{-1}\left(0\right)$ open because $\left(O\left(\xi_i^0\right)\times V\left(\mathcal{E}^0\right)\right)$ is $\mathbb{E}$ open. Moreover, it is a neighborhood of $pr^{-1}\left(\mathcal{E}^0\right)$, as verified below:

$$\begin{aligned}&pr^{-1}\left(\mathcal{E}^0\right)\overset{(6.14)}{=}\left\{\left(\xi_1^0,\mathcal{E}^0\right),...,\left(\xi_{r_{\mathcal{E}^0}}^0,\mathcal{E}^0\right)\right\}\subseteq\\&\subseteq\cup_{i=1}^{r_{\mathcal{E}^0}}\left(\left(O\left(\xi_i^0\right)\times V\left(\mathcal{E}^0\right)\right)\cap\widetilde{F}^{-1}\left(0\right)\right),\end{aligned} \tag{6.15}$$

where the last inclusion is proved in detail below. For any $i\in\{1,...,r_{\mathcal{E}^0}\}$, what follows is true. Since $O\left(\xi_i^0\right)$ is a neighborhood of ξ_i^0 and $V\left(\mathcal{E}^0\right)$ is a neighborhood of $\mathcal{E}^0$, we have that $\left(\xi_i^0,\mathcal{E}^0\right)\in O\left(\xi_i^0\right)\times V\left(\mathcal{E}^0\right)$. Moreover, from Conclusion A. above, we have that $\widetilde{F}\left(\xi_i^0,\mathcal{E}^0\right)=0$ and therefore $\left(\xi_i^0,\mathcal{E}^0\right)\in\widetilde{F}^{-1}\left(0\right)$.

Observe that

$$pr\text{ is proper and therefore closed,} \tag{6.16}$$

and

$$pr\text{ is onto and therefore }\mathcal{E}^0\in\text{Im }(pr). \tag{6.17}$$

Then, we can apply Proposition 9, page 11, in Villanacci and others (2002),[9] with the following identifications about assumptions and conclusions.

[9]Let $f:X\to Y$ be a closed map between topological spaces X and Y. Let y be an element of $f\left(X\right)$, and let U be an X open neighborhood of $f^{-1}\left(y\right)$. Then there exists a Y open neighborhood V of y such that $f^{-1}\left(V\right)\subseteq U$.

Assumptions:

top space X	top space Y	$f : X \to Y$	$f : X \to Y$ closed	$y \in f(X)$
$\widetilde{F}^{-1}(0)$	$\mathbb{E}$	$pr : \widetilde{F}^{-1}(0) \to \mathbb{E}$	(6.16)	(6.17)

and

U	$f^{-1}(y)$	U open neighborhood of $f^{-1}(y)$
$\cup_{i=1}^{r_{\mathcal{E}^0}} \left(\left(O\left(\xi_i^0\right) \times V\left(\mathcal{E}^0\right) \right) \cap \widetilde{F}^{-1}(0) \right)$	$pr^{-1}\left(\mathcal{E}^0\right)$	(6.15)

Then the Conclusion of quoted Proposition 9 is

$$\text{there exists a } Y \text{ open neighborhood } V \text{ of } y \text{ such that } f^{-1}(V) \subseteq U;$$

which, in the analyzed case, becomes

$$\begin{array}{|l|}
\hline
\text{there exists a } \mathbb{E} \text{ open neighborhood } V^*\left(\mathcal{E}^0\right) \text{ of } \mathcal{E}^0 \text{ such that} \\
\hline
\\
\hline
pr^{-1}\left(V^*\left(\mathcal{E}^0\right)\right) := \left\{ (\xi, \mathcal{E}) \in \Xi \times V^*\left(\mathcal{E}^0\right) : \widetilde{F}(\xi, \mathcal{E}) = 0 \right\} \subseteq \\
\hline
\\
\hline
\subseteq \cup_{i=1}^{r_{\mathcal{E}^0}} \left(\left(O\left(\xi_i^0\right) \times V\left(\mathcal{E}^0\right) \right) \cap \widetilde{F}^{-1}(0) \right) = \\
\hline
\\
\hline
= \cup_{i=1}^{r_{\mathcal{E}^0}} \left\{ (\xi, \mathcal{E}) \in O\left(\xi_i^0\right) \times V\left(\mathcal{E}^0\right) : \widetilde{F}(\xi, \mathcal{E}) = 0 \right\}. \\
\hline
\end{array} \tag{6.18}$$

For any $i \in \{1, ..., r_{\mathcal{E}^0}\}$, define

$$W^i = \left\{ (\xi, \mathcal{E}) \in O\left(\xi_i^0\right) \times V\left(\mathcal{E}^0\right) : \widetilde{F}(\xi, \mathcal{E}) = 0 \right\}$$

which is an $\widetilde{F}^{-1}(0)$ open neighborhood of $\left(\xi_i^0, \mathcal{E}^0\right)$. Therefore, from (6.18), we have

$$pr^{-1}\left(V^*\left(\mathcal{E}^0\right)\right) \subseteq \cup_{i=1}^{r_{\mathcal{E}^0}} W^i. \tag{6.19}$$

Moreover, define

$$V^{**}\left(\mathcal{E}^0\right) = V\left(\mathcal{E}^0\right) \cap V^*\left(\mathcal{E}^0\right), \tag{6.20}$$

which is an open neighborhood of $\mathcal{E}^0$ and for any $i \in \{1, ..., r_{\mathcal{E}^0}\}$

$$W_*^i := pr^{-1}\left(V^{**}\left(\mathcal{E}^0\right)\right) \cap W^i \overset{\text{def.}}{=}{}^{W^i} \left\{ (\xi, \mathcal{E}) \in O\left(\xi_i^0\right) \times V^{**}\left(\mathcal{E}^0\right) : \widetilde{F}(\xi, \mathcal{E}) = 0 \right\}, \tag{6.21}$$

which is a $\widetilde{F}^{-1}(0)$ open neighborhood of $\left(\xi_i^0, \mathcal{E}^0\right)$.

Observe that for any $i, j \in \{1, ..., r_{\mathcal{E}^0}\}$ with $i \neq j$, we have $W_*^i \cap W_*^j = \varnothing$ because $W^i \cap W^j = \varnothing$, which follows from the fact that $O\left(\xi_i^0\right) \cap O\left(\xi_j^0\right) = \varnothing$. Moreover,

$$pr^{-1}\left(V^{**}\left(\mathcal{E}^0\right)\right) \overset{(6.20)}{\subseteq} pr^{-1}\left(V^*\left(\mathcal{E}^0\right)\right) \overset{(6.19)}{\subseteq} \cup_{i=1}^{r_{\mathcal{E}^0}} W^i. \tag{6.22}$$

Therefore,

$$\begin{aligned}
&pr^{-1}\left(V^{**}\left(\mathcal{E}^0\right)\right) \overset{(6.22)}{=} pr^{-1}\left(V^{**}\left(\mathcal{E}^0\right)\right) \cap \left(\cup_{i=1}^{r_{\mathcal{E}^0}} W^i\right) = \\
&= \cup_{i=1}^{r_{\mathcal{E}^0}} \left(pr^{-1}\left(V^{**}\left(\mathcal{E}^0\right)\right) \cap W^i \right) \overset{(6.21)}{=} \cup_{i=1}^{r_{\mathcal{E}^0}} W_*^i.
\end{aligned} \tag{6.23}$$

Summarizing, we get the desired result 6. as follows.

$$\begin{aligned}
&pr^{-1}\left(V^{**}\left(\mathcal{E}^0\right)\right) := \left\{ (\xi, \mathcal{E}) \in \Xi \times V^{**}\left(\mathcal{E}^0\right) : \widetilde{F}(\xi, \mathcal{E}) = 0 \right\} \overset{(6.23)}{=} \cup_{i=1}^{r_{\mathcal{E}^0}} W_*^i \overset{(6.21)}{=} \\
&= \cup_{i=1}^{r_{\mathcal{E}^0}} \left\{ (\xi, \mathcal{E}) \in O\left(\xi_i^0\right) \times V^{**}\left(\mathcal{E}^0\right) : \widetilde{F}(\xi, \mathcal{E}) = 0 \right\} \overset{(1)}{=} \\
&= \cup_{i=1}^{r_{\mathcal{E}^0}} \left\{ (\xi, \mathcal{E}) \in O\left(\xi_i^0\right) \times V^{**}\left(\mathcal{E}^0\right) : \zeta_i(\mathcal{E}) = \xi \right\}
\end{aligned}$$

where (1) follows from 5., Claim 0 and the fact that $V^{**}\left(\mathcal{E}^0\right) = V\left(\mathcal{E}^0\right) \cap V^{*}\left(\mathcal{E}^0\right)$.
Claim.

$$\cup_{i=1}^{r_{\mathcal{E}^0}} \left\{(\xi, \mathcal{E}) \in O\left(\xi_i^0\right) \times V^{**}\left(\mathcal{E}^0\right) : \zeta_i(\mathcal{E}) = \xi\right\} \overset{(2)}{=}$$

$$\cup_{i=1}^{r_{\mathcal{E}^0}} \left\{(\xi, \mathcal{E}) \in \Xi \times V^{**}\left(\mathcal{E}^0\right) : \zeta_i(\mathcal{E}) = \xi\right\}.$$

Proof of the Claim.
(2) follows from the inclusion arguments below. We want to show that

$$\begin{aligned} A &:= \cup_{i=1}^{r_{\mathcal{E}^0}} \left\{(\xi, \mathcal{E}) \in O\left(\xi_i^0\right) \times V^{**}\left(\mathcal{E}^0\right) : \zeta_i(\mathcal{E}) = \xi\right\} = \\ &= \cup_{i=1}^{r_{\mathcal{E}^0}} \left\{(\xi, \mathcal{E}) \in \Xi \times V^{**}\left(\mathcal{E}^0\right) : \zeta_i(\mathcal{E}) = \xi\right\} := B \end{aligned} \tag{6.24}$$

[$\subseteq$] Obvious.
[$\supseteq$]
Take $(\xi, \mathcal{E}) \in B$.
Assumption. There exists $i \in \{1, ..., r_{\mathcal{E}^0}\}$ such that $(\xi, \mathcal{E}) \in \Xi \times V^{**}\left(\mathcal{E}^0\right)$ and $\zeta_i(\mathcal{E}) = \xi$.
Suppose our desired conclusion does not hold true, i.e.,
$\neg$ (there exists $j \in \{1, ..., r_{\mathcal{E}^0}\}$ such that $(\xi, \mathcal{E}) \in O\left(\xi_j^0\right) \times V^{**}\left(\mathcal{E}^0\right)$ and $\zeta_j(\mathcal{E}) = \xi$), i.e.,
for any $j \in \{1, ..., r_{\mathcal{E}^0}\}$, either

$$\text{a. } (\xi, \mathcal{E}) \notin O\left(\xi_j^0\right) \times V^{**}\left(\mathcal{E}^0\right) \text{ or}$$

$$\text{b. } \zeta_j(\mathcal{E}) \neq \xi.$$

Since by assumption, $\zeta_i(\mathcal{E}) = \xi$, then for $j = i$, b. does not hold and a. holds. Then, we have that $(\xi, \mathcal{E}) \notin O\left(\xi_i^0\right) \times V^{**}\left(\mathcal{E}^0\right)$; since by assumption $(\xi, \mathcal{E}) \in \Xi \times V^{**}\left(\mathcal{E}^0\right)$, then we have that

$$\xi \notin O\left(\xi_i^0\right). \tag{6.25}$$

Since, from (6.20), $V^{**}\left(\mathcal{E}^0\right) \subseteq V\left(\mathcal{E}^0\right)$, (6.25) contradicts Condition 2. in (6.12), as desired.
Property (6.13), i.e., Conclusion 6, follows up to considering $V\left(\mathcal{E}^0\right)$ in (6.13) as a smaller open neighborhood of $\mathcal{E}^0$ in $\mathcal{D}$ so that (6.24) holds.

Second proof (the one proposed in Gori (2007)).
Claim 1. There exists a $\mathbb{E}$ open neighborhood $V\left(\mathcal{E}^0\right)$ of $\mathcal{E}^0$ such that

$$\cup_{i=1}^{r_{\mathcal{E}^0}} \left\{(\xi, \mathcal{E}) \in O\left(\xi_i^0\right) \times V\left(\mathcal{E}^0\right) : \widetilde{F}(\xi, \mathcal{E}) = 0\right\} =$$

$$\left\{(\xi, \mathcal{E}) \in \Xi \times V\left(\mathcal{E}^0\right) : \widetilde{F}(\xi, \mathcal{E}) = 0\right\}.$$

Proof of Claim 1.
[$\subseteq$]
Obvious, because for any $i \in \{1, ..., r_{\mathcal{E}^0}\}$, $O\left(\xi_i^0\right) \subseteq \Xi$.
[$\supseteq$]
We want to show that

$$\exists \text{ a neighborhood } V\left(\mathcal{E}^0\right) \text{ of } \mathcal{E}^0 \text{ such that } \forall (\xi, \mathcal{E}),$$

$$\left\langle \ (\xi, \mathcal{E}) \in \Xi \times V\left(\mathcal{E}^0\right) \text{ and } \widetilde{F}(\xi, \mathcal{E}) = 0 \ \right\rangle \Rightarrow \ \xi \in \cup_{i=1}^{r_{\mathcal{E}^0}} O\left(\xi_i^0\right).$$

Suppose otherwise, i.e.,

$$\begin{aligned} &\text{for any } \mathbb{E} \text{ open neighborhood } \ N^0 \text{ of } \mathcal{E}^0, \\ &\exists \mathcal{E} \in N^0 \ \ \exists \xi \in \Xi \text{ such that } \widetilde{F}(\xi, \mathcal{E}) = 0 \text{ and } \xi \notin \cup_{i=1}^{r_{\mathcal{E}^0}} O\left(\xi_i^0\right). \end{aligned} \tag{6.26}$$

Let $\mathcal{E}^0 = (e^0, u^0, v^0)$. For any $k \in \mathbb{N}$ define

$$V^{[k]} = B_{\mathbb{R}^{CH}}\left(e^0, \frac{1}{k}\right) \times B_{\mathcal{U}^H}\left(u^0, \frac{1}{k}\right) \times B_{\mathcal{V}^B}\left(v^0, \frac{1}{k}\right)$$

which is a $\mathbb{E}$ open neighborhood N^0 of $\mathcal{E}^0$. Then $\exists \mathcal{E}^{[k]} \in V^{[k]}\left(\mathcal{E}^0\right)$ and $\exists \xi^{[k]} \in \Xi$ such that $\widetilde{F}\left(\xi^{[k]}, \mathcal{E}^{[k]}\right) = 0$ and $\xi^{[k]} \notin \cup_{i=1}^{r_{\mathcal{E}^0}} O\left(\xi_i^0\right)$. Moreover, by definition of $V^{[k]}$, we have that $\mathcal{E}^{[k]} \to \mathcal{E}^0$. Then, from Lemma 70 below, we have that, up to a subsequence, $\xi^{[k]} \to \xi^0$.

Then since $\widetilde{F}$ is continuous, $0 = \widetilde{F}\left(\xi^{[k]}, \mathcal{E}^{[k]}\right) \to \widetilde{F}\left(\xi^0, \mathcal{E}^0\right) = 0$. Therefore,

$$\xi^0 \in \left(\widetilde{F}_{\mathcal{E}^0}\right)^{-1}(0) \overset{\text{Part A.}}{=} \left\{\xi_i^0 : i \in \{1, ..., r_{\mathcal{E}^0}\}\right\} \subseteq \cup_{i=1}^{r_{\mathcal{E}^0}} O\left(\xi_i^0\right). \tag{6.27}$$

Since for any $k \in \mathbb{N}$, we have $\xi^{[k]} \in \cap_{i=1}^{r_{\mathcal{E}^0}} \left(O\left(\xi_i^0\right)\right)^C$, which is a Ξ closed set, then we have that

$$\xi^0 \in \cap_{i=1}^{r_{\mathcal{E}^0}} \left(O\left(\xi_i^0\right)\right)^C,$$

contradicting (6.27).

End of the proof of Claim 1.

