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Fairness, Justified Envy and Non-Discrimination with Consumption Externalities

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Abstract
We study fairness in finite exchange economies with consumption externalities using the framework of *A*-economies, in which each agent is associated with a reference group and may propose consumption plans for the agents belonging to that group. We introduce notions of coalitional fairness and justified envy adapted to this reference-group structure and relate them to *A*-equilibrium and the fuzzy *A*-core. Under the conditions ensuring fuzzy *A*-core equivalence, equilibrium, core, coalitional fairness and the justified *A*-envy-free core characterize the same allocations. We also introduce a notion of type appropriate to *A*-economies and show that, with Aubin coalitions, imposing a solidarity based non-discrimination requirement does not further restrict the fuzzy *A*-core. The analysis highlights how the reference structure governs both decentralized opportunities and the relevant comparisons underlying fairness.

Keywords: Exchange economies; consumption externalities; *A*-equilibrium; fuzzy *A*-core; coalitional fairness; justified envy; non-discrimination.

JEL classification: C71, D51, D62, D63.

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1 Introduction

This paper studies fairness in exchange economies with consumption externalities. In such economies, agents evaluate social consumption profiles rather than only their own consumption bundles. Their preferences may reflect altruism, inequality aversion, reciprocity or, more generally, interpersonal influence. Consumption externalities arise naturally also in collective models, where individual decisions depend on the consumption and welfare of other household members; see, for instance, Dufwenberg et al. (2011) and Gersbach and Haller (2001).

The presence of these externalities complicates both equilibrium and fairness analysis. Competitive allocations need not satisfy the usual efficiency properties, and the welfare consequences of assigning a bundle to one agent may depend on the consumption of others. The classical connections among competitive equilibrium, efficiency and fairness therefore need to be reconsidered in a framework that accommodates different forms of interpersonal influence while retaining a meaningful notion of decentralized choice.

A general framework of this kind is provided by the A-economy introduced by Vasil'ev (2016) and further developed by Graziano et al. (2026). In an A-economy, each individual i is associated with an exogenously given reference group A_i. The group identifies the agents for whom i may propose consumption bundles and whose aggregate resources determine the corresponding group budget. The family $A = (A_i)_{i \in N}$ may therefore describe heterogeneous patterns of interpersonal influence and control within the economy.

The flexibility of the family A allows several benchmark environments to be represented within the same framework. If $A_i = \{i\}$, one obtains a Walras-Nash economy with consumption externalities; see Borglin (1973) and Dufwenberg et al. (2011). If $A_i = N \setminus \{i\}$, the resulting structure reflects the mutual-support logic of a Berge-Zhukovskii environment; see Zhukovskii and Chikrii (1994), Abalo and Kostreva (2005) and Nessah et al. (2007). When the reference sets are generated by a partition, the framework encompasses family and collective household economies; see Haller (2000) and Gersbach and Haller (1999, 2001). More generally, the reference groups may differ across agents, allowing self-regarding, household and other-regarding forms of behavior to coexist within the same economy.

The family A also admits a network interpretation. A directed link from i to h indicates that $h \in A_i$. Unlike a standard social-interaction link[1], this relation identifies not only an agent whose consumption is relevant to i, but also a position for which i may propose a consumption bundle and whose resources enter the corresponding group budget. The reference network therefore combines interpersonal influence with a structure of choice and control. Its incidence properties also provide an interpretation of the balancedness condition used in the core equivalence results. This network and the associated hypergraph representation are discussed in Appendix B.

The equilibrium and cooperative properties of A-economies have been investigated through fuzzy coalition methods. The fuzzy A-core adapts Aubin's partial participation approach to an environment in which a coalition member proposes consumption plans for the members of her reference group. Under convexity and an appropriate balancedness condition on the family A, fuzzy A-core allocations coincide with A-equilibrium allocations. More recently, Graziano et al. (2026) extend this equivalence beyond convex preferences by allowing agents to participate simultaneously in several fuzzy coalitions.

The purpose of the present paper is to investigate a different implication of the structure underlying A-economies, namely its connection with fairness.

Fairness is particularly delicate in economies with consumption externalities. In the classical framework, envy is defined by asking whether an individual would prefer another agent's consumption bundle to her own. With interdependent preferences, however, the desirability of another bundle may depend on the consumption of other agents as well. Moreover, competitive allocations need not be Pareto efficient, so that the classical connection between competitive

[1] For a network approach to social interactions in general equilibrium, see Ghiglino and Goyal (2010).

equilibrium, efficiency and envy-freeness cannot in general be reproduced without modification. This also raises the question of how fairness itself should be formulated in the presence of consumption externalities, an issue that has received comparatively limited attention. An earlier and broader axiomatic perspective is provided by Moulin (1990), who introduces preference and group externalities by comparing each agent's actual welfare with welfare under suitably defined counterfactual scenarios. These externalities differ from the consumption externalities considered here, but the analysis examines how the resulting welfare bounds interact with no envy and resource monotonicity. More directly related to the present setting, Velez (2016) considers other-regarding preferences and argues that Foley's equity test, based on exchanging agents' consumption bundles, remains a meaningful fairness criterion in such an environment. Together, these contributions illustrate how counterfactual welfare comparisons and external effects may alter the formulation and interpretation of fairness requirements.

A related perspective is provided by Donnini and Pesce (2021a), who show that meaningful envy comparisons need not involve every pair of agents. They consider economies in which individuals are organized into classes of related agents, interpreted, for instance, as groups of neighbors. Fairness is evaluated locally by requiring absence of envy within the relevant classes, while suitable connectivity conditions allow local non-envy to imply fairness for the economy as a whole. This relational perspective is useful for A-economies, although the analogy is only partial. A neighborhood identifies the agents towards whom an individual directs an envy comparison. A reference group A_i also determines the agents whose consumption is involved in i's decision and the consumption profile that i may propose. Thus, the reference structure affects not only with whom fairness comparisons may be made, but also the appropriate object of comparison.

A second approach to fairness moves from individual to coalitional comparisons. Starting from Gabszewicz (1975), c-fairness asks whether a coalition could improve by obtaining the net trade assigned to another coalition. Aubin's partial-participation approach makes this comparison more flexible by allowing agents to participate only partially in cooperative arrangements. Building on this perspective, Basile et al. (2018) develop coalitional fairness with differentiated participation rates and show that it may also have an individual non-envy interpretation. In a related direction, Donnini and Pesce (2021b) formulate fairness through envy towards fuzzy coalitions and obtain characterizations involving competitive and Aubin-core allocations. This approach fits the A-economic framework naturally, since the fuzzy A-core already combines participation rates with the reference-group structure; see Vasil'ev (2016) and Graziano et al. (2026). It therefore raises the question whether an appropriate notion of Aubin coalitional fairness can characterize A-equilibrium allocations.

On the individual side, the problem is subtler. If agent i chooses a consumption profile for A_i, while agent j is associated with a possibly different reference group A_j, comparing only x_i and x_j ignores the interpersonal structure that defines the economy. Moreover, an observed difference between their situations need not represent an objectionable inequality, but may reflect different initial opportunities. This point is related to the analysis of no-envy and justified envy by Hervés-Beloso et al. (2025), who emphasize that fairness comparisons should take account of the circumstances that make two individuals meaningfully comparable. In an A-economy, an agent's opportunity is determined not only by her individual endowment, but also by the reference group over which she makes decisions and by its aggregate resources. We therefore compare agents whose reference groups have the same size and aggregate initial endowment. Since these groups may contain different agents, an additional identification is needed to interpret the consumption profile observed on A_j as a profile indexed by A_i. This leads to the notion of an admissible identification introduced in the paper.

We develop the coalitional and individual approaches jointly.

We begin by introducing a notion of *Aubin A-c-fairness*. Fuzzy coalitions compare net trades weighted according to their participation rates and to the reference-group structure.

We show that every A-equilibrium allocation is Aubin A-c-fair and that every Aubin A-c-fair allocation belongs to the fuzzy A-core. Consequently, under the assumptions ensuring fuzzy A-core equivalence, A-equilibrium allocations, fuzzy A-core allocations and Aubin A-c-fair allocations coincide. This provides a coalitional fairness characterization of A-equilibrium.

We then turn to individual fairness and introduce a notion of *justified A-envy*. Two agents are regarded as comparable when their reference groups have the same size and the same aggregate initial endowment. When the two groups differ, an admissible identification permits the consumption profile associated with one group to be interpreted as a consumption profile for the other. The identification leaves unchanged the agents who belong to both reference groups and operates only on the parts that distinguish them.

Our main definition is based on a replacement comparison, which is consistent with the sectional choice problem defining an A-equilibrium, although the resulting social consumption profile need not be feasible. We also introduce a feasible comparison obtained by exchanging the parts of the two reference groups that are not shared. When each agent's preferences depend only on consumption within her reference group, the replacement and swap comparisons are equivalent. We show that A-equilibrium allocations are justified A-envy-free and that Aubin A-c-fairness implies justified A-envy-freeness. Under the same preference restriction, these results also apply to the feasible swap version.

The benchmark cases illustrate the economic content of these comparisons. In a Walras-Nash economy, the replacement version of justified A-envy reduces to the familiar comparison between equally endowed agents, with other agents' consumption held fixed. The feasible swap version instead exchanges the bundles assigned to the two agents. In a family economy, the corresponding comparisons are made between households of the same size and with the same aggregate initial resources. In a Berge economy, the relevant objects are consumption profiles of the other agents rather than the decision maker's own bundle. The structure of A therefore determines not only individual behavior but also the economically meaningful object of a fairness comparison.

Beyond its role as a property of allocations, fairness may also be incorporated directly into cooperative analysis by restricting the blocking mechanism. Hervés-Beloso and Moreno-García (2026) pursue this idea through a family of non-discriminatory cores based on equity, status and solidarity criteria. Under these criteria, an agent excluded from a blocking coalition may interrupt the block when an agent of the same type inside the coalition receives a treatment that violates the relevant non-discrimination requirement.

Adapting this approach to an A-economy first requires a notion of type that reflects the reference-group structure. We introduce a notion of A-type based on the economic equivalence of reference-group opportunities and preferences. Agents of the same A-type need not have identical reference groups. Their reference groups must instead represent equivalent opportunities, and their preferences over the corresponding consumption plans must agree under a suitable isomorphism.