Therefore, we get

$$\left\{(\xi, \mathcal{E}) \in \Xi \times V\left(\mathcal{E}^0\right) : \widetilde{F}(\xi, \mathcal{E}) = 0\right\} \overset{\text{Claim 1.}}{=}$$

$$\cup_{i=1}^{r_{\mathcal{E}^0}} \left\{(\xi, \mathcal{E}) \in O\left(\xi_i^0\right) \times V\left(\mathcal{E}^0\right) : \widetilde{F}(\xi, \mathcal{E}) = 0\right\} \overset{\text{Conclusion } \mathbf{5}.}{=}$$

$$\cup_{i=1}^{r_{\mathcal{E}^0}} \left\{(\xi, \mathcal{E}) \in O\left(\xi_i^0\right) \times V\left(\mathcal{E}^0\right) : \xi = \zeta_i(\mathcal{E})\right\}$$

i.e., Conclusion **6**. ∎

Lemma 70 *Let $\left(e^{[k]}, u^{[k]}, v^{[k]}\right)_{k\in\mathbb{N}} \in \left(\mathbb{R}_{++}^{CH} \times \mathcal{U}^H \times \mathcal{V}^B\right)^\infty$ such that $\left(e^{[k]}, u^{[k]}, v^{[k]}\right) \overset{k}{\to} \mathcal{E}^0 \in \mathbb{R}_{++}^{CH} \times \mathcal{U}^H \times \mathcal{V}^B$ be given. Assume that for any $k \in \mathbb{N}$, there exists $\xi^{[k]} \in \Xi$ such that $\widetilde{F}\left(\xi^{[k]}, e^{[k]}, u^{[k]}, v^{[k]}\right) = 0$. Then, $\left(\xi^{[k]}\right)_{k\in\mathbb{N}}$ admits a convergent subsequence to $\xi^* \in \Xi$ such that $\widetilde{F}\left(\xi^*, \mathcal{E}^0\right) = 0$.*

Proof. It is an easy consequence of the properness of pr. Consider the compact set $C := \left\{\left(e^{[k]}, u^{[k]}, v^{[k]}\right) : k \in \mathbb{N}\right\} \cup \left\{\mathcal{E}^0\right\}$. Since pr is proper, then $pr^{-1}(C)$ is compact. Since

$$\text{for any } k \in \mathbb{N}, \text{ there exists } \xi^{[k]} \in \Xi \text{ such that } \widetilde{F}\left(\xi^{[k]}, e^{[k]}, u^{[k]}, v^{[k]}\right) = 0, \tag{6.28}$$

i.e., for any $k \in \mathbb{N}$, $\left(\xi^{[k]}, e^{[k]}, u^{[k]}, v^{[k]}\right) \in pr^{-1}\left(e^{[k]}, u^{[k]}, v^{[k]}\right)$, we have that $\left\{\left(\xi^{[k]}, e^{[k]}, u^{[k]}, v^{[k]}\right) : k \in \mathbb{N}\right\} \subseteq pr^{-1}(C)$ and therefore admits a subsequence, without loss of generality the sequence itself, which converges to (ξ^*, e^*, u^*, v^*). Since if a sequence converges, then any subsequence converges to the same limit, we do have that $(\xi^*, e^*, u^*, v^*) = \left(\xi^*, \mathcal{E}^0\right)$. Finally since $\widetilde{F}$ is continuous and using (6.28), we have that $0 = \lim_{k\to+\infty} \widetilde{F}\left(\xi^{[k]}, e^{[k]}, u^{[k]}, v^{[k]}\right) = \widetilde{F}\left(\xi^*, \mathcal{E}^0\right)$. ∎

Chapter 7

Appendix

7.1 Some remarks on the utility function

Proposition 71 *a. For any $h \in \mathcal{H}$, u_h is unbounded below and*
b. For any $h \in \mathcal{H}$ and $h' \in \mathcal{B}_h$, $v_{hh'}$ is bounded above .

Proof. a.
Suppose otherwise, i.e., u_h is bounded below, i.e.,

$$\exists k \in \mathbb{R} \text{ such that } \forall x_h \in \mathbb{R}^C_{++},\ u_h(x) \geq k. \tag{7.1}$$

Then, defined $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^C$, consider the sequence $\left(\frac{\mathbf{1}}{n}\right)_{n\in\mathbb{N}}$. Then, from (7.1),

$$\left\{\frac{\mathbf{1}}{n} : n \in \mathbb{N}\right\} \subseteq \left\{x \in \mathbb{R}^C_{++} : u_h(x_h) \geq k\right\}. \tag{7.2}$$

Obviously, $\frac{1}{n}\mathbf{1} \longrightarrow 0 \in \mathbb{R}^C$. Then,

$$0 \in \mathrm{Cl}_{\mathbb{R}^C}\left(\left\{\frac{\mathbf{1}}{n} : n \in \mathbb{N}\right\}\right) \overset{(7.2)}{\subseteq} \mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x \in \mathbb{R}^C_{++} : u_h(x_h) \geq k\right\}\right) \overset{\text{Assu. } 4}{\subseteq} \mathbb{R}^C_{++},$$

which is the desired contradiction.
b.
For simplicity, drop the subscript hh'. Take $x_0 \in (0,1)$. For any $x \in (x_0, 1)$, since v is concave and differentiable, then we have

$$v(x) \leq v(x_0) + v'(x_0)(x - x_0).$$

Since $x \in (x_0, 1)$, we have $x - x_0 < 1$ and since $v'(x_0) > 0$, we have

$$v'(x_0)(x - x_0) < v'(x_0).$$

Then,

$$\text{for any } x_0 \in (0,1),\ \text{ for any } x \in (x_0, 1)\ , \qquad v(x) < v(x_0) + v'(x_0).$$

Since v is strictly increasing, we have

$$\text{for any } x_0 \in (0,1),\ \text{ for any } x \in (0, x_0]\ ,\ v(x) < v(x_0) < v(x_0) + \overset{(>0)}{v'(x_0)}.$$

Summarizing, for any $x_0 \in (0,1)$ and for any $x \in (0,1)$, we have $v(x) < v(x_0) + v'(x_0)$. Therefore, for any $x \in (0,1)$, we have $v(x) < v(\frac{1}{2}) + v'(\frac{1}{2})$, as desired. ∎

Lemma 72 $\mathrm{Cl}_{\mathbb{R}^C}\left(u_h^{-1}([\underline{u}, +\infty))\right) \subseteq \mathbb{R}^C_{++} \ \Leftrightarrow\ \mathrm{Cl}_{\mathbb{R}^C}\left(u_h^{-1}([\underline{u}, +\infty))\right) \cap \partial\mathbb{R}^C_{++} = \varnothing.$

Proof. Preliminary Remarks.
1. Since $\mathbb{R}^C_{++}$ is open, then from Proposition 647 in Math 2 Notes, we have that

$$\mathbb{R}^C_{++} \cap \partial\left(\mathbb{R}^C_{++}\right) = \varnothing. \tag{7.3}$$

2.

$$\mathrm{Cl}_{\mathbb{R}^C}\left(\mathbb{R}^C_{++}\right) = \mathbb{R}^C_{+}. \tag{7.4}$$

[$\subseteq$] Since $\mathbb{R}^C_{++} \subseteq \mathbb{R}^C_{+}$, then $\mathrm{Cl}_{\mathbb{R}^C}\left(\mathbb{R}^C_{++}\right) = \mathrm{Cl}_{\mathbb{R}^C}\left(\mathbb{R}^C_{+}\right) \overset{\mathbb{R}^C_+ \text{ is closed}}{=} \mathbb{R}^C_{+}$.

[$\supseteq$] Taken $x \in \mathbb{R}^C_{+}$, consider the sequence $\left(x + \frac{1}{n}\mathbf{1}_C\right)_{n\in\mathbb{N}}$ whose elements belong to $\mathbb{R}^C_{++}$. Then $x + \frac{1}{n}\mathbf{1}_C \to x$, and therefore $x \in \mathrm{Ad}\left(\mathbb{R}^C_{++}\right) = \mathrm{Cl}_{\mathbb{R}^C}\left(\mathbb{R}^C_{++}\right)$.

3. If $u_h^{-1}\left([\underline{u}, +\infty)\right) = \varnothing$, then the desired result is trivially true.

End of Preliminary Remarks.

[$\Rightarrow$]

Observe that $\mathrm{Cl}_{\mathbb{R}^C}\left(u_h^{-1}\left([\underline{u}, +\infty)\right)\right) \cap \partial\mathbb{R}^C_{++} \overset{\text{Assu.}}{\subseteq} \mathbb{R}^C_{++} \cap \partial\mathbb{R}^C_{++} \overset{(7.3)}{=} \varnothing$.

[$\Leftarrow$]

Suppose otherwise, i.e., there exists $x \in \mathbb{R}^C$ such that

$$x \in \mathrm{Cl}_{\mathbb{R}^C}\left(u_h^{-1}\left([\underline{u}, +\infty)\right)\right) \tag{7.5}$$

and $x \notin \mathbb{R}^C_{++}$. Since $u_h^{-1}\left([\underline{u}, +\infty)\right) \subseteq \mathbb{R}^C_{++}$, then

$$\mathrm{Cl}_{\mathbb{R}^C}\left(u_h^{-1}\left([\underline{u}, +\infty)\right)\right) \subseteq \mathrm{Cl}_{\mathbb{R}^C}\left(\mathbb{R}^C_{++}\right) \overset{(7.4)}{=} \mathbb{R}^C_{+}.$$

Therefore

$$x \in \mathbb{R}^C_{+} \text{ and } x \notin \mathbb{R}^C_{++},$$

and then $x \in \partial\mathbb{R}^C_{++}$, and, then, from (7.5), we get the desired contradiction. ∎

Proposition 73 *Assume that u_h satisfies Assumptions u1, u2 and u3. Then,*

$$\left\langle \begin{array}{l} \textit{u4': for any } \underline{x} \in \mathbb{R}^C_{++},\ \mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}^C_{++} : u_h\left(x_h\right) \geq u\left(\underline{x}\right)\right\}\right) \subseteq \mathbb{R}^C_{++}, \textit{ and} \\ \\ \textit{u}_h \textit{ is unbounded below.} \end{array} \right\rangle$$

1.⇔

$\langle$ *u4. for any* $\alpha \in \mathbb{R}$, $\mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}^C_{++} : u_h\left(x_h\right) \geq \alpha\right\}\right) \subseteq \mathbb{R}^C_{++}$. $\rangle$

2. ⇔

$\langle$ *u4". for any* $\alpha \in \mathbb{R}$, $\left\{x_h \in \mathbb{R}^C_{++} : u_h\left(x_h\right) \geq \alpha\right\}$ *is* $\mathbb{R}^C$ *closed.* $\rangle$

Proof. 1. [$\Rightarrow$]

From Proposition 71.1, is u_h is unbounded below. Moreover, for any $\underline{x} \in \mathbb{R}^C_{++}$ defined $\underline{u} := u\left(\underline{x}\right)$, from u4, we have $\mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}^C_{++} : u_h\left(x_h\right) \geq \underline{u}\right\}\right) \subseteq \mathbb{R}^C_{++}$.

1. [$\Leftarrow$]

Take $k \in \mathbb{R}$. Since u_h is unbounded below, then there exists $x_{hk} \in \mathbb{R}^C_{++}$ such that $u_h\left(x_{hk}\right) < k$. Then,

$$\left\{x_h \in \mathbb{R}^C_{++} : u_h\left(x_h\right) \geq k\right\} \subseteq \left\{x_h \in \mathbb{R}^C_{++} : u_h\left(x_h\right) \geq u_h\left(x_{hk}\right)\right\}$$

and

$$\mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}^C_{++} : u_h\left(x_h\right) \geq k\right\}\right) \subseteq \mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}^C_{++} : u_h\left(x_h\right) \geq u_h\left(x_{hk}\right)\right\}\right) \subseteq \mathbb{R}^C_{++},$$

where the last inclusion follows from the assumption.

The proof of 2. ⇔ requires some results and it is provided in Proposition 77 below. ∎

Remark 74 *Observe that it is false that u4' implies u4. Indeed, $u : \mathbb{R}_{++} \longrightarrow \mathbb{R}$, $x \mapsto \log\left(1 + x\right)$ satisfies u4', but not u4. (Meanwhile $\widehat{u} : \mathbb{R}_{++} \longrightarrow \mathbb{R}$, $x \mapsto \log\left(x\right)$ satisfies both u4 and u4').*

Lemma 75 *Let a topological space $(X, \mathcal{T})$, a topological subspace $(Y, \mathcal{T}_Y)$ (i.e., $Y \subseteq X$ and $\mathcal{T}_Y$ be the relative topology) and a set $B \subseteq Y$. Then, $\mathrm{Cl}_{(Y,\mathcal{T}_Y)}\left(B\right) = \mathrm{Cl}_{(X,\mathcal{T})}\left(B\right) \cap Y$.*

Proof. See Theorem 17.4, page 95 in Munkres (2000). ■

Proposition 76 *Given a topological space $(X, \mathcal{T}_X)$ and sets $B, Y \subseteq X$, if*
1. $B \subseteq Y \subseteq X$,
2. B is Y-closed, and
3. $\mathrm{Cl}_{(X,\mathcal{T}_X)}(B) \subseteq Y$,
then B is X-closed.

Proof. From Assumption 1, we can use Lemma 75.

$$\mathrm{Cl}_{(X,\mathcal{T}_X)}(B) \overset{\text{Assu. } 3}{=} \mathrm{Cl}_{(X,\mathcal{T}_X)}(B) \cap Y \overset{\text{Lemma } 75}{=} \mathrm{Cl}_{(Y,\mathcal{T}_Y)}(B) \overset{\text{Assu. } 2}{=} B.$$

■

Proposition 77 *For any $\alpha \in \mathbb{R}$,*

$$\left\langle \ \mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \alpha\right\}\right) \subseteq \mathbb{R}^C_{++} \ \right\rangle \Leftrightarrow \left\langle \ \left\{x \in \mathbb{R}^C_{++} : u_h(x) \geq \alpha\right\} \text{ is } \mathbb{R}^C \text{ closed} \ \right\rangle$$

Proof. [⇒]
Take $X = \mathbb{R}^C$, $Y = \mathbb{R}^C_{++}$ and $B_\alpha := \left\{x_h \in \mathbb{R}^C_{++} : u(x_h) \geq \alpha\right\}$ where $u : \mathbb{R}^C_{++} \to \mathbb{R}$ is a continuous function and $\mathrm{Cl}_{\mathbb{R}^C}(B_\alpha) \subseteq \mathbb{R}^C_{++}$. Observe that
1. $B_\alpha \subseteq \mathbb{R}^C_{++} \subseteq \mathbb{R}^C$,
2. B_α is $\mathbb{R}^C_{++}$-closed, because u is continuous;
3. $\mathrm{Cl}_{\mathbb{R}^C}(B_\alpha) \subseteq \mathbb{R}^C_{++}$, by assumption.
Then, from the previous Proposition, $\left\{x_h \in \mathbb{R}^C_{++} : u(x_h) \geq \alpha\right\}$ is $\mathbb{R}^C$-closed.
[⇐] Recall $B_\alpha = \left\{x_h \in \mathbb{R}^C_{++} : u(x_h) \geq \alpha\right\}$ and the basic fact

$$B_\alpha \text{ is } \mathbb{R}^C \text{ closed } \Leftrightarrow \mathrm{Cl}_{\mathbb{R}^C}(B_\alpha) = B_\alpha. \tag{7.6}$$

Then,

$$\mathrm{Cl}_{\mathbb{R}^C}(B_\alpha) \overset{\text{Assu. and (7.6)}}{=} B_\alpha \overset{\text{Def. } B_\alpha}{\subseteq} \mathbb{R}^C_{++}.$$

■

7.2 Various basic facts

Definition 78 *Given two topological spaces $(X, \mathcal{T}_X)$ and $(Z, \mathcal{T}_Z)$, then*

$$\mathcal{B} = \{U \times V : U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Z\}$$

is a basis for a topology on $X \times Y$, called the box or product topology $\mathcal{T}_{X \times Z}$.

Remark 79 *Therefore if $(x, z) \in S \in \mathcal{T}_{X \times Z}$, then there exist $U_x \in \mathcal{T}_X$ and $V_z \in \mathcal{T}_Z$ such that $(x, z) \in U_x \times V_z \subseteq S$.*

Definition 80 *Given a topological space $(X, \mathcal{T}_X)$ and $x \in X$, the family of neighborhoods of x is denoted and defined as follows.*

$$\mathcal{N}_{(X,\mathcal{T}_X)}(x) = \{U \subseteq X : x \in U \text{ and } U \in \mathcal{T}_X\}.$$

Definition 81 *Given a set $A \subseteq (X, \mathcal{T}_X)$,the set of adherent points to A is denoted and defined as follows.*

$$\mathrm{Ad}_{(X,\mathcal{T}_X)}(A) = \left\{x \in X : \text{ for any } U \in \mathcal{N}_{(X,\mathcal{T}_X)}(x),\ U \cap A \neq \varnothing\right\}.$$

Proposition 82

$$\mathrm{Ad}_{(X,\mathcal{T}_X)}(A) = \mathrm{Cl}_{(X,\mathcal{T}_X)}(A).$$

Proof. See Math 2 notes. The proof is presented there for metric space, but it does generalize. ■

Proposition 83 *The intersection of a finite number of open and dense sets is open and dense.*

Proof. Take two open and dense sets V and W. We want to show that for any open nonempty set A, we do have $A \cap (V \cap W) \neq \varnothing$. Indeed, $A \cap (V \cap W) = (A \cap V) \cap W$. Since V is dense, then $A \cap V$ is open and nonempty. Then, since W is open and dense, then $(A \cap V) \cap W \neq \varnothing$, as desired. ■

Proposition 84 *The intersection of a finite (in fact countable) number of full measure sets has full measure.*

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a family of sets with full measure, i.e., such that for any $n \in \mathbb{N}$, $\mu\left(A_n^C\right) = 0$. We want to show that $\cap_{n\in\mathbb{N}} A_n$ has full measure, i.e., $\mu\left(\left(\cap_{n\in\mathbb{N}} A_n\right)^C\right) = 0$. Indeed, $\mu\left(\left(\cap_{n\in\mathbb{N}} A_n\right)^C\right) = \mu\left(\cup_{n\in\mathbb{N}} A_n^C\right) \leq \sum_{n\in\mathbb{N}} \mu\left(A_n^C\right) = 0$, where the weak inequality follows from countable subadditivity of measure. ■

Proposition 85 *(Corollary 2.3.9, page 64, in Webster (1984))If K is a convex subset of $\mathbb{R}^n$ such that $\operatorname{Int}(K) \neq \varnothing$, then $\operatorname{Cl}(\operatorname{Int} K) = \operatorname{Cl} K$.*

Proposition 86 *(Proposition 2.38, page 50 in Hu and Papageorgiou (1997)) A set-valued function $\varphi : X \rightrightarrows Y$ is lower hemi-continuous if and only if $\operatorname{Cl}(\varphi)$ is lower hemi-continuous.*

Proposition 87 *(in Hildebrand (1974), bottom page 23) If $\varphi : X \rightrightarrows Y$ is a set valued function which is closed graph and if $\varphi(X)$ is contained in a compact set, then φ is upper hemi-continuous.*

Proposition 88 *(Lemma 1, page 33, in Hildebrand (1974)) If a set-valued function φ of a metric space in $\mathbb{R}^n$ is non-empty valued, compact valued, convex valued, closed and lower hemi-continuous Then φ is upper hemi-continuous.*

Proposition 89 *Given $A \in \mathbb{M}(n,n), B \in \mathbb{M}(n,m), C \in \mathbb{M}(l,m)$ with $l \leq m$ and A nonsingular. Then,*

$$\operatorname{rank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} I_n & B \\ 0 & C \end{bmatrix}$$

Proof. Observe that

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}_{(n+l)\times(n+m)} \cdot \begin{bmatrix} A^{-1} & 0 \\ 0 & I_m \end{bmatrix}_{(n+m)\times(n+m)} = \begin{bmatrix} I_n & B \\ 0 & C \end{bmatrix}$$

and

$$\operatorname{rank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}_{(n+l)\times(n+m)} \cdot \begin{bmatrix} A^{-1} & 0 \\ 0 & I_m \end{bmatrix}_{(n+m)\times(n+m)} = \operatorname{rank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}_{(n+l)\times(n+m)},$$

where the equality follows from Proposition 22.2 in my notes on Differential Topology. ■

Proposition 90 *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that $(x_0, \mu_0) \in \mathbb{R}^2$ is such that*

$$g(x_0) > \mu_0.$$

Then there exists an open neighborhood $N \subseteq \mathbb{R}^2$ of (x_0, μ_0) such that for every $(x, \mu) \in N$, we have

$$g(x) > \mu.$$

Proof. Assume

$$g(x_0) > \mu_0.$$

Define

$$\delta := g(x_0) - \mu_0 > 0.$$

Since g is continuous at x_0, chosen

$$\varepsilon = \frac{\delta}{3},$$

there exists $\eta > 0$ such that

$$|x - x_0| < \eta \quad \Longrightarrow \quad |g(x) - g(x_0)| < \frac{\delta}{3}.$$

Hence,

$$g(x) > g(x_0) - \frac{\delta}{3}.$$

Now consider the open neighborhood

$$N = \left\{ (x,\mu) \in \mathbb{R}^2 : |x - x_0| < \eta,\ |\mu - \mu_0| < \frac{\delta}{3} \right\}.$$

For any $(x,\mu) \in N$,

$$g(x) > g(x_0) - \frac{\delta}{3} = \mu_0 + \delta - \frac{\delta}{3} = \mu_0 + \frac{2\delta}{3},$$

while

$$\mu < \mu_0 + \frac{\delta}{3}.$$

Therefore, for every $(x,\mu) \in N$,

$$g(x) - \mu > \left(\mu_0 + \frac{2\delta}{3}\right) - \left(\mu_0 + \frac{\delta}{3}\right) = \frac{\delta}{3} > 0,$$

and

$$g(x) > \mu.$$

Hence there exists an open neighborhood N of (x_0,μ_0) such that for all $(x,\mu) \in N$,

$$g(x) > \mu.$$

■

Proposition 91 *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, and $f\colon \mathbb{R}^2 \to \mathbb{R}$. We define*

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \qquad F(x,\mu) = \big(f(x,\mu), \min\{g(x),\mu\}\big),$$

and

$$F_1 : \mathbb{R}^2 \to \mathbb{R}^2, \qquad F_1(x,\mu) = \big(f(x,\mu),\mu\big).$$

Assume that $g(x_0) > \mu_0$. Then there exists an open neighborhood $N \subseteq \mathbb{R}^2$ of (x_0,μ_0) such that F and F_1 coincide on N.

Proof. Since $g(x_0) > \mu_0$ and g is continuous, from Proposition 90, there exists an open neighborhood N of (x_0,μ_0) such that

$$g(x) > \mu \qquad \text{for all } (x,\mu) \in N.$$

Therefore, for every $(x,\mu) \in N$,

$$\min\{g(x),\mu\} = \mu.$$

Hence,

$$F(x,\mu) = \big(f(x,\mu), \min\{g(x),\mu\}\big) = \big(f(x,\mu),\mu\big) = F_1(x,\mu).$$

Thus

$$F|_N = F_1|_N,$$

that is, F and F_1 coincide on N. ■

7.3 Some facts on the compact-open topology

Below, first we present some basic facts whose proofs are contained in Villanacci (2026). For facts not proven directly in that work, precise references to the underlying literature are provided there. For foundational concepts regarding the compact-open topology on spaces of continuous functions, we refer the reader to Munkres (2000), Hart (2003), and Schmeding (2022). We note, moreover, that Schmeding (2022) extends this framework to $\mathcal{C}^\infty$ functions.

Next, we describe in detail several results more specific to our analysis.

7.3.1 Basic results

Definition 92 *Let the following objects be given: $(X,\mathcal{T})$, $(Y,\mathcal{T}')$ topological spaces and the class $\mathcal{A}$ of compact subsets of X. The topology generated by the subbasis*

$$\mathcal{S}_{co}=\{F(A,G)\subseteq\mathcal{F}(X,Y):A\in\mathcal{A},G\in\mathcal{T}'\} \tag{7.7}$$

is called compact-open topology on $\mathcal{F}(X,Y)$ and it is denoted by $\mathcal{T}_{co}$.

Definition 93 *Let the following objects be given: a topological space $(X,\mathcal{T})$, a metric space (Y,d) and a sequence $(f_n)_{n\in\mathbb{N}}\in\mathcal{F}(X,Y)^{\infty}$. We say that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compact sets to $g\in\mathcal{F}(X,Y)$ if for any compact subset E of X, $\forall\varepsilon>0$, $\exists n_0:=n_0(E,\varepsilon)$ such that $\forall n>n_0$, $\forall\ x\in E$, $d(f_n(x),g(x))<\varepsilon$.*

Remark 94

$$\begin{array}{c}\textit{uniform}\\ \textit{convergence}\end{array}\quad\Rightarrow\quad\begin{array}{c}\textit{uniform convergence}\\ \textit{on compacta}\end{array}\quad\Rightarrow\quad\begin{array}{c}\textit{pointwise}\\ \textit{convergence}\end{array}$$

Theorem 95 *Let $(f_n)_{n\in\mathbb{N}}\in\mathcal{C}^0((X,\mathcal{T}),(Y,d))$ be given.*

(1) *$(f_n)_{n\in\mathbb{N}}$ converges uniformly on compacta to $g\in\mathcal{C}^0((X,\mathcal{T}),(Y,d))$*

$$\Leftrightarrow$$

(2) *$(f_n)_{n\in\mathbb{N}}$ converges to $g\in\mathcal{C}^0((X,\mathcal{T}),(Y,d))$ with respect to the compact-open topology.*

To define a metric on the compact-open topology we need to introduce the concept of hemicompactness.

Definition 96 *A topological space X is hemicompact if there exists a sequence of set K_n, $n\in\mathbb{N}$, contained in X such that*

1. *$X=\cup_{n=1}^{+\infty}K_n$, where for any $n\in\mathbb{N}$, $K_n\subseteq K_{n+1}$ and K_n is compact; and*
2. *If $K\subseteq X$ is a compact set, then there exists $N_K\in\mathbb{N}$ such that $K\subseteq K_{N_K}$.*

Proposition 97 *An open subset of an Euclidean subspace is hemicompact.*

Proposition 98 *Let X be a topological space and (Y,d') a metric space. Let $\mathcal{C}(X,Y)$ denote the set of continuous functions from X to Y. If X is hemicompact, then $\mathcal{C}(X,Y)$ with the compact-open topology is metrizable and the metric is defined as follows. Let the family $\{K_n:n\in\mathbb{N}\}$ presented above be given.*

For any $n\in\mathbb{N}$ and for any $f,g\in\mathcal{C}(K_n,Y)$,

$$d_{K_n}(f,g):=\max_{x\in K_n}d'(f(x),g(x))$$

Then a desired metric on $\mathcal{C}(X,Y)$ is

$$d(f,g)=\sum_{n=1}^{+\infty}\frac{1}{2^n}\min\{d_{K_n}(f,g),\ 1\}.$$

Proposition 99 *The metric*

$$d_1(f,g)=\sum_{n=1}^{+\infty}\min\left\{\frac{1}{2^n},d_{K_n}(f,g)\right\},$$

and the metric

$$d_2(f,g)=\sum_{n=1}^{+\infty}\frac{1}{2^n}\min\{d_{K_n}(f,g),\ 1\},$$

induce the same topology.

Definition 100 *Let X be a (real) vector space and let $\mathcal{T}$ be a topology on X. $(X,\mathcal{T})$ is a topological vector space if*

$$s:(X,\mathcal{T})\times(X,\mathcal{T})\to(X,\mathcal{T}),\ (x_1,x_2)\mapsto x_1+x_2$$

and

$$p:(\mathbb{R},\mathcal{T}_e)\times(X,\mathcal{T})\to(X,\mathcal{T}),\ (\lambda,x)\mapsto\lambda x$$

are continuous, where $\mathcal{T}_e$ denotes the Euclidean topology on $\mathbb{R}$.

Definition 101 *$(X, \mathcal{T})$ is a Frechet space if it is*

1. *a Topological vector space,*
2. *metrizable,*
3. *complete,*
4. *locally convex, i.e., every neighborhood of zero contains a convex neighborhood of zero.*

Proposition 102 *$\left(C^2\left(\mathbb{R}^n_{++}, \mathbb{R}\right), \mathcal{T}_2\right)$ is a Frechet space, where $\mathcal{T}_2$ is the topology induced by the metric d_2.*

Remark 103 *$\left(\mathcal{C}^2\left(\mathbb{R}^n_{++}, \mathbb{R}\right), \mathcal{T}_2\right)$ is not a normed space.*

7.3.2 Open neighborhoods

The following definitions and properties are adapted from Villanacci (2026), modified here to hold within the space $\mathcal{C}^3$.

We first introduce some notation. Let A be an open subset of $\mathbb{R}^n$, and equip $\mathcal{C}^3(A, \mathbb{R})$ with the compact-open topology. We denote by $\mathcal{T}^1_{co}$ the topology induced by the compact-open topology on $\mathcal{C}^3(A, \mathbb{R})$. We will consider also the space $\mathcal{C}^3((0,1), \mathbb{R})$ equipped with the compact-open topology, which we denote by $\mathcal{T}^2_{co}$.

We recall some notions about norms of higher-order derivatives and their relation with the compact-open topology.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be sufficiently differentiable. The meaning of $\|D^2 f(x)\|$ and $\|D^3 f(x)\|$ depends on which norm is imposed on the corresponding multilinear maps. The standard choice is the *operator norm*.

The second derivative $D^2 f(x)$ is a bilinear map

$$D^2 f(x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}.$$

Its operator norm is defined by

$$\|D^2 f(x)\| = \sup_{\|v\| \leq 1,\ \|w\| \leq 1} |D^2 f(x)[v, w]|.$$

Equivalently,

$$\|D^2 f(x)\| = \sup_{\|v\|=1, \|w\|=1} |v^\top D^2 f(x) w|.$$

Similarly, the third derivative

$$D^3 f(x) : (\mathbb{R}^n)^3 \to \mathbb{R}$$

is a trilinear map. Its operator norm is

$$\|D^3 f(x)\| = \sup_{\|v\|, \|w\|, \|z\| \leq 1} |D^3 f(x)[v, w, z]|,$$

where

$$D^3 f(x)[v, w, z] = \sum_{i,j,k=1}^{n} \frac{\partial^3 f(x)}{\partial x_i \partial x_j \partial x_k} v_i w_j z_k.$$

Hence, its norm is the supremum of the absolute value of this expression over unit vectors v, w, z.

Remark 104 (Relation with the compact-open topology) *When considering the space $\mathcal{C}^r(\mathbb{R}^C_{++}, \mathbb{R})$, we say that*

$$f_n \longrightarrow f$$

in the $\mathcal{C}^r$ compact-open topology if the derivatives of all orders up to r converge uniformly on every compact subset of $\mathbb{R}^C_{++}$.

In norm notation, this means that, for every compact $K \subset \mathbb{R}^C_{++}$,

$$\sup_{x \in K} \|D^j f_n(x) - D^j f(x)\| \longrightarrow 0, \qquad j = 0, \ldots, r,$$

where $\|D^j f(x)\|$ denotes the operator norm of the j-linear map $D^j f(x)$.

For $j = 0$, one interprets $D^0 f = f$. Thus, the corresponding term is simply

$$\sup_{x \in K} |f_n(x) - f(x)|.$$

Therefore, if the compact-open $\mathcal{C}^r$ topology is being defined formally, it is useful to specify the norm used on the spaces of multilinear maps.

Definition 105 *For any $f \in \mathcal{C}^3(A,\mathbb{R})$, compact subset K of A and any $\varepsilon \in \mathbb{R}_{++}$,*

$$V(f,K,\varepsilon) :=$$
$$= \left\{g \in \mathcal{C}^3(A,\mathbb{R}) : \forall x \in K \ \ |f(x)-g(x)| < \varepsilon, \|D(g-f)(x)\| < \varepsilon, \left\|D^2(g-f)(x)\right\| < \varepsilon, \left\|D^3(g-f)(x)\right\| < \varepsilon\right\}.$$

We are going to show that
1. $f \in V(f,K,\varepsilon)$;
2. $V(f,K,\varepsilon) \in \mathcal{T}^1_{co}$;
3. For any $f \in \mathcal{C}^3(A,\mathbb{R})$ and for any open neighborhood $\mathcal{O}_f$ of f, there exists a compact set $K \subseteq A$ and $\varepsilon \in \mathbb{R}_{++}$ such that $f \in V(f,K,\varepsilon) \subseteq \mathcal{O}_f$.

Then, defined

$$V(f) = \{V(f,K,\varepsilon) : \ K \text{ is a compact subset of } A \text{ and } \varepsilon \in \mathbb{R}_{++}\},$$

$V(f)$ is a local basis at f, and

$$\left\{V(f) : f \in \mathcal{C}^3(A,\mathbb{R})\right\}$$

is an open neighborhood system for $\mathcal{T}^1_{co}$.

Let us now present the proofs of the above mentioned results.

Remark 106 *Clearly, for any $f \in \mathcal{C}^3(A,\mathbb{R})$, compact subset K of A and any $\varepsilon \in \mathbb{R}_{++}$,*

$$f \in V(f,K,\varepsilon)$$

For $f \in \mathcal{C}^3(A,\mathbb{R})$ and $K \subseteq A$ compact, we define

$$\|f\|_{3,K} := \max_{x\in K}|f(x)| + \max_{x\in K}\|Df(x)\| + \max_{x\in K}\left\|D^2f(x)\right\| + \max_{x\in K}\left\|D^3f(x)\right\|.$$

We quote for the following result also Hart (2003) where it is claimed that the set $V(f,K,\varepsilon)$ is a basis for the compact-open topology in the case of continuous functions. For completeness, we provide with a proof on the case of $\mathcal{C}^3$ functions.

Lemma 107

$$V(f,K,\varepsilon) = \left\{g \in \mathcal{C}^3(A,\mathbb{R}) : \|g-f\|_{3,K} < \varepsilon\right\}.$$

Proof. Since f, g and their derivatives are continuous functions on K compact, from the extreme value theorem, there exist $x^*, \bar{x}, \tilde{x}, \underline{x} \in K$ such that

$$\|g-f\|_{3,K} = |(g-f)(x^*)| + \|D(g-f)(\bar{x})\| + \left\|D^2(g-f)(\tilde{x})\right\| + \left\|D^3(g-f)(\underline{x})\right\|.$$

Then, the two needed inclusions follow easily. In both cases, we show the contrapositive.

[$\subseteq$] If $\|g-f\|_{3,K} \geq \varepsilon$, then at least one of $|(g-f)(x^*)|$, $\|D(g-f)(\bar{x})\|$, $\left\|D^2(g-f)(\tilde{x})\right\|$, $\left\|D^3(g-f)(\underline{x})\right\|$ is greater than ε.

[$\supseteq$] If $\exists\, x \in K$ such that, without loss of generality, $|(g-f)(x)| \geq \varepsilon$, then $|(g-f)(x^*)| \geq \varepsilon$, contradicting the definition of x^*, since $\|g-f\|_{3,K} < \varepsilon$ implies $|(g-f)(x^*)| < \varepsilon$. ∎

Lemma 108 *Given a metric space (X,d), if $K \subseteq X$ is compact, $U \subseteq X$ is open and $K \subseteq U$, then there exists $r \in \mathbb{R}_{++}$ such that for any $x \in K$, $B(x,r) \subseteq U$.*

Proof. For any $x \in K$, since U is open and $x \in U$, there exists $r_x \in \mathbb{R}_{++}$ such that

$$B(x,r_x) \subseteq U.$$

The family $\{B(x, \frac{r_x}{2}) : x \in K\}$ is an open cover of K. Since K is compact, there exists a finite subcover $\{B(x_i, \frac{r_{x_i}}{2}) : i \in \{1,...,n\}\}$ such that

$$K \subseteq \bigcup_{i=1}^{n} B(x_i, \frac{r_{x_i}}{2}). \tag{7.8}$$

Observe that since for any $x \in K$, we have $B(x,r_x) \subseteq U$, we also have that for any $i \in \{1,...n\}$,

$$B(x_i,r_{x_i}) \subseteq U \tag{7.9}$$

Define

$$r := \min\left\{\frac{r_{x_i}}{2} : i \in \{1,...,n\}\right\} > 0. \tag{7.10}$$

Let $x \in K$. Then from (7.8), there exists $i \in \{1,...,n\}$ such that $x \in B(x_i, \frac{r_{x_i}}{2})$ and then

$$d(x, x_i) < \frac{r_{x_i}}{2}. \tag{7.11}$$

Let

$$y \in B(x, r); \tag{7.12}$$

we want to show that $y \in U$.