We then use A-types to formulate a non-discriminatory fuzzy A-core based on the solidarity criterion. The full support property of Aubin blocking implies that this restriction does not enlarge the fuzzy A-core: whenever an allocation can be blocked, it can be blocked with every agent participating at a positive rate. In the standard selfish benchmark, the same argument also shows that the distinction between weak and strict solidarity blocking disappears under Aubin participation. More generally, the equity, status and solidarity versions of the non-discriminatory Aubin core all coincide with the ordinary Aubin core.

Finally, two complementary aspects of the analysis are developed in the appendices. First, a direct cooperative proof of the fairness core equivalence clarifies the role of the balancedness assumption adopted in the paper. Balancedness provides the weights needed to aggregate net trades generated by possibly overlapping reference groups, while the divisibility of Aubin participation makes it possible to turn a violation of coalitional fairness into a fuzzy A-block,

without relying on equilibrium prices. Second, the reference structure is interpreted as a directed network (and as a hypergraph). These representations provide a structural interpretation of balancedness in terms of the incidence and covering properties of the family of reference groups.

The paper is organized as follows. Section 2 introduces the A-economic framework, the A-equilibrium, the fuzzy A-core, and the core equivalence result used in the analysis. Section 3 develops Aubin A-c-fairness and studies its relation with A-equilibrium and the fuzzy A-core. Section 4 introduces justified A-envy, develops both the replacement and feasible swap comparisons, studies their benchmark cases, and establishes their relations with A-equilibrium and coalitional fairness. Section 5 introduces A-types and the corresponding non-discriminatory fuzzy A-core, and shows how the full support property of Aubin blocking makes solidarity and related non-discrimination restrictions non-binding. Section 6 discusses possible extensions.

2 The A-economic framework

There is a finite set of agents $N = \{1, \dots, n\}$, with $n \geq 2$, a finite number l of commodities, and the common individual consumption space $\mathbb{R}^l_+$. Agent i has initial endowment $\omega_i \in \mathbb{R}^l_+$ and a preference relation $\succsim_i$ on $\mathbb{R}^{ln}_+$. Preferences may therefore exhibit consumption externalities: agent i's ranking may depend not only on her own consumption bundle, but also on the consumption of other agents.

To each agent i is exogenously associated a nonempty reference set $A_i \subseteq N$. The set A_i identifies the agents for whom i may propose consumption bundles. Reference sets may differ across agents and no self-inclusion requirement is imposed. In particular, i need not belong to A_i. Thus, different agents may act upon different subsets of the population. We write

$$A = (A_i)_{i \in N}, \qquad D_i := \{h \in N : i \in A_h\},$$

where D_i is the set of agents whose reference group contains agent i.

For every i,

$$X_{A_i} := \mathbb{R}^{l|A_i|}_+, \qquad x_{A_i} := (x_h)_{h \in A_i}.$$

When agent i proposes a possibly different consumption plan for the members of her reference group, we write

$$x_{(i)} = (x_{ih})_{h \in A_i} \in X_{A_i},$$

where x_{ih} denotes the bundle proposed by agent i for agent $h \in A_i$.

For an allocation x, define

$$P_{A_i}(x) := \left\{ z_{(i)} \in X_{A_i} : (z_{(i)}, x_{N \setminus A_i}) \succ_i x \right\}.$$

Thus, when evaluating a deviation on A_i, agent i keeps the consumption of agents outside A_i fixed at the status quo. Notice that this does not require preferences to be unaffected by consumption outside A_i. Such consumption may generate externalities, but it is held fixed when i evaluates alternative plans on her reference group. In this sense, deviations are evaluated according to a Nash-type conjecture outside A_i.

A vector $x = (x_i)_{i \in N}$ is feasible if

$$\sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i.$$

We denote the set of feasible allocations by $\mathcal{F}$, and write

$$E = \left\langle N, \mathbb{R}^l_+, (A_i, \succsim_i, \omega_i)_{i \in N} \right\rangle$$

for the economy.

2.1 Basic assumptions

We assume that the aggregate endowment satisfies

$$\sum_{i\in N} \omega_i \in \mathbb{R}^l_{++}.$$

For every agent i, the preference relation $\succsim_i$ is complete, transitive and continuous on $\mathbb{R}^{ln}_+$. Moreover, for every social consumption profile $x \in X := \mathbb{R}^{ln}_+$, preferences are strongly monotone on $X_{A_i} \times \{x_{N\setminus A_i}\}$.[2] No monotonicity with respect to agent i's own consumption is required when $i \notin A_i$.

The family $A = (A_i)_{i\in N}$ is assumed to satisfy

$$N = \bigcup_{i\in N} A_i$$

and to be balanced (in the sense of Bondareva) with full support. This means that

$$\exists\, \beta = (\beta_i)_{i\in N} \in \mathbb{R}^n_{++} \quad \text{such that} \quad \sum_{h\in D_i} \beta_h = 1, \qquad \forall i \in N. \tag{B}$$

Equivalently, if χ_{A_i} denotes the characteristic vector of A_i,

$$\sum_{i\in N} \beta_i \chi_{A_i} = \chi_N.$$

Remark 1. The family $A = (A_i)_{i\in N}$ can be represented as a directed network, with a link from i to h whenever $h \in A_i$. The balancedness condition with full support requires strictly positive weights on the reference groups such that every consumption position is covered with total weight one. It therefore imposes an incidence condition on the network generated by the family A. Appendix B develops this interpretation and gives the corresponding matrix and hypergraph formulations.

2.2 A-equilibrium

Given a strictly positive price vector p, define for each $i \in N$, the budget set

$$B_{A_i}(p,\omega) := \left\{ z_{(i)} \in X_{A_i} : p \cdot \sum_{h\in A_i} z_{ih} \le p \cdot \sum_{h\in A_i} \omega_h \right\}.$$

Hence agent i chooses a consumption profile for the members of A_i using the aggregate wealth of her reference group.

Definition 1 (A-equilibrium)**.** *A pair*

$$(x,p) \in \mathbb{R}^{ln}_+ \times \mathbb{R}^l_{++}$$

is an A-equilibrium if, for every $i \in N$,

$$x_{A_i} \in B_{A_i}(p,\omega), \qquad P_{A_i}(x) \cap B_{A_i}(p,\omega) = \varnothing,$$

and

$$\sum_{i\in N} x_i = \sum_{i\in N} \omega_i.$$

The set of A-equilibrium allocations is denoted by $\mathcal{W}(E)$.

The first two conditions express individual optimality with respect to the group budget, while the last one is the usual market clearing condition. In particular, in equilibrium, proposals concerning the same individual must be mutually consistent.

[2] More precisely, for every $x \in X$ and every $v \in \mathbb{R}^{l|A_i|}_+$, $v \neq 0$, $(x_{A_i} + v, x_{N\setminus A_i}) \succ_i x$.

2.3 Benchmark structures

The family A allows the model to encompass several equilibrium notions studied in the literature.

- Walras-Nash economy. If
$$A_i = \{i\} \qquad \text{for every } i \in N,$$
agent i chooses only her own consumption bundle, taking the consumption of all other agents as given. The A-equilibrium coincides with the competitive equilibrium à la Nash for an exchange economy with consumption externalities; see Borglin (1973). Related existence results are provided by Dufwenberg et al. (2011).

- Berge-Zhukovskii economy. If
$$A_i = N \setminus \{i\} \qquad \text{for every } i \in N,$$
agent i chooses a consumption profile for all the other agents while her own consumption is held fixed. This gives the general equilibrium counterpart of the Berge-Zhukovskii mutual support principle; see Zhukovskii and Chikrii (1994). For related results and more recent discussions, see also Abalo and Kostreva (2005), Nessah et al. (2007), and Haller (2024).

- Family economy. Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a partition of N into families, and let A_i be the family containing agent i. Members of the same family therefore share the same reference group and the same group budget. When preferences exhibit only intra-household externalities, this specification is related to the collective household models studied by Haller (2000) and Gersbach and Haller (1999, 2001) (see also Graziano et al., 2026 for a more explicit discussion of this connection in the A-economy framework).

- Total equilibrium. If
$$A_i = N \qquad \text{for every } i \in N,$$
every agent evaluates and chooses a consumption profile for the whole society and all agents face the same aggregate budget based on total initial resources.

Notice that Condition (B) is automatically satisfied in these benchmark cases. One can take $\beta_i = 1$ in the Walras-Nash case, $\beta_i = 1/(n-1)$ in the Berge-Zhukovskii case, $\beta_i = 1/|A_i|$ in the family economy, and $\beta_i = 1/n$ in the total-equilibrium case.

2.4 The fuzzy A-core

A fuzzy (or Aubin) coalition is a non-zero vector

$$\alpha = (\alpha_i)_{i \in N} \in [0,1]^n,$$

where α_i represents the participation rate of agent i. Its support is

$$\operatorname{supp}(\alpha) := \{i \in N : \alpha_i > 0\}.$$

For every $i \in N$, define

$$D_i(\alpha) := D_i \cap \operatorname{supp}(\alpha).$$

Definition 2 (Fuzzy A-core)**.** *Given $x \in \mathcal{F}$ and a fuzzy coalition α, we say that α A-improves upon x if there exist vectors*

$$z_{(h)} = (z_{hi})_{i \in A_h} \in X_{A_h}, \qquad h \in \operatorname{supp}(\alpha),$$

such that

$$z_{(h)} \in P_{A_h}(x), \qquad h \in \operatorname{supp}(\alpha),$$

and

$$\sum_{i\in N}\sum_{h\in D_i(\alpha)}\alpha_h z_{hi} \leq \sum_{i\in N}\sum_{h\in D_i(\alpha)}\alpha_h\omega_i.$$

The set of feasible allocations which cannot be A-improved upon by any fuzzy coalition is called the fuzzy (or Aubin) A-core and is denoted by

$$\mathcal{C}_f^A(E).$$

In the fuzzy blocking mechanism, every participating agent h proposes a consumption plan for the members of her reference-group A_h and uses the fraction α_h of their resources. Coalition members behave myopically in the sense that each of them formulates her reference-group plan independently of the plans proposed by the other members of the coalition, even when their reference groups overlap. Moreover, when evaluating an alternative plan on A_h, agent h keeps the consumption of agents outside A_h fixed at the status quo.

Under the assumptions stated above, convexity of preferences on the relevant A_i-sections yields the equivalence between the fuzzy A-core and A-equilibrium.