Observe that

$$d(y, x) \overset{(7.12)}{<} r \overset{(7.10)}{\leq} \frac{r_{x_i}}{2}. \tag{7.13}$$

Then

$$d(y, x_i) \leq d(y, x) + d(x, x_i) \overset{(7.13),(7.11)}{<} \frac{r_{x_i}}{2} + \frac{r_{x_i}}{2} = r_{x_i}.$$

Thus $y \in B(x_i, r_{x_i})$ and from (7.9), we have $y \in U$. ∎

Proposition 109 *For any $f \in \mathcal{C}^3(A, \mathbb{R})$ and for any open neighborhood $\mathcal{O}_f$ of f, there exists a compact set $K \subseteq A$ and $\varepsilon \in \mathbb{R}_{++}$ such that $f \in V(f, K, \varepsilon) \subseteq \mathcal{O}_f$.*

Proof. First of all observe that if $\mathcal{O}_f$ is a neighborhood of f, then from Proposition 107, Topology Notes,

$$\text{there exists } B \in \mathcal{T}_{co}^1 \text{ such that } f \in B \subseteq \mathcal{O}_f. \tag{7.14}$$

Then, by definition of $\mathcal{T}_{co}^1$, $\exists\, n \in \mathbb{N}$, $\exists\, K_1, ..., K_n$ compact and $\exists\, U_1, ..., U_n$ open such that such that

$$B = \cap_{i=1}^n F(K_i, U_i).$$

Define

$$K := \cup_{i=1}^n K_i$$

which is a compact set.

For any $i \in \{1,...,n\}$, $f(K_i)$ is a compact set contained in the open set U_i. Then, from Lemma 108, there exists $\varepsilon_i \in \mathbb{R}_{++}$ such that

$$\text{for any } x \in K_i,\ B(f(x), \varepsilon_i) \subseteq U_i. \tag{7.15}$$

Defined $\varepsilon = \min\{\varepsilon_1, ..., \varepsilon_n\} > 0$, we then have

$$\text{for any } i \in \{1,...,n\} \text{ and any } x \in K_i, \qquad B(f(x), \varepsilon) \subseteq U_i.$$

We now want to show that

$$V(f, K, \varepsilon) \subseteq B = \cap_{i=1}^n F(K_i, U_i). \tag{7.16}$$

Take $g \in V(f, K, \varepsilon)$; then by definition of $V(f, K, \varepsilon)$,

$$\text{for any } x \in K,\ |f(x) - g(x)| < \varepsilon,\ \|D(f-g)(x)\| < \varepsilon,\ \left\|D^2(f-g)(x)\right\| < \varepsilon,\ \left\|D^3(f-g)(x)\right\| < \varepsilon. \tag{7.17}$$

We want to show that for any $i \in \{1,...,n\}$, $g \in F(K_i, U_i)$ or for any $x \in K_i$, we have $g(x) \in U_i$. Indeed, for any $x \in K_i$, we have that $x \in K = \cup_{i=1}^n K_i$ and from (7.17),

$$|f(x) - g(x)| < \varepsilon \overset{\text{def. } \varepsilon}{\leq} \varepsilon_i$$

and similarly we have $\|D(f-g)(x)\| < \varepsilon_i$, $\left\|D^2(f-g)(x)\right\| < \varepsilon_i$, and $\left\|D^3(f-g)(x)\right\| < \varepsilon_i$. Hence

$$g(x) \in B(f(x), \varepsilon_i) \overset{(7.15)}{\subseteq} U_i,$$

as desired.

Finally, we get

$$f \overset{\text{Rmk } 106}{\in} V(f, K, \varepsilon) \overset{(7.16)}{\subseteq} B \overset{(7.14)}{\subseteq} \mathcal{O}_f.$$

which proves the desired claim. ∎

Proposition 110 *For any compact set $K \subseteq A$ and any $\varepsilon \in \mathbb{R}_{++}$, $V(f,K,\varepsilon)$ is open.*

Proof. We are going to prove that for any $g \in V(f,K,\varepsilon)$ there exists $B \in \mathcal{T}_{co}^1$ such that $g \in B \subseteq V(f,K,\varepsilon)$.

From Lemma 107,

$$V(f,K,\varepsilon) = \left\{ g \in \mathcal{C}^3(A,\mathbb{R}) : \ \|g-f\|_{3,K} < \varepsilon \right\}. \tag{7.18}$$

We set $\delta := \|g-f\|_{3,K}$ and define

$$\eta = \varepsilon - \delta > 0 \qquad \alpha = \frac{\eta}{3}.$$

Since g is continuous on K, by definition of continuity, for any $x \in K$ there exists $r_{x,\alpha} := r_x > 0$ such that

$$g(B(x,r_x)) \subseteq B(g(x),\alpha) \subseteq \mathbb{R}. \tag{7.19}$$

We remark that by continuity of Dg, D^2g and D^3g on K we have a similar property for each of the derivatives. Namely, there exist $\widetilde{r}_x, \overline{r}_x, \widehat{r}_x > 0$ such that

$$Dg(B(x,\widetilde{r}_x)) \subseteq B(Dg(x),\alpha) \subseteq \mathbb{R}^n, \quad D^2g(B(x,\overline{r}_x)) \subseteq B\left(D^2g(x),\alpha\right) \subseteq \mathbb{R}^{n^2},$$

and

$$D^3g(B(x,\widehat{r}_x)) \subseteq B\left(D^3g(x),\alpha\right) \subseteq \mathbb{R}^{n^3}.$$

Up to choose the minimum between the four radius, we can consider for all the balls involved r_x to be the common radius.

Observe that for any $x \in K$, we also have

$$B\left(x,\frac{r_x}{2}\right) \subseteq \mathrm{Cl}\left(B\left(x,\frac{r_x}{2}\right)\right) \subseteq B(x,r_x). \tag{7.20}$$

Since $\left\{B\left(x,\frac{r_x}{2}\right) \subseteq \mathbb{R}^n : x \in K\right\}$ is[1] an open cover of K there exists $n \in \mathbb{N}$, and for any $i \in \{1,...,n\}$, $x_i \in K$ and $r_i > 0$ such that $\left\{B\left(x_i,\frac{r_i}{2}\right) \subseteq \mathbb{R}^n : i = 1,...n\right\}$ is still a cover of K. Then,

$$K \subseteq \cup_{i=1}^n B\left(x_i,\frac{r_i}{2}\right) \subseteq \cup_{i=1}^n \mathrm{Cl}\left(B\left(x_i,\frac{r_i}{2}\right)\right);$$

and

$$K \overset{(1)}{\subseteq} \left(\cup_{i=1}^n \mathrm{Cl}\left(B\left(x_i,\frac{r_i}{2}\right)\right)\right) \cap K \overset{(2)}{=} \cup_{i=1}^n \left(\mathrm{Cl}\left(B\left(x_i,\frac{r_i}{2}\right)\right) \cap K\right) \overset{(3)}{\subseteq} K$$

where (1) follows from the fact that for arbitrary sets C, K, if $K \subseteq C$, then $K \subseteq C \cap K$;

(2) from the distributive property of set operations, i.e., "$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$";

(3) from the fact that $C \cap K \subseteq K$. Then, $K = \cup_{i=1}^n \left(\mathrm{Cl}\left(B\left(x_i,\frac{r_i}{2}\right)\right) \cap K\right)$. Then, defined, for any $i \in \{1,...,n\}$,

$$K_i = \mathrm{Cl}\left(B\left(x_i,\frac{r_i}{2}\right)\right) \cap K,$$

we have

$$K = \cup_{i=1}^n K_i. \tag{7.21}$$

Observe also that for any $i \in \{1,...,n\}$,

$$B\left(x_i,\frac{r_i}{2}\right) \subseteq \mathrm{Cl}\left(B\left(x_i,\frac{r_i}{2}\right)\right) \overset{\text{ift notes}}{=} \left\{x \in \mathbb{R}^n : \|x-x_i\| \leq \frac{r_i}{2}\right\} \subseteq B(x_i,r_i). \tag{7.22}$$

Then, we have

$$g(K_i) := g\left(\mathrm{Cl}\left(B\left(x_i,\frac{r_i}{2}\right)\right) \cap K\right) \subseteq g\left(\mathrm{Cl}\left(B\left(x_i,\frac{r_i}{2}\right)\right)\right) \overset{(7.22)}{\subseteq} g(B(x_i,r_i)) \overset{(7.19)}{\subseteq} B(g(x_i),\alpha) \tag{7.23}$$

Define

$$U_i = B(g(x_i),2\alpha) \subseteq \mathbb{R}, \quad V_i = B(Dg(x_i),2\alpha) \subseteq \mathbb{R}^n, \quad W_i = B\left(D^2g(x_i),2\alpha\right) \subseteq \mathbb{R}^{n^2},$$

[1] To be precise we should take $B\left(x,\frac{r_x}{2}\right) \cap A$. Indeed, since, by assumption, A is an Euclidean open set, such r_i does exist.

and

$$Z_i = B\left(D^3 g\left(x_i\right), 2\alpha\right) \subseteq \mathbb{R}^{n^3}.$$

Let $K_i := \mathrm{Cl}\left(B\left(x_i, \frac{r_i}{2}\right)\right) \cap K$ is compact because the closure of a set is closed, in any Hausdorff space compact sets are closed, and a closed subset of a compact set is compact. We now choose

$$B_1 = \cap_{i=1}^n F\left(K_i, U_i\right) = \left\{h \in \mathcal{C}^3\left(A, \mathbb{R}\right) : \forall i \in \{1, ..., n\}, h\left(K_i\right) \subseteq U_i\right\},$$

$$B_2 = \cap_{i=1}^n F\left(K_i, V_i\right) = \left\{h \in \mathcal{C}^3\left(A, \mathbb{R}\right) : \forall i \in \{1, ..., n\}, Dh\left(K_i\right) \subseteq V_i\right\},$$

$$B_3 = \cap_{i=1}^n F\left(K_i, W_i\right) = \left\{h \in \mathcal{C}^3\left(A, \mathbb{R}\right) : \forall i \in \{1, ..., n\}, D^2 h\left(K_i\right) \subseteq W_i\right\},$$

$$B_4 = \cap_{i=1}^n F\left(K_i, Z_i\right) = \left\{h \in \mathcal{C}^3\left(A, \mathbb{R}\right) : \forall i \in \{1, ..., n\}, D^3 h\left(K_i\right) \subseteq Z_i\right\},$$

and then

$$B := B_1 \cap B_2 \cap B_3 \cap B_4.$$

We are left with showing that $g \in B \subseteq V\left(f, K, \varepsilon\right)$.

1. $g \in B$.

Indeed, for any $i \in \{1, ..., n\}$,

$$g\left(K_i\right) \overset{(7.23)}{\subseteq} B\left(g\left(x_i\right), \alpha\right) \subseteq B\left(g\left(x_i\right), 2\alpha\right) = U_i.$$

2. $B \subseteq V\left(f, K, \varepsilon\right)$.

Take $h \in B$, we want to show that for any $x \in K$, $\left|h\left(x\right) - f\left(x\right)\right| < \varepsilon$. Since $x \in K$, from (7.21), there exists $i \in \{1, ..., n\}$ such that $x \in K_i$.

Since $h \in B$ and $x \in K_i$, we have $h\left(x\right) \in h\left(K_i\right) \subseteq U_i$ and then

$$\left|h\left(x\right) - g\left(x_i\right)\right| < 2\alpha; \tag{7.24}$$

From (7.23),

$$\left|g\left(x_i\right) - g\left(x\right)\right| < \alpha; \tag{7.25}$$

From (7.18), and since $g \in V\left(f, K, \varepsilon\right)$ and also $x_i \in K_i \subseteq K$, we have

$$\left|g\left(x_i\right) - f\left(x\right)\right| \leq \delta. \tag{7.26}$$

Then, from the above three inequalities, we get

$$\left|h\left(x\right) - f\left(x\right)\right| \leq \left|h\left(x\right) - g\left(x_i\right)\right| + \left|g\left(x_i\right) - g\left(x\right)\right| + \left|g\left(x_i\right) - f\left(x\right)\right|$$

$$< 2\alpha + \alpha + \delta = 3\alpha + \delta \overset{\text{def. } \alpha}{=} \eta + \delta \overset{\text{def. } \eta}{=} \varepsilon - \delta + \delta = \varepsilon.$$

In a similar manner as above, it is possible to prove that for any $x \in K$

$$\left\|D(f - g)\left(x\right)\right\| < \varepsilon, \ \left\|D^2(f - g)\left(x\right)\right\| < \varepsilon, \text{ and } \left\|D^3(f - g)\left(x\right)\right\| < \varepsilon$$

which implies $g \in V\left(f, K, \varepsilon\right)$ as desired. ■

7.3.3 Open neighborhoods and finite parameterizations

As described in Section 5.1, a crucial step in our proof of regularity is showing that Conditions 3.a and 3.b are satisfied. That is, for any utility function and any neighborhood of that function (in the compact-open topology), there exists a finite-dimensional parameterization belonging to that neighborhood. Below, we provide alternative proofs of this result using the approaches developed in the previous sections. We offer multiple proofs because their applicability varies across general equilibrium models: a specific approach may work in certain contexts where others fail.

The Sequence approach.

Proposition 111 *Let $f \in C^3((0,1),\mathbb{R})$ and let O_f be any compact-open open neighborhood of f. Then there exists $\delta > 0$ such that for any $\beta \in (-\delta, \delta)$,*

$$(1+\beta)f \in O_f.$$

Proof. We define the sequence

$$f_n := \left(1 + \frac{1}{n}\right) f \in \mathcal{C}^3((0,1),\mathbb{R})$$

We show the following:
Step 1. $(f_n)_{n\in\mathbb{N}}$ converges to f with respect to the compact-open topology.
Step 2. Desired result.
Step 1. We prove first that f_n converges uniformly on compact sets to f. Indeed, we want to show that

for any compact subset K of $(0,1)$, $\forall \varepsilon > 0$,

$\exists n_0 := n_0(K,\varepsilon)$ such that $\forall n \geq n_0$, $\forall\, x \in K$, $\left|\left(1+\frac{1}{n}\right) f(x) - f(x)\right| = \frac{1}{n}\left|f(x)\right| < \varepsilon$.

Let $B_K := \max_{x\in K} |f(x)|$. Then, for any $x \in K$, $\frac{1}{n}|f(x)| \leq \frac{1}{n} B_K$. Observe that $\frac{1}{n} B_K < \varepsilon$ if and only if $n > \frac{B_K}{\varepsilon}$. Choose $n_0(K,\varepsilon) = \frac{2B_K}{\varepsilon}$. Then,

$$\forall x \in K,\ \forall n \geq n_0(K,\varepsilon), \text{ we have } \frac{1}{n}|f(x)| \leq \frac{1}{n} B_K \leq \frac{1}{\frac{2B_K}{\varepsilon}} B_K = \frac{\varepsilon}{2} < \varepsilon,$$

as desired.

The convergence on compacta of the derivatives of f_n to the derivatives of f until the third order, follows exactly as above since all the derivatives of f_n, f are continuous until the third order.

Step 2.

From Theorem 515 in Villanacci (2026), we have that for any open neighborhood O_f of f, there exists $n_0' \in \mathbb{N}$ such that for any $n \geq n_0'$, $\quad \left(1+\frac{1}{n}\right) f \in O_f$.

We observe that $\left(\mathcal{C}^3((0,1),\mathbb{R}), \mathcal{T}_{co}^2\right)$ is locally convex, i.e., every neighborhood of zero contains a convex neighborhood of zero, because it is a Frechet space. Hence, every function $f \in \mathcal{C}^3((0,1),\mathbb{R})$ admits a basis of open convex neighborhoods. Therefore,

there exists an open convex neighborhood O_f^* of f contained in O_f and there exists $n_0 \in \mathbb{N}$ such that

$$\text{for any } n \geq n_0, \quad \left(1+\frac{1}{n}\right) f \in O_f^* \subseteq O_f. \tag{7.27}$$

Take $\delta^+ = \frac{1}{n_0}$, we want to show that for any $\alpha \in \left(0, \frac{1}{n_0}\right)$ we do have that $(1+\alpha) f \in O_f^*$. Indeed, since $0 < \alpha < \frac{1}{n_0}$, then there exists $n_\beta > n_0$ such that

$$\frac{1}{n_\alpha} \leq \alpha \leq \frac{1}{n_0}$$

and from (7.27), we have

$$\left(1+\frac{1}{n_\alpha}\right) f, \quad \left(1+\frac{1}{n_0}\right) f \in O_f.$$

Then, from the convexity of O_f^* we have $(1+\alpha)f \in O$, where $\alpha \in \left(0, \frac{1}{n_0}\right)$.

Considering the sequence $\left(\left(1-\frac{1}{n}\right) f\right)_{n\in\mathbb{N}\setminus\{1\}}$ and following the same argument as done above, we can show that there exists $\delta > 0$ such that if $\alpha \in (-\delta, +\delta)$, then $(1+\alpha)f \in O_f$, as desired. ∎

The Metric approach.

Since $\mathcal{C}^3$ is a metric space, obviously, the very natural choice is $B_{(\mathcal{C}^3,d)}(f,\varepsilon)$, where d is the distance introduced in the subsection *A Metric* in the chapter of the compact-open topology of *Topology Notes*. To get the desired result, we need some preliminary definitions and facts.

Proposition 112 *For any $p \in \mathbb{N}_0$ and any $n_0 \in \mathbb{N}_0$ such that $p > n_0$, we have*

1. If $\sum_{k=p}^{+\infty} a_k$ converges, then

$$\sum_{k=n_0}^{+\infty} a_k = \sum_{k=n_0}^{p-1} a_k + \sum_{k=p}^{+\infty} a_k\,, \tag{7.28}$$

and therefore $\sum_{k=n_0}^{+\infty} a_k$ converges.

2. If $\sum_{k=n_0}^{+\infty} a_k$ converges, then

$$\sum_{k=p}^{+\infty} a_k = \sum_{k=n_0}^{+\infty} a_k - \sum_{k=n_0}^{p-1} a_k, \tag{7.29}$$

and therefore $\sum_{k=p}^{+\infty} a_k$ converges.

3. If $\sum_{k=n_0}^{+\infty} a_k$ converges, then

$$\lim_{p\to+\infty} \sum_{k=p}^{+\infty} a_k = 0, \tag{7.30}$$

i.e., the "tail" of a convergent sequence converges to zero.

Proof. See Notes on Series. ∎

Remark 113 *From my Notes on series (see part on geometric series), we have*

$$\sum_{n=1}^{p} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^p\right)}{1-\frac{1}{2}} = \frac{\frac{1}{2}\left(1-\frac{1}{2^p}\right)}{\frac{1}{2}} = 1 - \frac{1}{2^p} := \beta_p \in (0,1)\,.$$

Lemma 114 *For any $v \in \mathcal{V}$, for any open (in the $\mathcal{C}^3$ compact-open topology) neighborhood $\mathcal{O}_v$ of v, there exists $\delta \in \mathbb{R}_{++}$ such that for any $\alpha \in (-\delta, +\delta)$, $\quad (1+\alpha)v \in \mathcal{O}_v$.*

Proof. We have the following metric:

$$\begin{aligned} &d\left((1+\alpha)\cdot v, v\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \min\left\{ \max_{x\in K_n} |\alpha v(x)| + \max_{x\in K_n} |\alpha\cdot v'(x)| + \max_{x\in K_n} |\alpha\cdot v''(x)| + \max_{x\in K_n} |\alpha\cdot v''(x)| + \max_{x\in K_n} |\alpha\cdot v'''(x)|, 1\right\}, \end{aligned}$$

where we used the fact that $|(1+\alpha)\cdot v(x) - v(x)| = |\alpha v(x)|$.
From the following property: for $a, b \in \mathbb{R}_{++}$

$$\min\{a+b,1\} \leq \min\{a,1\} + \min\{b,1\}$$

we can consider, for simplicity, the case

$$d\left((1+\alpha)\cdot v, v\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\left\{ \max_{x\in K_n} |\alpha v(x)|, 1\right\}.$$

The other cases follow also from the argument below because of the continuity of the derivatives of v.
Since, by assumption, $\mathcal{O}_v$ is an open neighborhood of v, then there exists $\varepsilon > 0$ such that $B(v,\varepsilon) \subseteq \mathcal{O}_v$. Therefore, it suffices to show that there exists $\delta > 0$ such that for any $\alpha \in (-\delta, \delta)$, we have that $d\left((1+\alpha)v\ ,\ v\right) < \varepsilon$. Observe that from Lemma 112.3, we have that

$$\lim_{p\to+\infty} \sum_{n=p+1}^{\infty} \frac{1}{2^n} \min\left\{ \max_{x\in K_n} |\alpha v(x)|, 1\right\} = 0,$$

i.e., for any $\varepsilon > 0$ there exists $P_\varepsilon \in \mathbb{N}$ such that for any $P \geq P_\varepsilon$,

$$\sum_{n=p+1}^{\infty} \frac{1}{2^n} \min\left\{\max_{x\in K_n} |\alpha v(x)|, 1\right\} < \frac{\varepsilon}{4}$$

From Lemma 112.2 and 112.1, we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \min\left\{\max_{x\in K_n} |\alpha v(x)|, 1\right\} =$$

$$= \sum_{n=1}^{p} \frac{1}{2^n} \min\left\{\max_{x\in K_n} |\alpha v(x)|, 1\right\} + \sum_{n=p+1}^{\infty} \frac{1}{2^n} \min\left\{\max_{x\in K_n} |\alpha v(x)|, 1\right\} <$$

$$< \sum_{n=1}^{p} \frac{1}{2^n} \min\left\{\max_{x\in K_n} |\alpha v(x)|, 1\right\} + \frac{\varepsilon}{4}$$

Observe that

$$\min\left\{\max_{x\in K_n} |\alpha v(x)|, 1\right\} \leq |\alpha| \max_{x\in K_n} |v(x)|.$$

Since v is continuous and for any $n \in \mathbb{N}$, K_n is compact, then for any $n \in \{1, ..., P\}$, there exists $v_n \in \mathbb{R}_+$ such that $v_n = \max_{x\in K_n} |v(x)|$.
Define

$$v^*_{P_\epsilon} = \max\{v_1, \ldots, v_{P_\epsilon}\} \geq 0$$

We distinguish two cases:
Case 1.
$v^*_{P_\epsilon} = 0$. This implies $v(x) = 0$ for all $x \in \bigcup_{n=1}^{P_\epsilon} K_n$. Consequently, $\max_{x\in K_n} |\alpha v(x)| = 0$ for all $n \leq P_\epsilon$ and any α. The finite sum is zero, and the total distance is just the tail which is bounded, i.e.,

$$d((1+\alpha)v, v) = 0 + \sum_{n=P_\epsilon+1}^{\infty} \frac{1}{2^n} \min\left\{\max_{x\in K_n} |\alpha v(x)|, 1\right\} < \frac{\epsilon}{4} < \epsilon$$

In this case, any $\delta > 0$ satisfies the condition.
Case 2.
$v^*_{P_\epsilon} > 0$. Then,

$$d((1+\alpha)v, v) < \sum_{n=1}^{P_\epsilon} \frac{1}{2^n} \min\left\{\max_{x\in K_n} |\alpha v(x)|, 1\right\} + \frac{\varepsilon}{4} \leq \sum_{n=1}^{P_\epsilon} \frac{1}{2^n} |\alpha| \cdot v^*_{P_\varepsilon} + \frac{\varepsilon}{4} =$$

$$= |\alpha| \cdot v^*_{P_\varepsilon} \sum_{n=1}^{P_\epsilon} \frac{1}{2^n} + \frac{\varepsilon}{4} \overset{\text{Rmk. } 113}{=} |\alpha| \cdot v^*_{P_\varepsilon} \cdot \beta_{P_\varepsilon} + \frac{\varepsilon}{4}.$$

Then, chosen $\delta = \frac{\varepsilon}{4v^*_{P_\varepsilon} \cdot \beta_{P_\varepsilon}} > 0$, we have that if $|\alpha| < \delta$, then

$$d((1+\alpha)\cdot v, v) < |\alpha| \cdot v^*_{P_\varepsilon} \cdot \beta_{P_\varepsilon} + \frac{\varepsilon}{4} < \frac{\varepsilon}{4v^*_{P_\varepsilon} \cdot \beta_{P_\varepsilon}} \cdot v^*_{P_\varepsilon} \cdot \beta_{P_\varepsilon} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} < \varepsilon,$$

as desired. ∎

Again on the distance approach

Proposition 115 *Given a metric space (X, d), let $\mathcal{M}(X, d)$ be the associated metric topology. Then the following family of subsets of X is a basis of $\mathcal{M}(X, d)$:*

$$\mathcal{B}(X, d) = \left\{B \subseteq X : \exists x \in X \ \exists r \in \mathbb{R}_{++} \text{ such that } B = B_{(X,d)}(x, r)\right\} = \left\{B_{(X,d)}(x, r) : x \in X \text{ and } r \in \mathbb{R}_{++}\right\}.$$

Proof. We use Proposition 107, page 36, in my Notes on Topology: Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{B}$ of subsets of X is an analytical basis for $\mathcal{T}$ if and only if

I .$\mathcal{B} \subseteq \mathcal{T}$, and

II. $\forall x \in X$, $\forall S \in \mathcal{T}$ such that $x \in S$, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq S$, i.e., for all points x in any $(X, \mathcal{T})$-open set S, there exists a basis element B such that $x \in B \subseteq S$.

Let's verify the above two conditions.

I. follows from the fact that $B_{(X,d)}(x, r) \in \mathcal{M}(X, d)$ - which is a fact from Math 2.

II. is just the definition of $\mathcal{M}(X, d)$: S is an (X, d) - open subset of X if for any $x \in S$ we do have that $x \in \text{Int}_{(X,d)}(S)$. ∎

Proposition 116 *Given a metric space (X,d). Let the box topology on $X\times X$ be denoted by $\mathcal{T}_{X\times X}$. If O is an open neighborhood of $(x_0,y_0)\in X\times X$, i.e., $(x_0,y_0)\in O$ and $O\in\mathcal{T}_{X\times X}$, then there exist $\delta_1,\delta_2\in\mathbb{R}_{++}$ such that $B_{(X,d)}(x_0,\delta_1)\times B_{(X,d)}(y_0,\delta_2)$ is a $\mathcal{T}_{X\times X}$ open neighborhood of (x_0,y_0) contained in O.*

Proof. From Proposition 380, page 114, in my Notes on Topology the following family of subsets of $X\times X$ is a basis of $\mathcal{T}_{X\times X}$

$$\mathcal{B}_{X\times X}=\{A\subseteq X\times X:\exists C,D\in\mathcal{B}(X,d)\text{ such that }A=C\times D\}$$

From Proposition 115 and condition II. defining a basis, since $(x_0,y_0)\in O\in\mathcal{T}_{X\times X}$, there exist $x,y\in X,r_x,r_y\in\mathbb{R}_{++}$ such that $(x_0,y_0)\in O\subseteq B_{(X,d)}(x,r_x)\times B_{(X,d)}(y,r_y)$ and therefore there exist $\delta_1,\delta_2\in\mathbb{R}_{++}$ such that $B_{(X,d)}(x_0,\delta_1)\subseteq B_{(X,d)}(x,r_x)$ and $B_{(X,d)}(y_0,\delta_2)\subseteq B_{(X,d)}(y,r_y)$. Summarizing, there exist $\delta_1,\delta_2\in\mathbb{R}_{++}$ such that $B_{(X,d)}(x_0,\delta_1)\times B_{(X,d)}(y_0,\delta_2)\subseteq O$. Finally, $B_{(X,d)}(x_0,\delta_1)\times B_{(X,d)}(y_0,\delta_2)$ is a $\mathcal{T}_{X\times X}$ open set because, from Proposition 115, $B_{(X,d)}(x_0,\delta_1)\times B_{(X,d)}(y_0,\delta_2)\in\mathcal{B}_{X\times X}$ and part I in the characterization of basis, presented in Proposition 107, page 36, $\mathcal{B}_{X\times X}\subseteq\mathcal{M}(X,d)$. ■

Lemma 117 *Given a metric space (X,d), then for any $(x,y),(x_0,y_0)\in X\times X$,*

$$|d(x,y)-d(x_0,y_0)|\leq d(x,x_0)+d(y,y_0)$$

Proof. Observe that using the triangle inequality twice, we have

$$d(x,y)\leq d(x,x_0)+d(x_0,y)\leq d(x,x_0)+d(x_0,y_0)+d(y_0,y)$$

and then

$$d(x,y)-d(x_0,y_0)\leq d(x,x_0)+d(y_0,y). \tag{7.31}$$

With a completely symmetric argument, interchanging x with x_0 and y with y_0, we get

$$d(x_0,y_0)-d(x,y)\leq d(x_0,x)+d(y,y_0),$$

and

$$d(x,y)-d(x_0,y_0)\geq -(d(x_0,x)+d(y,y_0)). \tag{7.32}$$

From(7.31), (7.32) and the fact that for any $a,b\in\mathbb{R}$, $-b\leq a\leq b\Leftrightarrow|a|\leq b$, we get the desired result. ■

Proposition 118 *Given a metric space (X,d), then $d:X\times X\to\mathbb{R}$ is (box-topology, Euclidean topology on $\mathbb{R}$) continuous.*

Proof. We want to show that for any $(x_0,y_0)\in X\times X$, for any $\varepsilon\in\mathbb{R}_{++}$ there exist a box topology open neighborhood $O_{(x_0,y_0)}$ of (x_0,y_0) such that if $(x,y)\in O_{(x_0,y_0)}$, then $|d(x,y)-d(x_0,y_0)|<\varepsilon$.

From Proposition 116, there exist $\delta_1,\delta_2\in\mathbb{R}_{++}$ such that $B_{(X,d)}(x_0,\delta_1)\times B_{(X,d)}(y_0,\delta_2)$ is a $\mathcal{T}_{X\times X}$ open neighborhood of (x_0,y_0) and therefore $B_{(X,d)}(x_0,\delta_1)\times B_{(X,d)}(y_0,\delta_2)\subseteq O$.

Therefore, it suffices to show that for any $(x_0,y_0)\in X\times X$, for any $\varepsilon\in\mathbb{R}_{++}$ there exist $\delta_1,\delta_2\in\mathbb{R}_{++}$ for any $x\in B_{(X,d)}(x_0,\delta_1)$ and $y\in B_{(X,d)}(y_0,\delta_2)$ we have $|d(x,y)-d(x_0,y_0)|<\varepsilon$, or that

for any $(x_0,y_0)\in X\times X$, for any $\varepsilon\in\mathbb{R}_{++}$

there exist $\delta_1,\delta_2\in\mathbb{R}_{++}$ such if $d(x,x_0)<\delta_1$ and $d(y,y_0)<\delta_2$,

then $|d(x,y)-d(x_0,y_0)|<\varepsilon$.

Choose $\delta_1=\delta_2=\frac{\varepsilon}{2}$. Then,

$$|d(x,y)-d(x_0,y_0)|\overset{\text{Lemma 117}}{\leq}d(x,x_0)+d(y,y_0)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

as desired. ■

Proposition 119 *Let $f \in \mathcal{C}^3((0,1),\mathbb{R})$ and let O_f be any compact-open open neighborhood of f. Then there exists $\delta > 0$ such that for any $\alpha \in (-\delta, \delta)$,*

$$(1+\alpha)f \in O_f.$$

Proof. Since continuity implies sequential continuity and from Step 1. in Proposition 111 the sequence $\left(\left(1+\frac{1}{n}\right)f\right)_{n\in\mathbb{N}}$ converges uniformly on compact sets to f, we have

$$\lim_{n\to+\infty} d\left(\left(1+\frac{1}{n}\right)f - f\right) = d(f,f) = 0,$$

i.e., $\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}$ such that $\forall n \geq N_\varepsilon$, $\left|d\left(\left(1+\frac{1}{n}\right)f - f\right)\right| < \varepsilon$. By the convergence with respect to the compact-open topology in $\mathcal{C}^3((0,1),\mathbb{R})$, we have also

$$\lim_{n\to+\infty} d\left(\left(1+\frac{1}{n}\right)f' - f'\right) = \lim_{n\to+\infty} d\left(\left(1+\frac{1}{n}\right)f'' - f''\right) = \lim_{n\to+\infty} d\left(\left(1+\frac{1}{n}\right)f''' - f'''\right) = 0,$$

and hence $\left(1+\frac{1}{n}\right)f \in B(f,\varepsilon) \subseteq O_f$, as desired.
Hence, following the same argument of Proposition 111, we can find $\delta > 0$ such that for every $\alpha \in (0,\delta)$ we have $(1+\alpha)f \in B(f,\varepsilon) \subseteq O_f$.

Considering the sequence $\left(\left(1-\frac{1}{n}\right)f\right)_{n\in\mathbb{N}\setminus\{1\}}$ and following the same argument as done above, we can show that there exists $\delta > 0$ such that for every $\alpha \in (-\delta, +\delta)$, then $(1+\alpha)f \in O_f$, as desired. ∎

Open neighborhood system approach

As an application of the results presented in the Section 7.3.2, we get what follows.

Proposition 120 *Let $f \in \mathcal{C}^3((0,1),\mathbb{R})$ and let O_f be any open neighborhood of f. Then there exists $\delta > 0$ such that for all $\alpha \in (-\delta, \delta)$,*

$$(1+\alpha)f \in O_f.$$

Proof. Since O_f is open and contains f, then there exists a compact subset K of $(0,1)$ and $\varepsilon > 0$ such that $V(f,K,\varepsilon) \subseteq \mathcal{O}_f$. Then it is enough to show that there exists $\delta > 0$ such that if $|\alpha| < \delta$, then

$$(1+\alpha)f \in V(f,K,\varepsilon).$$

Let $M = \|f\|_{3,K}$. The condition $(1+\alpha)f \in V(f,K,\varepsilon)$ means that

$$\begin{aligned}
&\max_{x\in K}|(1+\alpha)f(x) - f(x)| + \max_{x\in K}|(1+\alpha)f'(x) - f'(x)| + \max_{x\in K}|(1+\alpha)f''(x) - f''(x)| + \max_{x\in K}|(1+\alpha)f'''(x) - f'''(x)| \\
&= |\alpha| \cdot \left(\max_{x\in K}|f(x)| + \max_{x\in K}|f'(x)| + \max_{x\in K}|f''(x)| + \max_{x\in K}|f'''(x)|\right) \\
&= |\alpha| M \leq \varepsilon,
\end{aligned}$$

If $M = 0$, we are done. Otherwise, define

$$\delta = \frac{\varepsilon}{M} > 0.$$

Thus, for any $\alpha \in \mathbb{R}$ such that $|\alpha| < \delta$, we obtain

$$\|f\|_{3,K} = |\alpha| M < \frac{\varepsilon}{M} M < \varepsilon,$$

as desired. ∎

Definition 121 *Fixed $h \in \mathcal{H}$, consider $u_h \in \mathcal{U}$ and $\alpha_h = (\alpha_h^c)_{c\in\mathcal{C}} \in \mathbb{R}^C_{++}$ and define u_{α_h} as*

$$u_{\alpha_h} : x_h \longmapsto u_h(x_h) + \sum_{c\in\mathcal{C}} \alpha_h^c \log x_h^c. \tag{7.33}$$

Proposition 122 *For any $u_h \in \mathcal{U}$ and for any (compact-open topology) open neighborhood N_{u_h} of u_h, there exists $\varepsilon_h \in \mathbb{R}_{++}$ such that for any $\alpha_h \in \mathbb{R}^C_{++} \cap B_{\mathbb{R}^C}(0, \varepsilon_h)$, we have that $u_h + l_{\alpha_h} \in N_{u_h}$.*

Proof. From Proposition 565 in Villanacci (2026), - see Section 7.3.2 cut and pasted here from those Topology Notes - there exists a compact set $K \subseteq \mathbb{R}^C_{++}$ and $\varepsilon \in \mathbb{R}_{++}$ such that $u_h \in V(u_h, K, \varepsilon) \subseteq N_{u_h}$. Recall that

$$V(u_h, K, \varepsilon) := \left\{ g \in \mathcal{C}^3\left(\mathbb{R}^C_{++}, \mathbb{R}\right) : \|g - u_h\|_{3,K} < \varepsilon \right\},$$

where

$$\|f\|_{3,K} := \max_{x \in K} |f(x)| + \max_{x \in K} \|Df(x)\| + \max_{x \in K} \left\|D^2 f(x)\right\| + \max_{x \in K} \left\|D^3 f(x)\right\|$$

for any $f \in \mathcal{C}^3\left(\mathbb{R}^C_{++}, \mathbb{R}\right)$.