Theorem 1 (A-core equivalence under convexity)**.** *Assume the previous hypotheses and suppose, in addition, that agent i's preferences are convex on each A_i-section. Then*

$$W(E) = C_f^A(E).$$

Remark 2. The fuzzy A-core equivalence result under convexity was established by Vasil'ev (2016). Graziano et al. (2026) extend this result to A-economies with consumption externalities and non-convex preferences. Following Husseinov (1994), they enlarge the blocking mechanism by allowing agents to participate simultaneously in a finite collection of fuzzy coalitions, represented by a fuzzy coalition matrix. The resulting generalized fuzzy A-core, $C_h^A(E)$, satisfies

$$W(E) = C_h^A(E)$$

without any convexity assumption. Moreover, under convexity, $C_h^A(E) = C_f^A(E)$. The same paper also establishes an analogous equivalence theorem for economies with Arrowian markets for consumption externalities, where generalized fuzzy core allocations are decentralized through market and personalized prices.

3 Aubin coalitional fairness in A-economies

The starting point is the classical idea of c-fairness due to Gabszewicz (1975): an allocation is discriminatory if one coalition could make all its members better off by achieving the net trade assigned to another coalition. In an A-economy this idea cannot be transferred literally. A member h of an envious coalition does not in general propose only, or even necessarily, her own consumption bundle. Instead, she proposes a consumption profile for the agents in A_h. Consequently, both the improving plan and the resources used to evaluate its feasibility must reflect the reference-group structure.

The Aubin approach adds a second ingredient. A coalition member may participate only with a fraction of her resources. Participation rates therefore provide a natural way of weighting the net trades generated by the different reference groups. This motivates the following A-weighted net trade.

For any fuzzy coalition $\gamma = (\gamma_i)_{i\in N} \in [0,1]^n$, define

$$D_i(\gamma) := D_i \cap \operatorname{supp}(\gamma).$$

Let $\alpha \in [0,1]^n \setminus \{0\}$ be a non-zero fuzzy coalition and, for every $h \in \operatorname{supp}(\alpha)$, let

$$z_{(h)} = (z_{hi})_{i\in A_h} \in X_{A_h}.$$

The A-weighted net trade associated with the plans z is

$$\begin{aligned} T_A(\alpha, z) &:= \sum_{i \in N} \sum_{h \in D_i(\alpha)} \alpha_h (z_{hi} - \omega_i) \\ &= \sum_{h \in \mathrm{supp}(\alpha)} \alpha_h \sum_{i \in A_h} (z_{hi} - \omega_i). \end{aligned}$$

For any fuzzy coalition $\beta \in [0,1]^n$, possibly the null coalition, the A-weighted net trade assigned to β at an allocation x, interpreted as the status quo, is

$$\begin{aligned} T_A(\beta, x) &:= \sum_{i \in N} \sum_{h \in D_i(\beta)} \beta_h (x_i - \omega_i) \\ &= \sum_{h \in \mathrm{supp}(\beta)} \beta_h \sum_{i \in A_h} (x_i - \omega_i). \end{aligned}$$

In particular, $T_A(0, x) = 0$.

Definition 3 (Aubin A-c-fairness)**.** *A feasible allocation x is Aubin A-c-fair if there do not exist fuzzy coalitions*

$$\alpha, \beta \in [0,1]^n, \qquad \alpha \neq 0, \qquad \mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta) = \varnothing,$$

and plans

$$z_{(h)} \in X_{A_h}, \qquad h \in \mathrm{supp}(\alpha),$$

such that

$$z_{(h)} \in P_{A_h}(x) \qquad \textit{for every } h \in \mathrm{supp}(\alpha),$$

and

$$T_A(\alpha, z) \leq T_A(\beta, x).$$

The null coalition $\beta = 0$ is allowed.

Let $C^A_{\mathrm{fair}}(E)$ denote the set of Aubin A-c-fair allocations. Coalition α is potentially envious of β if all its members can improve while using no more than the A-weighted net trade assigned to β. Aubin A-c-fairness can therefore be interpreted as ruling out envy between the opportunities available to fuzzy coalitions, where these opportunities are measured by the net trades associated with their participation rates and the reference structure.

Viewed in this way, the definition adapts the notion of coalitional fairness with participation rates introduced by Basile et al. (2018) to the reference structure of an A-economy[3]. While Basile et al. (2018) allow the envious and envied coalitions to have overlapping supports and different participation rates, we initially require their supports to be disjoint. The extension to overlapping participation is discussed in Section 6.

The distinctive feature of our notion is that the relevant net trades are generated by reference-group proposals. For every $h \in \mathrm{supp}(\alpha)$, $z_{(h)}$ is a plan for the whole reference-group A_h, rather than merely a replacement for h's individual bundle, and the participation rate α_h weights the net trade associated with that proposal.

Proposition 1.

$$\mathcal{C}^A_{\mathrm{fair}}(E) \subseteq \mathcal{C}^A_f(E).$$

Proof. If $x \notin \mathcal{C}^A_f(E)$, some non-zero fuzzy coalition α has improving plans z with $T_A(\alpha, z) \leq 0$. Taking $\beta = 0$ violates Aubin A-c-fairness. □

[3]See also Donnini and Pesce (2021b) for a related fuzzy coalition approach to fairness.

Proposition 2. *Every A-equilibrium allocation is Aubin A-c-fair:*

$$W(E) \subseteq C^A_{\text{fair}}(E).$$

Proof. Let (x, p) be an A-equilibrium and suppose, by contradiction, that α, β, z violate Aubin A-c-fairness.

For every $h \in \text{supp}(\alpha)$, we have $z_{(h)} \in P_{A_h}(x)$. Equilibrium optimality implies

$$P_{A_h}(x) \cap B_{A_h}(p, \omega) = \varnothing,$$

and therefore $z_{(h)} \notin B_{A_h}(p, \omega)$. Hence

$$p \cdot \sum_{i \in A_h} (z_{hi} - \omega_i) > 0.$$

Multiplying by $\alpha_h > 0$ and summing over $h \in \text{supp}(\alpha)$, we obtain

$$p \cdot T_A(\alpha, z) > 0.$$

On the other hand, equilibrium feasibility on each reference-group budget gives

$$p \cdot \sum_{i \in A_h} (x_i - \omega_i) \leq 0$$

for every $h \in \text{supp}(\beta)$. Multiplying by β_h and summing yields

$$p \cdot T_A(\beta, x) \leq 0.$$

Since the violation of Aubin A-c-fairness requires

$$T_A(\alpha, z) \leq T_A(\beta, x),$$

we obtain

$$p \cdot T_A(\alpha, z) \leq p \cdot T_A(\beta, x) \leq 0,$$

contradicting $p \cdot T_A(\alpha, z) > 0$. Thus x is Aubin A-c-fair. □

Propositions 1 and 2 locate Aubin A-c-fairness between the decentralized and cooperative solution concepts of the A-economy. Indeed,

$$W(E) \subseteq C^A_{\text{fair}}(E) \subseteq C^A_f(E).$$

Thus, whenever A-equilibrium and the fuzzy A-core coincide, Aubin A-c-fairness provides an intermediate fairness characterization of the same set of allocations. This yields the following result which is a consequence of previous propositions.

Theorem 2 (Aubin A-c-fairness and the fuzzy A-core)**.** *Assume that*

- *the aggregate endowment satisfies*

$$\sum_{i \in N} \omega_i \in \mathbb{R}^l_{++};$$

- *the family $A = (A_i)_{i \in N}$ satisfies*

$$N = \bigcup_{i \in N} A_i$$

and is balanced with full support, i.e., there exists

$$\beta = (\beta_i)_{i \in N} \in \mathbb{R}^n_{++}$$

such that

$$\sum_{h \in D_i} \beta_h = 1, \qquad \forall i \in N;$$

- *for every $i \in N$, the preference relation $\succsim_i$ is complete, transitive and continuous on $\mathbb{R}^{ln}_+$, strongly monotone on A_i-sections, and convex on*

$$X_{A_i} \times \{x_{N \setminus A_i}\}$$

for every x.

Then

$$\mathcal{W}(E) = \mathcal{C}^A_{\text{fair}}(E) = \mathcal{C}^A_f(E).$$

In particular,

$$\mathcal{C}^A_{\text{fair}}(E) = \mathcal{C}^A_f(E).$$

Remark 3. The preceding equality between Aubin A-c-fair allocations and fuzzy A-core allocations has been obtained indirectly, by combining the inclusions

$$\mathcal{W}(E) \subseteq \mathcal{C}^A_{\text{fair}}(E) \subseteq \mathcal{C}^A_f(E)$$

with the fuzzy A-core equivalence theorem. A direct cooperative proof is also available under the additional endowment condition stated in Appendix A, without passing through equilibrium prices. The key observation is that balancedness provides strictly positive weights $b = (b_h)_{h \in N}$ such that

$$T_A(b, x) = \sum_{i \in N} (x_i - \omega_i)$$

for every social consumption profile x. Thus, the A-weighted net trades generated by possibly overlapping reference groups can be aggregated so as to reproduce the ordinary aggregate net trade of the economy. Together with the divisibility of participation provided by Aubin coalitions, this identity makes it possible to transform a violation of Aubin A-c-fairness directly into a fuzzy A-block. The argument is given in Appendix A.

4 Justified A-envy

We now turn from coalitional to individual fairness. As developed below, in an A-economy this requires identifying both the agents who can be meaningfully compared and the appropriate object of comparison.

The notion of justified envy proposed by Hervés-Beloso et al. (2025) is particularly suitable as a starting point for A-economies. Their approach recognizes that, with heterogeneous initial endowments, ordinary envy may merely reflect different initial opportunities. They therefore restrict justified envy to comparisons between agents with the same initial resources.

This idea fits the A-economic framework naturally, but the relevant opportunity set is now group-based. In an A-equilibrium, agent i chooses a consumption profile for A_i subject to a budget determined by the aggregate endowment of A_i. Hence the natural counterpart of equality of individual endowments is equality of aggregate endowments of the reference groups. Moreover, because agent i evaluates a profile indexed by A_i, rather than a single bundle, the profile associated with another reference-group must first be made comparable with a profile on A_i.

More generally, fairness comparisons in the presence of consumption externalities may be formulated by considering suitable transformations of a consumption profile, such as substitutions, transpositions or permutations; see Thomson (2011). The reference-group structure of an A-economy raises a related but distinct identification problem, since the consumption profiles to be compared may be indexed by different sets of agents. Substitution based comparisons need not preserve feasibility, since consumption assigned to one position may be reproduced at another. Transposition based comparisons avoid this difficulty by exchanging consumption bundles rather than reproducing them. We first develop a *substitution-based* notion adapted to the choice structure of an A-economy and return below to a feasible *swap-based* variant.

Definition 4 (A-comparability)**.** *Agents i and j are A-comparable if*

$$|A_i| = |A_j|, \qquad \sum_{h \in A_i} \omega_h = \sum_{h \in A_j} \omega_h.$$

Definition 5 (Admissible identification)**.** *For A-comparable agents i, j, an admissible identification is a bijection*

$$\sigma_{ij} : A_i \to A_j$$

such that $\sigma_{ij}(h) = h$ for every $h \in A_i \cap A_j$.

Since $|A_i| = |A_j|$, such an identification exists. Define

$$x_{A_j}^{\sigma_{ij}} = (x_{\sigma_{ij}(h)})_{h \in A_i} \in X_{A_i}.$$

The bundle $x_{A_j}^{\sigma_{ij}}$ is the consumption profile observed on A_j, reindexed as a consumption profile on A_i according to the admissible identification σ_{ij}. In particular, it is true that

$$\sum_{h \in A_i} x_{\sigma_{ij}(h)} = \sum_{k \in A_j} x_k.$$

The requirement that an admissible identification fix the agents in $A_i \cap A_j$ is meant to preserve the usual "standing in the other's shoes" interpretation underlying envy comparisons. Agents belonging to both reference groups represent the common part of the two situations being compared and therefore retain their identity and consumption. The identification operates only on the non-common parts $A_i \setminus A_j$ and $A_j \setminus A_i$, which capture the difference between the two reference-group positions. This restriction also preserves the directed nature of envy. As Thomson (2011) observes, an arbitrary permutation of a social consumption profile may express an agent's view about how consumption should be distributed among others, without representing envy towards a particular individual. The admissible identification considered here does not generate such an arbitrary redistribution. It translates the reference-group profile associated with a specified agent j into the coordinates of i's reference group, while leaving their common positions unchanged. Possible inequalities arising from permutations within the common part concern instead a distinct dimension of positional or within-group fairness.

The role of admissible identifications is particularly transparent in a family economy. If agents i and j belong to the same family, then $A_i = A_j$, and admissibility forces σ_{ij} to be the identity. The transferred profile therefore coincides with the status quo, and justified A-envy cannot arise between members of the same family, since there is no distinct reference-group opportunity to compare with their own. Under the present criterion, economically meaningful comparisons are therefore inter-family comparisons. If i and j belong to two different families of the same size and with the same aggregate initial endowment, agent i asks whether she would prefer the consumption pattern observed in j's family if that pattern were reproduced within her own family. Since distinct families form disjoint elements of the partition, there may be several admissible identifications between their members. As formalized below, justified A-envy arises whenever at least one such identification yields a strictly preferred family consumption profile. Thus, in the family benchmark, it can be interpreted as envy across households endowed with comparable initial opportunities.

Definition 6 (Justified A-envy)**.** *Agent i justifiably A-envies j at x if i, j are A-comparable and, for some admissible σ_{ij},*

$$(x_{A_j}^{\sigma_{ij}}, x_{N \setminus A_i}) \succ_i x.$$

A feasible allocation is justified A-envy-free if no such pair exists. Denote the set of such allocations by $\mathcal{E}_A(E)$.

Since preferences are complete, the previous definition is equivalently written as follows: x is justified A-envy-free if, for every pair of A-comparable agents i, j and every admissible identification σ_{ij},

$$x \succsim_i (x_{A_j}^{\sigma_{ij}}, x_{N \setminus A_i}).$$

Thus our definition is demanding in terms of envy-freeness: once two agents are A-comparable, the status quo must be weakly preferred under every admissible way of transferring the profile of A_j to A_i. This definition reflects the reference-group structure of the A-economic environment. Equal aggregate endowments preserve the central idea of justified envy: comparisons concern agents facing analogous initial opportunities. Equal reference-group size and the admissible identification reflect the second distinctive feature of the model, namely that preferences are evaluated over reference-group profiles. Thus, agent i asks whether she would prefer the consumption pattern observed in a group of the same size and with the same aggregate resources, once that pattern is expressed on her own reference-group. Agents outside A_i remain at the status quo.

Remark 4. The preceding notion differs from the swapping test proposed by Velez (2016) for economies with other-regarding preferences. Under that test, agent i compares the status quo with the allocation obtained by exchanging her consumption bundle with that of another agent j, while leaving all other consumptions unchanged. The present definition instead transfers a reference-group consumption profile to another reference group while holding consumption outside the receiving group fixed. It is therefore aligned with the choice problem of an A-economy, but the resulting comparison need not preserve feasibility. A feasible group-based extension of the swapping test is introduced below.

Remark 5. The preceding remark concerns how the consumption profile used in the envy comparison is constructed. A separate issue is how the admissible identifications enter the criterion for declaring envy. Our definition declares that i justifiably A-envies j if there exists at least one admissible identification under which the transferred profile is strictly preferred. Consequently, absence of envy requires weak preference for the status quo for every admissible identification. A more cautious criterion for declaring envy would require the transferred profile to be strictly preferred for every admissible identification. Under that alternative notion, envy would be ruled out whenever there exists at least one admissible identification for which the status quo is weakly preferred. We do not adopt this weaker notion here. The two notions coincide whenever the admissible identification is unique, as in the Walrasian-Nash and Berge-Zhukovskii benchmark cases. They also coincide when agent i's preferences on the A_i-section are invariant under permutations of the positions in A_i. Under this invariance condition, all admissible identifications generate consumption profiles that are indifferent for i. Moreover, allowing arbitrary bijections from A_i to A_j, rather than requiring them to preserve the intersection, would not alter the resulting notion of justified A-envy.

4.1 Benchmark cases

The meaning of justified A-envy becomes particularly transparent in the main benchmark structures introduced in Section 2. Depending on the reference structure, the relevant object of comparison may be an individual consumption bundle or a consumption profile associated with a group of agents.

Remark 6 (Walras-Nash benchmark)**.** Suppose that

$$A_i = \{i\} \qquad \text{for every } i \in N.$$

Since every reference-group is a singleton, the equal cardinality requirement is automatically satisfied and the admissible identification between A_i and A_j is unique.

Moreover, A-comparability reduces to

$$\omega_i = \omega_j.$$

Hence agent i justifiably A-envies agent j precisely when

$$(x_j, x_{-i}) \succ_i x.$$

Thus, in the presence of consumption externalities, agent i asks whether she would prefer to receive j's consumption bundle while abstracting from any simultaneous change in the consumption of the other agents, whose bundles are held fixed.

If preferences are selfish, the comparison further reduces to

$$x_j \succ_i x_i,$$

recovering the standard justified-envy comparison between equally endowed agents.

Remark 7 (Berge-Zhukovskii benchmark)**.** Suppose that

$$A_i = N \setminus \{i\} \qquad \text{for every } i \in N.$$

All reference groups have cardinality $n-1$, so the cardinality condition for A-comparability is automatically satisfied. Moreover,

$$\sum_{h \neq i} \omega_h = \sum_{h \neq j} \omega_h \quad \Longleftrightarrow \quad \omega_i = \omega_j.$$

Thus, comparability again reduces to equality of the individual endowments of i and j.

For two distinct agents i and j,

$$A_i \cap A_j = N \setminus \{i, j\}.$$

An admissible identification therefore fixes every agent belonging to this intersection and maps the only remaining element $j \in A_i$ into $i \in A_j$. Hence the admissible identification is unique.

Justified A-envy consequently has a genuinely other-regarding interpretation. Under the unique admissible identification, all agents in $N \setminus \{i, j\}$ retain their consumption bundles, while the position occupied by j in A_i is assigned the bundle x_i, since j is identified with i in A_j. Agent i then evaluates this transferred reference-group profile, keeping her own consumption x_i fixed outside A_i. Thus, she does not ask whether she would prefer j's consumption bundle to her own, but whether she would prefer the consumption profile of the other agents that results from placing her own bundle in j's position. In this sense, the comparison preserves the usual "standing in the other's shoes" interpretation of envy while adapting it to the mutual-support structure of a Berge economy.

Remark 8 (Family-economy benchmark)**.** Suppose that the reference sets are generated by a partition

$$\mathcal{H} = \{H_1, \ldots, H_k\},$$

with A_i equal to the family containing agent i. As discussed above, justified A-envy cannot arise between members of the same family. Members of distinct families are A-comparable when their families have the same cardinality and the same aggregate initial endowment. In that case, agent i justifiably A-envies agent j whenever at least one admissible identification makes the consumption profile observed in j's family, once transferred to i's family, strictly preferable to the status quo. Thus, justified A-envy takes the form of envy across households endowed with comparable initial opportunities.

Remark 9 (Total-equilibrium benchmark)**.** Suppose that

$$A_i = N \qquad \text{for every } i \in N.$$

Every pair of agents is automatically A-comparable. However,

$$A_i = A_j = N,$$

and admissibility requires

$$\sigma_{ij} = \mathrm{id}_N .$$

Therefore

$$x_{A_j}^{\sigma_{ij}} = x.$$

Since $N \setminus A_i = \varnothing$, justified A-envy would require

$$x \succ_i x,$$

which is impossible. Hence every feasible allocation is justified A-envy-free in the total-equilibrium benchmark. The notion is therefore vacuous in this case. This reflects the fact that justified A-envy is designed to compare distinct, although comparable, reference-group consumption profiles; when every reference-group coincides with the whole society, there is no distinct profile to transfer.

Having introduced justified A-envy and examined its behavior in the main benchmark cases, we now turn to its relation with the decentralized solution of the A-economy. The first question is whether the opportunity based comparison underlying justified A-envy is respected by A-equilibrium allocations.