Observe that for any K compact of $\mathbb{R}^{CH}_{++}$, $\alpha_h \in \mathbb{R}^C_{++}$, there exists $L^K \in \mathbb{R}_{++}$ such that

$$\begin{aligned} &\|U_{\alpha_h} - u_h\|_{3,K} = \left\|\textstyle\sum_{c=1}^C \alpha_h^c \cdot \log(x_h^c)\right\|_{3,K} \overset{(1)}{\leq} \\ &\leq \max_{x \in K} \textstyle\sum_{c=1}^C \alpha_h^c \cdot |\log(x_h^c)| + \sum_{j=1}^3 \left(\sum_{c=1}^C \alpha_h^c \cdot \left\|D^j \log(x_h^c)\right\|\right) \overset{(2)}{\leq} \\ &\leq \alpha^* \cdot \left\|\textstyle\sum_{c=1}^C |\log(x_h^c)|\right\|_{3,K} \overset{(3)}{=} \alpha^* \cdot L^K, \end{aligned} \tag{7.34}$$

where (1) follows from the triangle inequality and the fact that $\alpha_h \in \mathbb{R}^C_{++}$, (2) from $\alpha^* := \max\{\alpha_h^c : c = 1, ..., C\} \in \mathbb{R}_{++}$, and (3) from the fact that since K is compact and $\sum_{c=1}^C |\log(x_h^c)|$ can be used to defined a $\mathcal{C}^3$ function : $\mathbb{R}^C_{++} \to \mathbb{R}$; observe also that $L^K \in \mathbb{R}_+$. (Observe that if $K = \{\mathbf{1}\} \subseteq \mathbb{R}^C$, then $L^K = 0$.)

If $L^K = 0$, then for any $\alpha_h \in \mathbb{R}^C_{++}$, we have that $U_{\alpha_h} \in V(u_h, K, \varepsilon)$.

Assume now that $L^K > 0$. Take $\alpha_h >> 0$ and such that $\alpha_h \leq \frac{\varepsilon}{2L^K} \cdot \mathbf{1}_C$. Then, $\max\{\alpha_h^c : c = 1, ..., C\} \leq \frac{\varepsilon}{2L^K} := \varepsilon_h$. Then, for any $\alpha_h \in \left(0_{\mathbb{R}^C}, \frac{\varepsilon}{2L^K} \cdot \mathbf{1}_C\right] \subseteq \mathbb{R}^C_{++}$, we have

$$\|U_{\alpha_h} - u_h\|_{3,K} \overset{(7.34)}{\leq} (\max\{\alpha_h^c : c = 1, ..., C\}) \cdot L^K \leq \frac{\varepsilon}{2L^K} \cdot L^K = \frac{\varepsilon}{2} < \varepsilon.$$

Since $\mathbb{R}^C_{++} \cap B_{\mathbb{R}^C}(0, \varepsilon_h) \subseteq (0_{\mathbb{R}^C}, \varepsilon_h \cdot \mathbf{1}_C]$,[2] we have that for any $\alpha_h \in \mathbb{R}^C_{++} \cap B_{\mathbb{R}^C}(0, \varepsilon_h)$, $u_h + l_{\alpha_h} \in V(u_h, K, \varepsilon) \subseteq N_{u_h}$, as desired (we did use Condition 1 to insure that $u_h + l_{\alpha_h} \in \mathcal{U}$). ∎

7.4 Borderline situations are rare

The motivation of this section is what follows. In general equilibrium analysis, we often want to prove that 0 is a regular value for a smooth function F and some of the component functions of F do include a min function - say as a consequence of the fact that F involves Karush-Kuhn-Tucker conditions. An obvious immediate problem is that the min function is not even differentiable when both the constraint and the multiplier are equal to zero, which can be called a "borderline" cases. We therefore show that borderline cases occur in a closed and measure zero subset D^* of the economy space.

We need the following definition and results whose proofs are presented in Villanacci and others (2002).

Theorem 123 *(Theorem 26, page 151) Let M, Ω and N be $\mathcal{C}^1$ manifolds of dimensions m, p and n, respectively. Let $f : M \times \Omega \to N$ be a $\mathcal{C}^1$ function.*

If y is a regular value for f, then there exists a full measure subset Ω^* of Ω such that for any $\omega \in \Omega^*$, y is a regular value for f_ω (where, $f_\omega : M \to N$, $x \mapsto f(x, \omega)$).

If $\dim M < \dim N$, then for any $\omega \in \Omega^*$, $f_\omega^{-1}(y) = \varnothing$.

Definition 124 *Let (X, d) and (Y, d') be metric spaces, and let $f : X \to Y$ be continuous. We say that f is proper if for any compact subset K of Y, we have that $f^{-1}(K)$ is compact.*

Proposition 125 *(Theorem 19, page 90) Let (X, d) and (Y, d') be metric spaces, and let $f : X \to Y$ be continuous. If f is proper, then f is closed.*

[2] Observe that

1. $x \in \mathbb{R}^C_{++} \cap B_{\mathbb{R}^C}(0, \varepsilon) \Leftrightarrow$ 1.a. $x >> 0$, and 1.b. $\|x\| < \varepsilon$., and
2. $x \in (0_{\mathbb{R}^C}, \varepsilon \cdot \mathbf{1}_C] \subseteq \mathbb{R}^C \Leftrightarrow$ for any $c \in \{1, ..., C\}$, $0 < x^c < \varepsilon$.

We are going to show that $\neg 2 \Rightarrow \neg 1$.

$\neg 2 \Leftrightarrow \exists c' \in \{1, ..., C\}$ such that $x^{c'} \notin (0, \varepsilon)$. If $x^{c'} \leq 0$, then we contradict 1.a.; therefore, $x >> 0$. Now, if $x^{c'} \geq \varepsilon$, we have $\|x\|^2 := \sum_{c=1}^C (x^c)^2 \geq \varepsilon^2$ and then $\|x\| \geq \varepsilon$, contradicting 1.b.

7.4.1 The case of one min function

Let the following objects be given. A nonempty open set $\Xi \subseteq \mathbb{R}^n$, a nonempty open set $\Omega \subseteq \mathbb{R}^k$, a $\mathcal{C}^1$ function $\varphi : \Xi \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^n$, $(\xi, \lambda, \omega) \mapsto \varphi(\xi, \lambda, \omega)$, a $\mathcal{C}^1$ function $g : \Xi \times \Omega \longrightarrow \mathbb{R}$, $(\xi, \omega) \mapsto g(\xi, \omega)$.

Define the functions

$$F : \Xi \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^{n+1}, \qquad (\xi, \lambda, \omega) \mapsto \begin{pmatrix} \varphi(\xi, \lambda, \omega) \\ \min\{\lambda, g(\xi, \omega)\} \end{pmatrix}, \tag{7.35}$$

$$F^0 : \Xi \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^{n+2}, \qquad (\xi, \lambda, \omega) \mapsto \begin{pmatrix} \varphi(\xi, \lambda, \omega) \\ g(\xi, \omega) \\ \lambda \end{pmatrix},$$

Assumption 1. (existence) For any $\omega \in \Omega$, $(F_\omega)^{-1}(0) \neq \varnothing$.

Assumption 2. (properness) $pr : F^{-1}(0) \longrightarrow \Omega$, $(x, \lambda, \omega) \mapsto \omega$ is proper.

Assumption 3. 0 is a regular valued for F^0.

Proposition 126 *Defined*

$$\Omega^* := \left\{ \omega \in \Omega : \forall (\xi, \lambda) \in (F_\omega)^{-1}(0), \text{ either } \langle \lambda > 0 \text{ and } g(\xi, \omega) = 0 \ \rangle \text{ or } \langle \ g(\xi, \omega) > 0 \text{ and } \lambda = 0 \rangle \ \right\},$$

if Assumption 1 and 2 hold, then Ω^* is open; if Assumption 1 and 3 hold, then Ω^* is full measure.

Proof. Observe that

$$\begin{aligned} &\Omega^* = \\ &= \{\omega \in \Omega : \forall (\xi, \omega) \in \Xi \times \mathbb{R} \text{ s. th. } F(\xi, \lambda, \omega) = 0, \text{ either } \langle \lambda > 0 \text{ and } g(\xi, \omega) = 0 \ \rangle \text{ or } \langle \ g(\xi, \omega) > 0 \text{ and } \lambda = 0 \rangle\} \overset{(1)}{=} \\ &= \{\omega \in \Omega : \forall (\xi, \lambda) \in \Xi \times \mathbb{R} \text{ such that } F(\xi, \lambda, \omega) = 0, \text{ either } \lambda \neq 0 \text{ or } g(\xi, \omega) \neq 0\} := B, \end{aligned}$$

where (1) follows from what said below.

$\Omega^* \subseteq B$: obvious: $a > b \Rightarrow a \neq b$.

$B \subseteq \Omega^*$: we want to show that if $F(\xi, \lambda, \omega) = 0$ and either $\lambda \neq 0$ or $g(\xi, \omega) \neq 0$, then either $\langle \lambda > 0$ and $g(\xi, \omega) = 0 \ \rangle$ or $\langle \ g(\xi, \omega) > 0$ and $\lambda = 0 \rangle$. Indeed, if $F(\xi, \lambda, \omega) = 0$, then $\min\{g(\xi, \omega), \lambda\} = 0$ and

$$\begin{array}{lcl} \min\{g(\xi, \omega), \lambda\} = 0 & \Rightarrow & \left[\begin{array}{l} g(\xi, \omega) > 0 \text{ and } \lambda = 0, \text{ or} \\ g(\xi, \omega) = 0 \text{ and } \lambda > 0, \text{ or} \\ g(\xi, \omega) = 0 \text{ and } \lambda = 0 \\ \text{and} \\ \lambda \neq 0 \text{ or } g(\xi, \omega) \neq 0 \end{array} \right] \Rightarrow \left[\begin{array}{l} g(\xi, \omega) > 0 \text{ and } \lambda = 0, \text{ or} \\ g(\xi, \omega) = 0 \text{ and } \lambda > 0 \end{array} \right] \end{array} \tag{7.36}$$

Openness (a topological argument).

Define

$$C = \{(\xi, \lambda, \omega) \in \Xi \times \mathbb{R} \times \Omega : \ F(\xi, \lambda, \omega) = 0, \ \lambda = 0 \text{ and } g(\xi, \omega) = 0\} \subseteq F^{-1}(0).$$

Observe that since $C \subseteq (F)^{-1}(0)$, then $pr(C)$ is well defined. Moreover, C is closed and then, from Assumption 2, $pr(C)$ is closed. Observe that

$$\begin{aligned} &pr(C) := \left\{\omega \in \Omega : \exists \left(\overline{\xi}, \overline{\lambda}, \overline{\omega}\right) \in C \text{ such that } pr\left(\overline{\xi}, \overline{\lambda}, \overline{\omega}\right) = \omega\right\} \overset{(1)}{=} \\ &= \left\{\omega \in \Omega : \exists \left(\overline{\xi}, \overline{\lambda}\right) \in \Xi \times \mathbb{R} \text{ such that } \left(\overline{\xi}, \overline{\lambda}, \omega\right) \in C\right\} = \\ &= \left\{\omega \in \Omega : \ \exists \left(\overline{\xi}, \overline{\lambda}\right) \in \Xi \times \mathbb{R} \text{ such that } F\left(\overline{\xi}, \overline{\lambda}, \omega\right) = 0, \ \overline{\lambda} = 0 \text{ and } g\left(\overline{\xi}, \omega\right) = 0\right\}, \end{aligned}$$

where (1) follows from the fact that $pr\left(\overline{\xi}, \overline{\lambda}, \overline{\omega}\right) = \overline{\omega}$. Then the following set is open:

$$\Omega \backslash pr(C) = \{\omega \in \Omega : \forall \ (\xi, \lambda) \in \Xi \times \mathbb{R} \text{ either } F(\xi, \lambda, \omega) \neq 0 \text{ or } \lambda \neq 0 \text{ or } g(\xi, \omega) \neq 0\}.$$

We are left with showing that

$$\Omega \backslash pr\left(C\right) = \Omega^{*},$$

which is done below.

Recall that

$$\Omega^{*} = \{\omega \in \Omega : \forall\left(\xi,\lambda\right) \in \Xi \times \mathbb{R} \text{ such that } F\left(\xi,\lambda,\omega\right) = 0, \text{ either } \lambda \neq 0 \text{ or } g\left(\xi,\omega\right) \neq 0\} =$$

$$= \{\omega \in \Omega : \forall\left(\xi,\lambda\right) \in \Xi \times \mathbb{R}\ ,\ F\left(\xi,\lambda,\omega\right) = 0\ \Rightarrow \text{either } \lambda \neq 0 \text{ or } g\left(\xi,\omega\right) \neq 0\}$$

$\Omega \backslash pr\left(C\right) \subseteq \Omega^{*}$:

We want to show that if $\omega \in \Omega \backslash pr\left(C\right)$, then for any $\left(\xi,\lambda\right) \in \Xi \times \mathbb{R}$ such that $F\left(\xi,\lambda,\omega\right) = 0$, either $\lambda \neq 0$ or $g\left(\xi,\omega\right) \neq 0$.

Since $\omega \in \Omega \setminus pr\left(C\right)$, if $F\left(\xi,\lambda,\omega\right) = 0$, then either $\lambda \neq 0$ or $g\left(\xi,\omega\right) \neq 0$, as desired.

$\Omega^{*} \subseteq \Omega \backslash pr\left(C\right)$:

Take $\omega \in \Omega^{*}$. Observe that

$$\left(\xi,\lambda\right) \in \Xi \times \mathbb{R} \quad \Rightarrow \quad \begin{array}{l} \lceil \quad F\left(\xi,\lambda,\omega\right) \neq 0, \\ \mid \quad \text{or} \\ \lfloor \quad F\left(\xi,\lambda,\omega\right) = 0 \end{array} \quad \overset{\omega \in \Omega^{*}}{\Rightarrow} \quad \text{either } \lambda \neq 0 \text{ or } g\left(\xi,\omega\right) \neq 0$$

Then, for any $\left(x,\lambda\right) \in X \times \mathbb{R}^{m}$, either $F\left(x,\lambda,\omega\right) \neq 0$ or $\lambda \neq 0$ or $g\left(x,\omega\right) \neq 0$, as desired.

The desired result also follows from what said below.

$$\begin{array}{l} \text{Assume} \\ p\left(\xi,\lambda,\omega\right) \equiv \left\langle F\left(\xi,\lambda,\omega\right) \neq 0\right\rangle, \\ q\left(\xi,\lambda,\omega\right) \equiv \left\langle \text{either } \lambda \neq 0 \text{ or } g\left(\xi,\omega\right) \neq 0\right\rangle. \\ \text{Define} \\ \Omega \backslash pr\left(C\right) = \{\omega \in \Omega : \forall\ \left(\xi,\lambda\right) \in \Xi \times \mathbb{R}^{m},\ p\left(\xi,\lambda,\omega\right)\ \vee\ q\left(\xi,\lambda,\omega\right)\} \\ \Omega^{*} = \{\omega \in \Omega : \forall\ \left(\xi,\lambda\right) \in \Xi \times \mathbb{R}^{m},\ \left(\neg p\left(\xi,\lambda,\omega\right)\right)\ \Rightarrow\ q\left(\xi,\lambda,\omega\right)\} \\ \text{Observe} \\ \left\langle \left(\neg p\right)\ \Rightarrow\ q\right\rangle \equiv \left\langle\ \left(\neg\left(\neg p\right) \vee q\right)\right\rangle \equiv \left\langle p \vee q\right\rangle. \end{array} \tag{7.37}$$

Full measure (differential topology argument).

We want to apply Theorem 123, with the following identifications

M	Ω	N	m	p	n	$f : M \times \Omega \to N$
$\Xi \times \mathbb{R}$	Ω	$\mathbb{R}^{n+2}$	$n+1$	k	$n+2$	$F^{0} : \Xi \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^{n+2}$, $\left(\xi,\lambda,\omega\right) \mapsto \begin{pmatrix} \varphi\left(\xi,\lambda,\omega\right) \\ g\left(\xi,\omega\right) \\ \lambda \end{pmatrix}$

Assumptions:

Theorem:	Our case:
f is $\mathcal{C}^{1}$	By assumption
y is a regular value for f	Assumption 3 with $y = 0$
$m = \dim M < \dim N = n$	$n+1 < n+2$

Conclusion:

Theorem:	Our case:
$\exists$ full measure set Ω^{*} s. th. $\forall \omega \in \Omega^{*}$, $f_{\omega}^{-1}\left(y\right) = \varnothing$;	$\exists$ full measure set T s. th. $\forall \omega \in T$, $\left(F_{\omega}^{0}\right)^{-1}\left(0\right) = \varnothing$.