Proposition 3. *Every A-equilibrium allocation is justified A-envy-free:*

$$\mathcal{W}(E) \subseteq \mathcal{E}_A(E).$$

Proof. Let (x, p) be an A-equilibrium and suppose i justifiably A-envies j. Then $x_{A_j}^{\sigma_{ij}} \in P_{A_i}(x)$ for some admissible identification. Since $x_{A_j} \in B_{A_j}(p, \omega)$,

$$p \cdot \sum_{k \in A_j} x_k \leq p \cdot \sum_{k \in A_j} \omega_k.$$

Comparability and preservation of aggregate consumption imply

$$p \cdot \sum_{h \in A_i} x_{\sigma_{ij}(h)} \leq p \cdot \sum_{h \in A_i} \omega_h,$$

so $x_{A_j}^{\sigma_{ij}} \in B_{A_i}(p, \omega)$, contradicting equilibrium optimality. □

One consequence of the preceding result is that, whenever an A-equilibrium exists, a justified A-envy-free allocation exists. Hence, in every class of A-economies admitting an equilibrium,[4] justified A-envy-free allocations are obtained without a separate existence argument.

More importantly, the proposition shows that the same reference-group structure that determines individual choice and budget opportunities in an A-equilibrium also determines the relevant notion of comparable opportunities for justified envy.

This establishes a connection between justified A-envy and decentralized choice. We next ask whether the same individual fairness property can also be recovered from the cooperative side of the model. This leads to a direct connection between justified A-envy and the coalitional fairness notion introduced in Section 3.

[4]Existence of A-equilibria is discussed in Graziano et al. (2026).

4.2 Coalitional fairness and justified A-envy

The connection is particularly direct: coalitional fairness already rules out the individual comparisons underlying justified A-envy.

Proposition 4.

$$\mathcal{C}^A_{\text{fair}}(E) \subseteq \mathcal{E}_A(E).$$

Proof. If $x \notin \mathcal{E}_A(E)$, choose A-comparable i, j and admissible σ_{ij} with $x^{\sigma_{ij}}_{A_j} \in P_{A_i}(x)$. Let α and β be the singleton fuzzy coalitions concentrated respectively on i and j, both with weight one, and put $z_{(i)} = x^{\sigma_{ij}}_{A_j}$. Then

$$T_A(\alpha, z) = \sum_{h \in A_i} (x_{\sigma_{ij}(h)} - \omega_h) = \sum_{k \in A_j} (x_k - \omega_k) = T_A(\beta, x),$$

where the second equality follows from bijectivity and A-comparability. Hence x violates Aubin A-c-fairness. □

The proofs of Propositions 3 and 4 show that the overlap preserving restriction imposed on admissible identifications is not needed for either inclusion. Both arguments use only the bijectivity of the identification and the equality of aggregate reference-group endowments. Consequently, both inclusions remain valid under the more demanding notion of justified A-envy obtained by admitting all bijections from A_i to A_j.

4.3 The justified A-envy-free fuzzy core

The preceding results allow us to combine the cooperative and individual fairness requirements in a single solution concept, extending to the A-economic framework the justified envy-free core proposed by Hervés-Beloso et al. (2025).

Definition 7. *The justified A-envy-free fuzzy core is*

$$\mathcal{C}^A_{EF}(E) := \mathcal{C}^A_f(E) \cap \mathcal{E}_A(E).$$

Theorem 3 (Fairness inclusions)**.** *For every A-economy,*

$$\mathcal{W}(E) \subseteq \mathcal{C}^A_{\text{fair}}(E) \subseteq \mathcal{C}^A_{EF}(E) \subseteq \mathcal{C}^A_f(E).$$

The first inclusion follows from equilibrium optimality and price support. The second combines the two implications established above: Aubin A-c-fairness implies both fuzzy A-core membership and justified A-envy-freeness. The last inclusion follows immediately from the definition of the justified A-envy-free fuzzy core.

Corollary 1 (Fairness characterization of A-equilibria)**.** *Assume the hypotheses of Theorem 1. Then*

$$W(E) = C^A_{\text{fair}}(E) = C^A_{EF}(E) = C^A_f(E).$$

Thus, whenever the A-core equivalence theorem applies, A-equilibrium allocations admit simultaneously a stability interpretation, a coalitional-fairness interpretation and an individual justified-envy-free interpretation. This contrasts with the corresponding chain of inclusions obtained by Hervés-Beloso et al. (2025) with ordinary coalitions, which may in general be strict. In the present Aubin framework, under the conditions ensuring fuzzy A-core equivalence, these inclusions collapse to equalities. In particular, this remains true in the standard selfish benchmark, where $A_i = \{i\}$ and preferences depend only on individual consumption.

4.4 A feasible swap-based comparison

The justified A-envy notion introduced above is based on a replacement operation. The consumption profile observed on A_j is reindexed on A_i, while consumption outside A_i remains fixed. This is consistent with the sectional choice problem defining an A-equilibrium, but the resulting social consumption profile need not be feasible. We now introduce a related comparison that exchanges the non-common parts of the two reference groups and therefore preserves feasibility.

Let i and j be A-comparable and let

$$\sigma_{ij} : A_i \longrightarrow A_j$$

be an admissible identification. Since σ_{ij} fixes $A_i \cap A_j$, it maps $A_i \setminus A_j$ bijectively onto $A_j \setminus A_i$. Define the permutation

$$\pi_{ij}^{\sigma} : N \longrightarrow N$$

by

$$\pi_{ij}^{\sigma}(h) = \begin{cases} \sigma_{ij}(h), & h \in A_i \setminus A_j, \\ \sigma_{ij}^{-1}(h), & h \in A_j \setminus A_i, \\ h, & h \notin (A_i \setminus A_j) \cup (A_j \setminus A_i). \end{cases}$$

For a social consumption profile x, let

$$x^{\sigma,ij} := \left(x_{\pi_{ij}^{\sigma}(h)}\right)_{h \in N}.$$

Thus, the consumption bundles assigned to $A_i \setminus A_j$ and $A_j \setminus A_i$ are exchanged according to σ_{ij}, while the bundles assigned to $A_i \cap A_j$ and to $N \setminus (A_i \cup A_j)$ remain fixed. Since π_{ij}^{σ} is a permutation of N,

$$\sum_{h \in N} x_h^{\sigma,ij} = \sum_{h \in N} x_h.$$

Consequently, $x^{\sigma,ij}$ is feasible whenever x is feasible.

Fixing $A_i \cap A_j$ is not required for feasibility, since any bijection from A_i to A_j can be extended to a permutation of N. It ensures, however, that the swap affects only $A_i \setminus A_j$ and $A_j \setminus A_i$, while their common positions remain unchanged, consistently with the interpretation given above. Under Reference-Group Dependence, the equivalence between the replacement and swap comparisons would continue to hold if more general permutation extensions were admitted.

Definition 8 (Swap-based justified A-envy)**.** *Agent i swap-based justifiably A-envies j at x if i and j are A-comparable and, for some admissible identification σ_{ij},*

$$x^{\sigma,ij} \succ_i x.$$

A feasible allocation is swap-based justified A-envy-free if no such pair exists. Denote the set of these allocations by $\mathcal{E}_A^{\mathrm{sw}}(E)$.

This definition extends the logic of the swapping test to reference-group opportunities. In the Walras-Nash benchmark, where $A_i = \{i\}$, it exchanges the consumption bundles of i and j, exactly as in Velez (2016). In the Berge-Zhukovskii benchmark, $A_i = N \setminus \{i\}$ and $A_j = N \setminus \{j\}$, so the two non-common parts are respectively $\{j\}$ and $\{i\}$, and the same individual swap is obtained. In a family economy, the comparison exchanges the consumption profiles assigned to two distinct comparable families. If $A_i = A_j$, the non-common parts are empty and the swap leaves the allocation unchanged.

The replacement and swap based comparisons coincide under a natural restriction on the scope of consumption externalities.

Reference-Group Dependence. For every $i \in N$ and every $x, y \in X$,

$$x_{A_i} = y_{A_i} \quad \Longrightarrow \quad x \sim_i y.$$

Thus, the consumption profile on A_i contains all the coordinates relevant to agent i's welfare. The condition allows preferences to distinguish among the identities and consumption bundles of the members of A_i; it only rules out consumption effects from positions outside A_i.[5]

Proposition 5 (Replacement-swap equivalence)**.** *Under Reference-Group Dependence,*

$$EA(E) = EA^{\mathrm{sw}}(E).$$

Proof. Let i, j be A-comparable and let σ_{ij} be an admissible identification. By construction,

$$\left(x^{\sigma,ij}\right)_{A_i} = x^{\sigma_{ij}}_{A_j}.$$

Hence Reference-Group Dependence implies

$$x^{\sigma,ij} \sim_i \left(x^{\sigma_{ij}}_{A_j}, x_{N\setminus A_i}\right).$$

It follows that

$$x^{\sigma,ij} \succ_i x \quad \Longleftrightarrow \quad \left(x^{\sigma_{ij}}_{A_j}, x_{N\setminus A_i}\right) \succ_i x.$$

The same admissible identifications enter the two definitions, so the corresponding envy-free sets coincide. □

Define the corresponding refinement of the fuzzy A-core by

$$\mathcal{C}^A_{EF,\mathrm{sw}}(E) := \mathcal{C}^A_f(E) \cap \mathcal{E}^{\mathrm{sw}}_A(E).$$

Under Reference-Group Dependence, the preceding proposition gives

$$\mathcal{E}_A(E) = \mathcal{E}^{\mathrm{sw}}_A(E)$$

and hence

$$\mathcal{C}^A_{EF}(E) = \mathcal{C}^A_{EF,\mathrm{sw}}(E).$$

Consequently, the fairness results established for the replacement comparison carry over to the feasible swap comparison. In particular,

$$\mathcal{W}(E) \subseteq \mathcal{C}^A_{\mathrm{fair}}(E) \subseteq \mathcal{C}^A_{EF,\mathrm{sw}}(E) \subseteq \mathcal{C}^A_f(E).$$

If, in addition, the hypotheses of Theorem 1 hold, all these inclusions become equalities:

$$\mathcal{W}(E) = \mathcal{C}^A_{\mathrm{fair}}(E) = \mathcal{C}^A_{EF,\mathrm{sw}}(E) = \mathcal{C}^A_f(E).$$

[5] Reference-Group Dependence is stronger than an anonymity condition of the type considered by Velez (2016). Such a condition may make the assignment of a given collection of outside bundles irrelevant, but replacement and swapping may generate different collections. Reference-Group Dependence instead makes consumption outside A_i irrelevant to agent i's preferences.

5 Non-discrimination and A-types

Some cooperative notions incorporate fairness directly into the blocking mechanism by restricting the way in which agents of the same type may be treated by a coalition. In a standard exchange economy, two agents have the same type when they have the same preferences and the same initial endowment. We denote by

$$T(i) = \{j \in N : \omega_j = \omega_i \text{ and } \succsim_j = \succsim_i\}$$

the type class of agent i.

Building on this notion, Hervés-Beloso and Moreno-García (2026) introduce a family of non-discriminatory cores in which normative constraints based on type identity apply to proposed coalitional deviations. Their equity, status and solidarity criteria specify different circumstances in which an agent excluded from a blocking coalition may interrupt the block because of the treatment received by an agent of the same type inside the coalition. Under the solidarity criterion, interruption occurs whenever the agent inside the coalition is strictly improved. The underlying principle is that a coalitional improvement should not selectively benefit some members of a type while excluding otherwise identical agents.

Adapting this idea to an A-economy raises two questions. First, the standard notion of type must be modified because the opportunities and choice objects of an agent depend on her reference group. Second, the implications of the solidarity restriction must be reconsidered when agents can participate through Aubin coalitions. We address these questions in turn.

5.1 A-types

In a standard exchange economy, type identity is naturally described in terms of individual preferences and endowments. In an A-economy, however, the relevant choice object of agent i is a consumption plan for the members of A_i, and the corresponding budget depends on the aggregate endowment of that reference group. Type identity should therefore reflect both the opportunities represented by the reference group and the preferences induced on the corresponding A_i-section.

Two agents of the same A-type need not have the same reference set. Indeed, requiring $A_i = A_j$ would be inappropriate even in the Walrasian-Nash benchmark, where $A_i = \{i\}$ for every i. We therefore compare reference groups up to an isomorphism.

For two reference groups A_i and A_j satisfying

$$|A_i| = |A_j|,$$

a *reference group isomorphism* from A_i to A_j is a bijection

$$\sigma_{ij} : A_i \longrightarrow A_j.$$

It induces a relabelling map

$$\tau_{ij} : X_{A_i} \longrightarrow X_{A_j}$$

defined by

$$\left[\tau_{ij}(z_{A_i})\right]_{\sigma_{ij}(h)} = z_h \qquad \text{for every } h \in A_i.$$

Thus, if $h \in A_i$ is identified with $k = \sigma_{ij}(h) \in A_j$, the bundle assigned to position h in z_{A_i} is assigned to position k in $\tau_{ij}(z_{A_i})$.

Unlike an admissible identification in the definition of justified A-envy, a reference group isomorphism is not required to fix the agents in $A_i \cap A_j$. The two mappings serve different purposes. In an envy comparison, common agents represent the common part of the two situations being compared and therefore retain their identity. Here the mapping identifies

economically equivalent positions in two reference groups. Allowing arbitrary bijections also makes reference group isomorphisms stable under composition, as required by the notion of type introduced below.

For every social consumption profile $x \in X$, let $\succsim_i^x$ denote the preference relation induced by $\succsim_i$ on the A_i-section through x. Thus,

$$z_{A_i} \succsim_i^x y_{A_i}$$

if and only if

$$(z_{A_i}, x_{N\setminus A_i}) \succsim_i (y_{A_i}, x_{N\setminus A_i}).$$

Hence, $\succsim_i^x$ describes how agent i ranks alternative consumption plans for the members of A_i, keeping consumption outside A_i fixed at x.

Definition 9 (A-type equivalence)**.** *For $i, j \in N$, write*

$$i \sim_A j$$

if

$$|A_i| = |A_j|,$$
$$\sum_{h\in A_i} \omega_h = \sum_{h\in A_j} \omega_h,$$

and there exists a reference group isomorphism

$$\sigma_{ij} : A_i \longrightarrow A_j$$

such that, for every $x \in X$ and every $z_{A_i}, y_{A_i} \in X_{A_i}$,

$$z_{A_i} \succsim_i^x y_{A_i} \quad \Longleftrightarrow \quad \tau_{ij}(z_{A_i}) \succsim_j^x \tau_{ij}(y_{A_i}).$$

The use of at least one reference group isomorphism is deliberate. Requiring preference equivalence under every isomorphism would impose anonymity within each reference group. The present definition instead allows positions to play different economic roles, provided that these roles can be matched through an isomorphism that preserves preferences. Equivalence is also required only on the sections relevant to the A-economic choice problem, not on the whole social consumption space. Agents of the same A-type must therefore rank corresponding reference group plans in the same way for every fixed outside profile, while remaining free to respond to consumption outside their respective reference groups.

Proposition 6. *The relation $\sim_A$ is an equivalence relation on N.*

Proof. Reflexivity follows by taking

$$\sigma_{ii} = \mathrm{id}_{A_i}.$$

For symmetry, suppose that $i \sim_A j$ through σ_{ij}. Since σ_{ij} is a bijection,

$$\sigma_{ji} := \sigma_{ij}^{-1}$$

is a reference group isomorphism from A_j to A_i, and

$$\tau_{ji} = \tau_{ij}^{-1}.$$

The preference equivalence in the definition can therefore be read in both directions, and hence $j \sim_A i$.

For transitivity, suppose that $i \sim_A j$ through σ_{ij} and that $j \sim_A k$ through σ_{jk}. Define

$$\sigma_{ik} := \sigma_{jk} \circ \sigma_{ij}.$$

This is a bijection from A_i to A_k, and

$$\tau_{ik} = \tau_{jk} \circ \tau_{ij}.$$

For every $x \in X$ and every $z_{A_i}, y_{A_i} \in X_{A_i}$, preference equivalence between i and j, followed by preference equivalence between j and k, gives

$$z_{A_i} \succsim_i^x y_{A_i} \quad \Longleftrightarrow \quad \tau_{ik}(z_{A_i}) \succsim_k^x \tau_{ik}(y_{A_i}).$$

Equality of reference group cardinalities and aggregate endowments is also transitive. Hence $i \sim_A k$. □

Definition 10 (A-type)**.** *The A-type of agent i is her equivalence class under $\sim_A$:*

$$T_A(i) := [i]_{\sim_A}.$$

Example 1. *Suppose that there is one commodity and*

$$A_1 = \{1, 2\}, \qquad A_3 = \{3, 4\},$$

with

$$\omega_1 + \omega_2 = \omega_3 + \omega_4.$$

Consider the reference group isomorphism defined by

$$\sigma_{13}(1) = 3, \qquad \sigma_{13}(2) = 4.$$

If, for every social profile x, the preferences induced on the relevant sections are represented by

$$u_1^x(a, b) = a + 2b \qquad \textit{and} \qquad u_3^x(c, d) = c + 2d,$$

then σ_{13} preserves the ranking of corresponding consumption plans, and agents 1 *and* 3 *have the same A-type. The alternative isomorphism matching* 1 *with* 4 *and* 2 *with* 3 *need not preserve these rankings. Thus, A-type equivalence requires the existence of a suitable isomorphism preserving preferences, but does not impose anonymity across positions within a reference group.*

The definition specializes naturally to the main benchmark structures. In the Walrasian-Nash case, $A_i = \{i\}$, so equality of aggregate reference group endowments reduces to

$$\omega_i = \omega_j.$$

The reference group isomorphism is unique, and the preference condition requires i and j to rank alternative individual consumption bundles in the same way for every fixed outside profile. Under the separability condition of Borglin (1973), these sectional rankings define preferences over individual consumption. With selfish preferences, the standard notion of type is recovered.

In a family economy, agents associated with different families may have the same A-type when their families have the same size and aggregate endowment and their preferences over family consumption profiles are isomorphic. In the Berge-Zhukovskii case,

$$A_i = N \setminus \{i\},$$

reference groups have the same cardinality and equality of their aggregate endowments is equivalent to

$$\omega_i = \omega_j.$$

Agents of the same A-type must additionally have isomorphic preferences on the corresponding $(n-1)$-agent sections. Finally, when

$$A_i = N \qquad \text{for every } i,$$

cardinality and aggregate resources are automatically the same, and type identity depends on whether preferences over the social consumption profile agree under a suitable permutation of positions.

5.2 The non-discriminatory fuzzy A-core

We can now adapt the solidarity criterion to the A-economic framework. In a fuzzy A-block, every agent with a positive participation rate is strictly improved. Therefore, an agent outside the support is a solidarity based interrupter whenever an agent of the same A-type belongs to the support. Absence of solidarity based interrupters is consequently equivalent to

$$\alpha_i > 0 \quad \Longrightarrow \quad T_A(i) \subseteq \operatorname{supp}(\alpha).$$

Thus, under the strict blocking convention used in this paper, the support of a non-discriminatory fuzzy coalition must be a union of complete A-types.

Definition 11 (Non-discriminatory fuzzy A-core)**.** *A fuzzy coalition*

$$\alpha = (\alpha_i)_{i \in N} \in [0,1]^n \setminus \{0\}$$

is A-non-discriminatory if

$$\alpha_i > 0 \quad \Longrightarrow \quad T_A(i) \subseteq \operatorname{supp}(\alpha).$$

A feasible allocation x belongs to the non-discriminatory fuzzy A-core if it cannot be A-improved upon by any A-non-discriminatory fuzzy coalition. We denote this set by

$$\mathcal{C}^A_{\mathrm{ND,f}}(E).$$

The non-discrimination requirement restricts the set of admissible blocking coalitions. Therefore,

$$\mathcal{C}^A_f(E) \subseteq \mathcal{C}^A_{\mathrm{ND,f}}(E).$$

With ordinary coalitions in a finite economy, this restriction may enlarge the core; see Hervés-Beloso and Moreno-García (2026). With Aubin coalitions, however, the full support property established for A-economies makes the solidarity restriction non-binding.