Then we can conclude that

$$T := \left\{\omega \in \Omega : \left(F_{\omega}^{0}\right)^{-1}\left(0\right) = \varnothing\right\} = \left\{\omega \in \Omega : \forall\left(\xi,\lambda\right) \in \Xi \times \mathbb{R},\ \ F^{0}\left(\xi,\lambda,\omega\right) \neq 0\right\}$$

has full measure. We are left with showing that $T = \Omega^*$ or $T^C = (\Omega^*)^C$. Recall that

$$\Omega^* := \left\{\omega \in \Omega : \forall\,(\xi,\lambda) \in (F_\omega)^{-1}(0)\text{, either } \langle \lambda > 0 \text{ and } g(\xi,\omega) = 0 \ \rangle \text{ or } \langle\ g(\xi,\omega) > 0 \text{ and } \lambda = 0\rangle\ \right\}.$$

$$(\Omega^*)^C = \{\omega \in \Omega : \exists\,(\xi,\lambda) \in \Xi \times \mathbb{R} \text{ such that } F(\xi,\lambda,\omega) = 0,\ \lambda = 0,\ g(\xi,\omega) = 0\} \overset{\text{Def. } F}{=}$$

$$= \left\{ \begin{array}{l} \omega \in \Omega : \exists\,(\xi,\lambda) \in \Xi \times \mathbb{R} \text{ such that } \varphi(\xi,\omega) = 0,\ \min\{\lambda, g(\xi,\omega)\} = 0, \\ \neg\,\langle \lambda > 0 \text{ and } g(\xi,\omega) = 0 \ \rangle, \neg\ \langle\ g(\xi,\omega) > 0 \text{ and } \lambda = 0\rangle \end{array} \right\} =$$

$$= \{\omega \in \Omega : \exists\,(\xi,\lambda) \in \Xi \times \mathbb{R} \text{ such that } \varphi(\xi,\omega) = 0,\ \lambda = 0,\ g(\xi,\omega) = 0\}.$$

where the last equality follows from the fact that

$$\min\{\lambda, g(\xi,\omega)\} = \quad 0 \quad \Leftrightarrow \left\langle \begin{array}{l} \text{either } \langle \lambda > 0 \text{ and } g(\xi,\omega) = 0 \ \rangle \\ \text{or } \langle\ g(\xi,\omega) > 0 \text{ and } \lambda = 0\rangle \\ \text{or } \langle\ g(\xi,\omega) = 0 \text{ and } \lambda = 0\rangle \end{array} \right\rangle.$$

$$T^C = \left\{\omega \in \Omega : \exists\,(\xi,\lambda) \in \Xi \times \mathbb{R} \text{ such that } \ F^0(\xi,\lambda,\omega) = 0\right\} \overset{\text{Def. } F^0}{=}$$

$$= \{\omega \in \Omega : \exists\,(\xi,\lambda) \in \Xi \times \mathbb{R} \text{ such that } \varphi(\xi,\lambda,\omega) = 0,\ \lambda = 0,\ g(\xi,\omega) = 0\}$$

■

7.4.2 The case of several min functions

Let the following objects be given. A nonempty open set $\Xi \subseteq \mathbb{R}^n$, a nonempty open set $\Omega \subseteq \mathbb{R}^k$, a $\mathcal{C}^1$ function $\varphi : \Xi \times \mathbb{R}^m \times \Omega \longrightarrow \mathbb{R}^n$, $(\xi,\lambda,\omega) \mapsto \varphi(\xi,\lambda,\omega)$, a $\mathcal{C}^1$ function $g : \Xi \times \Omega \longrightarrow \mathbb{R}^m$, $(\xi,\omega) \mapsto g(\xi,\omega)$.

Define $M := \{1, ..., m\}$,

$$F : \Xi \times \mathbb{R}^m \times \Omega \longrightarrow \mathbb{R}^{n+m}, \qquad (\xi,\lambda,\omega) \mapsto \left(\begin{array}{l} \varphi(\xi,\lambda,\omega) \\ (\min\{\lambda_j, g_j(\xi,\omega)\})_{j\in M} \end{array} \right);$$

$P(M) =$

$\{\ \left(M^{+0}, M^{0+}, \{j\}\right) \in (\mathcal{P}(M))^3 : j \in M$ and

$\left\{M^{+0}, M^{0+}\right\}$ is a generalized partition, i.e., a partition with possible empty elements of $M \setminus \{j\}$ $\}$

The $+0$ subscript means that multiplier is strictly positive and associated constraint is zero and similarly for the subscript $0+$. Observe that defined $m^{+0} = \#M^{+0}$ and $m^{0+} = \#M^{=+}$, we have $m^{0+}, m^{+0} \in \{0, 1...., m-1\}$ and $m^{0+} + m^{+0} = m - 1$. Then, for any $\left(M^{+0}, M^{0+}, \{j\}\right) \in P(M)$,

$$F_{(M^{+0},M^{0+},\{j\})} : \Xi \times \mathbb{R}^m \times \Omega \longrightarrow \mathbb{R}^n \times \mathbb{R}^{m^{0+}+m^{+0}+2=m+1},$$

$$(\xi,\lambda,\omega) \mapsto \left(\begin{array}{l} \varphi(\xi,\lambda,\omega) \\ (g_k(\xi,\omega))_{k\in M^{+0}} \\ (\lambda_l)_{l\in M^{0+}} \\ g_j(\xi,\omega) \\ \lambda_j \end{array} \right)$$

Assumption 1. (existence) For any $\omega \in \Omega$, $(F_\omega)^{-1}(0) \neq \varnothing$.
Assumption 2. (properness) $pr : F^{-1}(0) \longrightarrow \Pi$, $(\xi,\lambda,\omega) \mapsto \omega$ is proper.
Assumption 3. For any $\left(M^{+0}, M^{0+}, \{j\}\right) \in P(M)$, 0 is a regular value for $F_{(M^{+0},M^{0+},\{j\})}$.

Proposition 127 *Define*

$$\Omega^* := \left\{\omega \in \Omega : \forall\,(\xi,\lambda) \in (F_\omega)^{-1}(0), \forall j \in M, \textit{ either } \langle \lambda_j > 0 \textit{ and } g_j(\xi,\omega) = 0 \ \rangle \textit{ or } \langle\ g_j(\xi,\omega) > 0 \textit{ and } \lambda_j = 0\rangle\right\},$$

if Assumption 1 and 2 hold, then Ω^* is open; if Assumption 1 and 3 hold, then Ω^* is full measure.

Remark 128 *Assumption 3 says that in the analysis of the borderline cases, it is sufficient to consider only one borderline case at the time.*

Proof. of Proposition 127.
Openness.
Observe that using the same argument presented in the proof of Proposition 126, we have that

$$\Omega^* := \left\{\omega \in \Omega : \forall\,(\xi,\lambda) \in (F_\omega)^{-1}(0)\,, \forall j \in M, \text{ either } \lambda_j \neq 0 \text{ or } g_j(\xi,\omega) \neq 0\right\} =$$

$$= \{\omega \in \Omega : \forall\,(\xi,\lambda) \in \Xi \times \mathbb{R}^m \text{ such that } F(\xi,\lambda,\omega) = 0,\ \forall j \in M, \text{ either } \lambda_j \neq 0 \text{ or } g_j(\xi,\omega) \neq 0\}$$

Define

$$C = \{(\xi,\lambda,\omega) \in \Xi \times \mathbb{R}^m \times \Omega :\ F(\xi,\lambda,\omega) = 0 \text{ and } \exists j \in M \text{ such that } \lambda_j = 0 \text{ and } g_j(\xi,\omega) = 0\} \subseteq F^{-1}(0)\,.$$

Observe that since $C \subseteq F^{-1}(0)$, then $pr(C)$ is well defined. Moreover, C is closed and then, from Assumption 2, $pr(C)$ is closed. Indeed,

$$pr(C) := \left\{\omega \in \Omega : \exists\left(\overline{\xi},\overline{\lambda},\overline{\omega}\right) \in C \text{ such that } \overline{\omega} = pr\left(\overline{\xi},\overline{\lambda},\overline{\omega}\right) = \omega\right\} =$$

$$= \left\{\omega \in \Omega :\ \exists\left(\overline{\xi},\overline{\lambda}\right) \in \Xi \times \mathbb{R}^m \text{ such that } F\left(\overline{\xi},\overline{\lambda},\omega\right) = 0 \text{ and } \exists j \in M \text{ such that } \overline{\lambda}_j = 0 \text{ and } g_j\left(\overline{\xi},\omega\right) = 0\right\}.$$

Then the following set is open:

$$\Omega\backslash pr(C) = \{\omega \in \Omega : \forall\ (\xi,\lambda) \in \Xi \times \mathbb{R}^m \text{ either } F(\xi,\lambda,\omega) \neq 0 \text{ or for any } j \in M, \text{ either } \lambda_j \neq 0 \text{ or } g_j(\xi,\omega) \neq 0\}$$

We are left with showing that

$$\Omega\backslash pr(C) = \Omega^*,$$

which is done below.[3]
Recall that

$$\Omega^* = \{\omega \in \Omega : \forall\,(\xi,\lambda) \in \Xi \times \mathbb{R}^m \text{ such that } F(\xi,\lambda,\omega) = 0,\ \forall j \in M, \text{ either } \lambda_j \neq 0 \text{ or } g_j(\xi,\omega) \neq 0\}$$

$$= \{\omega \in \Omega : \forall\,(\xi,\lambda) \in \Xi \times \mathbb{R}^m\ ,\ F(\xi,\lambda,\omega) = 0\ \Rightarrow\ \forall j \in M, \text{ either } \lambda_j \neq 0 \text{ or } g_j(\xi,\omega) \neq 0\}$$

$\Omega\backslash pr(C) \subseteq \Omega^*$:
We want to show that if $\omega \in \Omega\backslash pr(C)$, then for any $(\xi,\lambda) \in \Xi \times \mathbb{R}^m$ such that $F(\xi,\lambda,\omega) = 0$, either $\lambda \neq 0$ or $g(\xi,\omega) \neq 0$.
Since $\omega \in \Omega \setminus pr(C)$, if $F(\xi,\lambda,\omega) = 0$, then either $\lambda \neq 0$ or $g(\xi,\omega) \neq 0$, as desired.
$\Omega^* \subseteq \Omega\backslash pr(C)$:
Take $\omega \in \Omega^*$. Observe that

$$(\xi,\lambda) \in \Xi \times \mathbb{R}^m \quad \Rightarrow \quad \begin{cases} F(\xi,\lambda,\omega) \neq 0, \\ \text{or} \\ F(\xi,\lambda,\omega) = 0 \quad \overset{\omega\in\Omega^*}{\Rightarrow} \quad \forall j \in M, \text{ either } \lambda \neq 0 \text{ or } g(\xi,\omega) \neq 0 \end{cases}$$

Then, for any $(\xi,\lambda) \in \Xi \times \mathbb{R}^m$, either $F(\xi,\lambda,\omega) \neq 0$ or $\forall j \in M$, either $\lambda \neq 0$ or $g(\xi,\omega) \neq 0$, as desired.
The desired result also follows from what said in (7.37), with

$$q(\xi,\lambda,\omega) \equiv \langle \text{ either } \lambda_j \neq 0 \text{ or } g_j(\xi,\omega) \neq 0,\ \forall j \in M\rangle.$$

Full measure.
We want to apply Theorem 123, with the following identifications

M	Ω	N	m	p	n	$f : M \times \Omega \to N$
$\Xi \times \mathbb{R}^m$	Ω	$\mathbb{R}^{n+m+1}$	$n+m$	k	$n++m+1$	$F_{(M^{+0},M^{0+},\{j\})} : \Xi \times \mathbb{R}^m \times \Omega \longrightarrow \mathbb{R}^n \times \mathbb{R}^{m+1}$

[3]The proof is almost identical to the proof of the previous proposition.

Assumptions:

Theorem:	Our case:
f is $\mathcal{C}^1$	By assumption
y is a regular value for f	Assumption 3 with $y=0$
$m=\dim M<\dim N=n$	$n+m<n+m+1$

Conclusion:

Theorem:	Our case:
$\exists$ full measure set Ω^* s. th. $\forall\omega\in\Omega^*$, $f_\omega^{-1}(y)=\varnothing$;	$\exists$ full measure set $T_{(M^{+0},M^{0+},\{j\})}$ s. th. $\forall\omega\in T_{(M^{+0},M^{0+},\{j\})}$, $\left(F_{M^{+0},M^{0+},\{j\}\mid\omega}\right)^{-1}(0)=\varnothing$.

Then,

$$T_{(M^{+0},M^{0+},\{j\})}:=\left\{\omega\in\Omega:\left(F_{M^{+0},M^{0+},\{j\}\mid\omega}\right)^{-1}(0)=\varnothing\right\}=$$

$$=\left\{\omega\in\Omega:\forall\,(\xi,\lambda)\in\Xi\times\mathbb{R}^m,\ \ F_{M^{+0},M^{0+},\{j\}\mid\omega}(\xi,\lambda,\omega)\neq 0\right\}$$

has full measure. It is enough to show that $\cap_{(M^{+0},M^{0+},\{j\})\in P(M)}T_{(M^{+0},M^{0+},\{j\})}\subseteq\Omega^*$. Indeed, we are going to show

$$(\Omega^*)^C\subseteq\cup_{(M^{+0},M^{0+},\{j\})\in P(M)}\left(T_{(M^{+0},M^{0+},\{j\})}\right)^C.$$

Defined

$$Z_{(M^{+0},M^{0+},\{j\})}:=\Omega\setminus T_{(M^{+0},M^{0+},\{j\})}=\left\{\omega\in\Omega:\left(F_{M^{+0},M^{0+},\{j\}\mid\omega}\right)^{-1}(0)\neq\varnothing\right\}=$$

$$=\left\{\omega\in\Omega:\exists\,(\xi,\lambda)\in\Xi\times\mathbb{R}^m\text{ such that }\ F_{M^{+0},M^{0+},\{j\}\mid\omega}(\xi,\lambda,\omega)=0\right\},$$

we want to show

$$(\Omega^*)^C\subseteq\cup_{(M^{+0},M^{0+},\{j\})\in P(M)}Z_{(M^{+0},M^{0+},\{j\})}.$$

Indeed,

$$(\Omega^*)^C=\left\{\omega\in\Omega:\exists\,(\xi,\lambda)\in(F_\omega)^{-1}(0)\ \ \exists j\in M\text{ such that }\lambda_j=0\text{ and }g_j(\xi,\omega)=0\right\}\overset{\text{Def. }F}{=}$$

$$\left\{\omega\in\Omega:\begin{array}{l}\exists\,(\xi,\lambda)\in\Xi\times\mathbb{R}^m,\ \exists\left(\widehat{M}^{0+},\widehat{M}^{+0},\widehat{M}^{00}\right)\in\mathcal{P}(M),\exists j\in\widehat{M}^{00}\text{ such that}\\ \left(\begin{array}{ll}\varphi(\xi,\lambda,\omega)=0 & \\ \left(\begin{array}{l}\lambda_k=0\\ g_k(\xi,\omega)>0\end{array}\right) & \text{if }k\in\widehat{M}^{0+}\\ \left(\begin{array}{l}\lambda_l>0\\ g_l(\xi,\omega)=0\end{array}\right) & \text{if }l\in\widehat{M}^{+0}\\ \left(\begin{array}{l}\lambda_{j'}=0\\ g_{j'}(\xi,\omega)=0\end{array}\right) & \text{if }j'\in\widehat{M}^{00}\end{array}\right)\end{array}\right\}\overset{(*)}{\subseteq}$$

$$\left\{\omega\in\Omega:\begin{array}{l}\exists\,(\xi,\lambda)\in\Xi\times\mathbb{R}^m,\ \exists\left(\widehat{M}^{0+},\widehat{M}^{+0},\widehat{M}^{00}\right)\in\mathcal{P}(M),\text{ such that}\\ \left(\begin{array}{ll}\varphi(\xi,\lambda,\omega)=0 & \\ \lambda_k=0 & \text{if }k\in\widehat{M}^{0+}\\ g_l(\xi,\omega)=0 & \text{if }l\in\widehat{M}^{+0}\\ g_{j'}(\xi,\omega)=0 & \text{if }j'\in\widehat{M}^{00}\setminus\{j\}\\ \left(\begin{array}{l}\lambda_j=0\\ g_j(\xi,\omega)=0\end{array}\right) & \end{array}\right)\end{array}\right\}=$$

$$\left\{\omega\in\Omega:\begin{array}{l}\exists\,(\xi,\lambda)\in\Xi\times\mathbb{R}^m,\ \exists\left(\widehat{M}^{0+},\widehat{M}^{+0},\widehat{M}^{00}\right)\in\mathcal{P}(M)\text{ such that}\\ \left(\begin{array}{ll}\varphi(\xi,\lambda,\omega)=0 & \\ \lambda_k=0 & \text{if }k\in\widehat{M}^{0+}\\ & \\ g_l(\xi,\omega)=0 & \text{if }l\in\widehat{M}^{+0}\cup\left(\widehat{M}^{00}\overset{\downarrow}{\setminus}\{j\}\right)\\ \left(\begin{array}{l}\lambda_j=0\\ g_j(\xi,\omega)=0\end{array}\right) & \end{array}\right)\end{array}\right\}=$$

$$= \left\{ \omega \in \Omega : \begin{array}{c} \exists\,(\xi,\lambda) \in \Xi \times \mathbb{R}^m \;\; \exists \left(\widehat{M}^{0+}, \widehat{M}^{+0} \cup \left(\widehat{M}^{00} \backslash \{j\} \right), \{j\} \right) \in P(M) \quad \text{such that} \\ \left(F_\omega^{\widehat{M}^{0+}, \widehat{M}^{+0} \cup \left(\widehat{M}^{00} \backslash \{j\} \right), \{j\}} \right)^{-1} (0) \neq \varnothing \end{array} \right\} =$$

$$= \left\{ \omega \in \Omega : \begin{array}{c} \exists \left(\widehat{M}^{0+}, \widehat{M}^{+0} \cup \left(\widehat{M}^{00} \backslash \{j\} \right), \{j\} \right) \in P(M) \\ \exists\,(\xi,\lambda) \in \Xi \times \mathbb{R}^m \text{such that} \\ \left(F_\omega^{\widehat{M}^{0+}, \widehat{M}^{+0} \cup \left(\widehat{M}^{00} \backslash \{j\} \right), \{j\}} \right)^{-1} (0) \neq \varnothing \end{array} \right\}$$

$$= \left\{ \omega \in \Omega : \exists \left(\widehat{M}^{0+}, \widehat{M}^{+0} \cup \left(\widehat{M}^{00} \backslash \{j\} \right), \{j\} \right) \in P(M) \quad \text{such that } \omega \in Z_{\left(\widehat{M}^{0+}, \widehat{M}^{+0} \cup \left(\widehat{M}^{00} \backslash \{j\} \right), \{j\} \right)} \right\}$$

$$= \cup_{(M^{+0}, M^{0+}, \{j\}) \in P(M)} Z_{(M^{+0}, M^{0+}, \{j\})},$$

as desired.