Recall that a fuzzy coalition α has full support if

$$\operatorname{supp}(\alpha) = N,$$

or, equivalently,

$$\alpha_i > 0 \qquad \text{for every } i \in N.$$

Graziano et al. (2025) show that, under continuity, strong monotonicity on the A_i-sections and strictly positive individual endowments, the Aubin A-core is unchanged when blocking is restricted to fuzzy coalitions with full support. Thus, whenever an allocation is Aubin blocked, it admits a block in which every agent participates with a strictly positive rate. Every such block is automatically A-non-discriminatory, since

$$T_A(i) \subseteq N = \operatorname{supp}(\alpha) \qquad \text{for every } i \in N.$$

Proposition 7 (Non-discrimination and Aubin blocking)**.** *Suppose that, for every $i \in N$,*

$$\omega_i \in \mathbb{R}^l_{++},$$

and that preferences are complete, transitive and continuous on $\mathbb{R}^{ln}_+$ and strongly monotone on the A_i-sections. Then

$$\mathcal{C}^A_{\mathrm{ND,f}}(E) = \mathcal{C}^A_f(E).$$

Proof. Since every A-non-discriminatory fuzzy coalition is a fuzzy coalition,

$$\mathcal{C}_f^A(E) \subseteq \mathcal{C}_{\mathrm{ND,f}}^A(E).$$

For the reverse inclusion, suppose that

$$x \notin \mathcal{C}_f^A(E).$$

By the full support characterization of the Aubin A-core, there exists a fuzzy coalition α which A-improves upon x and satisfies

$$\mathrm{supp}(\alpha) = N.$$

Hence,

$$T_A(i) \subseteq N = \mathrm{supp}(\alpha) \qquad \text{for every } i \in N.$$

The coalition is therefore A-non-discriminatory, and

$$x \notin \mathcal{C}_{\mathrm{ND,f}}^A(E).$$

This proves

$$\mathcal{C}_{\mathrm{ND,f}}^A(E) \subseteq \mathcal{C}_f^A(E).$$

□

The proposition highlights the role of divisible participation. The non-discrimination requirement rules out selective blocking within an A-type. The full support property implies, however, that such selective coalitions are not needed to determine the fuzzy A-core. The equality does not depend on the particular criteria used to define A-types, although the type notion introduced above retains independent economic content whenever comparisons or collective requirements depend on equivalent positions in the reference structure.

Remark 10. In the standard selfish benchmark,

$$A_i = \{i\} \qquad \text{for every } i \in N,$$

and each agent's preferences depend only on her own consumption. The A-type notion then reduces to the usual notion of type. The preceding result also clarifies the distinction between weak and strict dominance in the solidarity criterion.

Let $\mathcal{C}_f^w(E)$ and $\mathcal{C}_f^s(E)$ denote the Aubin cores defined through weak and strict blocking, respectively. Let $\mathcal{C}_{\mathrm{ND,f}}^w(E)$ denote the core based on weak Aubin blocking, where an agent outside the support is a solidarity based interrupter whenever a participant of the same type is strictly improved. Let $\mathcal{C}_{\mathrm{ND,f}}^s(E)$ denote the corresponding core under strict Aubin blocking. In the strict version, absence of interrupters is equivalent to requiring the support of the blocking coalition to be a union of complete type classes.

Under continuity and strong monotonicity, weak and strict Aubin blocking generate the same core:

$$\mathcal{C}_f^w(E) = \mathcal{C}_f^s(E).$$

A sufficiently small transfer from a strictly improved participant to the participants who are indifferent preserves the former's strict improvement and makes the latter strictly better off.

Moreover,

$$\mathcal{C}_f^w(E) \subseteq \mathcal{C}_{\mathrm{ND,f}}^w(E) \subseteq \mathcal{C}_{\mathrm{ND,f}}^s(E).$$

If individual endowments are strictly positive, the full support characterization gives the reverse inclusion. Every allocation outside $\mathcal{C}_f^s(E)$ admits a strict Aubin block with full support, and such a block is automatically non-discriminatory. Therefore,

$$\mathcal{C}_{\mathrm{ND,f}}^s(E) \subseteq \mathcal{C}_f^s(E).$$

It follows that

$$\mathcal{C}_f^w(E) = \mathcal{C}_f^s(E) = \mathcal{C}_{\mathrm{ND,f}}^w(E) = \mathcal{C}_{\mathrm{ND,f}}^s(E).$$

This contrasts with the case of ordinary coalitions studied by Hervés-Beloso and Moreno-García (2026). There, weak and strict solidarity blocking need not be interchangeable, because making an initially indifferent participant strictly better off may create new interrupters.

The same full support argument applies to the intermediate equity based and status based non-discriminatory cores introduced by those authors. A blocking coalition with full support has no excluded agents and therefore no discriminatory interrupters, irrespective of the criterion used to identify them. Consequently, under Aubin participation, all the corresponding non-discriminatory Aubin cores coincide with the ordinary Aubin core. Thus, distinctions that are substantive under ordinary coalition formation become non-binding when divisible participation and full support blocking are available.

Combining the previous result with fuzzy A-core equivalence yields a non-discriminatory characterization of A-equilibrium allocations.

Theorem 4 (Non-discriminatory characterization of A-equilibria)**.** *Suppose that, for every $i \in N$,*

$$\omega_i \in \mathbb{R}^l_{++},$$

that preferences are complete, transitive and continuous on $\mathbb{R}^{ln}_+$, strongly monotone and convex on the A_i-sections, and that the family $A = (A_i)_{i\in N}$ satisfies the balancedness condition. Then

$$\mathcal{W}(E) = \mathcal{C}_f^A(E) = \mathcal{C}_{\mathrm{ND,f}}^A(E).$$

6 Concluding remarks

The analysis suggests a few directions for further research. A first one concerns differentiated participation. Our definition of Aubin A-c-fairness requires the envious and the envied fuzzy coalitions to have disjoint supports. Basile et al. (2018) consider a more general participation structure in which an agent may belong simultaneously to both coalitions, with coalition specific participation rates subject to a no-overuse condition. Extending this approach to A-economies is less immediate, since participation generates weighted claims on the resources of the agents in the relevant reference groups. An appropriate no-overuse condition would therefore have to account for possible overlaps among reference groups.

A related question arises beyond convex preferences. Graziano et al. (2026), following Husseinov's approach, obtain core equivalence in non-convex A-economies by allowing simultaneous participation in a finite collection of fuzzy coalitions. Although the role of multiple participation is different in the two settings, the parallel suggests investigating whether a suitably generalized notion of Aubin A-c-fairness could provide a fairness characterization also in the non-convex case.

The results also point to a close connection between coalitional and individual fairness. In the present framework, Aubin A-c-fairness rules out justified A-envy, extending to reference-group comparisons the link between coalitional fairness and individual non-envy found in standard exchange economies. The relevant individual comparisons are now determined jointly by comparable reference-group opportunities and admissible identifications of the corresponding consumption profiles.

Finally, the inclusion of A-equilibrium allocations in the justified A-envy-free set links the existence of fair allocations to equilibrium existence. Under the conditions ensuring fuzzy A-core equivalence, equilibrium, cooperative stability, coalitional fairness and justified A-envy-freeness characterize the same allocations. It would be interesting to investigate which of these connections survive in classes of A-economies in which equilibrium or core equivalence cannot be established.

A A direct proof of the fairness core equivalence

We provide here a direct proof of the equivalence between Aubin A-c-fairness and the fuzzy A-core. The argument does not pass through equilibrium prices and makes explicit the role played by balancedness.

Recall that, for a fuzzy coalition α and a family of reference-group plans $z = (z_{(h)})_{h\in\mathrm{supp}(\alpha)}$,

$$T_A(\alpha, z) = \sum_{h\in N} \alpha_h \left(\sum_{i\in A_h} z_{hi} - \sum_{i\in A_h} \omega_i \right),$$

where terms corresponding to agents outside $\mathrm{supp}(\alpha)$ are understood to be zero.

We first isolate the net trade aggregation property implied by balancedness.

Proposition 8 (Balancedness and aggregate net trades)**.** *The family $A = (A_i)_{i\in N}$ is balanced with full support if and only if there exists*

$$b = (b_h)_{h\in N} \in (0, 1]^n$$

such that

$$T^A(b, x) = \sum_{i\in N} (x_i - \omega_i)$$

for every $x \in \mathbb{R}^{\ell n}_+$.

Proof. Notice first that every full support balancing vector belongs to $(0, 1]^n$. Moreover, for every $b \in [0, 1]^n$,

$$\begin{aligned} T_A(b, x) &= \sum_{h\in N} b_h \sum_{i\in A_h} (x_i - \omega_i) \\ &= \sum_{i\in N} \left(\sum_{h\in D_i} b_h \right) (x_i - \omega_i). \end{aligned}$$

Hence, if

$$\sum_{h\in D_i} b_h = 1 \qquad \text{for every } i \in N,$$

then

$$T_A(b, x) = \sum_{i\in N} (x_i - \omega_i)$$

for every x.

Conversely, suppose that the latter identity holds for every $x \in \mathbb{R}^{\ell n}_+$. Using the expression for $T^A(b, x)$ above, we have

$$T^A(b, x) = \sum_{i\in N} \left(\sum_{h\in D_i} b_h \right) (x_i - \omega_i) = \sum_{i\in N} (x_i - \omega_i)$$

for every $x \in \mathbb{R}^{\ell n}_+$.

Fix $i \in N$ and choose x such that $x_j = \omega_j$ for every $j \neq i$. Then

$$\left(\sum_{h\in D_i} b_h \right) (x_i - \omega_i) = x_i - \omega_i.$$

Choosing $x_i \neq \omega_i$ yields

$$\sum_{h\in D_i} b_h = 1.$$

Since i was arbitrary, this holds for every $i \in N$. Thus b is a full support balancing vector. □

Remark 11. The proposition clarifies the role of balancedness in aggregating net trades: the balancing weights ensure that each individual net trade enters the A-weighted aggregate net trade with total coefficient one. In the selfish benchmark $A_i = \{i\}$, this property is automatic; balancedness extends it to possibly overlapping reference structures.

We can now prove the fairness core equivalence directly.

Proposition 9 (Direct fairness core equivalence)**.** *Assume that the family $\mathcal{A} = (A_i)_{i\in N}$ is balanced with full support. Suppose that preferences are complete, transitive and continuous on $\mathbb{R}^{\ell n}_+$, strongly monotone and convex on the A_i-sections. Assume, in addition, that*

$$\sum_{i\in A_h} \omega_i \in \mathbb{R}^{\ell}_{++} \qquad \text{for every } h \in N.$$

Then

$$C^A_{\text{fair}}(E) = C^A_f(E).$$

Proof. The inclusion

$$C^A_{\text{fair}}(E) \subseteq C^A_f(E)$$

follows from Proposition 1. It remains to prove the reverse inclusion.

Let $x \in C^A_f(E)$ and suppose, by contradiction, that $x \notin C^A_{\text{fair}}(E)$. Then there exist fuzzy coalitions $\alpha, \beta \in [0,1]^n$, with $\alpha \neq 0$ and disjoint supports, and plans

$$z_{(h)} \in P_{A_h}(x), \qquad h \in \text{supp}(\alpha),$$

such that

$$T_A(\alpha, z) \leq T_A(\beta, x). \tag{A.1}$$

Let $b \in \mathbb{R}^n_{++}$ be a balancing vector. By the preceding proposition and feasibility of x,

$$T_A(b, x) = \sum_{i\in N}(x_i - \omega_i) \leq 0. \tag{A.2}$$

Since $\text{supp}(\alpha)$ is finite, continuity and $z_{(h)} \in P_{A_h}(x)$ allow us to choose a common $\varepsilon \in (0,1)$, sufficiently small, such that for every $h \in \text{supp}(\alpha)$,

$$z^\varepsilon(h) := (1-\varepsilon)z_{(h)} \in P_{A_h}(x). \tag{A.3}$$

Writing

$$\Omega_h := \sum_{i\in A_h} \omega_i,$$

we obtain

$$T_A(\alpha, z^\varepsilon) = (1-\varepsilon)T_A(\alpha, z) - \varepsilon \sum_{h\in N} \alpha_h \Omega_h.$$

Hence, by (A.1),

$$T_A(\alpha, z^\varepsilon) \leq (1-\varepsilon)T_A(\beta, x) - \varepsilon \sum_{h\in N} \alpha_h \Omega_h. \tag{A.4}$$

Choose $\rho, \lambda > 0$ sufficiently small that

$$\rho b_h < \frac{1}{2} \qquad \text{for every } h \in N,$$

$$\lambda(1-\varepsilon)\beta_h < \frac{\rho b_h}{2} \qquad \text{for every } h \in \text{supp}(\beta),$$

and

$$\lambda < \frac{1}{2}.$$

Define

$$r_h := \rho b_h - \lambda(1-\varepsilon)\beta_h, \qquad h \in N.$$

Then

$$r_h > \frac{\rho b_h}{2} > 0 \qquad \text{for every } h \in N.$$

Let

$$\gamma_h := \lambda\alpha_h + r_h.$$

Since $\alpha_h \leq 1$,

$$0 < \gamma_h \leq \lambda + \rho b_h < 1 \qquad \text{for every } h \in N.$$

Thus $\gamma \in (0,1)^n$ is a fuzzy coalition with full support.

By linearity,

$$T_A(r,x) = \rho T_A(b,x) - \lambda(1-\varepsilon)T_A(\beta,x).$$

Multiplying (A.4) by λ, adding the preceding equality and using (A.2), we obtain

$$\lambda T_A(\alpha, z^\varepsilon) + T_A(r,x) \leq -\lambda\varepsilon \sum_{h\in N} \alpha_h \Omega_h. \tag{A.5}$$

Set

$$d := \lambda\varepsilon \sum_{h\in N} \alpha_h \Omega_h.$$

By the strict positivity of the aggregate endowment of every reference group and since $\alpha \neq 0$,

$$d \in \mathbb{R}^\ell_{++}. \tag{A.6}$$

We now combine, for each agent, the improving participation $\lambda\alpha_h$ with the residual participation r_h. For $h \in \operatorname{supp}(\alpha)$, define

$$\bar{z}_{(h)} := \frac{\lambda\alpha_h}{\gamma_h} z^\varepsilon(h) + \frac{r_h}{\gamma_h} x_{A_h},$$

and, for $h \notin \operatorname{supp}(\alpha)$, let

$$\bar{z}_{(h)} := x_{A_h}.$$

Since preferences are convex on the A_h-section and $z^\varepsilon(h) \in P_{A_h}(x)$, we have

$$\left(\bar{z}_{(h)}, x_{N\setminus A_h}\right) \succeq_h x \qquad \text{for every } h \in N. \tag{A.7}$$

Moreover, by construction,

$$T_A(\gamma, \bar{z}) = \lambda T_A(\alpha, z^\varepsilon) + T_A(r,x) \leq -d. \tag{A.8}$$

It remains to turn the weak improvements in (A.7) into strict ones. For every $h \in N$, choose

$$q(h) \in \mathbb{R}^{\ell|A_h|}_{++}$$

and, for $\eta > 0$, set

$$\widehat{z}_{(h)} := \bar{z}_{(h)} + \eta q(h).$$

Strong monotonicity on the A_h-section and transitivity imply

$$\widehat{z}_{(h)} \in P_{A_h}(x) \qquad \text{for every } h \in N.$$

Furthermore,

$$T_A(\gamma, \widehat{z}) \leq -d + \eta c,$$

where

$$c := \sum_{h \in N} \gamma_h \sum_{i \in A_h} q_{hi} \in \mathbb{R}^{\ell}_{++}.$$

Since $d \gg 0$, choose $\eta > 0$ sufficiently small that

$$\eta c \leq d.$$

Then

$$T_A(\gamma, \widehat{z}) \leq 0.$$

Thus γ A-improves upon x, contradicting $x \in C_f^A(E)$. Therefore

$$C_f^A(E) \subseteq C_{\text{fair}}^A(E),$$

and the two sets coincide. □

Remark 12. The proof uses balancedness directly, without invoking supporting prices. The balancing vector provides the aggregation mechanism needed to combine a violation of Aubin A-c-fairness with the divisibility of Aubin participation and construct a fuzzy A-block. In this sense, balancedness compensates for the possible overlap of reference groups. Although the present argument is formulated in the finite agent setting considered here, its proof strategy is inspired by Basile et al. (2018), suitably adapted to the reference-group structure of an A-economy.

B A network interpretation of the reference structure

This appendix provides a network interpretation of the family of reference sets

$$A = (A_i)_{i \in N}$$

and of the balancedness condition used in the paper. The purpose is mainly interpretative. The reference structure can be represented as a directed network in which an edge from i to j indicates that the consumption of j enters the reference group of i.

It is useful to distinguish this reference relation from the welfare relations represented by individual preferences. The network records which consumption positions belong to each set A_i. It does not specify whether a change in the consumption of a referenced agent is beneficial or harmful to i, nor how strongly it affects i's welfare. These features remain encoded in the preference relation $\succeq_i$.

B.1 The reference network and balancedness

Let $G = (g_{ij})_{i,j \in N}$ be the incidence matrix of the reference structure, where

$$g_{ij} = \begin{cases} 1, & \text{if } j \in A_i, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, row i identifies the agents whose consumption positions belong to A_i. Equivalently, the directed reference network contains an edge from i to j whenever $j \in A_i$.

For each $j \in N$, define

$$D_j := \{i \in N : j \in A_i\}.$$

The set D_j contains the agents whose reference groups include j. Its cardinality is the number of reference sets in which agent j appears.

A vector

$$\beta = (\beta_i)_{i \in N} \in \mathbb{R}^N_{++}$$

is a full support balancing vector for the reference structure if

$$\sum_{i:j\in A_i} \beta_i = 1 \qquad \text{for every } j \in N.$$

In matrix form, this condition is

$$G^\top \beta = \mathbf{1}.$$

Accordingly, balancedness with full support requires strictly positive weights on the reference groups such that every individual position is covered with total weight one.

This condition depends on the incidence pattern of the family $(A_i)_{i\in N}$, rather than on symmetry of the reference relation or equality of the cardinalities of the reference groups. For example, if every agent belongs to exactly d reference sets, balancedness with full support follows by taking

$$\beta_i = \frac{1}{d} \qquad \text{for every } i \in N.$$

Constant incidence is sufficient, but it is not necessary.

Remark 13. Balancedness with full support is compatible with asymmetric reference relations. Consider three agents and let

$$A_1 = \{1, 2\}, \qquad A_2 = \{2, 3\}, \qquad A_3 = \{1, 3\}.$$

The reference network is balanced by

$$\beta_1 = \beta_2 = \beta_3 = \frac{1}{2},$$

since each agent belongs to exactly two reference sets.

Not every reference network admits a full support balancing vector. For example, if

$$A_1 = \{1, 2\}, \qquad A_2 = \{2\},$$

the balancing equations are

$$\beta_1 = 1, \qquad \beta_1 + \beta_2 = 1.$$

They imply $\beta_2 = 0$, contradicting the strict positivity required by full support. The distinction between reference structures that do or do not admit a full support balancing vector is central to the role of balancedness in core and equilibrium arguments; see also Graziano et al. (2026).

B.2 Hypergraph and decentralization interpretations

The same structure can be represented as a hypergraph whose vertices are the agents and whose hyperedges are the reference sets A_i. If χ_{A_i} denotes the incidence vector of A_i, the balancedness condition becomes

$$\sum_{i\in N} \beta_i \chi_{A_i} = \chi_N.$$

Thus, the reference sets, weighted by a strictly positive balancing vector, form a fractional exact cover of the population.

Balancedness also explains how individual budget equalities defined on overlapping reference groups can be aggregated without counting the same consumption position or endowment more than once. Suppose that prices are represented by $p \in \mathbb{R}^\ell_+$ and that, for each $i \in N$, the local budget equality is

$$p \cdot \sum_{j\in A_i} x_j = p \cdot \sum_{j\in A_i} \omega_j.$$

Multiplying the equality for agent i by β_i and summing over i gives

$$\sum_{i\in N} \beta_i p \cdot \sum_{j\in A_i} x_j = \sum_{i\in N} \beta_i p \cdot \sum_{j\in A_i} \omega_j.$$

Changing the order of summation yields

$$p \cdot \sum_{j\in N} \left(\sum_{i:j\in A_i} \beta_i \right) x_j = p \cdot \sum_{j\in N} \left(\sum_{i:j\in A_i} \beta_i \right) \omega_j.$$

Under balancedness,

$$\sum_{i:j\in A_i} \beta_i = 1 \qquad \text{for every } j \in N,$$

and therefore

$$p \cdot \sum_{j\in N} x_j = p \cdot \sum_{j\in N} \omega_j.$$

The balancing weights consequently reconcile the overlapping local descriptions with the aggregate resource relation. In this sense, balancedness is not merely a regularity condition on the reference network. It is the incidence property that allows the sectional choice problems associated with the sets A_i to be connected to a single aggregate resource constraint.

The network interpretation also clarifies the fairness notions used in the paper. Comparability concerns the structure of two reference groups and the possibility of identifying their positions, whereas the balancing condition concerns the system of reference groups as a whole. The former supports interpersonal fairness comparisons; the latter permits aggregation and decentralization. They are therefore distinct properties of the same underlying reference structure.

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