Observe that inclusion $(*)$ follows from the fact the set after that symbol is obtained from the set before it just eliminating some conditions to be satisfied. ∎

7.5 On openness and density

Lemma 129 *Let X and Y be topological spaces. If for any $x \in X$, there exists a dense subset $D_x \subseteq Y$ such that for any $y \in D_x$ a property $p(x,y)$ holds, then there exists a dense subset D of $X \times Y$, defined as*

$$D = \{(x,y) \in X \times Y : y \in D_x\},$$

such that for any $(x,y) \in D$, the property $p(x,y)$ holds.

Proof. It is a direct consequence of the two propositions below. ∎

Proposition 130 *Let X and Y be topological spaces. Suppose that for every $x \in X$ there exists a dense subset $D_x \subseteq Y$. Define*

$$D = \{(x,y) \in X \times Y : y \in D_x\}.$$

Then D is dense in $X \times Y$.

Proof. Let $W \subseteq X \times Y$ be a nonempty open set. In the product topology, there exist nonempty open sets $U \subseteq X$ and $V \subseteq Y$ such that

$$U \times V \subseteq W.$$

Choose any $x_0 \in U$. Since D_{x_0} is dense in Y, we have

$$V \cap D_{x_0} \neq \varnothing.$$

Let $y_0 \in V \cap D_{x_0}$. Then

$$(x_0, y_0) \in U \times V \subseteq W.$$

Moreover, $y_0 \in D_{x_0}$, hence $(x_0, y_0) \in D$. Therefore,

$$W \cap D \neq \varnothing.$$

Since every nonempty open subset of $X \times Y$ intersects D, the set D is dense in $X \times Y$. ∎

Proposition 131 *Let X and Y be topological spaces. Suppose that for every $x \in X$ there exists a subset $D_x \subseteq Y$ such that for any $y \in D_x$, the property $p(x,y)$ holds. Then, for any*

$$(x,y) \in D := \{(x,y) \in X \times Y : y \in D_x\},$$

we have that $p(x,y)$ holds.

Proof. By assumption, for every $x \in X$ there exists a subset $D_x \subseteq Y$ such that $y \in D_x \implies p(x,y)$.
Now take any $(x,y) \in D$. By the definition of D, we have $y \in D_x$.
Applying the given assumption for this x, it follows that $p(x,y)$ holds.
Therefore, every $(x,y) \in D$ satisfies $p(x,y)$. ∎

Proposition 132 *If:*

1. $\widetilde{F} : \Xi \times \mathbb{E} \to \mathbb{R}^n$, $(\xi, \mathcal{E}) \mapsto \widetilde{F}(\xi, \mathcal{E})$ *is a continuous function,*
2. *the natural projection function* $pr : \widetilde{F}^{-1}(0) \to \mathbb{E}$ *is proper,*
3. *the function* $g : \Xi \times \mathbb{E} \to \mathbb{R}$ *is continuous,*

then the set

$$\mathcal{O} := \left\{ \mathcal{E} \in \mathbb{E} : \textit{for any } (\xi, \mathcal{E}) \in \widetilde{F}^{-1}(0),\ g(\xi, \mathcal{E}) > 0 \right\}$$

is $\mathcal{E}$*-open.*

Proof. Define the sets

$$A := \left\{ (\xi, \mathcal{E}) \in \widetilde{F}^{-1}(0) : g(\xi, \mathcal{E}) > 0 \right\},$$
$$B := \widetilde{F}^{-1}(0) \setminus A = \left\{ (\xi, \mathcal{E}) \in \widetilde{F}^{-1}(0) : g(\xi, \mathcal{E}) \leq 0 \right\}$$

where B is a $\widetilde{F}^{-1}(0)$-closed set because g is continuous.

Then $pr(B)$ is $\mathbb{E}$-closed because pr is proper, and therefore

$$\mathbb{E} \setminus pr(B)$$

is $\mathbb{E}$-open.

To get the desired result, we have to show that $\mathcal{O} = \mathbb{E} \setminus pr(B)$, as verified below. Observe that:

$$\begin{aligned}
pr(B) &:= \left\{ \mathcal{E} \in \mathbb{E} : \exists (\widetilde{\xi}, \widetilde{\mathcal{E}}) \in B \text{ such that } pr(\widetilde{\xi}, \widetilde{\mathcal{E}}) = \mathcal{E} \right\} \overset{\text{def. } pr}{=} \\
&= \left\{ \mathcal{E} \in \mathbb{E} : \exists (\widetilde{\xi}, \widetilde{\mathcal{E}}) \in B \text{ such that } \widetilde{\mathcal{E}} = \mathcal{E} \right\} = \\
&= \left\{ \mathcal{E} \in \mathbb{E} : \exists (\widetilde{\xi}, \mathcal{E}) \in B \right\} \overset{\text{def. } B}{=} \\
&= \left\{ \mathcal{E} \in \mathbb{E} : \exists (\widetilde{\xi}, \mathcal{E}) \in \left\{ (\xi, \mathcal{E}) \in \widetilde{F}^{-1}(0) : g(\xi, \mathcal{E}) \leq 0 \right\} \right\} = \\
&= \left\{ \mathcal{E} \in \mathbb{E} : \exists (\widetilde{\xi}, \mathcal{E}) \in \widetilde{F}^{-1}(0) \text{ such that } g(\widetilde{\xi}, \mathcal{E}) \leq 0 \right\}
\end{aligned}$$

and then

$$\mathbb{E} \setminus pr(B) = \left\{ \mathcal{E} \in \mathbb{E} : \forall (\widetilde{\xi}, \mathcal{E}) \in \widetilde{F}^{-1}(0),\ g(\widetilde{\xi}, \mathcal{E}) > 0 \right\}.$$

∎

Proposition 133 *Let* $g(\xi, \mathcal{E})$ *and* μ *be a generic constraint function and its associated multiplier, respectively. Suppose that for any* $(\xi, \mathcal{E}) \in \widetilde{F}^{-1}(0)$*,*

$$g(\xi, \mathcal{E}) \geq 0, \qquad \mu \geq 0, \qquad \min\{g(\xi, \mathcal{E}), \mu\} = 0.$$

Then the following sets are open:

$$\mathcal{O}_1 := \left\{ \mathcal{E} \in \mathbb{E} : \textit{for any } (\xi, \mathcal{E}) \in \widetilde{F}^{-1}(0),\ g(\xi, \mathcal{E}) > 0 \textit{ and } \mu = 0 \right\},$$

$$\mathcal{O}_2 := \left\{ \mathcal{E} \in \mathbb{E} : \textit{for any } (\xi, \mathcal{E}) \in \widetilde{F}^{-1}(0),\ g(\xi, \mathcal{E}) = 0 \textit{ and } \mu > 0 \right\}.$$

Proof. Define

$$A_1 := \left\{(\xi,\mathcal{E}) \in \widetilde{F}^{-1}(0) : g(\xi,\mathcal{E}) > 0 \text{ and } \mu = 0\right\},$$
$$B_1 := \left\{(\xi,\mathcal{E}) \in \widetilde{F}^{-1}(0) : \neg\left\langle g(\xi,\mathcal{E}) > 0 \text{ and } \mu = 0\right\rangle\right\}.$$

Observe that $(\xi,\mathcal{E}) \in \widetilde{F}^{-1}(0)$ implies that $\min\{g(\xi,\mathcal{E}),\mu\} = 0$. Moreover,

$$\left\langle \begin{array}{l} \min\{g(\xi,\mathcal{E}),\mu\} = 0 \\ \wedge \\ \neg\left\langle g(\xi,\mathcal{E}) > 0 \text{ and } \mu = 0\right\rangle \end{array} \right\rangle \iff$$
$$\iff \left\langle \begin{array}{l} \left\langle \begin{array}{l} \text{either } g(\xi,\mathcal{E}) > 0 \text{ and } \mu = 0, \\ \text{or } g(\xi,\mathcal{E}) = 0 \text{ and } \mu > 0, \\ \text{or } g(\xi,\mathcal{E}) = 0 \text{ and } \mu = 0 \end{array} \right\rangle \\ \wedge \\ \neg\left\langle g(\xi,\mathcal{E}) > 0 \text{ and } \mu = 0\right\rangle \end{array} \right\rangle$$
$$\iff \left\langle \begin{array}{l} \text{either } g(\xi,\mathcal{E}) = 0 \text{ and } \mu > 0, \\ \text{or } g(\xi,\mathcal{E}) = 0 \text{ and } \mu = 0 \end{array} \right\rangle \iff \left\langle g(\xi,\mathcal{E}) = 0 \text{ and } \mu \geq 0\right\rangle.$$

Summarizing, if $(\xi,\mathcal{E}) \in \widetilde{F}^{-1}(0)$, then

$$\neg\left\langle g(\xi,\mathcal{E}) > 0 \text{ and } \mu = 0\right\rangle \iff \left\langle g(\xi,\mathcal{E}) = 0 \text{ and } \mu \geq 0\right\rangle. \tag{7.38}$$

Then,

$$B_1 := \left\{(\xi,\mathcal{E}) \in \widetilde{F}^{-1}(0) : g(\xi,\mathcal{E}) = 0 \text{ and } \mu \geq 0\right\}$$

and B_1 is a closed subset of $\widetilde{F}^{-1}(0)$. Then we can follow the standard proof. Since the projection pr is proper, then $pr(B_1)$ is closed and $\mathbb{E} \setminus pr(B_1)$ is open, see Proposition 132.

We want to show that $\mathcal{O}_1 = \mathbb{E} \setminus pr(B_1)$, which is done below.

$$pr(B_1) = \left\{\mathcal{E} \in \mathbb{E} : \exists(\widetilde{\xi},\mathcal{E}) \in \widetilde{F}^{-1}(0) \text{ such that } g(\widetilde{\xi},\mathcal{E}) = 0 \text{ and } \mu \geq 0\right\},$$

and

$$\mathbb{E} \setminus pr(B_1) = \left\{\mathcal{E} \in \mathbb{E} : \forall(\widetilde{\xi},\mathcal{E}) \in \widetilde{F}^{-1}(0),\ \neg\left\langle g(\widetilde{\xi},\mathcal{E}) = 0 \text{ and } \mu \geq 0\right\rangle\right\}$$
$$\overset{(7.38)}{=} \left\{\mathcal{E} \in \mathbb{E} : \forall(\widetilde{\xi},\mathcal{E}) \in \widetilde{F}^{-1}(0),\ g(\widetilde{\xi},\mathcal{E}) > 0 \text{ and } \mu = 0\right\} \overset{\text{def. } \mathcal{O}_1}{=} \mathcal{O}_1.$$

The proof for $\mathcal{O}_2$ is perfectly symmetric. ∎

7.6 Michal-Bastiani differentiability and Glöckner's Theorem

We refer to Schmeding (2022) for further details and results concerning Michal–Bastiani differentiability and Glöckner's theorem.

Michal-Bastiani differentiability and differentiability

We recall the definition of Michal-Bastiani differentiability for functions defined on open subsets of $\mathbb{R}^n$.

Definition 134 *Let E and W be two locally convex and topological real vector spaces.*

A function $f\colon V \to W$ defined on $V \subseteq E$ an open subset of E, is said to be Michal-Bastiani differentiable if:

1. *$f \in \mathcal{C}^0(V,W)$;*
2. *the function*

$$d_M f : V \times E \to W, \qquad (w,s) \longmapsto \lim_{t\to 0} \frac{f(w+ts) - f(w)}{t}$$

is well defined, i.e., the limit always exists and it is component-wise finite;

3. $d_M f$ is continuous.

If $f\colon V \to W$ is Michal-Bastiani differentiable, then we set $f \in \mathcal{C}^1_M(V,W)$.

We also need to define $\mathcal{C}^1$ function in the case in which the domain is not necessarily an open set.

Definition 135 *Given a function $f : X \subseteq E \to \mathbb{R}^n$, $v \longmapsto f(v)$, where X is not necessarily open, we say that*

1. $f \in \mathcal{C}^0(X,\mathbb{R}^n)$ if f is continuous on X;

2. $f \in \mathcal{C}^1_M(X,\mathbb{R}^n)$ if for any $v_0 \in X$ there exists an E open neighborhood $N(v_0)$ and a function $\overline{f} : N(v_0) \to \mathbb{R}^n$ such that

1. for any $v \in N(v_0) \cap X$, $g(v) = \overline{f}(v)$;

2. $\overline{f} \in \mathcal{C}^1_M(N(v_0),\mathbb{R}^n)$.

Remark 136 *Given an open subset $\mathcal{O}$ of $\mathbb{R}^n$, an open subset V of a topological vector space E and a function $f : \mathcal{O} \times V \to \mathbb{R}^n$, $(x,v) \longmapsto f(x,v)$, with standard notation, we define*

$$\text{for any } v \in V,\ f_v := f_{\{v\}} : \mathcal{O} \to \mathbb{R}^n, \quad x \longmapsto f_v(x) = f(x,v).$$

If $f_v \in \mathcal{C}^1(\mathcal{O} \subseteq \mathbb{R}^n, \mathbb{R}^n)$, then from basic analysis, for any $x_0 \in \mathcal{O}$, there exists a linear function

$$d(f_v)_{x_0} : \mathbb{R}^n \to \mathbb{R}^n, u \longmapsto d(f_v)_{x_0}(u)$$

such that

$$\forall w \in \mathbb{R}^n,\ d(f_v)_{x_0}(w) = [D_x(f_v)(x_0)] \cdot w = (f_v)'(x_0;w) := \lim_{t\to 0} \frac{f(x_0+tw,v) - f(x_0,v)}{t}.$$

From basic analysis, $d(f_v)$ is an invertible linear function if and only if the Jacobian $[D_x(f_v)(x_0)]$ is an invertible matrix.

Proposition 137 *Let an open subset A of $\mathbb{R}^n$ and a continuous function $f : A \longmapsto \mathbb{R}^m$ be given. Then, the following statements are equivalent:*

1. f is $\mathcal{C}^1$;

2. $d_M f$ is continuous, i.e., f is $\mathcal{C}^1_M$;

3. f is differentiable and f' is continuous.

Proposition 138 *Let an $\mathbb{R}^n$ open set $\mathcal{O}$ and an E open set V be given. If $f : \mathcal{O}\times V \to \mathbb{R}^n$ is Michal-Bastiani differentiable then for any $v \in \mathcal{V}$, $f_v \in \mathcal{C}^1(\mathcal{O} \subseteq \mathbb{R}^n, \mathbb{R}^n)$.*

Proof. Since f and $d_M f$ are continuous by assumption, then from Proposition 220 in my Notes on Topology, for any $v \in V$, we have that f_v and $d_M(f_v)$ are continuous, i.e., f_v is Michal-Bastiani differentiable. Then the desired result follows from Proposition 137. ∎

The implicit function theorem by Glöckner

Theorem 139 *Given an open subset $\mathcal{O}$ of $\mathbb{R}^n$, an open subset V of a topological vector space E and a function $f : \mathcal{O} \times V \to \mathbb{R}^n$, $(x,v) \longmapsto f(x,v)$, assume that*

1. $f \in \mathcal{C}^1_M(\mathcal{O}\times V, \mathbb{R}^n)$ and

there exists $(x_0,v_0) \in \mathcal{O} \times V$ such that

2. $f(x_0,v_0) = 0$, and

3. $d(f_{v_0})_{x_0} : \mathbb{R}^n \to \mathbb{R}^n$ is invertible,

then[4]

there exist an open neighborhood $\mathcal{O}(x_0) \subseteq \mathcal{O} \subseteq \mathbb{R}^n$ of x_0, and open neighborhood $V(v_0) \subseteq V \subseteq E$ of v_0 and a function $\zeta : V(v_0) \to \mathcal{O}(x_0)$ such that

1. $\zeta \in \mathcal{C}^1(V(v_0),\mathbb{R}^n)$;

2. $\zeta(V(v_0)) \subseteq \mathcal{O}(x_0)$;

3. $\zeta(v_0) = x_0$;

4. $\{(x,v) \in \mathcal{O}(x_0) \times V(v_0) : f(x,v) = 0\} = \text{graph } \zeta := \{(x,v) \in \mathcal{O}(x_0) \times V(v_0) : x = \zeta(v)\}$.

[4]Indeed, the standard conclusion of the finite dimensional version of the Implicit Function Theorem holds true.

Remark 140 *We emphasize that the general version of the implicit function theorem presented above is essential for our analysis, rather than the classical version (see, for instance, Lang, 1983, p. 131, or Coleman, 2012, p. 166). This is because one of the factors in the Cartesian product constituting the domain of the equilibrium function is the space of three times continuously differentiable utility functions endowed with the $\mathcal{C}^3$ compact-open topology. Since this topology is not induced by a norm, the classical implicit function theorem is not applicable in our setting. By contrast, the space of utility functions equipped with the $\mathcal{C}^3$ compact-open topology is a topological vector space, so Glöckner's implicit function theorem applies and provides the appropriate framework for our analysis.*

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